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A GENERALIZATION OF THE WANG SEQUENCE

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Abstract

The classical Wang sequence (for cohomology of fiber bundles over a sphere) is extended to a more generalized setting, given by gluing together two disc bundles over manifolds along their boundaries.

1. The result

Given an Euclidean vector bundle ξ over a manifold N let $D(\xi)$ and $S(\xi) = \partial D(\xi)$ be the unit disc bundle and the unit sphere bundle of ξ, respectively. That is,

$$D(\xi) = \{v \in \xi \mid \|v\|^2 \leqslant 1\}, S(\xi) = \{v \in \xi \mid \|v\|^2 = 1\}.$$

A smooth manifold M is said to have *focal genus* 2 if it admits a decomposition of the form

$$M = D(\xi_1) \cup_g D(\xi_2), \tag{1}$$

where ξ_t is an oriented Euclidean vector bundle on a manifold N_t, $t = 1, 2$, and where $g : S(\xi_2) \to S(\xi_1)$ is a diffeomorphism. We identify each N_t as a submanifold of M via the obvious embedding $i_t : N_t \overset{\sigma_t}{\to} D(\xi_t) \subset M$, and call it a *focal submanifold* of M, where σ_t is the zero section of the bundle ξ_t.

Let $p_t : S(\xi_t) \to N_t$ be the bundle projection, and put

$$g_t = \begin{cases} g : S(\xi_2) \to S(\xi_1) & \text{for } t = 1; \\ g^{-1} : S(\xi_1) \to S(\xi_2) & \text{for } t = 2. \end{cases}$$

For each $t \in \{1, 2\}$ let $\bar{t}$ be its complement in $\{1, 2\}$.

In this paper all homologies and cohomologies are over integer coefficients. For a manifold N write $1 \in H^0(N)$ for the multiplicative unit. Our main result is

Theorem 1.1. *Let M be a manifold having focal genus 2 and with focal submanifolds N_t, $t = 1, 2$. For each $t \in \{1, 2\}$ there is an exact sequences*

$$\cdots \to H^{r-1}(N_t) \overset{\theta_t}{\to} H^{r-\dim \xi_{\bar{t}}}(N_{\bar{t}}) \overset{\alpha_t}{\to} H^r(M) \overset{i_t^*}{\to} H^r(N_t) \overset{\theta_t}{\to} \cdots, \tag{2}$$

where

i) $\alpha_t(i_{\bar{t}}^*(x) \cup y) = x \cup \alpha_t(y)$ *for* $x \in H^*(M)$, $y \in H^*(N_{\bar{t}})$;

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ii) $\theta_t = \beta_{\bar{t}} \circ g_t^* \circ p_t^*$ *with* $\beta_t : H^*(S(\xi_t)) \to H^*(N_t)$ *the connecting homomorphism in the Gysin sequence of the fibration* p_t *[**8**, p.143].*

Moreover, if both N_1*,* N_2 *are oriented, and if* g *is orientation reversing, then with respect to the induced orientation on* M

iii) $\alpha_t(1)$ *is the Poincare dual of the cycle class* $i_{\bar{t}*}[N_{\bar{t}}] \in H_*(M)$*.*

The decomposition for θ_t in ii) is useful to derive its multiplicative properties. To explain this we examine the case where the oriented bundle ξ_2 has a decomposition $\xi_2 = \eta \oplus \varepsilon$ with ε the trivial $\mathbb{R}$–bundle on N_2. Let $s : N_2 \to S(\xi_2) = S(\eta \oplus \varepsilon)$ be the section defined by $s(b) = (0,1)$, and write $i_b : p_2^{-1}(b) \to S(\xi_2)$ for the inclusion of the fibre sphere $p_2^{-1}(b)$, $b \in N_2$. It is shown in [**3**] that there is a unique class $\alpha \in H^{\dim \xi_2 - 1}(S(\xi_2))$ such that $s^*(\alpha) = 0$ and $\langle i_b^*(\alpha), [p_2^{-1}(b)] \rangle = 1$, $b \in N_2$. Moreover,

