Numerical Calculation of Twisted Adjoint L-Values Attached to Modular Forms

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CONTENTS

Doi, Hida, and Ishii have shown that the values of twisted adjoint L-functions $L(1, \mathrm{Ad}(f) \otimes \chi)$ attached to modular forms f are closely connected with discriminants of Hecke fields. Goto has given a numerical example of this L-value for an elliptic cusp form f of level 1 and weight 20. We shall show a method to calculate the L-values, which is more effective than Goto's, and give new numerical examples.

1. INTRODUCTION

In [Doi et al. 1998], theoretical and experimental evidences have been given for the divisibility of the discriminant of Hilbert modular Hecke fields by the twisted adjoint L-values of elliptic cusp forms at $s = 1$. Goto [1998] has given a numerical example of this L-value for an elliptic cusp form of level 1 and weight 20. Using Hida's identity (Theorem 2.1), he reduced the computation of the L-value to those of Rankin products, which are more accessible by a work of Shimura. However, by high-dimensionality of the space concerned, his calculation was limited to weight 20 and level 1.

In this paper, we shall show a way to reduce the dimension of the space involved by employing an involution operator. By extending the computation to higher weight, one can look into the case where we have two base-change lifts: one from level 1 and another from the "Neben" space of the character χ below. There could be two possibilties:

(1) The Hecke fields of non-lifts is split into more than 1 pieces; one for each lift so that the discriminant is divisible by the $L(1, \mathrm{Ad}(f) \otimes \chi)$ of each f;
(2) The Hecke field of the non-lift is a single field with discriminat divisible by the product of two

1058-6458/2000 $0.50 per page
Experimental Mathematics **9**:1, page 67

L-values: one from level 1 and another from the Neben space.

In the limit of our computation, (2) is always the case. This is a unique new finding in our work (apart from the computational advantage).

We denote by $S_k(N)$ the space of cusp forms of weight k and level N. For a prime N such that $N \equiv 1 \pmod 4$, we let $F = \mathbb{Q}(\sqrt{N})$, and write χ for the Legendre symbol $\left(\frac{N}{}\right)$. For a primitive form $f \in S_k(1)$ and the character χ, we denote by $L(s, \mathrm{Ad}(f) \otimes \chi)$ the twisted adjoint L-function of f, which is, up to the shift: $s \mapsto s + k - 1$, the symmetric square L-function defined in [Shimura 1975]. By [Sturm 1989], it is known the L-value $L(1, \mathrm{Ad}(f) \otimes \chi)/\big(\pi^{k+2}\langle f, f\rangle\big)$ is algebraic and contained in K_f, where K_f is the algebraic number field generated by the Fourier coefficients of f and $\langle f, f\rangle$ is the normalized Petersson inner product.

For $f = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z} \in S_k(1)$, we denote by f^χ the twist of f by χ, that is,

$$f^\chi(z) = \sum_{n=1}^{\infty} \chi(n)a(n)e^{2\pi i n z}.$$

We remark that $f^\chi \in S_k^0(N^2)$, which is the space generated by the primitive forms of level N^2. We denote by $\widehat{f}$ the base-change lift to $\mathrm{GL}(2)_{/F}$ of the elliptic cusp form f. As in [Goto 1998] we can see that the calculation of the L-value $L(1, \mathrm{Ad}(f) \otimes \chi)$ is reduced to that of $D(k-2; \widehat{f}, \widehat{g}\,)$, $D(k-2; f, g)$, and $D(k-2; f^\chi, g)$. We shall show mainly the way to calculate the value of $D(k-2; f^\chi, g)/\big(\pi^k\langle f^\chi, f^\chi\rangle\big)$. This calculation is reduced to that of

$$\langle f^\chi, h_0\rangle / \langle f^\chi, f^\chi\rangle$$

by [Shimura 1976]. Here h_0 is a cusp form in $S_k(N^2)$, which is defined by (2–4) and (2–5) in the text.

Taking the involution $\tau_{N^2} = \left(\begin{smallmatrix} 0 & -1 \\ N^2 & 0 \end{smallmatrix}\right)$, we put $h_0^* = (h_0 + h_0\|_k \tau_{N^2})/2$. Then we can show h_0^* is the element of $\big(S_k^0(N^2)^-\big)^\perp$, and $\langle f^\chi, h_0\rangle = \langle f^\chi, h_0^*\rangle$ (see Lemma 4.1). We note that if $f \in S_k(1)$ then f^χ is contained in the subspace $S_k^0(N^2)^+$ of $S_k^0(N^2)$. Here see (3–1) for $S_k^0(N^2)^\varepsilon$ ($\varepsilon = +, -$). So we can calculate the value in the case of higher weight.

