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ON THE EXISTENCE OF A v_2^{32}-SELF MAP ON $M(1,4)$ AT THE PRIME 2

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(*communicated by Donald M. Davis*)

Abstract

Let $M(1)$ be the mod 2 Moore spectrum. J.F. Adams proved that $M(1)$ admits a minimal v_1-self map $v_1^4 \colon \Sigma^8 M(1) \to M(1)$. Let $M(1,4)$ be the cofiber of this self-map. The purpose of this paper is to prove that $M(1,4)$ admits a minimal v_2-self map of the form $v_2^{32} \colon \Sigma^{192} M(1,4) \to M(1,4)$. The existence of this map implies the existence of many 192-periodic families of elements in the stable homotopy groups of spheres.

Contents

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1. Introduction

Fix a prime p. The p-component of the stable homotopy groups of spheres admits a filtration called the chromatic filtration. Elements in the nth layer of this filtration fit into infinite v_n-periodic families. Theoretically, this process is well understood, thanks to the Nilpotence and Periodicity Theorems of Devinatz, Hopkins, and Smith [**HS98, DHS88**].

It is difficult in practice, however, to explicitly identify v_n-periodic elements, and to determine their periods. One useful technique is to inductively form cofiber sequences:

$$S \xrightarrow{p^{i_0}} S \to M(i_0),$$
$$\Sigma^{2i_1(p-1)} M(i_0) \xrightarrow{v_1^{i_1}} M(i_0) \to M(i_0, i_1),$$
$$\vdots$$
$$\Sigma^{2i_n(p^n-1)} M(i_0, \dots, i_{n-1}) \xrightarrow{v_n^{i_n}} M(i_0, \dots, i_{n-1}) \to M(i_0, \dots, i_n).$$

The maps v_k^i are v_k-self maps. The Periodicity Theorem guarantees their existence for large i. The reader is warned that there are potentially many non-homotopic v_k^i-self maps, so the homotopy types of the spectra $M(i_0, \dots, i_n)$ are not determined merely from the sequence $(i_0, \dots, i_n)$.

It is challenging to determine the minimal sequence $(i_0, i_1, \dots, i_n)$. This minimal sequence determines the periods of the primary constituents of the v_n-periodic families in the stable homotopy groups of spheres. We refer the reader to [**Rav86**, Ch. 5.5], [**Rav92**], and [**Beh07**] for a more detailed discussion.

We give a brief synopsis of what is known concerning the minimal sequence of integers $(i_0, \dots, i_n)$ so that the spectrum $M(i_0, \dots, i_n)$ exists at a given prime p. For $p \geqslant 3$, it is known that the complex $M(1,1)$ is minimal [**Ada66**], for $p \geqslant 5$, the complex $M(1,1,1)$ is minimal [**Smi70**], and for $p \geqslant 7$, the complex $M(1,1,1,1)$ is minimal [**Tod71**]. For $p = 2$, the complex $M(1,4)$ is minimal [**Ada66**], and for $p = 3$, the complex $M(1,1,9)$ is minimal [**BP04**].

In [**DM81**], it was argued that the complex $M(1,4,8)$ is minimal at the prime 2, i.e., that there is a v_2-self map:

$$\Sigma^{48} M(1,4) \xrightarrow{v_2^8} M(1,4).$$

The result is incorrect: the image of v_2^8 in the Adams-Novikov spectral sequence for *tmf* is not a permanent cycle [**HM, Bau08**]. In fact the first multiple of v_2 which is a permanent cycle in this spectral sequence is v_2^{32}. The purpose of this paper is to prove the following theorem:

Theorem 1.1. *There is a v_2^{32}-self map*

$$v \colon \Sigma^{192} M(1,4) \to M(1,4).$$

Corollary 1.2. *At the prime* 2*, the complex $M(1,4,32)$ is minimal.*

Remark 1.3. A v_2^{32}-self map is, by definition, a map v whose induced map

$$v_*\colon K(2)_*M(1,4) \to K(2)_*M(1,4)$$

is given by multiplication by v_2^{32}. In particular, the map v, and all of its iterates, must be essential. Since there is a map of ring spectra

$$tmf \to K(2)$$

under which the periodicity generator $v_2^{32} \in \pi_{192}(tmf_2)$ maps to $v_2^{32} \in \pi_{192}K(2)$, to prove Theorem 1.1, it suffices to prove that there exists a self-map v such that

$$v_*\colon tmf_*M(1,4) \to tmf_*M(1,4)$$

is given by multiplication by v_2^{32}.

Remark 1.4. The fourth author reports that methods similar to those described in this paper show that the spectra A_1 and $M(2,4)$ also admit v_2^{32}-self maps. Here, A_1 is a spectrum whose cohomology is a free module of rank 1 over the subalgebra $A(1)$ of the Steenrod algebra (see [**DM81**]).

The self-map of Theorem 1.1 produces many v_2^{32}-periodic infinite families of elements in the stable homotopy groups of spheres. These families are discussed in detail in [**HM**]. In fact, all of the results of [**DM81**] and [**Mah81**] concerning v_2-periodic families are valid with v_2^8 replaced by v_2^{32}.

Organization of the paper

In Section 2, we reduce Theorem 1.1 to showing that there exists a homotopy element

$$v \in \pi_{192}(M(1,4) \wedge DM(1))$$

with Hurewitz image $v_2^{32} \in tmf_{192}(M(1,4) \wedge DM(1))$. Here, $DM(1)$ is the Spanier-Whitehead dual of the spectrum $M(1)$.

In Section 3 we construct modified Adams spectral sequences (MASSs) of the form

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, H(1,4) \otimes DH(1,4)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)), \tag{1.5}$$

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, H(1,4) \otimes H_*(X)) \Rightarrow \pi_{t-s}(M(1,4) \wedge X), \tag{1.6}$$

where A_* is the dual Steenrod algebra, $H(1,4)$ and $DH(1,4)$ are objects in the derived category of A_*-comodules, and Ext_{A_*} is a group of homomorphisms in the derived category. We show that (1.5) is a spectral sequence of algebras, and that (1.6) is a spectral sequence of modules over (1.5).

In Section 4 we prove that there exists an element

$$v_2^8 \in \mathrm{Ext}^{8,56}_{A_*}(\mathbb{F}_2, H(1,4) \otimes DH(1,4)).$$

In Section 5, we give a general overview of the theory of generalized Brown-Gitler A_*-comodules $M_i(j)$. We describe a spectral sequence which computes Ext_{A_*} in terms of $\mathrm{Ext}_{A(i)}$ of tensor products of these comodules. The case of interest is where $i = 2$, and the spectral sequence is an algebraic version of the tmf-resolution.

In Section 6 we compute

$$\mathrm{Ext}^{*,*}_{A(2)_*}(H(1,4) \otimes M_2(1)^{\otimes k})$$

for $k \leqslant 3$.

In Section 7 we establish vanishing lines for the Ext groups appearing in the algebraic *tmf*-resolution. These vanishing lines imply that the only targets of a potential differential supported by v_2^{32} are detected in the algebraic *tmf*-resolution by the Ext groups computed in Section 6.

In Section 8, we completely compute the MASS for $tmf \wedge M(1,4)$.

In Section 9, we show that in the MASS for $M(1,4) \wedge DM(1,4)$, the differential $d_2(v_2^8)$ is central. This allows us to deduce that $d_2(v_2^{16}) = 0$. We then argue that the differential $d_3(v_2^{16})$ is central, which implies that $d_3(v_2^{32}) = 0$. We just need to show that v_2^{32} is a permanent cycle.

In Section 10, we show that $\bar{\kappa}^6$ is killed in the E_3-term of the MASS for $M(1,4) \wedge DM(1,4)$.

In Section 11, we prove the main theorem. We identify possible targets of $d_r(v_2^{32})$ in the MASS for $M(1,4) \wedge DM(1)$ using the results of Sections 6 and 7, and then eliminate these possibilities using the differentials computed in Sections 8 and 10.

Conventions

In this paper we shall always be implicitly working in the stable homotopy category localized at the prime 2. All homology and cohomology groups in this paper are implicitly taken with $\mathbb{F}_2$ coefficients.

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2. Generalized Moore spectra

Let $M(1)$ be the mod 2 Moore spectrum. There are many v_1-self-maps

$$\Sigma^8 M(1) \to M(1),$$

however, low dimensional calculations indicate that there is precisely one with Adams filtration 4. We shall call this map v_1^4, and its cofiber will be denoted $M(1,4)$.

It is useful to regard the desired self-map v of Theorem 1.1 as an element of the homotopy group $\pi_{192}(M(1,4) \wedge DM(1,4))$. The proof of the theorem is simplified by the following splitting result:

Proposition 2.1 (Davis-Mahowald [**DM81**, Lem. 3.2])**.** *The projection*

$$M(1,4) \wedge DM(1,4) \to M(1,4) \wedge DM(1)$$

is a split surjection.

Corollary 2.2. *An element $x \in \pi_k(M(1,4))$ extends to a self-map*

$$\widetilde{x} \colon \Sigma^k M(1,4) \to M(1,4)$$

if and only if $2x = 0$.

To prove Theorem 1.1, it therefore suffices to construct an appropriate element $v' \in \pi_{192}(M(1,4) \wedge DM(1))$.

3. Modified Adams spectral sequences

For a graded Hopf algebra Γ over a field k, let $\mathcal{D}_\Gamma$ denote the derived category of Γ-comodules. For objects M and N of $\mathcal{D}_\Gamma$, we define groups

$$\mathrm{Ext}_\Gamma^{s,t}(M,N) = \mathcal{D}_\Gamma(\Sigma^t M, N[s])$$

as a group of maps in the derived category. Here $\Sigma^t M$ denotes the t-fold shift with respect to the internal grading of M, and $N[s]$ denotes the s-fold shift with respect to the triangulated structure of $\mathcal{D}_\Gamma$. This reduces to the usual definition of Ext_Γ when M and N are Γ-comodules. We shall frequently use the abbreviation

$$\mathrm{Ext}_\Gamma^{*,*}(M) := \mathrm{Ext}_\Gamma^{*,*}(k, M).$$

For a left Γ-comodule M and a right Γ-comodule N, let $C^*(N,\Gamma,M)$ denote the reduced cobar complex with

$$C^s(N,\Gamma,M) = N \otimes \overline{\Gamma}^{\otimes s} \otimes M.$$

Here, $\overline{\Gamma}$ is the cokernel of the unit $k \to \Gamma$. Then $C^*(\Gamma,\Gamma,M)$ is an injective resolution for M in the category of Γ-comodules, and

$$\mathrm{Ext}_\Gamma^{s,t}(M) = H^s(C^*(k,\Gamma,M))_t.$$

We refer the reader to [**Rav86**, Appendix 1] for details.

Let A_* denote the dual Steenrod algebra. Let $H(1) = H_*(M(1))$ be the homology of the mod 2 Moore spectrum. There is a triangle in $\mathcal{D}_{A_*}$:

$$\Sigma\mathbb{F}_2[-1] \xrightarrow{h_0} \mathbb{F}_2 \to H(1) \to \Sigma\mathbb{F}_2. \tag{3.1}$$

Let $v_1^4 \colon \Sigma^{12} H(1)[-4] \to H(1)$ be the unique non-zero element of $\mathrm{Ext}_{A_*}^{4,12}(H(1),H(1))$, which detects the v_1-self map of $M(1)$ in Adams filtration 4. Let $H(1,4)$ denote the cofiber

$$\Sigma^{12} H(1)[-4] \xrightarrow{v_1^4} H(1) \to H(1,4) \to \Sigma^{12} H(1)[-3].$$

Let

$$DM(1,4) = F(M(1,4), S) \simeq \Sigma^{-10} M(1,4)$$

denote the Spanier-Whitehead dual of $M(1,4)$, and let

$$DH(1,4) = \mathrm{Hom}_{\mathbb{F}_2}(H(1,4), \mathbb{F}_2) \cong \Sigma^{-13} H(1,4)[3]$$

denote the corresponding object in $\mathcal{D}_{A_*}$.

Proposition 3.2. *Let X be a finite complex. Then there are* modified Adams spectral sequences *(MASSs) of the form:*

$$E_2^{s,t}(M(1,4) \wedge X) = \mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes H_*(X)) \Rightarrow \pi_{t-s}(M(1,4) \wedge X),$$

$$E_2^{s,t}(M(1,4) \wedge DM(1,4)) = \mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1,4)) \\ \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)).$$

Proof. Consider the canonical Adams resolution of $M(1)$:

$$\begin{tikzcd} M(1) \arrow[r, equal] & M(1)_0 \arrow[d] & M(1)_1 \arrow[l] \arrow[d] & M(1)_2 \arrow[l] \arrow[d] & \cdots , \arrow[l] \\ & K(1)_0 & K(1)_1 & K(1)_2 \end{tikzcd}$$

where

$$M(1)_i = \overline{H}^{\wedge i} \wedge M(1),$$
$$K(1)_i = H \wedge \overline{H}^{\wedge i} \wedge M(1).$$

Here H denotes the Eilenberg-MacLane spectrum $H\mathbb{F}_2$, and $\overline{H}$ denotes the fiber of the unit $S \to H$. Since the self-map $v_1^4 \colon \Sigma^8 M(1) \to M(1)$ has Adams filtration 4, there exists a lift:

$$\begin{tikzcd} & M(1)_4 \arrow[d] \\ \Sigma^8 M(1) \arrow[ur, dotted, "\widetilde{v_1^4}"] \arrow[r, "v_1^4"'] & M(1). \end{tikzcd}$$

The lift $\widetilde{v_1^4}$ induces a map of Adams resolutions:

$$\begin{tikzcd} \Sigma^8 M(1)_0 \arrow[r, equal] \arrow[d, "v_1^4"'] & \cdots \arrow[r, equal] & \Sigma^8 M(1)_0 \arrow[d, "(\widetilde{v_1^4})_0"'] & \Sigma^8 M(1)_1 \arrow[l] \arrow[d, "(\widetilde{v_1^4})_1"'] & \cdots \arrow[l] \\ M(1)_0 & \cdots \arrow[l] & M(1)_4 \arrow[l] & M(1)_5 \arrow[l] & \cdots , \arrow[l] \end{tikzcd} \tag{3.3}$$

where the maps $(\widetilde{v_1^4})_i$ are given by

$$(\widetilde{v_1^4})_i \colon \Sigma^8 M(1)_i = \Sigma^8 \overline{H}^{\wedge i} \wedge M(1) \xrightarrow{1 \wedge \widetilde{v_1^4}} \overline{H}^{\wedge i} \wedge \overline{H}^{\wedge 4} \wedge M(1) = M(1)_{i+4}.$$

The mapping cones of the vertical maps of (3.3)

$$\Sigma^8 M(1)_{i-4} \xrightarrow{(\widetilde{v_1^4})_i} M(1)_i \to M(1,4)_i$$

form a resolution:

$$\begin{tikzcd} M(1,4) \arrow[r, equal] & M(1,4)_0 \arrow[d] & M(1,4)_1 \arrow[l] \arrow[d] & M(1,4)_2 \arrow[l] \arrow[d] & \cdots . \arrow[l] \\ & K(1,4)_0 & K(1,4)_1 & K(1,4)_2 & \end{tikzcd}$$

Smashing this resolution with X, we obtain a spectral sequence

$$E_1^{s,t}(M(1,4)\wedge X) = \pi_{t-s}(K(1,4)_s \wedge X) \Rightarrow \pi_{t-s}(M(1,4)\wedge X). \tag{3.4}$$

By the 3×3 Lemma, the cofibers $K(1,4)_i$ fit into cofiber sequences

$$\Sigma^8 K(1)_{i-4} \xrightarrow{(\overline{v_1^4})_i} K(1)_i \to K(1,4)_i. \tag{3.5}$$

Here we take $K(1)_{i-4} = *$ if $i < 4$, and $(\overline{v_1^4})_i$ is the map induced by smashing $(\widetilde{v_1^4})_i$ with H.