Lemma 1.2. *([**3**, Lemma 4]). As an* $H^*(N_2)$*–module,* $H^*(S(\xi_2))$ *has the basis* $\{1, \alpha\}$ *subject to the single relation*

$$\alpha^2 + e(\eta)\alpha = 0,$$

where $e(\eta) \in H^*(N_2)$ *is the Euler class of the oriented bundle* η*.*

With respect to the $H^*(N_2)$–module structure on $H^*(S(\xi_2))$ the composed ring map

$$H^*(N_1) \overset{p_1^*}{\to} H^*(S(\xi_1)) \overset{g_1^*}{\to} H^*(S(\xi_2)) = H^*(N_2)\{1, \alpha\}$$

gives rise to two additive maps $\varphi, \psi : H^*(N_1) \to H^*(N_2)$ characterized by

$$g_1^* \circ p_1^*(x) = \varphi(x) + \psi(x)\alpha, x \in H^*(N_1). \tag{3}$$

Corollary 1.3. *Let* $M = D(\xi_1) \cup_g D(\xi_2)$ *be a manifold having focal genus 2 and with* $\xi_2 = \eta \oplus \varepsilon$*. Then*

$$\theta_1 = \psi : H^*(N_1) \to H^*(N_2).$$

In particular

$$\theta_1(xy) = \varphi(x)\theta_1(y) + (-1)^{(k-1)|y|}\theta_1(x)\varphi(y) + (-1)^{(k-1)|y| - k^2}\theta_1(x)\theta_1(y)e(\eta).$$

Remark 1.4. The author is grateful to Fuquan Fang and Zizhou Tang for informing him that a manifold $M = D(\xi_1) \cup_g D(\xi_2)$ with focal genus 2 coincides with the notion of the *double mapping cylinder* of the pairs of maps $p_1 : S(\xi_1) \to N_1$, $p_2 g_2 : S(\xi_1) \to N_2$ due to Grove and Halperin [**4**]. Letting F be a path component of the homotopy fiber of the inclusion $S(\xi_1) \subset M$, then a classification on F up to fundamental group, integral cohomology and rational homotopy type has been obtained by Grove and Halperin in [**4**, Theorem D].

The name "double mapping cylinder" could indicate the much more general construction of the adjoint space $M_f \cup M_g$, where M_f and M_g are the mapping cylinders of two maps $f : X \to Y$ and $g : X \to Z$ between topological spaces, and where the union is over the common subspace $X \times 1 \subset M_f$, M_g. With the term "focal genus 2" we hope to emphasize the geometric decomposition (1) on M given by two disc bundles over the focal submanifolds N_t, $t = 1, 2$.

2. Examples and applications

Manifolds having focal genus 2 are many. We choose to mention those that arise naturally in geometry.

2.1. Let M be a compact manifold on which a Lie group G acts with *cohomogeneity one* (i.e., the principal orbits G/H are of co-dimension 1). Then the orbit space M/G is either a circle or an interval. In the latter case the two endpoints of the interval correspond to nonprincipal orbits $N_1,\ N_2 \subset M$ and the manifold M is obtained by gluing the normal disc bundles over N_1 and N_2 along their common boundaries.

We refer to [**4, 5, 7**] for examples of manifolds admitting cohomogeneity one action of Lie groups.

2.2. Let M be a Riemannian manifold that admits a *transnormal function* $f : M \to \mathbb{R}$ (i.e., there is a function $b : \mathbb{R} \to \mathbb{R}$ so that $\|df\|^2 = b(f)$ [**10**]), and let $N_1,\ N_2 \subset M$ be the focal manifolds of f corresponding to the maximum and minimum of f. Theorem A in [**10**] concludes that, again, M is obtained by gluing the normal disc bundles over N_1 and N_2 along their common boundaries.

The idea of *transnormal function* is a natural generalization of the classical isoparametric functions, see [**1, 6, 10**].

2.3. Let $\pi : E \to M$ be a smooth fibration over a base manifold $M = D(\xi_1) \cup_g D(\xi_2)$ having focal genus 2. We set $E_t = \pi^{-1}(N_t)$ and put $\pi_t = \pi \mid E_t : E_t \to N_t$, $t \in \{1, 2\}$. Then a focal genus 2 structure on E with focal submanifolds E_t is seen from

$$E = D(\pi_1^*\xi_1) \cup_G D(\pi_2^*\xi_2),$$

where $\pi_t^*\xi_t$ is the pullback of ξ_t via π_t and where the diffeomorphism G is a bundle map over g.