In fact, in the last section, we shall give numerical examples of the special values of $D(k-2; f^\chi, E_{4,1}^*)$ and the twisted adjoint L-values of $L(1, \mathrm{Ad}(f) \otimes \chi)$ for $N = 5$ and $k = 22, 24$.

2. PRELIMINARIES

Though Hida's identity and some facts on the holomorphic projection have been described in [Goto 1998; Shimura 1976], for reader's convenience, we recall them in this section.

We write $e(z) = e^{2\pi i z}$. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field, and the complex number field, respectively. The upper half complex plane is denoted by

$$\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Let k and l be positive integers. For a cusp form f and a modular form g with Fourier expansions

$$f(z) = \sum_{n=1}^{\infty} a(n)\boldsymbol{e}(nz) \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b(n)\boldsymbol{e}(nz),$$

we put

$$D(s; f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s} \quad \text{for } s \in \mathbb{C}.$$

We let N, χ and F be as in the introduction. For a primitive form $f \in S_k(1)$ having the Fourier expansion $f(z) = \sum_{n=1}^{\infty} a(n)\boldsymbol{e}(nz)$, we define the twisted adjoint L-function associated with f by

$$\begin{aligned} L(s, &\mathrm{Ad}(f) \otimes \chi) \\ &= \prod_p \big((1 - \chi(p)\alpha_p\beta_p^{-1}p^{-s})(1 - \chi(p)p^{-s}) \\ &\qquad \cdot (1 - \chi(p)\alpha_p^{-1}\beta_p p^{-s})\big)^{-1} \end{aligned}$$

for $s \in \mathbb{C}$, where the product is taken over all rational primes p, and

$$\alpha_p + \beta_p = a(p), \quad \alpha_p\beta_p = p^{k-1}.$$

Theorem 2.1 (Hida). *We suppose $k > l$, and let $f \in S_k(1)$ and $g \in G_l(1)$ be primitive forms. Then for an integer m such that*

$$\tfrac{1}{2}(k + l) - 1 < m < k, \tag{2–1}$$

the following identity holds:

$$\frac{D(m; \widehat{f}, \widehat{g})}{N^{1/2}\pi^{2k}\langle \widehat{f}, \widehat{f}\rangle} = c \cdot \frac{D(m; f, g)}{\pi^k\langle f, f\rangle} \cdot \frac{D(m; f^\chi, g)}{\pi^k\langle f^\chi, f^\chi\rangle} \cdot \left(\frac{L(1, \mathrm{Ad}(f) \otimes \chi)}{\pi^{k+2}\langle f, f\rangle}\right)^{-1},$$

where

$$c = \frac{2^{2k}(N+1)(N^{m'} - 1)B_{2,\chi}B_{m'}}{N^2\Gamma(k)B_{m',\chi}} \cdot \frac{\langle f^\chi, f^\chi\rangle}{\langle f, f\rangle}.$$

with $m' = 2(m+1) - k - l$. *Here* $B_{m'}$ *is the* m'*-th Bernoulli number and* $B_{m',\chi}$ *is the* m'*-th generalized Bernoulli number associated with* χ.

In our case, we know that

$$\frac{\langle f^\chi, f^\chi\rangle}{\langle f, f\rangle} = \left(N^{(k/2)-1}(N+1) + a(N)\right) \cdot \left(N^{(k/2)-1}(N+1) - a(N)\right) \cdot \frac{N-1}{N^k(N+1)};$$

see [Shimura 1976, Proof of Proposition 1]. Here $a(N)$ is the N-th Fourier coefficient of a primitive form f in $S_k(1)$.

We consider the case $k > l+2$ and put $m = k-2$, then m satisfies the condition (2–1).

Now we assume that l is an even integer greater than 2. For $z \in \mathfrak{H}$, we define

$$E_{l,1}(z) = 2\zeta(l) + 2\frac{(2\pi i)^l}{(l-1)!}\sum_{n=1}^{\infty}\sigma_{l-1}(n)\boldsymbol{e}(nz),$$
$$E^*_{l,1}(z) = 2^{-1}\zeta(l)^{-1}E_{l,1}(z),$$

where ζ is the Riemann zeta function and $\sigma_{l-1}(n) = \sum_{0<d|n} d^{l-1}$.