Using the A_*-comodule structure of $H(1)$ together with the fact that the composite

$$S^8 \hookrightarrow \Sigma^8 M(1) \xrightarrow{v_1^4} M(1)$$

has Adams filtration 4, one may easily check that the map

$$\Sigma^{12} H(1) = \Sigma^{12}\pi_* K(1)_0 \xrightarrow{(\overline{v_1^4})_0} \Sigma^4 \pi_* K(1)_4 = C^4(\mathbb{F}_2, A_*, H(1))$$

is injective. It follows that the maps

$$\Sigma^{12} C^{i-4}(\mathbb{F}_2, A_*, H(1)) = \Sigma^{8+i}\pi_* K(1)_{i-4} \xrightarrow{(\overline{v_1^4})_i} \Sigma^i \pi_* K(1)_i = C^i(\mathbb{F}_2, A_*, H(1))$$

are injective for all i. We conclude that the cofiber sequences (3.5) give rise to short exact sequences

$$\begin{aligned} 0 \to \Sigma^{12} C^{i-4}(\mathbb{F}_2, A_*, H(1)\otimes H_*X) \xrightarrow{(\overline{v_1^4})_i} & C^i(\mathbb{F}_2, A_*, H(1)\otimes H_*X) \\ & \to \Sigma^i \pi_*(K(1,4)_i \wedge X) \to 0. \end{aligned}$$

In the derived category D_{A_*} we have a map of triangles:

$$\begin{tikzcd} \Sigma^{12} C^{*-4}(A_*, A_*, H(1)) \arrow[r, "(\overline{v_1^4})_*"] \arrow[d, "\simeq"] & C^*(A_*, A_*, H(1)) \arrow[r] \arrow[d, "\simeq"] & Q(1,4)_* \arrow[d, "\simeq"] \\ \Sigma^{12} H(1)[-4] \arrow[r, "v_1^4"'] & H(1) \arrow[r] & H(1,4), \end{tikzcd}$$

where $Q(1,4)_i$ is the cokernel of the inclusion

$$\Sigma^{12} C^{i-4}(A_*, A_*, H(1)) \xrightarrow{(\overline{v_1^4})_i} C^i(A_*, A_*, H(1)).$$

Since we have isomorphisms of cochain complexes

$$\pi_*(K(1,4)_* \wedge X) \cong \mathrm{Hom}_{A_*}(\mathbb{F}_2, Q(1,4)_* \otimes H_*X),$$

we deduce that the E_2-term of the spectral sequence (3.4) is given by

$$E_2^{s,t}(M(1,4)\wedge X) = \mathrm{Ext}^{s,t}_{A_*}(H(1,4)\otimes H_*X).$$

Consider the Adams resolution for the Spanier-Whitehead dual $DM(1)$:

$$\begin{tikzcd}
DM(1) \arrow[r, equal] & DM(1)_0 \arrow[d] & DM(1)_1 \arrow[l] \arrow[d] & DM(1)_2 \arrow[l] \arrow[d] & \cdots \arrow[l], \\
 & KD(1)_0 & KD(1)_1 & KD(1)_2 &
\end{tikzcd}$$

where

$$\begin{aligned}
DM(1)_i &= F(M(1), \overline{H}^{\wedge i}), \\
KD(1)_i &= F(M(1), H\wedge \overline{H}^{\wedge i}).
\end{aligned}$$

Define maps $(D\widetilde{v_1^4})_i$ to be the composites

$$\begin{aligned}
(D\widetilde{v_1^4})_i\colon DM(1)_i = F(M(1), \overline{H}^{\wedge i}) &\xrightarrow{u} F(\overline{H}^{\wedge 4}\wedge M(1), \overline{H}^{\wedge i+4}) \\
&\xrightarrow{(\widetilde{v_1^4})^*} F(\Sigma^8 M(1), \overline{H}^{\wedge i+4}) = \Sigma^{-8}DM(1)_{i+4},
\end{aligned}$$

where u is the unit of the adjunction. These maps assemble to give a map of Adams resolutions:

$$\begin{tikzcd}
DM(1)_0 \arrow[r, equal] \arrow[d, "Dv_1^4"'] & \cdots \arrow[r, equal] & DM(1)_0 \arrow[d, "(D\widetilde{v_1^4})_0"'] & M(1)_1 \arrow[l] \arrow[d, "(D\widetilde{v_1^4})_1"'] & \cdots \arrow[l] \\
\Sigma^{-8}DM(1)_0 & \cdots \arrow[l] & \Sigma^{-8}DM(1)_4 \arrow[l] & \Sigma^{-8}DM(1)_5 \arrow[l] & \cdots \arrow[l].
\end{tikzcd}$$

Letting $DM(1,4)_i$ denote the homotopy fibers of the vertical maps of (3):

$$DM(1,4)_i \to DM(1)_i \xrightarrow{(D\widetilde{v_1^4})_i} \Sigma^{-8}DM(1)_{i+4},$$

we obtain a modified Adams resolution of $DM(1,4)$:

$$\begin{tikzcd}
DM(1,4) \arrow[r, equal] & DM(1,4)_{-4} \arrow[d] & DM(1,4)_{-3} \arrow[l] \arrow[d] & DM(1,4)_{-2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & KD(1,4)_{-4} & KD(1,4)_{-3} & KD(1,4)_{-2} &
\end{tikzcd}$$

and a corresponding modified Adams spectral sequence

$$E_2^{s,t}(DM(1,4)) = \mathrm{Ext}^{s,t}_{A_*}(DH(1,4)) \Rightarrow \pi_{t-s}(DM(1,4)).$$

By taking iterated mapping cylinders, we may assume that the maps

$$\begin{aligned}
M(1,4)_{i+1} &\to M(1,4)_i, \\
DM(1,4)_{i+1} &\to DM(1,4)_i
\end{aligned}$$

are inclusions of subcomplexes. Taking the smash product of resolutions [**BMMS86**, Ch. IV, Def. 4.2]

$$\{(M(1,4) \wedge DM(1,4))_i\} = \{M(1,4)_i\} \wedge \{DM(1,4)_i\}$$

gives the spectral sequence

$$\mathrm{Ext}^{s,t}_{A_*}(H(1,4) \otimes DH(1,4)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1,4)). \qquad \square$$

Proposition 3.6. *The modified Adams spectral sequence $\{E_r(M(1,4) \wedge DM(1,4))\}$ is a spectral sequence of algebras, and the spectral sequence $\{E_r(M(1,4) \wedge X)\}$ is a spectral sequence of modules over $\{E_r(M(1,4) \wedge DM(1,4))\}$.*

Proof. The canonical Adams resolution for the sphere spectrum is given by $\{\overline{H}^{\wedge i}\}$. The canonical evaluation maps

$$\begin{aligned} &DM(1,4)_i \wedge M(1,4)_j \\ &\quad = (DM(1)_i \times_{(D\widetilde{v_1^4})_i} (\Sigma^{-8} DM(1)_{i+4})^I) \wedge (M(1)_j \cup_{(\widetilde{v_1^4})_j} C\Sigma^8 M(1)_{j-4} \to \overline{H}^{\wedge i+j} \end{aligned}$$

induce maps of modified Adams resolutions

$$\begin{aligned} \{(M(1,4)\wedge & DM(1,4))_i\} \wedge \{(M(1,4) \wedge DM(1,4))_i\} \\ &= \{M(1,4)_i\} \wedge \{(DM(1,4) \wedge M(1,4))_i\} \wedge \{DM(1,4)_i\} \\ &\to \{M(1,4)_i\} \wedge \{\overline{H}^{\wedge i}\} \wedge \{DM(1,4)_i\} \\ &= \{(M(1,4) \wedge DM(1,4))_i\}, \end{aligned}$$

$$\begin{aligned} \{(M(1,4)\wedge & DM(1,4))_i\} \wedge \{M(1,4)_i \wedge X\} \\ &= \{M(1,4)_i\} \wedge \{(DM(1,4) \wedge M(1,4))_i\} \wedge \{\overline{H}^{\wedge i} \wedge X\} \\ &\to \{M(1,4)_i\} \wedge \{\overline{H}^{\wedge i}\} \wedge \{\overline{H}^{\wedge i} \wedge X\} \\ &= \{M(1,4)_i \wedge X\}. \end{aligned}$$

These maps induce the desired pairings on the corresponding MASSs. $\square$

4. v_2^8-periodicity in Ext_{A_*}

A similar (but easier) argument to Proposition 2.1 proves the following lemma:

Lemma 4.1. *The morphism*

$$H(1,4) \wedge DH(1,4) \to H(1,4) \wedge DH(1)$$

is a split surjection.

Corollary 4.2. *An element $x \in \mathrm{Ext}^{s,t}_{A_*}(H(1,4))$ lifts to give an element*

$$\widetilde{x} \in \mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1,4))$$

if an only if $h_0 x = 0$.

A computation of $\mathrm{Ext}_{A(2)_*}(H(1,4))$ appears in Figure 8.1. Note that it is v_2^8-periodic.

Proposition 4.3. *There exists an element*

$$\widetilde{v_2^8} \in \mathrm{Ext}_{A_*}^{8,56}(H(1,4) \otimes DH(1,4)),$$

which maps to the element $v_2^8 \in \mathrm{Ext}_{A(2)_*}^{8,56}(H(1,4))$ *under the composite*

$$\mathrm{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1,4)) \to \mathrm{Ext}_{A_*}^{*,*}(H(1,4)) \to \mathrm{Ext}_{A(2)_*}(H(1,4)).$$

Proof. In the May spectral sequence for $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$, the element v_2^8 is detected by $b_{3,0}^4$. Using Nakamura's formula [**Nak72**], and the calculations of [**Tan70**], we see that in the May spectral sequence for $\mathrm{Ext}_{A_*}(\mathbb{F}_2)$, there are differentials:

$$d_8(b_{3,0}^4) = b_{2,0}^4 h_5,$$
$$d_4(b_{2,0}^2 h_5) = h_0^4 h_3 h_5.$$

In the May spectral sequence, v_1^4 multiplication corresponds to multiplication by $b_{2,0}^2$. It follows that an element of $\mathrm{Ext}_{A_*}^{8,56}(H(1,4))$ which maps to

$$v_2^8 \in \mathrm{Ext}_{A(2)_*}^{8,56}(H(1,4))$$

must have image $h_0^3 h_3 h_5$ under the composite

$$\mathrm{Ext}_{A_*}^{8,56}(H(1,4)) \xrightarrow{\delta_{v_1^4}} \mathrm{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{\delta_{v_0}} \mathrm{Ext}_{A_*}^{5,43}(\mathbb{F}_2).$$

Since the element $h_0^3 h_3 h_5 \in \mathrm{Ext}_{A_*}^{5,43}(\mathbb{F}_2)$ is killed by h_0 multiplication, it lifts to an element $h_0^3 h_3 h_5[1] \in \mathrm{Ext}_{A_*}^{5,44}(H(1))$. Consider the exact sequence

$$\mathrm{Ext}_{A_*}^{8,56}(H(1)) \to \mathrm{Ext}_{A_*}^{8,56}(H(1,4)) \to \mathrm{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{v_1^4} \mathrm{Ext}_{A_*}^{9,56}(H(1)).$$

A computer calculation of $\mathrm{Ext}_{A_*}^{*,*}(H(1))$ using Bruner's programs [**Bru93**] reveals that:

1. $\mathrm{Ext}_{A_*}^{9,56}(H(1)) = 0$,
2. Every element $x \in \mathrm{Ext}_{A_*}^{5,44}(H(1))$ satisfies $h_0 x = 0$,
3. Every element $y \in \mathrm{Ext}_{A_*}^{9,57}(H(1))$ satisfies $y = h_0 z$ for some $z \in \mathrm{Ext}_{A_*}^{8,56}(H(1))$.

These three facts allow us to deduce that there exists an element $w \in \mathrm{Ext}_{A_*}^{8,56}(H(1,4))$ which maps to $h_0^3 h_3 h_5[1]$, and for which we have $h_0 w = 0$. By Corollary 4.2, the element w lifts to the desired element $\widetilde{v_2^8}$ in $\mathrm{Ext}_{A_*}^{8,56}(H(1,4) \otimes DH(1,4))$. □

We shall abusively refer to the element $\widetilde{v_2^8} \in \mathrm{Ext}_{A_*}^{8,56}(H(1,4) \otimes DH(1,4))$ as v_2^8.