2.4. Let $p : E \to S^n$ be a smooth fibration over the n–sphere S^n with fiber F. A focal genus 2 structure on E is afforded by

$$E = F \times D^n \cup_G F \times D^n$$

by **2.3**, where the clutching diffeomorphism $G : F \times S^{n-1} \to F \times S^{n-1}$ has the form $G(z, v) = (\beta(v)z, v)$ for some map $\beta : (S^{n-1}, s_0) \to (Diff(F), id)$, and where $Diff(F)$ is the group of diffeomorphism on F. It follows from $\beta(s_0) = id$ that the corresponding homomorphism φ in (3) acts identically on $H^*(F)$. It follows also from the triviality of ξ_2 that $e(\eta) = 0$. Thus Theorem 1.1, together with Corollary 1.3, implies that

Corollary 2.1. *Let $p : E \to S^n$ be a smooth fibration over the n–sphere S^n with fiber F. There is an exact sequence*

$$\cdots \to H^{r-1}(F) \xrightarrow{\theta} H^{r-n}(F) \xrightarrow{\alpha} H^r(E) \xrightarrow{i^*} H^r(F) \xrightarrow{\theta} \cdots \tag{4}$$

in which
i) $\alpha(i^*(x) \cup y) = x \cup \alpha(y)$;
ii) $\theta(xy) = \theta(x)y + (-1)^{(n-1)\dim x} x\theta(y)$.

In 1949 H.-C. Wang [**9**] studied fibrations $p : E \to S^n$ with the typical fiber F a finite simplicial complex, and obtained an exact sequence (in analogue to (4)) in the homology groups of total space E and the fiber F. The sequence (4) was later obtained as a consequence of the spectral sequence of a fibration and the property ii) was first observed by Leray [**2**, p.442]. Property i) is also practical and a proof is given in [**11**, p.336].

3. Proof of Theorem 1.1

For a smooth submanifold $i : N \hookrightarrow M$ in a Riemannian manifold M with oriented normal bundle ξ, identify the unit disc bundle $D(\xi)$ of ξ as a tubular neighborhood $t : D(\xi) \to M$ of the embedding. It induces the *Excision isomorphism*

$$t^* : H^*(M, M \setminus N) \overset{\cong}{\to} H^*(D(\xi), D(\xi) \setminus N)$$

and satisfies also $i = t \circ \sigma$, where $\sigma : N \to D(\xi)$ is the zero section. Since σ is a homotopy inverse of the bundle projection $p : D(\xi) \to N$ we have

$$t^* = p^* \circ i^* : H^*(M) \to H^*(D(\xi)).$$

Let $U \in H^k(D(\xi); D(\xi) \setminus N)$ be the Thom class of the oriented bundle ξ, $k = \dim_{\mathbb{R}} \xi$. Cup product with U yields the *Thom isomorphism* [**8**, p.97]

$$T : H^*(N) \underset{\cong}{\overset{p^*}{\to}} H^*(D(\xi)) \underset{\cong}{\overset{\cup U}{\to}} H^*(D(\xi); D(\xi) \setminus N), \quad T(\beta) = p^*(\beta) \cup U.$$

Write ψ for the composed isomorphism

$$\psi = (t^*)^{-1} \circ T : H^*(N) \overset{\cong}{\to} H^*(M, M \setminus N)$$

and let $s : (M, \emptyset) \to (M, M \setminus N)$ be the obvious inclusion of the topological pairs.

Lemma 3.1. *For any $x \in H^*(M)$, $y \in H^*(N)$ one has in $H^*(M)$ the relation*

$$x \cup s^*\psi(y) = s^*\psi(i^*(x) \cup y).$$

Proof. Consider the commutative diagram induced by t

$$\begin{array}{ccc} H^*(M) \times H^*(M, M \setminus N) & \overset{t^* \times t^*}{\to} & H^*(D(\xi)) \times H^*(D(\xi); D(\xi) \setminus N) \\ \cup \downarrow & & \downarrow \cup \\ H^*(M, M \setminus N) & \underset{\cong}{\overset{t^*}{\to}} & H^*(D(\xi); D(\xi) \setminus N) \end{array},$$

where the vertical maps are the cup products. It follows that for any $x \in H^*(M)$, $y \in H^*(N)$ one has in $H^*(D(\xi); D(\xi) \setminus N)$ that

$$\begin{aligned} t^*(x \cup \psi(y)) &= t^*(x) \cup t^*(\psi(y)) \\ &= (t^*(x) \cup p^*(y)) \cup U (\text{by the definition of } \psi) \\ &= p^*(i^*(x) \cup y) \cup U (\text{since} t^* = p^* \circ i^*) \\ &= T(i^*(x) \cup y) (\text{by the definition of } T). \end{aligned}$$

Applying $s^* \circ t^{*-1}$ to the both ends of the equalities verifies Lemma 3.1. □

Proof of Theorem 1.1. Let $M = D(\xi_1) \cup_g D(\xi_2)$ be a manifold having focal genus 2 and with focal submanifolds $i_t : N_t \hookrightarrow M$, $t = 1, 2$. Clearly, a tubular neighborhood of i_t can be taken as $D(\xi_t) \subset M$.