For $\lambda > 0$, we put

$$E^*_{\lambda,N^2}(z) = (1-N^{-\lambda})^{-1}\left(E^*_{\lambda,1}(N^2z) - N^{-\lambda}E^*_{\lambda,1}(Nz)\right) \tag{2–2}$$

and define a differential operator δ_λ by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right) \quad (z = x+iy).$$

We assume that $k > l+4$ and f belongs to $S_k(N^2)$. We put $\lambda = k-l-2$. Since $l > 2$ and $E^*_{l,1} \in G_l(1)$, by [Shimura 1976, Theorem 2], we obtain

$$D(k-2; f, E^*_{l,1}) = t \cdot \pi^k \langle f, E^*_{l,1}\delta_\lambda E^*_{\lambda,N^2}\rangle. \tag{2–3}$$

Here

$$t = t(N,k,l) = -\frac{4^{k-1}N(N+1)}{3(k-l-2)\left((k-3)!\right)}.$$

By [Shimura 1976], there exist unique elements $h_0 \in G_k(N^2)$ and $h_1 \in G_{k-2}(N^2)$, satisfying the following identity:

$$E^*_{l,1}\delta_\lambda E^*_{\lambda,N^2} = h_0 + \delta_{k-2}h_1. \tag{2–4}$$

Then we have

$$\langle f, E^*_{l,1}\delta_\lambda E^*_{\lambda,N^2}\rangle = \langle f, h_0\rangle \quad \text{for every } f \in S_k(N^2) \tag{2–5}$$

and

$$h_0 = \frac{1}{2\pi i(k-2)} \cdot \left(l \cdot E^*_{l,1} \cdot \frac{\partial}{\partial z}E^*_{\lambda,N^2} - \lambda \cdot \frac{\partial}{\partial z}E^*_{l,1} \cdot E^*_{\lambda,N^2}\right), \tag{2–6}$$

which is a cusp form, because a derivative of any (slowly increasing) holomorphic modular form is fast decreasing.

3. THE INVOLUTION τ

For an integer M, we let W be the subspace of $S_k(M)$ generated by all the functions of the form $f^{(m)}(z) = f(mz)$, for $f(z) \in S_k(M')$ with $M' \mid M$, $M' \neq M$, and m is a positive integer dividing M/M'. Then we write $S^0_k(M)$ for the orthogonal complement of W in $S_k(M)$ with respect to the Petersson inner product.

Hereafter in this section we assume that N is a rational prime and k is an even integer. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \{\alpha \in \mathrm{GL}_2(\mathbb{R}) \mid \det(\alpha) > 0\}$, we put

$$f\|_k\alpha(z) = (\det(\alpha))^{k/2}(cz+d)^{-k}f(\alpha(z)),$$

where z is the variable on $\mathfrak{H}$, so

$$\alpha(z) = (az+b)/(cz+d).$$

Now for an integer M, we let $\{f_1, \dots, f_d\}$ be the basis of consisting of primitive forms of $S^0_k(M)$. Using this basis, for an integer M', we denote by $S^0_k(M)^{(M')}$ the space generated by $\{f_1^{(M')}, \dots, f_d^{(M')}\}$. Then we know that the space $S_k(N^2)$ can be the following direct sum of subspaces as follows (see [Miyake 1989]):

$$S_k(N^2) = S^0_k(N^2) \oplus S^0_k(N) \oplus S^0_k(1) \oplus W,$$
$$W = S^0_k(N)^{(N)} \oplus S^0_k(1)^{(N^2)} \oplus S^0_k(1)^{(N)}.$$

We remark that the space $S^0_k(N^2)$ is orthogonal to its complement $S^0_k(N) \oplus S^0_k(1) \oplus W$. For an integer M, we put $\tau_M = \left(\begin{smallmatrix} 0 & -1 \\ M & 0\end{smallmatrix}\right)$. Using τ_{N^2}, $S^0_k(N^2)$ can be decomposed as follows:

$$S^0_k(N^2) = S^0_k(N^2)^+ \oplus S^0_k(N^2)^-.$$

Here

$$S^0_k(N^2)^\varepsilon = \{f \in S^0_k(N^2) \mid f\|_k\tau_{N^2} = \varepsilon f\}, \tag{3–1}$$

where $\varepsilon = +, -$. Now, by [Atkin and Lehner 1970], the operator "$f \mapsto f\|_k\tau_{N^2}$" induces an involution of $S_k(N^2)$. Since k is even, for $f \in S^0_k(N^2)$, we have $(f\|_k\tau_{N^2})\big\|_k\tau_{N^2} = (-1)^k f = f$. We obtain:

Lemma 3.1. *The operator "$f \mapsto f\|_k\tau_{N^2}$" induces the following isomorphisms*:

$$\begin{gathered} S_k^0(N^2)^\varepsilon \xrightarrow{\sim} S_k^0(N^2)^\varepsilon \quad (\varepsilon = +,-), \\ S_k^0(N) \xrightarrow{\sim} S_k^0(N)^{(N)}, \quad S_k^0(N)^{(N)} \xrightarrow{\sim} S_k^0(N), \\ S_k^0(1) \xrightarrow{\sim} S_k^0(1)^{(N^2)}, \quad S_k^0(1)^{(N^2)} \xrightarrow{\sim} S_k^0(1), \\ S_k^0(1)^{(N)} \xrightarrow{\sim} S_k^0(1)^{(N)}. \end{gathered}$$