5. Brown-Gitler comodules

Definitions

Let $A(i)_*$ denote the quotient of the dual Steenrod algebra dual to the subalgebra $A(i)$ of the Steenrod algebra. There is an isomorphism

$$A(i)_* \cong \mathbb{F}_p[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3 \ldots, \bar{\xi}_{i+1}]/(\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \bar{\xi}_3^{2^{i-1}}, \ldots, \bar{\xi}_{i+1}^2).$$

Here, $\bar{\xi}_i$ denotes the conjugate of ξ_i. We define a filtration on A_* which induces a filtration on the A_*-subcomodule

$$(A/\!\!/A(i))_* = A_* \Box_{A(i)_*} \mathbb{F}_2 \cong \mathbb{F}_2[\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \dots, \bar{\xi}_{i+1}^2, \bar{\xi}_{i+2}, \dots].$$

Our filtration is an increasing filtration of algebras given on generators by $|\bar{\xi}_j| = 2^{j-1}$. In particular, every element of $(A/\!\!/A(i))_*$ has filtration divisible by 2^{i+1}. The Brown-Gitler comodule $N_i(j)$ is the subspace of $(A/\!\!/A(i))_*$ spanned by all elements of filtration less than or equal to $2^{i+1}j$. Using the coproduct formula

$$\psi(\bar{\xi}_k) = \sum_{k_1+k_2=k} \bar{\xi}_{k_1} \otimes \bar{\xi}_{k_2}^{2^{k_1}}, \tag{5.1}$$

the submodule $N_i(j)$ is easily seen to be an A_*-subcomodule. Thus we have an increasing sequence of A_*-comodules:

$$\mathbb{F}_2 \cong N_i(0) \subset N_i(1) \subset N_i(2) \subset \cdots \subset (A/\!\!/A(i))_*.$$

Define a map of ungraded rings

$$\phi_i \colon (A/\!\!/A(i))_* \to (A/\!\!/A(i-1))_*$$

whose effect on generators is given by:

$$\phi_i(\bar{\xi}_k^{2^l}) = \begin{cases} \bar{\xi}_{k-1}^{2^l}, & k > 1, \\ 1, & k = 1. \end{cases}$$

Lemma 5.2. *The map ϕ_i is a map of ungraded $A(i)_*$-comodules.*

Proof. As an $A(i)_*$-comodule algebra, $(A/\!\!/A(i))_*$ is generated by the elements

$$\{\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \dots\}.$$

It therefore suffices to check that ϕ_i commutes with the coaction on on these generators. This is easily checked using the coproduct formula (5.1) and the relations in $A(i)_*$. □

Let $M_i(j)$ denote the subspace of $(A/\!\!/A(i))_*$ spanned by the monomials of filtration exactly $2^{i+1}j$.

Lemma 5.3. *The map ϕ_i maps the subspace $M_i(j)$ isomorphically onto the A_*-subcomodule $N_{i-1}(j) \subset (A/\!\!/A(i-1))_*$.*

Proof. The subspace of $M_i(j)$ spanned by monomials of the form $\bar{\xi}_1^{2^{i+1}s}x$, where x is a monomial involving $\bar{\xi}_k^{2^l}$ for $k > 1$, is mapped isomorphically onto the subspace $M_{i-1}(j-s) \subset N_{i-1}(j)$. □

Using Lemma 5.2, we have the following corollaries:

Corollary 5.4. *The subspace $M_i(j) \subset (A/\!\!/A(i))_*$ is an $A(i)_*$-subcomodule.*

Corollary 5.5. *There is an isomorphism of (graded) $A(i)_*$-comodules*

$$M_i(j) \cong \Sigma^{2^{i+1}j} N_{i-1}(j).$$

Corollary 5.6. *There is a splitting of $A(i)_*$-comodules*

$$(A/\!\!/A(i))_* \cong \bigoplus_{j \geqslant 0} M_i(j).$$

Remark 5.7. The comodule $N_{-1}(j)$ (respectively $N_0(j)$, $N_1(j)$) is isomorphic as an A_*-comodule to the homology of the jth $\mathbb{Z}/2$ (respectively integral, *bo*) Brown-Gitler spectrum. It is not known in general if the comodules $N_i(j)$ are realizable for $i > 1$.

Algebraic resolutions

We now describe an algebraic analog of an Adams resolution. For $i = -1$ (respectively $i = 0, 1, 2$), this algebraic resolution will correspond to the $H\mathbb{F}_2$ (respectively $H\mathbb{Z}$, *bo*, *tmf*) Adams resolution.

Let X be an object of the derived category $\mathcal{D}_{A_*}$. We define $T_i(X)^\bullet$ to be the following cosimplicial object:

$$(A/\!\!/A(i))_* \otimes X \underset{1\otimes u}{\overset{u\otimes 1}{\rightrightarrows}} (A/\!\!/A(i))_*^{\otimes 2} \otimes X \overset{\substack{u\otimes 1\otimes 1 \\ 1\otimes u\otimes 1 \\ 1\otimes 1\otimes u}}{\longrightarrow} (A/\!\!/A(i))_*^{\otimes 3} \otimes X \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots$$

Here, u is the unit

$$\mathbb{F}_2 \to (A/\!\!/A(i))_*.$$

Since $(A/\!\!/A(i))_*$ is an algebra, the canonical map

$$X \to \mathrm{Tot}(T^i(X)^\bullet)$$

is a quasi-isomorphism (see, for instance, [**Wei94**, Prop. 8.6.8]). We therefore have a Bousfield-Kan spectral sequence

$$E_1^{s,t,n} = \mathrm{Ext}^{s,t}_{A_*}((A/\!\!/A(i))_* \otimes \overline{(A/\!\!/A(i))}_*^{\otimes n} \otimes X[-n]) \Rightarrow \mathrm{Ext}^{s,t}_{A_*}(X), \qquad (5.8)$$

where

$$\overline{(A/\!\!/A(i))}_* = \mathrm{coker}\left(\mathbb{F}_2 \xrightarrow{u} (A/\!\!/A(i))_*\right).$$

The E_1-term can be simplified using a change of rings isomorphism, together with the splitting of Corollary 5.6:

$$\begin{aligned} E_1^{s,t,n} &= \mathrm{Ext}^{s,t}_{A_*}((A/\!\!/A(i))_* \otimes \overline{(A/\!\!/A(i))}_*^{\otimes n} \otimes X[-n]) \\ &\cong \mathrm{Ext}^{s,t}_{A(i)_*}(\overline{(A/\!\!/A(i))}_*^{\otimes n} \otimes X[-n]) \\ &\cong \bigoplus_{j_1,\ldots,j_n \geqslant 1} \mathrm{Ext}^{s,t}_{A(i)_*}(M_i(j_1) \otimes \cdots \otimes M_i(j_n) \otimes X[-n]). \end{aligned}$$

We shall call this spectral sequence (5.8) the $A/\!\!/A(i)$*-resolution* for X. In this paper we are only be interested in the case where $i = 2$. In this case, we shall refer to the $A/\!\!/A(2)$-resolution as the *algebraic tmf-resolution*.

Lemma 5.9. *Let R be a monoid in the derived category $\mathcal{D}_{A_*}$. Then the $A/\!\!/A(i)$-resolution for R is a spectral sequence of algebras. If M is an R-module, then the $A/\!\!/A(i)$-resolution for M is a spectral sequence of modules over the $A/\!\!/A(i)$-resolution for R.*

6. Ext computations

In this section we describe $\mathrm{Ext}^{s,t}_{A(2)_*}(M)$ for various objects $M \in \mathcal{D}_{A(2)_*}$. We first explain the computations, and then describe the methodology used to produce these computations. Charts displaying these Ext groups can be found in the following figures:

Figure 6.1: $\mathrm{Ext}^{*,*}_{A(2)_*}(\mathbb{F}_2)$ and $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1))$,
Figure 6.2: $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 2})$ and $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 3})$,
Figure 6.3: $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1) \otimes H(1))$ and $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1))$,
Figure 6.4: $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1))$ and $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1) \otimes H(1,4))$,
Figure 6.5: $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1,4))$ and $\mathrm{Ext}^{*,*}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1,4))$.

In each of these charts, the indexing has been modified to put the bottom generator of $M_2(1)^{\otimes k}$ in internal degree 0. The meaning of the notation in each of these charts is explained below.

$\mathrm{Ext}_{\mathbf{A(2)_*}}(\mathbb{F}_2)$

All of the elements are $c_4 = v_1^4$-periodic, and v_2^8-periodic. Exactly one v_1^4 multiple of each element is displayed with the $\bullet$ replaced by a $\circ$. Observe the wedge pattern beginning in $t-s = 35$. This pattern is infinite, propagated horizontally by $h_{2,1}$-multiplication and vertically by v_1-multiplication. Here, $h_{2,1}$ is the name of the generator in the May spectral sequence of bidegree $(t-s,s) = (5,1)$, and $h_{2,1}^4 = g$.

$\mathrm{Ext}_{\mathbf{A(2)_*}}(\mathbf{M_2(1)^{\otimes k}})$, for $\mathbf{k = 1,2,3}$

Every element is v_2^8-periodic. However, unlike $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$, not every element of these Ext groups is v_1^4-periodic. Rather, it is the case that either an element $x \in \mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k})$ satisfies $v_1^4 x = 0$, or it is v_1^4-periodic. Each of the v_1^4-periodic elements fit into families which look like shifted and truncated copies of $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2)$, and are labeled with a $\circ$. We have only included the beginning of these v_1^4-periodic patterns in the chart. The other generators are labeled with a $\bullet$. A $\square$ indicates a polynomial algebra $\mathbb{F}_2[h_{2,1}]$.

$\mathrm{Ext}_{\mathbf{A(2)_*}}(\mathbf{M_2(1)^{\otimes k}} \otimes \mathbf{H(1)})$, for $\mathbf{k = 1,2,3}$

The notation in these charts is identical to that in the charts for $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k})$ with the exception that the v_1^4-periodic patterns are truncated shifted copies of $\mathrm{Ext}_{A(1)_*}(H(1))$.

$\mathrm{Ext}_{\mathbf{A(2)_*}}(\mathbf{M_2(1)^{\otimes k}} \otimes \mathbf{H(1,4)})$, for $\mathbf{k = 1,2,3}$

Because we have taken the cofiber of v_1^4, none of the elements are v_1^4 periodic in these charts. The generators of the first v_2^8-periodic pattern are denoted with a $\bullet$ or a $\square$, where again a $\square$ denotes a polynomial algebra on $h_{2,1}$. In these charts, however, it is not the case that every element is v_2^8-periodic: some elements in the first lightening flash in the 0-stem fail to be v_2^8-periodic. We have conveyed this information by displaying the elements in the next v_2^8-pattern with $\circ$. With the exception of these first few generators, all of the other generators are v_2^8-periodic.

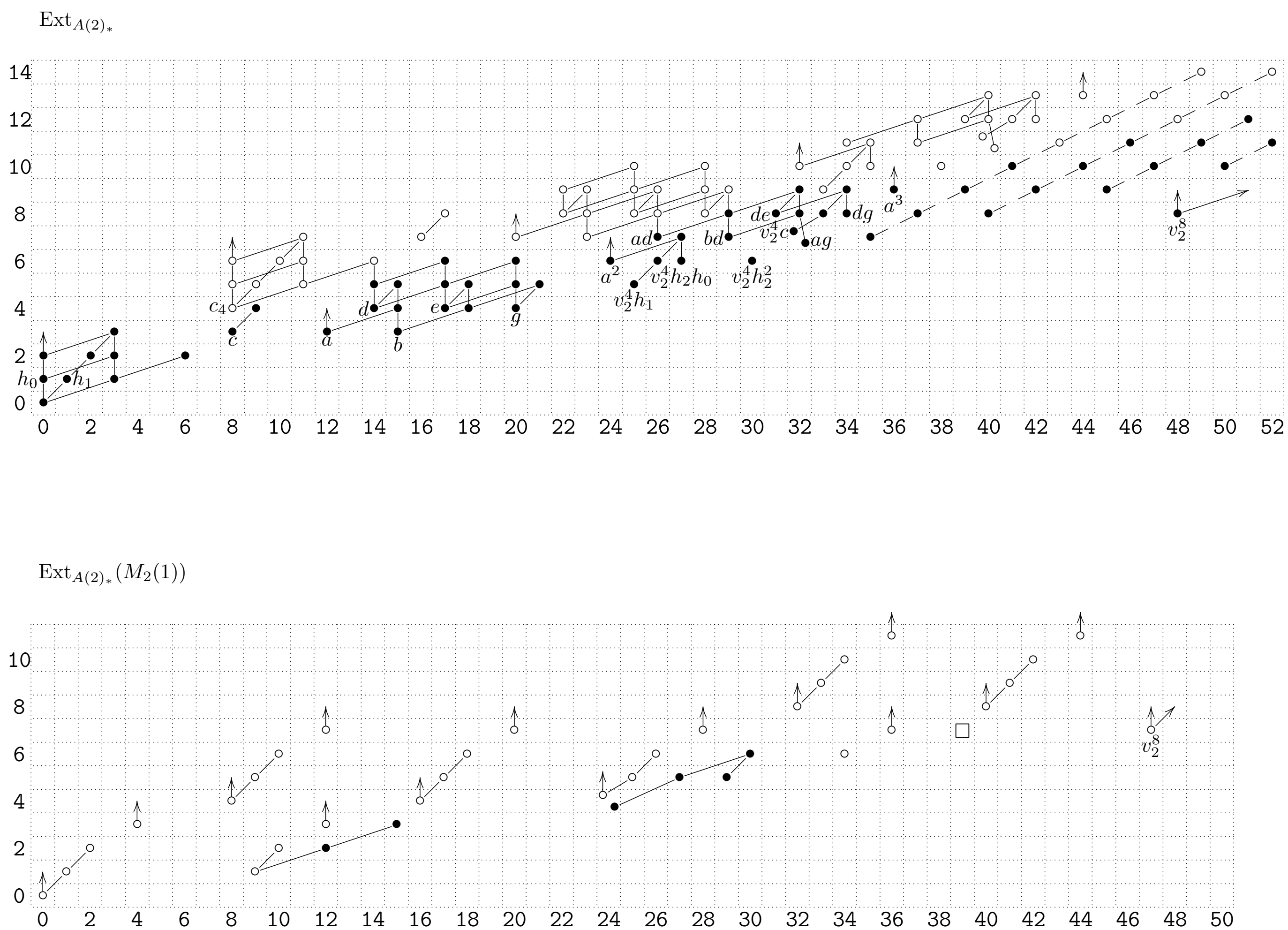

$\mathrm{Ext}_{A(2)_*}$
h_0
h_1
c_4
c
a
d
b
e
g
a^2
$v_2^4h_1$
$v_2^4h_2h_0$
ad
bd
$v_2^4h_2^2$
de
v_2^4c
ag
dg
a^3
v_2^8
$\mathrm{Ext}_{A(2)_*}(M_2(1))$
v_2^8

Figure 6.1.