Assume that $t = 1$ (consequently, $\bar{t} = 2$) and that $\dim \xi_2 = k$. Let

$$j : (M, \emptyset) \to (M, N_1), h : (M, N_1) \to (M, M\backslash N_2)$$

and $s : (M, \emptyset) \to (M, M\backslash N_2)$ be the obvious inclusions of the topological pairs. Since N_1 is a strong deformation retract of $M\backslash N_2$ the map h induces an isomorphism on cohomology.

Define the map α_1 by the commutativity of the diagram

$$\begin{array}{ccc} H^r(M, N_1) & \xrightarrow{j^*} & H^r(M) \\ h^* \uparrow\cong & s^* \nearrow & \uparrow \alpha_1 \\ H^r(M, M\backslash N_2) & \underset{\cong}{\xleftarrow{\psi}} & H^{r-k}(N_2) \end{array}, \tag{5}$$

where the commutativity of the upper triangle follows from $s = h \circ j$. In view of the isomorphism $h^* \circ \psi$ we may replace in the cohomology exact sequence

$$\cdots \to H^{r-1}(N_1) \xrightarrow{\partial} H^r(M, N_1) \xrightarrow{j^*} H^r(M) \xrightarrow{i_1^*} H^r(N_1) \to \cdots$$

of the pair (M, N_1) the relative cohomology group $H^r(M, N_1)$ by $H^{r-k}(N_2)$ to obtain

$$\cdots \to H^{r-1}(N_1) \xrightarrow{\theta_1} H^{r-k}(N_2) \xrightarrow{\alpha_1} H^r(M) \xrightarrow{i_1^*} H^r(N_1) \to \cdots,$$

where $\theta_1 = \psi^{-1} \circ (h^*)^{-1} \circ \partial$. This establishes the exactness of the sequence (2).

Property iii) of the Theorem is essentially shown in the proof of [**8**, Theorem 11.3], and property i) comes directly from Lemma 3.1.

It remains to establish the decomposition ii) for the additive map $\theta_1 = \psi^{-1} \circ (h^*)^{-1} \circ \partial$ in (2). The inclusion $D(\xi_2) \hookrightarrow M$ gives rise to a relative homeomorphism $f : (D(\xi_2), S(\xi_2)) \hookrightarrow (M, D(\xi_1))$ which satisfies $f \mid S(\xi_2) = g_1$ and induces also the (vertical) exact ladder

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{r-1}(N_1) & \underset{\cong}{\xleftarrow{\sigma_1^*}} & H^{r-1}(D(\xi_1)) & \xrightarrow{g_1^*} & H^{r-1}(S(\xi_2)) & & \\ \partial \downarrow & & \partial_1 \downarrow & & \partial_2 \downarrow & \beta_2 \searrow & \\ H^r(M, N_1) & \underset{\cong}{\xleftarrow{h'^*}} & H^r(M, D(\xi_1)) & \underset{\cong}{\xrightarrow{f^*}} & H^r(D(\xi_2), S(\xi_2)) & \underset{\cong}{\xleftarrow{T}} & H^{r-k}(N_2) \\ \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \end{array},$$

where $h' : (M, N_1) \to (M, D(\xi_1))$ is the obvious inclusion with $h' \mid N_1 = \sigma_1$, and where β_2 is the connecting homomorphism in the Gysin exact sequence of the sphere bundle $S(\xi_2) \to N_2$. Property ii) follows from commutativity of the above diagrams, as well as the following obvious relations

$$\begin{aligned} & g_1^* \circ (\sigma_1^*)^{-1} = g_1^* \circ p_1^*; \\ & \theta_1 = T^{-1} \circ f^* \circ (h'^*)^{-1} \circ \partial; \\ & \beta_2 = T^{-1} \circ \partial_2, \end{aligned}$$

where T is the Thom isomorphism of the bundle ξ_2. These complete the proof.

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