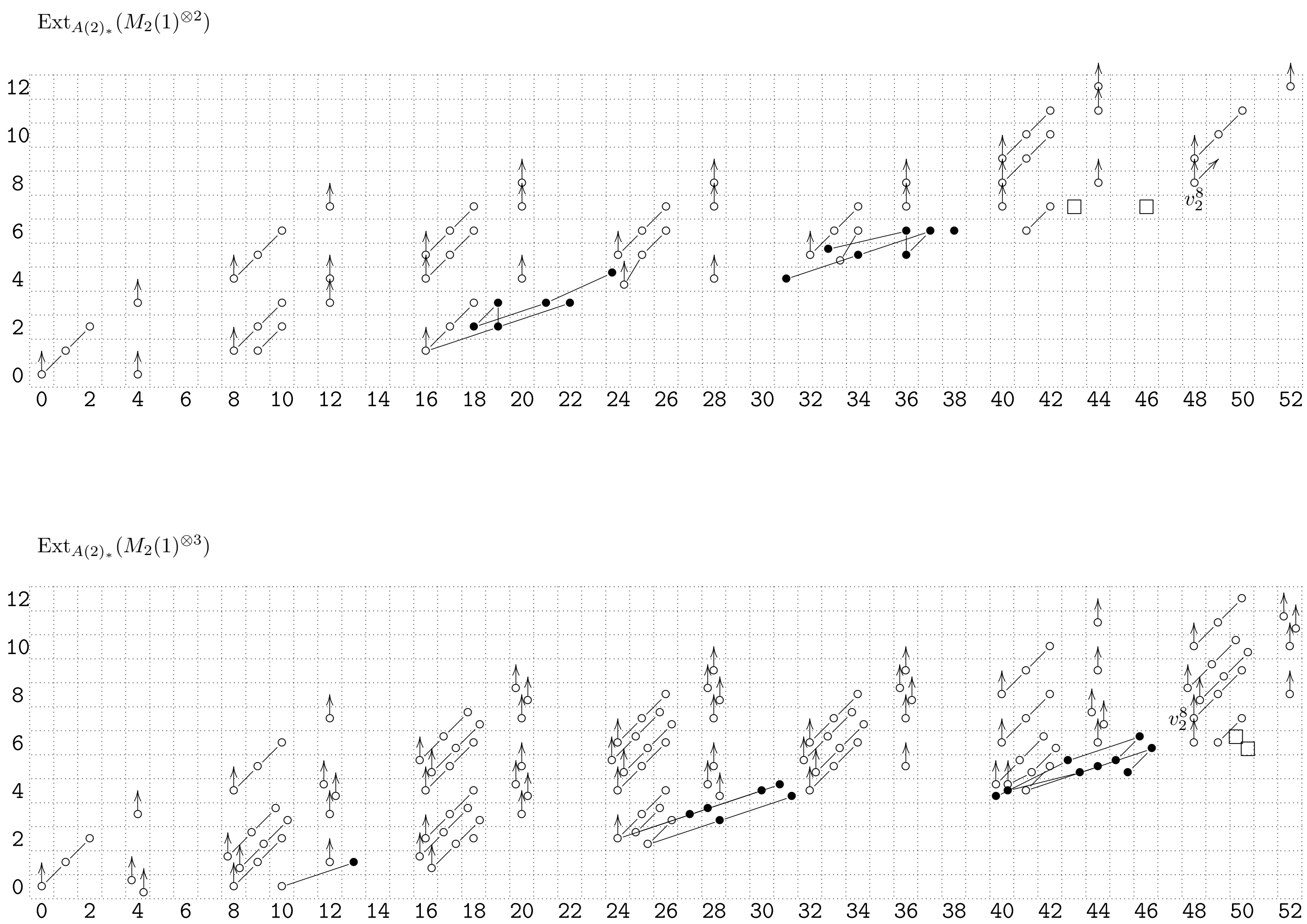
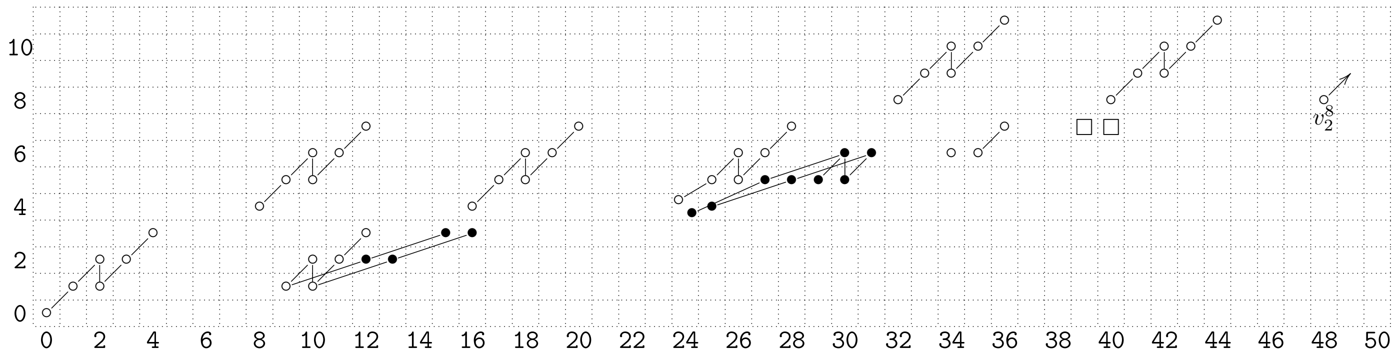


Figure 6.2.

$\mathrm{Ext}_{A(2)_*}(M_2(1) \otimes H(1))$



$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1))$

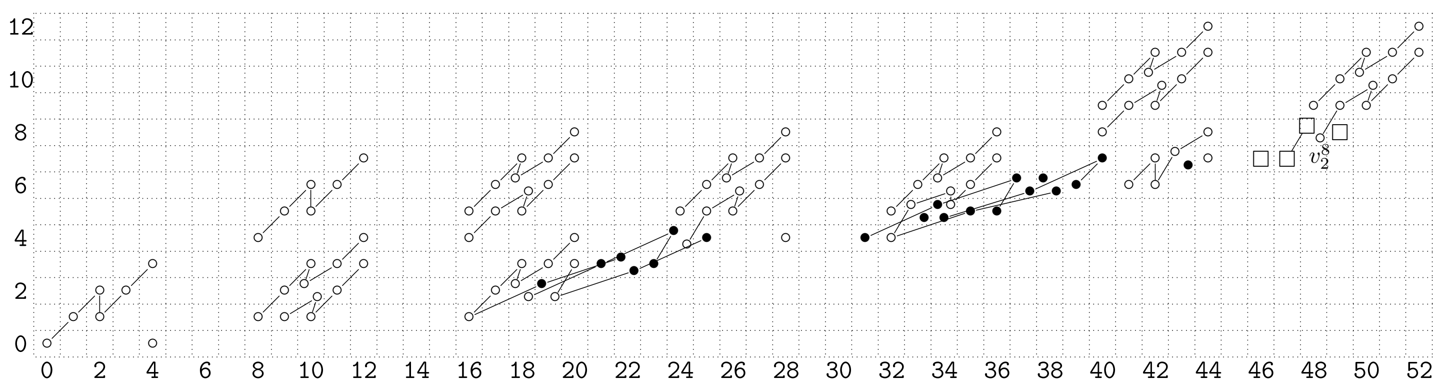
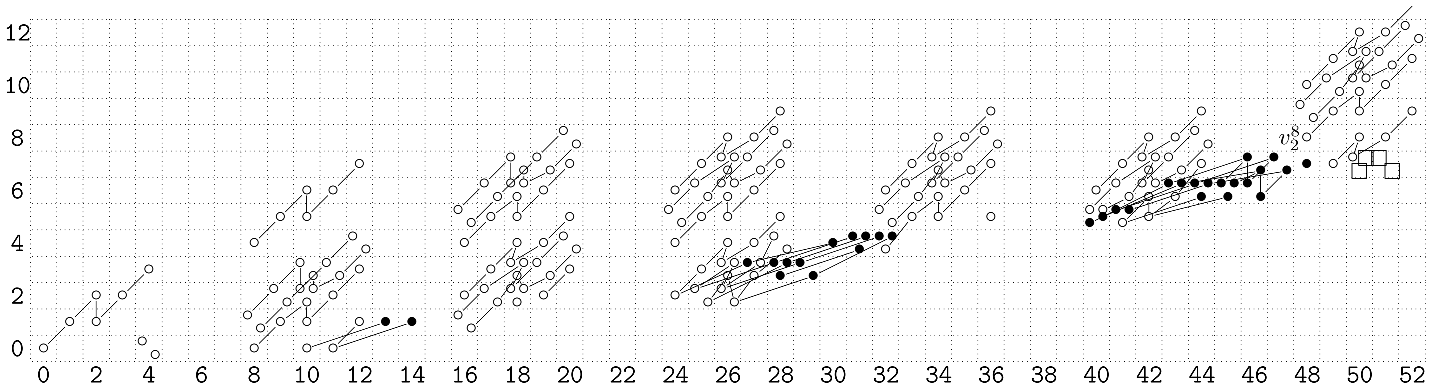


Figure 6.3.

$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1))$



$\mathrm{Ext}_{A(2)_*}(M_2(1) \otimes H(1,4))$

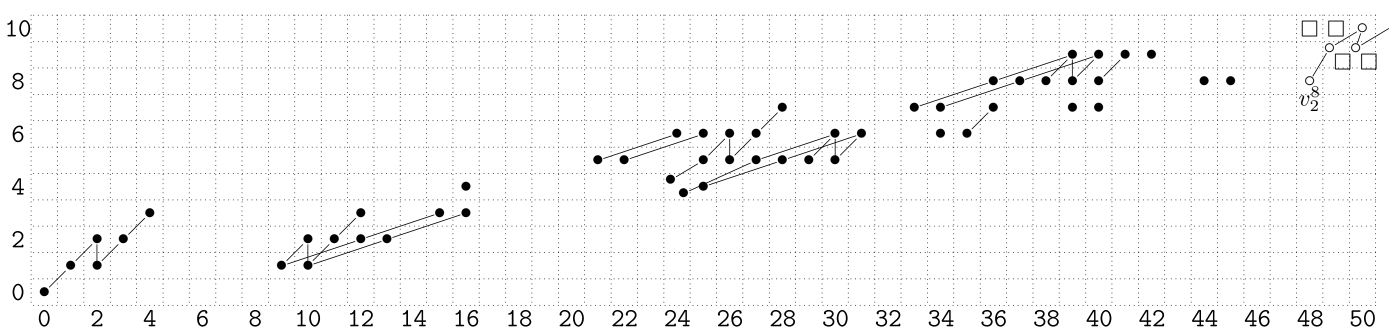
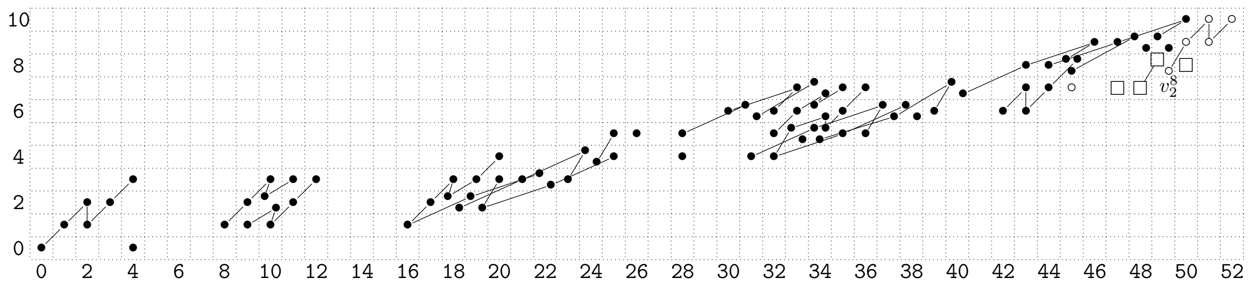


Figure 6.4.

$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1,4))$



$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1,4))$

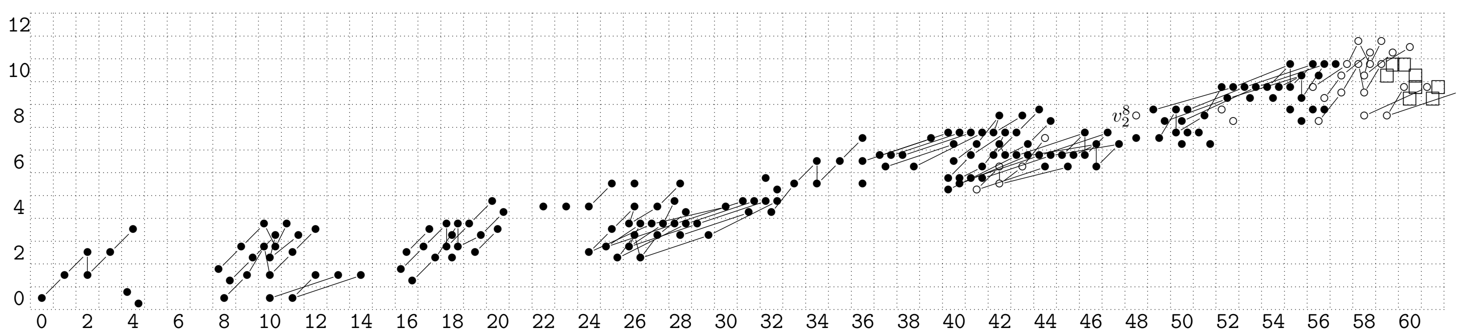
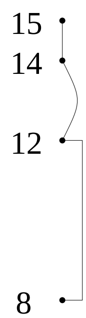


Figure 6.5.

Methodology

We explain how these charts were produced. The computation of $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ is well-known (see, for instance, [**DM82**]). The $A(2)_*$ comodule $M_2(1)$ can be described by the following diagram of generators:



Here, the dual action of the Steenrod algebra is encoded with a straight line denoting Sq^1_*, a curved line denoting Sq^2_*, and the bracket denoting Sq^4_*. A computation of $\mathrm{Ext}_{A(2)_*}(M_2(1))$ can be found in [**DM82**]. The computation of $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2})$ was obtained from $\mathrm{Ext}_{A(2)_*}(M_2(1))$ by inductively working up the skeletal filtration of the second factor of $M_2(1)$. The computation of $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3})$ was then obtained from $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2})$ by inductively working up the skeletal filtration of the third factor of $M_2(1)$. Along the way, because $H(1)$ occurs as a subcomodule of $M_2(1)$, we have computed $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1))$ for $k = 1, 2$. We then use the long exact sequence induced by the triangle (3.1) to obtain $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1))$.

Each of these manual computations was independently verified by R.R. Bruner's computer program for computing Ext [**Bru93**]. This computer program constructs minimal resolutions of modules over the subalgebra $A(2)$. We also used the computer program to gain complete understanding of v_1^4-periodicity in these Ext groups, as we now explain. Note that there is an element

$$v_1 \in \mathrm{Ext}^{1,3}_{A(2)_*}(H(1) \otimes H_* C\eta),$$

where $C\eta$ is the cofiber of $\eta \in \pi_1^s$. We used Bruner's programs to compute minimal resolutions for

$$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1) \otimes H_* C\eta), \quad k = 1, 2, 3,$$

and read off all of the v_1-multiplicative structure in these Ext groups from the minimal resolutions. We then used an η-Bockstein spectral sequence to recover the v_1^4-multiplicative structure on

$$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1)), \qquad k = 1, 2, 3.$$

From this, the computation of

$$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1,4)), \quad k = 1, 2, 3$$

was easily determined by the long exact sequence arising from the triangle (3).

7. Reducing the computation to $M_2(1)^{\otimes k}$ for $k \leqslant 3$

Inductive short exact sequences

We will construct some short exact sequences that relate the various Brown-Gitler comodules $N_1(j)$. We have an isomorphism

$$(A(2)/\!\!/A(1))_* \cong \Lambda[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3].$$

Observe that there is an isomorphism of $\mathbb{F}_2$-vector spaces

$$\tau : (A/\!\!/A(1))_* \xrightarrow{\cong} (A/\!\!/A(2))_* \otimes (A(2)/\!\!/A(1))_*$$

given on the monomial basis by

$$\tau(\bar{\xi}_1^{8i_1+4\epsilon_1}\bar{\xi}_2^{4i_2+2\epsilon_2}\bar{\xi}_3^{2i_3+\epsilon_3}\bar{\xi}_4^{i_4}\cdots) = \bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2}\bar{\xi}_3^{2i_3}\bar{\xi}_4^{i_4}\cdots \otimes \bar{\xi}_1^{4\epsilon_1}\bar{\xi}_2^{2\epsilon_2}\bar{\xi}_3^{\epsilon_3}$$

for $i_j \geqslant 0$ and $\epsilon_j = 0, 1$. The map τ is *not* an isomorphism of $A(2)_*$-comodules. For instance, in $(A/\!\!/A(1))_*$ we have the coaction

$$\psi(\bar{\xi}_1^4\bar{\xi}_2^2) = \bar{\xi}_1^4\bar{\xi}_2^2 \otimes 1 + \bar{\xi}_1^4 \otimes \bar{\xi}_2^2 + \bar{\xi}_2^2 \otimes \bar{\xi}_1^4 + 1 \otimes \bar{\xi}_1^4\bar{\xi}_2^2 + \bar{\xi}_1^6 \otimes \bar{\xi}_1^4 + \bar{\xi}_1^2 \otimes \bar{\xi}_1^8$$

whereas in $(A/\!\!/A(2))_* \otimes (A/\!\!/A(1))_*$ we have

$$\psi(1 \otimes \bar{\xi}_1^4\bar{\xi}_2^2) = \bar{\xi}_1^4\bar{\xi}_2^2 \otimes 1 \otimes 1 + \bar{\xi}_1^4 \otimes 1 \otimes \bar{\xi}_2^2 + \bar{\xi}_2^2 \otimes 1 \otimes \bar{\xi}_1^4 + 1 \otimes 1 \otimes \bar{\xi}_1^4\bar{\xi}_2^2 + \bar{\xi}_1^6 \otimes 1 \otimes \bar{\xi}_1^4.$$

However, there is a decreasing filtration

$$(A/\!\!/A(1))_* = F^0(A/\!\!/A(1))_* \supset F^1(A/\!\!/A(1))_* \supset \cdots$$

of $A(2)_*$-comodules such that τ induces an isomorphism of the associated graded $A(2)_*$-comodules

$$\tau \colon E^0(A/\!\!/A(1))_* \xrightarrow{\cong} (A/\!\!/A(2))_* \otimes (A(2)/\!\!/A(1))_*.$$

The decreasing filtration is given as follows: under the isomorphism

$$(A/\!\!/A(2))_* \cong \bigoplus_k M_2(k)$$

of $A(2)_*$-comodules given by Corollary 5.6, we define

$$F^j(A/\!\!/A(1))_* := \tau^{-1}\left(\Big(\bigoplus_{k=j}^{\infty} M_2(k)\Big) \otimes (A(2)/\!\!/A(1))_*\right).$$

Using the coproduct formula (5.1) this is easily verified to be a decreasing filtration by $A(2)_*$-comodules — the coaction preserves or raises the filtration.

Consider the quotients

$$Q^j(A/\!\!/A(1))_* := (A/\!\!/A(1))_*/F^{j+1}(A/\!\!/A(1))_*.$$

The map τ induces isomorphisms of $\mathbb{F}_2$-vector spaces

$$\tau \colon Q^j(A/\!\!/A(1))_* \xrightarrow{\cong} N_2(j) \otimes (A(2)/\!\!/A(1))_*.$$

Furthermore, the filtration $\{F^k(A/\!\!/A(1))_*\}$ projects to a finite decreasing filtration of $Q^j(A/\!\!/A(1))_*$ by $A(2)_*$-comodules, such that τ induces an isomorphism of associated graded $A(2)_*$-comodules

$$\tau\colon E^0Q^j(A/\!\!/A(1))_* \xrightarrow{\cong} N_2(j) \otimes (A(2)/\!\!/A(1))_*. \tag{7.1}$$

Lemma 7.2. *There is a short exact sequence of $A(2)_*$-comodules:*

$$0 \to \Sigma^{8j} N_1(j) \otimes N_1(1) \to N_1(2j+1) \to Q^{j-1}(A/\!\!/A(1))_* \to 0.$$

Lemma 7.3. *There is an exact sequence of $A(2)_*$-comodules:*

$$0 \to \Sigma^{8j} N_1(j) \to N_1(2j) \to Q^{j-1}(A/\!\!/A(1))_* \to \Sigma^{8j+9} N_1(j-1) \to 0.$$

Proof of Lemma 7.2. Since the elements of $(A(2)/\!\!/A(1))_*$ have Brown-Gitler filtration at most 12, the image of the composite

$$N_2(j-1) \otimes (A(2)/\!\!/A(1))_* \hookrightarrow (A/\!\!/A(2))_* \otimes (A(2)/\!\!/A(1))_* \xrightarrow{\tau^{-1}} (A/\!\!/A(1))_*$$

lies in $N_1(2j+1)$, giving a surjection of $A(2)_*$-comodules

$$\rho\colon N_1(2j+1) \twoheadrightarrow Q^{j-1}(A/\!\!/A(1))_*.$$

As $\mathbb{F}_2$-vector spaces, we have

$$\tau(N_1(2j+1)) = N_2(j-1) \otimes (A(2)/\!\!/A(1))_* \oplus M_2(j) \otimes N_1(1),$$

where the Brown-Gitler comodule $N_1(1)$ is identified as the $A(2)_*$-subcomodule

$$N_1(1) = \mathbb{F}_2\{1, \bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3\} \subset (A(2)/\!\!/A(1))_*.$$

We deduce that the kernel of ρ is

$$M_2(j) \otimes N_1(1) \cong \Sigma^{8j} N_1(j) \otimes N_1(1). \qquad \square$$

Proof of Lemma 7.3. As an $\mathbb{F}_2$-vector space, the image of $N_1(2j)$ in $(A/\!\!/A(2))_* \otimes (A(2)/\!\!/A(1))_*$ under the isomorphism τ is given by

$$\tau(N_1(2j)) \cong \begin{pmatrix} N_2(j-2) \otimes (A(2)/\!\!/A(1))_* \\ \oplus \\ M_2(j-1) \otimes \mathbb{F}_2\{1, \bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_1^4\bar{\xi}_2^2, \bar{\xi}_1^4\bar{\xi}_3, \bar{\xi}_2^2\bar{\xi}_3\} \\ \oplus \\ M_2(j) \otimes \mathbb{F}_2\{1\} \end{pmatrix}.$$

Thus, at least on the level of $\mathbb{F}_2$-vector spaces, we have an exact sequence

$$\begin{aligned} 0 \to M_2(j) \otimes \mathbb{F}_2\{1\} &\xrightarrow{\alpha} N_1(2j) \xrightarrow{\beta} Q^{j-1}(A/\!\!/A(1))_* \\ &\xrightarrow{\gamma} M_2(j-1) \otimes \mathbb{F}_2\{\bar{\xi}_1^4\bar{\xi}_2^2\bar{\xi}_3\} \to 0. \end{aligned}$$

We just need to prove that these are maps of $A(2)_*$-comodules. The map γ is clearly a map of $A(2)_*$-comodules. We have the following diagram of inclusions of $A(2)_*$-comodules:

$$\begin{tikzcd} M_2(j) \otimes \mathbb{F}_2\{1\} \arrow[r, hook, "\alpha"] \arrow[d, hook] & N_1(2j) \arrow[r, hook, "\delta"] \arrow[d, hook] & Q^j(A/\!\!/A(1))_* . \\ (A/\!\!/A(2))_* \arrow[r, hook] & (A/\!\!/A(1))_* \arrow[ur, two heads] & \end{tikzcd} \tag{7.4}$$

In particular, the map α is a map of $A(2)_*$-comodules. Let K be the cokernel of α. Then we get an induced map of short exact sequences of $A(2)_*$-comodules:

$$\begin{tikzcd}
M_2(j)\otimes \mathbb{F}_2\{1\} \arrow[r, "\alpha"] \arrow[d] & N_1(2j) \arrow[r, "\beta_1"] \arrow[d, "\delta"] & K \arrow[d, dotted, "\beta_2"] \\
M_2(j)\otimes (A(2)/\!\!/A(1))_* \arrow[r] & Q^j(A/\!\!/A(1))_* \arrow[r] & Q^{j-1}(A/\!\!/A(1))_*.
\end{tikzcd}$$

We deduce that the map β is a map of $A(2)_*$-comodules, because it is given by the composite $\beta_2\circ\beta_1$ of $A(2)_*$-comodule maps. ☐

Vanishing lines

We reduce the computations needed to those of $M_2(1)^{\otimes k}$ for $k \leqslant 3$ using vanishing lines for modified Adams E_2 terms. Note that after a finite range, $\mathrm{Ext}_{A(2)_*}(H(1,4))$ has a vanishing line of slope $1/5$.

Lemma 7.5. *We have*

$$\mathrm{Ext}^{s,t}_{A(2)_*}(N_1(j)\otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+a_j}{6}, \frac{(t-s)+b_j}{5}\right\},$$

and the constants a_j and b_j are inductively defined by

$$\begin{aligned}
a_0 &= 21,\\
b_0 &= 9,\\
a_1 &= 15,\\
b_1 &= 2,\\
a_{2j} &= \max\{a_{j-1}-8j-2, a_j-8j\},\\
b_{2j} &= \max\{b_{j-1}-8j-3, b_j-8j\},\\
a_{2j+1} &= a_j-8j,\\
b_{2j+1} &= b_j-8j.
\end{aligned}$$

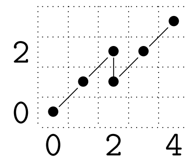
Proof. The case of $j=0,1$ is obtained by examining Figures 6.4 and 8.1. The case of $j\geqslant 2$ is established by induction using Lemmas 7.2 and 7.3. The terms involving $Q^j(A/\!\!/A(1))_*$ are handled using the spectral sequence

$$\mathrm{Ext}^{s,t}_{A(2)_*}(N_2(j)\otimes(A(2)/\!\!/A(1))_*\otimes H(1,4)) \Rightarrow \mathrm{Ext}^{s,t}_{A(2)_*}(Q^j(A/\!\!/A(1))_*\otimes H(1,4))$$

induced from (7.1), and the change-of-rings isomorphism

$$\mathrm{Ext}^{s,t}_{A(2)_*}(N_2(j)\otimes(A(2)/\!\!/A(1))_*\otimes H(1,4)) \cong \mathrm{Ext}^{s,t}_{A(1)_*}(N_2(j)\otimes H(1,4)).$$

The only non-zero values values of $\mathrm{Ext}^{s,t}_{A(1)_*}(H(1,4))$ are displayed below.



In particular, we see that $\mathrm{Ext}^{s,t}_{A(1)_*}(H(1,4))$ is zero for $s > \frac{(t-s)+17}{7}$. $\square$

We extract the following estimate:

Lemma 7.6. *Suppose that $j_1, \dots, j_n$ is a sequence of positive integers such that for some i, $j_i \geqslant 2$. Then we have*

$$\mathrm{Ext}^{s,t}_{A(2)_*}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+2}{6}, \frac{(t-s)-12}{5}\right\}.$$

Proof. Assume that $n = 1$, and set j equal to $j_1 \geqslant 2$. By Lemma 7.5, we have

$$\mathrm{Ext}_{A(2)_*}(M_2(j)[-1] \otimes H(1,4)) = 0$$

if

$$s > \max\left\{\frac{(t-s)-8j+8+17}{7}, \frac{(t-s)-8j+7+a_j}{6}, \frac{(t-s)-8j+6+b_j}{5}\right\}.$$

It therefore suffices to prove that the following inequalities are satisfied:

$$\begin{aligned} 17 &\geqslant 17 - 8j + 8, \\ 2 &\geqslant a_j - 8j + 7, \\ -12 &\geqslant b_j - 8j + 6. \end{aligned}$$

The inequalities are true for $j = 2, 3$. By induction, these inequalities hold for all j.

We now induct on n. We may as well assume that $j_1 \geqslant 2$. Assume that

$$\mathrm{Ext}^{s,t}_{A(2)_*}(M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n+1] \otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+2}{6}, \frac{(t-s)-12}{5}\right\}.$$

By filtering the A_*-comodule $M_2(j_n)$ by degree, we obtain an Atiyah-Hirzebruch type spectral sequence which converges to

$$\mathrm{Ext}^{s,t}_{A(2)_*}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1,4))$$

and whose E_1-page is given by

$$\bigoplus_x \mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^{|x|} M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1,4)),$$

where x ranges over an $\mathbb{F}_2$-basis of $M_2(j_n)$. The smallest value $|x|$ can take is 8, in the case $j_1 = 1$. By our inductive hypothesis, we have

$$\mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^8 M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+1}{6}, \frac{(t-s)-14}{5}\right\}.$$

This verifies the inductive step. □

Lemma 7.7. *Suppose that n is greater than 3. Then we have*

$$\mathrm{Ext}^{s,t}_{A(2)_*}(M_2(1)^{\otimes n}[-n] \otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+4}{6}, \frac{(t-s)-17}{5}\right\}.$$

Proof. Examining Figure 6.5, we see that

$$\mathrm{Ext}^{s,t}_{A(2)_*}(N_1(1)^{\otimes 3} \otimes H(1,4)) = 0$$

for

$$s > \max\left\{\frac{(t-s)+17}{7}, \frac{(t-s)+8}{6}, \frac{(t-s)-9}{5}\right\}.$$

The lemma follows from induction on n, using Atiyah-Hirzebruch spectral sequences as in the proof of Lemma 7.7. □

8. The modified Adams spectral sequence for $tmf_* M(1,4)$

In this section we describe a complete computation of the MASS

$$\mathrm{Ext}^{s,t}_{A(2)_*}(H(1,4)) \Rightarrow \pi_{t-s}(tmf \wedge M(1,4)).$$

The spectral sequence is displayed in four pages in Figures 8.1 and 8.2. The entire spectral sequence is v_2^{32}-periodic.

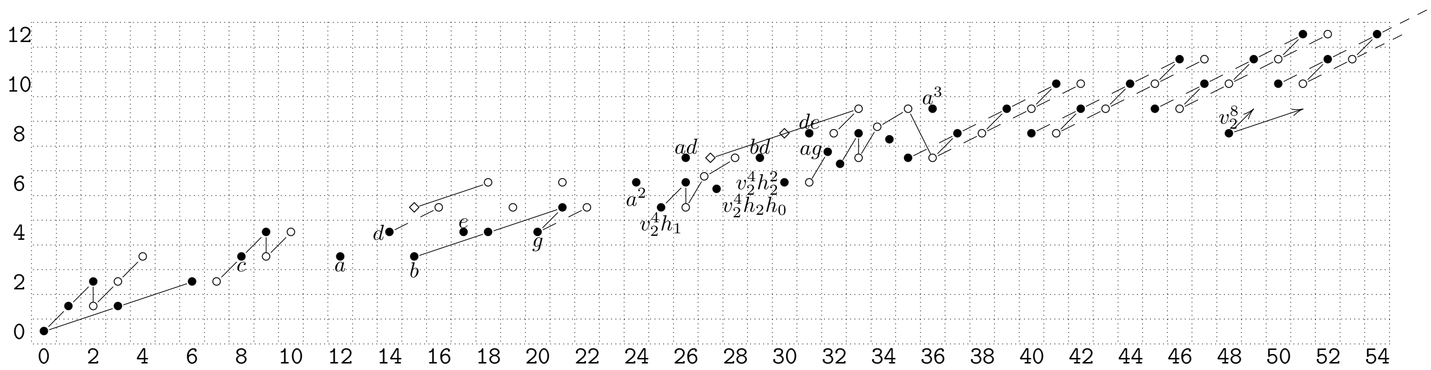
We first explain what is happening in these charts. Then we explain the methodology used to produce these differentials.

Page 1: dimensions 0–48

The truncated wedge beginning in $t - s = 35$ is infinite, and propagated by g-multiplication. The entire chart is periodic under v_2^8-multiplication. Classes born on the 0-cell of $M(1)$ are denoted with a $\bullet$, and classes born on the 1-cell of $M(1)$ are denoted with a $\circ$. Although multiplication by $c_4 = v_1^4$ is faithful in $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$, it is not faithful in $\mathrm{Ext}_{A(2)_*}(M(1))$. We therefore get some classes coming from the 9-cell of $M(1,4)$, which we denote with a $\diamond$.

There are only two possible Adams differentials through $t - s = 47$, and only one of them actually occurs. This differential is indicated on the chart.

MASS for $tmf_*M(1,4)$, p1:



MASS for $tmf_*M(1,4)$, p2:

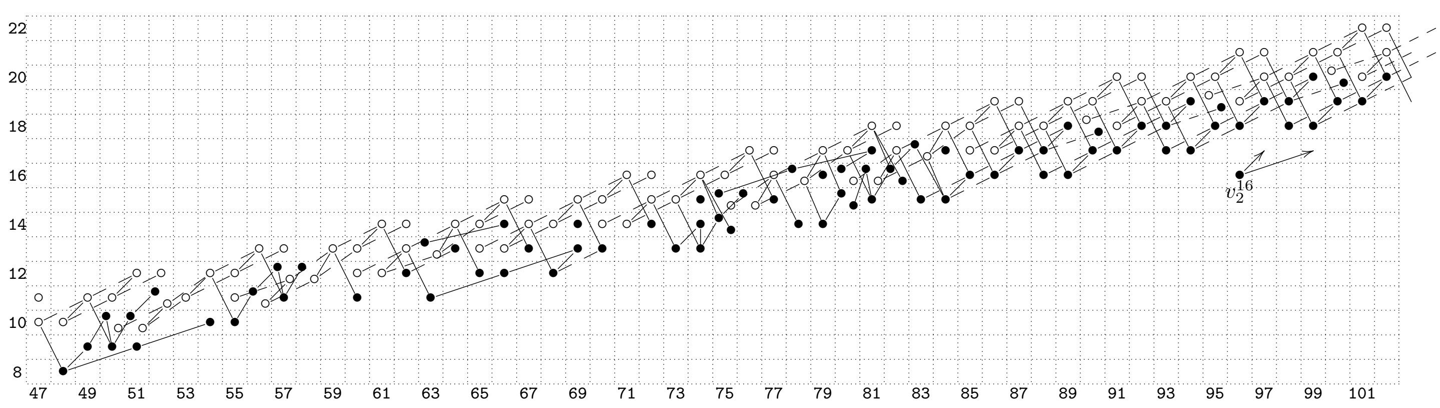
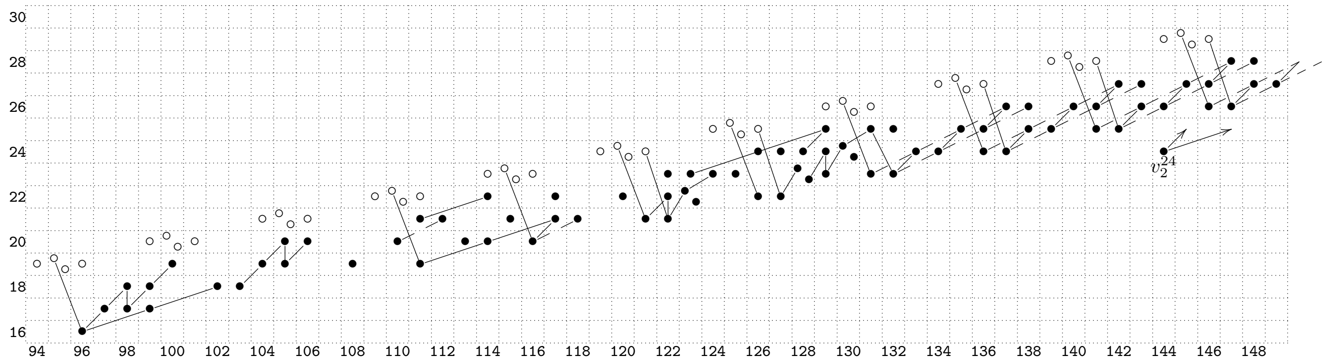


Figure 8.1.

MASS for $tmf_*M(1,4)$, p3:



MASS for $tmf_*M(1,4)$, p4:

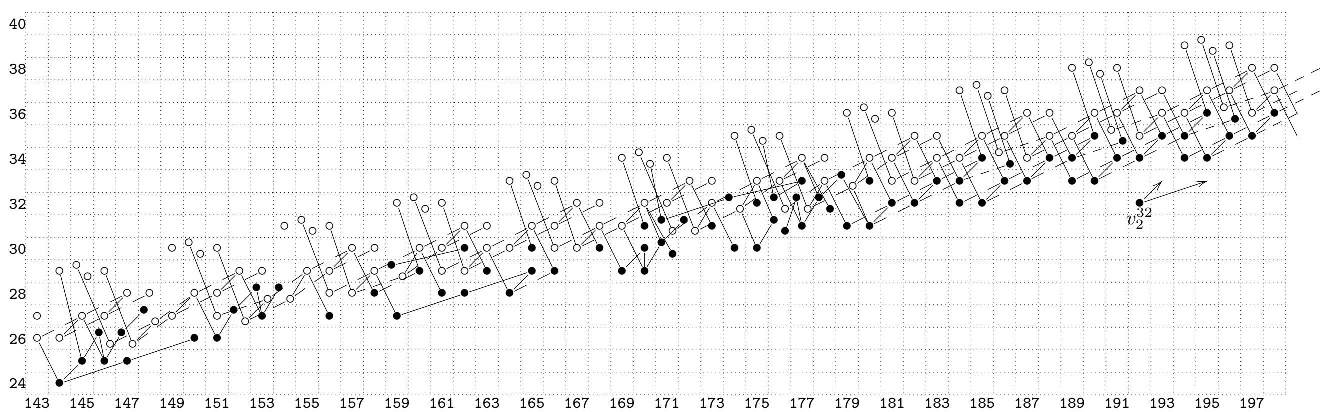
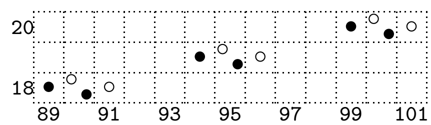


Figure 8.2.

Page 2: dimensions 48–96

We move to the region between the occurrence of v_2^8 and v_2^{16}. There are numerous d_2 differentials in this range, displayed in the chart. In this chart, the classes propagated by v_2^8 are denoted with a $\bullet$, and the classes coming from the truncated wedge starting in $t-s=35$ are denoted with a $\circ$. Note that beginning in $t-s=91$, we just have the following pattern:



Page 3: dimensions 96–144

We now move up to the region between v_2^{16} and v_2^{24}. We propagate only the $h_{2,1}$-periodic pattern from the previous page (denoted with $\circ$); everything else is either the source or target of a d_2. We denote the elements propagated by v_2^{16} multiplication with a $\bullet$.

Page 4: dimensions 144–192

We now introduce the differentials supported by v_2^{24} and its multiples. We see that eventually we get a small gap in homotopy between the 180 stem and the 192 stem. Then the pattern repeats with v_2^{32}-periodicity.

Methodology

In [**HM**], the structure of the Adams spectral sequence

$$\mathrm{Ext}^{*,*}_{A(2)_*}(\mathbb{F}_2) \Rightarrow \pi_* tmf_2$$

is completely determined. The Adams spectral sequence for $tmf_* M(1,4)$ is a module over the Adams spectral sequence for tmf_*, and all of the differentials for $tmf_* M(1,4)$ were deduced from this structure. These computations were double-checked against the Atiyah-Hirzebruch spectral sequence

$$H^*(M(1,4), tmf_*) \Rightarrow tmf_* M(1,4)$$

using the known values of tmf_*. As a further consistency check, a combination of Gross-Hopkins duality [**HG94**] and Mahowald-Rezk [**MR99**] duality shows that $tmf_* M(1,4)$ is, up to a shift, Pontryagin self-dual, and this is consistent with our computations.

9. $d_2(v_2^8)$ and $d_3(v_2^{16})$

In this section we will lift the differentials $d_2(v_2^8)$ and $d_3(v_2^{16})$ from the MASS for $tmf_* M(1,4)$ to the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$. We will observe that both $d_2(v_2^8)$ and $d_3(v_2^{16})$ are central, and hence, using the fact that the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$ is a spectral sequence of algebras, we will deduce that $d_r(v_2^{32})$ is zero for $r < 4$.

Lemma 9.1. *In the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$, there is a differential*

$$d_2(v_2^8) = \widetilde{e_0 r},$$

where $\widetilde{e_0 r}$ is the image of the element $e_0 r$ under the map

$$\mathrm{Ext}_{A_*}^{10,57}(\mathbb{F}_2) \to \mathrm{Ext}_{A_*}^{10,57}(H(1,4) \otimes DH(1,4)).$$

Proof. By Proposition 2.1 it suffices to establish that $d_2(v_2^8) = \widetilde{e_0 r}$ in the MASS

$$\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1)).$$

The differential $d_2(v_2^8)$ in the Adams spectral sequence for *tmf* maps to a differential $d_2(v_2^8) = \widetilde{\widetilde{e_0 r}}$ under the map of (M)ASSs

$$\begin{tikzcd}
\mathrm{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} tmf \arrow[d] \\
\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] & tmf_{t-s}(M(1,4) \wedge DM(1)),
\end{tikzcd}$$

where $\widetilde{\widetilde{e_0 r}}$ is the image of $e_0 r$ under the composite

$$\mathrm{Ext}_{A_*}^{10,57}(\mathbb{F}_2) \to \mathrm{Ext}_{A_*}^{10,57}(H(1,4) \otimes DH(1)) \to \mathrm{Ext}_{A(2)_*}^{10,57}(H(1,4) \otimes DH(1)).$$

We wish to lift the differential $d_2(v_2^8) = \widetilde{\widetilde{e_0 r}}$ to $d_2(v_2^8) = \widetilde{e_0 r}$ using the map of MASSs:

$$\begin{tikzcd}
\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(M(1,4) \wedge DM(1)) \arrow[d] \\
\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] & tmf_{t-s}(M(1,4) \wedge DM(1)).
\end{tikzcd}$$

However, using

$$\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4)) = \begin{cases} e_0 r[0] & (t-s, s) = (47, 10), \\ e_0 r[1] & (t-s, s) = (48, 10) \end{cases}$$

and

$$\mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2) = \begin{cases} e_0 r & (t-s, s) = (47, 10), \\ 0 & (t-s, s) = (46, 10), (48, 10), (55, 5), (56, 5), (57, 5), \end{cases}$$

we may deduce that the map

$$\mathrm{Ext}_{A_*}^{10,57}(H(1,4) \otimes DH(1)) \to \mathrm{Ext}_{A(2)_*}^{10,57}(H(1,4) \otimes DH(1))$$

is an isomorphism. This suffices to show that the differential $d_2(v_2^8)$ lifts as desired. $\square$

Since $d_2(v_2^8)$ is central, Proposition 3.6 gives the following corollary:

Corollary 9.2. *In the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$, we have $d_2(v_2^{16}) = 0$.*

We now investigate $d_3(v_2^{16})$.

Lemma 9.3. *In the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$, the element $d_3(v_2^{16})$ is in the image of*

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \to \mathrm{Ext}_{A_*}^{19,114}(H(1,4) \otimes DH(1,4)).$$

In particular, $d_3(v_2^{16})$ is central.

Proof. By Proposition 2.1 it suffices to establish that in the MASS for $\pi_*(M(1,4) \wedge DM(1))$, the element $y = d_3(v_2^{16})$ is in the image of the map

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \to \mathrm{Ext}_{A_*}^{19,114}(H(1,4) \otimes DH(1)).$$

The differential $d_3(v_2^{16})$ in the ASS for *tmf* maps to a differential $d_3(v_2^{16}) = z$ under the map of (M)ASSs

$$\begin{tikzcd}
\mathrm{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} tmf \arrow[d] \\
\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] & tmf_{t-s}(M(1,4) \wedge DM(1)),
\end{tikzcd}$$

where z is in the image of

$$\mathrm{Ext}_{A(2)_*}^{19,114}(\mathbb{F}_2) \to \mathrm{Ext}_{A(2)_*}^{19,114}(H(1,4) \otimes DH(1)).$$

Using the map of spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(M(1,4) \wedge DM(1)) \arrow[d] \\
\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) \arrow[r, Rightarrow] & tmf_{t-s}(M(1,4) \wedge DM(1)),
\end{tikzcd}$$

we see that y maps to z. Therefore z detects y in the algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1))$. Since the algebraic *tmf*-resolution is functorial, we deduce that y is in the image of the map

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \xrightarrow{i_*} \mathrm{Ext}_{A_*}^{19,114}(H(1,4) \otimes DH(1))$$

modulo higher terms of the algebraic *tmf*-resolution: that is to say, there exists an element

$$x \in \mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2)$$

such that $y - i_*(x)$ is detected in a higher filtration of the algebraic *tmf*-resolution.

We are left with showing that $w = y - i_*(x) = 0$. Suppose not. Using our vanishing lines from Section 7 and our $\mathrm{Ext}_{A(2)_*}$ computations from Section 6, we deduce that

w is detected in the algebraic *tmf*-resolution by an element

$$\overline{w} \in \mathrm{Ext}^{19,114}_{A(2)_*}(M_2(1) \otimes H(1,4) \otimes DH(1)[-1])$$

and the image of $\overline{w}$ under the map

$$\mathrm{Ext}^{19,114}_{A(2)_*}(M_2(1) \otimes H(1,4) \otimes DH(1)[-1]) \xrightarrow{1 \otimes p_* \otimes 1} \mathrm{Ext}^{19,114}_{A(2)_*}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])$$

is non-trivial, where p_* is the projection

$$H(1,4) \to \Sigma^{12} H(1)[-3]$$

in the derived category of A_*-comodules induced by the projection

$$p \colon M(1,4) \to \Sigma^9 M(1).$$

We deduce that in the MASS for $M(1) \wedge DM(1)$ there is a differential

$$d_3((p_* \otimes 1)(v_2^{16})) = (p_* \otimes 1)(w).$$

We will verify the following claim:

Claim 9.4. The element $(p_* \otimes 1)(w)$ is non-trivial in the E_3-page of the MASS for $M(1) \wedge DM(1)$.

Assuming Claim 9.4, we deduce that $d_3((p_* \otimes 1)(v_2^{16}))$ is non-trivial. However, the image of v_2^{16} under the map

$$\mathrm{Ext}^{16,112}_{A_*}(H(1,4) \otimes DH(1)) \xrightarrow{p_* \otimes 1} \mathrm{Ext}^{16,112}_{A_*}(\Sigma^{12} H(1) \otimes DH(1)[-3])$$

may be computed using the May spectral sequence. In the May spectral sequence, the element v_2^{16} is detected by $b_{3,0}^8$. Applying Nakamura's formula [**Nak72**] to the May spectral sequence differential $d_8(b_{3,0}^4) = h_5 b_{2,0}^4$ in the proof of Proposition 4.3 gives

$$d_{16}(b_{3,0}^8) = h_6 b_{2,0}^8$$

from which it follows that

$$(p_* \otimes 1)(v_2^{16}) = h_6 b_{2,0}^6.$$

The element $b_{2,0}^6$ detects the cube of the Adams map:

$$v_1^{12} = (v_1^4)^3 \in \pi_{24}(M(1) \wedge DM(1)).$$

Since this homotopy element has order 2, the Adams differential

$$d_2(h_6) = h_0 h_5^2$$

implies that the element $h_6 b_{2,0}^6$ detects the Toda bracket of the composite

$$S^{86} \xrightarrow{\theta_5} S^{24} \xrightarrow{2} S^{24} \xrightarrow{v_1^{12}} M(1) \wedge DM(1).$$

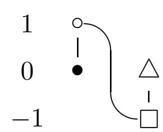
In particular, $h_6 b_{2,0}^6$ is a permanent cycle in the MASS for $M(1) \wedge DM(1)$, which contradicts the existence of a non-trivial differential $d_3((p_* \otimes 1)(v_2^{16}))$. Thus the assumption that $w \neq 0$ gives rise to a contradiction, and we conclude that $w = 0$, as desired.

We are left with verifying Claim 9.4. We will verify this claim by establishing:

1. The element
$$(1 \otimes p_* \otimes 1)(\overline{w}) \in \mathrm{Ext}^{19,114}_{A(2)_*}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])$$
is not the target of a differential in the algebraic *tmf*-resolution for
$$\mathrm{Ext}^{*,*}_{A_*}(H(1) \otimes DH(1)).$$
2. The element
$$(p_* \otimes 1)(w) \in \mathrm{Ext}^{19,114}_{A_*}(\Sigma^{12} H(1) \otimes DH(1)[-3])$$
is not the target of a d_2 differential in the MASS for $M(1) \wedge DM(1)$.

Item (1) above is verified by observing that
$$\mathrm{Ext}^{s,t}_{A(2)_*}(\mathbb{F}_2) = 0 \qquad (t-s, s) = (86, 15), (87, 15), (88, 15)$$
and so there are no possible contributions to
$$\mathrm{Ext}^{87,15}_{A(2)_*}(H(1) \otimes DH(1)),$$
and this is the only possible source for a differential in the algebraic *tmf*-resolution.

We now verify (2). The A_*-comodule $H(1) \otimes DH(1)$ has the following diagram of generators:


(9.5)

Here the straight lines encode the action of Sq^1_* and the curved line denotes a Sq^2_*. Using Bruner's computer generated $\mathrm{Ext}_{A_*}(\mathbb{F}_2)$ charts [**Bru93**], we compute the vicinity of $(p_* \otimes 1)(w)$ in $\mathrm{Ext}^{*,*}_{A_*}(H(1) \otimes DH(1))$ in Table 9.1.

$s \backslash t-s$	86	87
16	$\circ\circ$ $\triangle$ $(p_* \otimes 1)(w)$ $\bullet\bullet$	$*$
15	$*$	$*$
14	$*$	$\circ\ b_{87}$ $\square a_{87}$

Table 9.1. $\mathrm{Ext}^{s,t}_{A_*}(H(1) \otimes DH(1))$ near $(p_* \otimes 1)(w)$

In this table, entries marked with $*$ are not computed, otherwise, elements are denoted by the generator (as in (9.5)) that supports it. The only possible sources for a non-trivial d_2 are a_{87} and b_{87}.

The element a_{87} is the image of an element

$$a_{88} \in \mathrm{Ext}_{A_*}^{14,102}(\mathbb{F}_2)$$

under the inclusion of the bottom generator

$$\Sigma^{-1}\mathbb{F}_2 \to H(1) \otimes DH(1).$$

Since $\mathrm{Ext}_{A_*}^{16,103}(\mathbb{F}_2) = 0$, we deduce that $d_2(a_{88}) = 0$ in the ASS for π_*S. The map of MASSs induced from the inclusion of the bottom cell of $M(1) \wedge DM(1)$ gives $d_2(a_{87}) = 0$.

We now turn our attention to b_{87}. Table 9.2 shows the portion of $\mathrm{Ext}_{A_*}(H(1) \otimes DH(1))$ mapped to the vicinity of Table 9.1 under h_2-multiplication.

$s \backslash t-s$	83	84
15	c_{83} $\bullet$ $c'_{83}\square\square c''_{83}$	$*$
14	$*$	$*$
13	$*$	$\circ\ b_{84}$ $\square$

Table 9.2. $\mathrm{Ext}_{A_*}^{s,t}(H(1) \otimes DH(1))$ near $h_2^{-1}b_{87}$

Using the h_2 multiplicative structure in Bruner's tables [**Bru93**], we deduce that

$$\begin{aligned} h_2 b_{84} &= b_{87}, \\ h_2 c_{83} &= 0, \\ h_2 c'_{83} &= 0, \\ h_2 c''_{83} &= 0. \end{aligned}$$

Since h_2 is a permanent cycle in the ASS for the sphere, we have

$$d_2(b_{87}) = d_2(h_2 b_{84}) = h_2 d_2(b_{84}) = 0.$$

This completes our proof of Claim 9.4. $\square$

Proposition 3.6 gives the following corollary:

Corollary 9.6. *In the MASS for $\pi_*(M(1,4) \wedge DM(1,4))$, we have $d_3(v_2^{32}) = 0$.*

10. Calculation of an Adams differential

The image of the element $\bar{\kappa} \in \pi_{20}(S)_2$ in $\pi_{20}(M(1,4) \wedge DM(1,4))$ gives rise to a self-map

$$\widetilde{\kappa} : M(1,4) \to M(1,4).$$

The element $g \in \mathrm{Ext}_{A_*}^{4,24}(\mathbb{F}_2)$, which detects $\bar{\kappa}$ maps to a permanent cycle

$$\widetilde{g} \in \mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1,4))$$

which detects $\widetilde{\kappa} \in \pi_{20}(M(1,4) \wedge DM(1,4))$ in the MASS. The purpose of this section is to prove the following theorem:

Theorem 10.1.

1. *The element* $v_2^{20}h_1 \in \mathrm{Ext}^{21,142}_{A(2)_*}(H(1,4))$ *lifts to an element*
$$\widetilde{v_2^{20}h_1} \in \mathrm{Ext}^{21,142}_{A_*}(H(1,4) \otimes DH(1,4)).$$
2. *There is a differential*
$$d_3(\widetilde{v_2^{20}h_1}) = \widetilde{g}^6 + R$$
in the MASS for $M(1,4) \wedge DM(1,4)$, *where* R *is an element of filtration greater than* 0 *in the algebraic tmf-resolution.*

Proof. Table 10.1 displays a small portion of the E_1-page of the algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}(H(1,4) \wedge DH(1))$.

$s\backslash t-s$	120	121
24	$\bullet\ \bullet\ \bullet\ g^6$ $\circ\circ$	$\bullet\bullet$ $\circ$
23	$a_{120}\ \bullet\ \bullet b_{120}$ $\circ\ x_{120}$ $\odot\ y_{120}$	$\bullet\bullet$ $\circ\circ$ $\odot\odot$
22	$\bullet$ $\odot\ z_{120}$	$\odot$ $*$
21	$\bullet$ $\circ$ $\odot\ \odot\ \odot\ \odot\ \odot$ $\circledcirc\ \circledcirc\ \circledcirc\circledcirc$ $*$	$v_2^{20}h_1\ \bullet\ \bullet$ $\odot\ \odot\ \odot\odot$ $\circledcirc\circledcirc$ $*$

Table 10.1. The algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1))$ near $v_2^{20}h_1$

In this and all future tables depicting the algebraic *tmf*-resolution, we have the following key:

$\bullet$ = generator of $\mathrm{Ext}_{A(2)_*}(H(1,4) \otimes DH(1))$,
$\circ$ = generator of $\mathrm{Ext}_{A(2)_*}(M_2(1)[-1] \otimes H(1,4) \otimes DH(1))$,
$\odot$ = generator of $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2}[-2] \otimes H(1,4) \otimes DH(1))$,
$\circledcirc$ = generator of $\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3}[-3] \otimes H(1,4) \otimes DH(1))$,
$*$ = potential contribution from
$$\mathrm{Ext}_{A(2)_*}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1,4) \otimes DH(1))$$
where either for some i, $j_i > 1$, or $n > 3$.

We shall refer to all differentials in the algebraic *tmf*-resolution as d_1 differentials. Differentials in the MASS
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1))$$
will be referred to by d_r for $r \geqslant 2$.

In order to prove (1), we must show that the element $v_2^{20}h_1$ in Table 10.1 does not support a non-trivial d_1. There is one possible target z_{120} in $(t-s,s) = (120,22)$, but we will argue shortly that this possibility cannot occur. Assuming for the moment that $d_1(v_2^{20}h_1) = 0$, we would conclude that $v_2^{20}h_1$ lifts to an element

$$\widetilde{v_2^{20}h_1} \in \mathrm{Ext}_{A_*}^{21,142}(H(1,4) \otimes DH(1,4)).$$

The composite

$$H(1,4) \wedge DH(1,4) \to H(1,4) \to \mathit{tmf} \wedge H(1,4)$$

induces a map of MASSs:

$$\begin{tikzcd}
\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1,4)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(M(1,4) \wedge DM(1,4)) \arrow[d] \\
\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4)) \arrow[r, Rightarrow] & \pi_{t-s}(\mathit{tmf} \wedge M(1,4)).
\end{tikzcd}$$

In the MASS for $\mathit{tmf} \wedge M(1,4)$, there is a differential

$$d_3(v_2^{20}h_1) = g^6.$$

In order to prove (2), we need to lift this differential to the MASS for $M(1,4) \wedge DM(1,4)$. By Proposition 2.1, it suffices to lift this differential to the MASS for $M(1,4) \wedge DM(1)$:

$$\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1)).$$

The obstruction to lifting this differential is that $\widetilde{v_2^{20}h_1}$ could support a d_2 in the MASS for $M(1,4) \wedge DM(1)$. In fact, Table 10.1 demonstrates that there are four possible targets for such a d_2 in $(t-s,s) = (120,23)$: these are labeled a_{120}, b_{120}, x_{120}, y_{120}.

We now argue (1) and (2) by showing that the element $v_2^{20}h_1$ in Table 10.1 cannot support a non-trivial d_1 or d_2. We will need Tables 10.2 and 10.3, which depict the tmf-resolution in the vicinities of $gv_2^{20}h_1$ and $gv_2^4h_1$, respectively.

Write $d_1(v_2^{20}h_1) = c \cdot z_{120}$ for $c \in \mathbb{F}_2$. Then we have

$$d_1(gv_2^{20}h_1) = c \cdot gz_{120}.$$

Table 10.3 shows that $d_1(gv_2^4h_1) = 0$. Multiplying by the d_2-cycle v_2^{16} of Corollary 9.2, we deduce that we must have $d_1(gv_2^{20}h_1) = 0$. Thus c equals 0, and we have proven (1).

Write

$$d_2(v_2^{20}h_1) = c_1a_{120} + c_2b_{120} + c_3x_{120} + c_4y_{120}$$

for $c_i \in \mathbb{F}_2$. The image of $v_2^{20}h_1$ in $\mathrm{Ext}_{A(2)_*}(H(1,4))$ is a d_2-cycle in the MASS

$$\mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4)) \to \pi_{t-s}(\mathit{tmf} \wedge M(1,4)).$$

$s \backslash t-s$	140	141
27	$ga_{120} \bullet \bullet gb_{120}$ $\circ\ gx_{120}$ $\odot\ gy_{120}$	$\bullet$ $\circ\circ$ $\odot\odot$
26	$\bullet\bullet$ $\odot\ gz_{120}$	$\bullet\ x_{141}$ $\odot$ $*$
25	$\bullet$ $\circ\ \circ\ \circ$ $\odot\ \odot\ \odot$ $\circledcirc\ \circledcirc\ \circledcirc\circledcirc$ $*$	$gv_2^{20}h_1 \bullet \bullet$ $\circ\circ$ $\odot\odot$ $\circledcirc\circledcirc$ $*$

Table 10.2. The algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1))$ near $gv_2^{20}h_1$

$s \backslash t-s$	44	45
11		$\bullet$
10	$\bullet\bullet$	$\bullet\ v_2^{-16}x_{141}$
9	$\bullet$ $\circ\circ$	$gv_2^4h_1 \bullet \bullet$ $\circ\circ$ $*$

Table 10.3. The algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1))$ near $gv_2^4h_1$

We therefore deduce that $c_1 = c_2 = 0$. We wish to show that $d_2(v_2^{20}h_1) = 0$, i.e., that it is contained in the image of d_1. We have

$$d_2(gv_2^{20}h_1) = c_3 gx_{120} + c_4 gy_{120}.$$

Examining Table 10.3, we see that $d_2(gv_2^4h_1) = 0$. Since v_2^{16} is a d_2-cycle, we deduce that $d_2(gv_2^{20}h_1) = 0$. This means that

$$c_3 gx_{120} + c_4 gx_{120}$$

is in the target of a d_1. With the exception of the element x_{141}, all of the generators in $(t-s, s) = (141, 26)$ are g-periodic. Thus we have

$$d_1(E_1^{26,167}) = g \cdot d_1(E_1^{22,143}) + \mathbb{F}_2\{d_1(x_{141})\},$$

where $E_1^{s,t}$ is the E_1-term of the algebraic *tmf*-resolution for

$$\mathrm{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1)).$$

However, we see from Table 10.3 that $d_1(v_2^{-16}x_{141}) = 0$, so it follows that $d_1(x_{141}) = 0$. We may therefore deduce the vanishing of $d_2(v_2^{20}h_1)$ from the vanishing of $d_2(gv_2^{20}h_1)$. We have proven (2). $\square$

11. Proof of the main theorem

By Proposition 4.3 and Lemma 5.9, the element

$$v_2^{32} \in \mathrm{Ext}^{32,224}_{A(2)_*}(H(1,4) \otimes DH(1))$$

is a permanent cycle in the algebraic *tmf*-resolution, and it detects an element

$$v_2^{32} \in \mathrm{Ext}^{32,224}_{A_*}(H(1,4) \otimes DH(1)).$$

By Corollary 9.6, the element v_2^{32} persists to the E_4-page of the MASS for $M(1,4) \wedge DM(1)$. By Proposition 2.1, our main theorem (Theorem 1.1) is a consequence of the following lemma:

Lemma 11.1. *The element*

$$v_2^{32} \in \mathrm{Ext}^{32,224}_{A_*}(H(1,4) \otimes DH(1))$$

cannot support a non-trivial d_r in the MASS for $M(1,4) \wedge DM(1)$ for $r \geqslant 4$.

Proof. We shall make use of the following tables: Table 11.1 depicts the algebraic *tmf*-resolution for $\mathrm{Ext}^{*,*}_{A_*}(H(1,4) \otimes DH(1))$ in the region where all possible targets of $d_r(v_2^{32})$ can lie, for $r \geqslant 4$. Note that there are no non-zero elements in the algebraic *tmf*-resolution that can contribute to $\mathrm{Ext}^{s,191+s}_{A_*}(H(1,4) \otimes DH(1))$ for $s > 40$. Table 11.2 depicts a region of the algebraic *tmf*-resolution which maps to the region of Table 11.1 under g^6-multiplication. The notation in these tables is explained in Section 10. The subgroups labeled G_{191} and G_{71} are the subgroups generated by the contributions in the algebraic *tmf*-resolution labeled with a $*$.

$s \backslash t-s$	190	191
40	$\bullet$	$\bullet\bullet$
39	$\bullet$	$\bullet$
38	$\bullet\ \bullet\ \bullet$ $g^6 b_{70} \circ \circ g^6 c_{70}$	$\bullet\bullet$ $g^6 f_{71} \circ$
37	$\bullet\bullet$ $\circ$ $g^6 a_{70} \odot$	$v_2^8 k_{143} \bullet \bullet v_2^8 l_{143}$ $g^6 d_{71} \circ \circ g^6 e_{71}$ $g^6 b_{71} \odot \odot g^6 c_{71}$
36	$\bullet\bullet$ $\odot$	$\bullet$ $g^6 a_{71} \odot$ $G_{191}\ *$

Table 11.1. The algebraic *tmf*-resolution for $\mathrm{Ext}^{*,*}_{A_*}(H(1,4) \otimes DH(1))$ in the vicinity of $(t-s, s) = (191, 36)$

$s\backslash t-s$	70	71
15	$\bullet$	$\bullet$
14	$\bullet\bullet$ $b_{70} \circ \circ c_{70}$	$\bullet\bullet$ $f_{71} \circ$
13	$\bullet$ $\circ\circ$ $a_{70} \odot$	$d_{71} \circ \circ \circ e_{71}$ $b_{71} \odot \odot c_{71}$
12	$\odot$ $*$	$a_{71} \odot \odot \odot \odot \odot$ $G_{71} *$

Table 11.2. The algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}^{*,*}(H(1,4)\otimes DH(1))$ in the vicinity of $(t-s,s)=(71,12)$

The element $v_2^{32} \in \mathrm{Ext}_{A(2)_*}^{32,224}(H(1,4))$ detects a non-trivial permanent cycle of order 2 in $tmf_{192}M(1,4)$. We deduce that the element

$$v_2^{32} \in \mathrm{Ext}_{A(2)_*}(H(1,4)\otimes DH(1))$$

is a permanent cycle in the MASS for $tmf \wedge M(1,4)\wedge DM(1)$. Consider the map of MASSs

$$\begin{tikzcd} \mathrm{Ext}_{A_*}^{s,t}(H(1,4)\otimes DH(1)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(M(1,4)\wedge DM(1)) \arrow[d] \\ \mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4)\otimes DH(1)) \arrow[r, Rightarrow] & tmf_{t-s}(M(1,4)\wedge DM(1)) \end{tikzcd} \tag{11.2}$$

induced by the map

$$M(1,4)\wedge DM(1) \to tmf \wedge M(1,4)\wedge DM(1).$$

Because v_2^{32} is a permanent cycle in the MASS for $tmf\wedge M(1,4)\wedge DM(1)$, we deduce that in the MASS for $M(1,4)\wedge DM(1)$, the differential $d_r(v_2^{32})$ cannot hit an element coming from $\mathrm{Ext}_{A(2)_*}$ in the algebraic *tmf*-resolution (these elements are represented by a $\bullet$ in Table 11.1). Thus the only possible targets for $d_r(v_2^{32})$ in Table 11.1 are

$$g^6a_{71}, g^6b_{71}, g^6c_{71}, g^6d_{71}, g^6e_{71}, g^6f_{71} \tag{11.3}$$

or an element of the group G_{191}. We claim that none of these elements persist to detect a non-trivial element of the E_4-page of the MASS.

Each of the elements in (11.3) is in the image of multiplication by g^6. Because the groups G_{71} and G_{191} lie on the edge of the slope $1/5$ vanishing line of Lemmas 7.6 and 7.7, each of the elements in G_{191} are of the form g^6y for $y\in G_{71}$.

Suppose that x is a linear combination of the elements

$$a_{71}, b_{71}, c_{71}, d_{71}, e_{71}, f_{71} \tag{11.4}$$

and the elements in G_{71}. We must show that g^6x cannot be the non-trivial image of $d_r(v_2^{32})$ for $r \geqslant 4$.

If x is a d_r-cycle for $r \leqslant 3$, then x persists to E_4. Using the multiplicative structure of the MASS (Proposition 3.6) together with the fact that $g^6 = 0$ in the E_4-page of the MASS for $M(1,4) \wedge DM(1,4)$ (Theorem 10.1)[1], we deduce that $g^6 x$ is zero in E_4. It therefore cannot be a non-trivial target for $d_r(v_2^{32})$.

Suppose, however, that $d_r(x)$ is non-trivial for some $r \leqslant 3$. Since differentials in the algebraic *tmf* resolution must increase filtration, we deduce that the only possible targets for $d_r(x)$ are linear combinations of

$$a_{70}, b_{70}, c_{70}$$

and $\bullet$'s in Table 11.2 for which $t - s = 70$ and $s \geqslant 14$. However, each of these $\bullet$'s map to non-trivial permanent cycles under the map of spectral sequences (11.2), and therefore cannot be the target of MASS differentials. The only remaining possibilities are

$$\begin{aligned} &\text{Case (1):} \qquad d_1(x) = a_{70}, \\ &\text{Case (2):} \qquad d_2(x) = t_1 b_{70} + t_2 c_{70}, \end{aligned}$$

for $(0,0) \neq (t_1, t_2) \in \mathbb{F}_2 \oplus \mathbb{F}_2$. Using Theorem 10.1 we see that in these cases we would respectively have:

$$\begin{aligned} &\text{Case (1):} \qquad d_1(g^6 x) = g^6 a_{70}, \\ &\text{Case (2):} \qquad d_2(g^6 x) = t_1 g^6 b_{70} + t_2 g^6 c_{70}. \end{aligned}$$

If we are in Case (1), we are done: the differential $d_r(v_2^{32})$ cannot be detected by $g^6 x$ because $g^6 x$ does not persist to E_2. If we are in Case (2), however, we must verify that $t_1 g^6 b_{70} + t_2 g^6 c_{70}$ is not in the image of a d_1-differential. The only possibility is

$$d_1(s_1 v_2^8 k_{143} + s_2 v_2^8 l_{143}) = t_1 g^6 b_{70} + t_2 g^6 c_{70}. \tag{11.5}$$

The algebraic *tmf*-resolution for $\mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1))$ in the vicinity of the elements k_{143} and l_{143} is displayed below.

$s \backslash t-s$	142	143
30	$\bullet$	$\bullet$
29	$\bullet\bullet$	$k_{143} \bullet \bullet l_{143}$

We see that k_{143} and l_{143} must be d_1-cycles. By Proposition 4.3 and Lemma 5.9, we deduce that $v_2^8 k_{143}$ and $v_2^8 l_{143}$ must be d_1-cycles. Thus Possibility (11.5) cannot occur, and we deduce that in Case (2), $d_2(g^6 x)$ does not vanish. We conclude that in Case (2), $g^6 x$ cannot persist to E_4 and therefore it cannot be the target of $d_r(v_2^{32})$ for $r \geqslant 4$. $\square$

[1] This statement must be interpreted with care — Theorem 10.1 asserts that there is an element in E_2 of the MASS for $M(1,4) \wedge DM(1,4)$ which is detected by $\widetilde{g}^6$ in the algebraic *tmf*-resolution, and which is the target of a d_3 in the MASS.

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