We let $c(m,f)$ denote the m-th Fourier coefficient of f at ∞. By (2–4), (2–5), Lemma 3.1 and simple calculation, we obtain:

Proposition 3.2. *For the element $h_0 \in G_k(N^2)$ which is defined in* (2–4), *we have the m-th Fourier coefficients of h_0 and $h_0\|_k\tau_{N^2}$*:

$$\begin{aligned} c(m,h_0) &= C\Big(mB_l a(m) + m(N^\lambda-1)B_\lambda\sigma_{l-1}(m) \\ &\qquad + 2\sum_{i=1}^{m-1} c(i)\sigma_{l-1}(i)a(m-i)\Big), \\ c(m,h_0\|_k\tau_{N^2}) & \\ &= N^l C\Big(mB_l a'(m) + m(N^\lambda-1)B_\lambda\sigma_{l-1}(m) \\ &\qquad + 2\sum_{i=1}^{m-1} c(i)\sigma_{l-1}(i/N^2)a'(m-i)\Big), \end{aligned}$$

where

$$\begin{aligned} a(n) &= \sigma_{\lambda-1}(n/N) - N^\lambda\sigma_{\lambda-1}(n/N^2), \\ a'(n) &= \sigma_{\lambda-1}(n/N) - \sigma_{\lambda-1}(n), \\ C &= \frac{2l\lambda}{(k-2)(N^\lambda-1)B_l B_\lambda}, \\ c(i) &= c_m(i) = (k-2)i - lm, \\ \sigma_{\lambda-1}(n) &= \begin{cases} \sum_{0<d\mid n} d^{\lambda-1} & \textit{if } n \textit{ is an integer}, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned}$$

4. AN EXPLICIT FORMULA OF $D(k-2; f, E^*_{l,1})$

We continue with the notation of Sections 2 and 3. We shall describe an explicit formula of $D(k-2; f, E^*_{l,1})$ for a primitive form $f \in S_k^0(N^2)^+$.

Lemma 4.1. *Set $h_0^* = (h_0 + h_0\|_k\tau_{N^2})/2$. Then h_0^* is contained in the orthogonal complement of $S_k^0(N^2)^-$ in $S_k(N^2)$, and we have*

$$\langle f, h_0\rangle = \langle f, h_0^*\rangle \tag{4–1}$$

for every primitive form $f \in S_k^0(N^2)^+$.

Proof. We let $\{f_1^+, \dots, f_{d^+}^+\}$ and $\{f_1^-, \dots, f_{d^-}^-\}$ be the basis of consisting of primitive forms of $S_k^0(N^2)^+$ and $S_k^0(N^2)^-$, respectively. We put

$$(f_1, \dots, f_{d_0}) = (f_1^+, \dots, f_{d^+}^+, f_1^-, \dots, f_{d^-}^-).$$

Then $\{f_1, \dots, f_{d_0}\}$ is the basis of consisting of primitive forms of $S_k^0(N^2)$. We put simply $f = f_1$. Since $h_0 \in S_k(N^2)$, we have

$$h_0 = x_1 f + x_2 f_2 + \dots + x_{d_0} f_{d_0} + g.$$

Here $x_i \in \mathbb{C}$ $(i = 1, \dots, d_0)$ and $g \in S_k^0(N) \oplus S_k^0(1) \oplus W$. Then, by Lemma 3.1, we have

$$\begin{aligned} h_0\|_k\tau_{N^2} &= x_1 f + x_2 f_2 + \dots + x_{d^+} f_{d^+} \\ &\qquad - \big(x_{d^++1} f_{d^++1} + \dots + x_{d_0} f_{d_0}\big) + \tilde g, \end{aligned}$$

where $\tilde g \in S_k^0(N) \oplus S_k^0(1) \oplus W$. So, we obtain

$$h_0^* = x_1 f + x_2 f_2 + \dots + x_{d^+} f_{d^+} + (g + \tilde g)/2,$$

where $(g+\tilde g)/2 \in S_k^0(N) \oplus S_k^0(1) \oplus W$. Thus h_0^* is contained in the orthogonal complement of $S_k^0(N^2)^-$ in $S_k(N^2)$. Therefore we obtain (4–1). □

We put $d = \dim S_k^0(N^2)^+ \oplus S_k^0(N) \oplus S_k^0(1)$. Then, by the same arguments as [Goto 1998], and also our Proposition 3.2 and Lemma 4.1, we obtain:

Proposition 4.2. *We let p be a prime $\neq N$ and l be an even integer > 2. We assume that $k > l+4$ and f is a primitive form belonging to $S_k^0(N^2)^+$. And we denote by α the p-th Fourier coefficient of f. Then*

$$\frac{D(k-2; f, E^*_{l,1})}{\pi^k\langle f, f\rangle} = t \cdot \frac{\psi(\alpha)}{\varphi(\alpha)},$$

where

$$\begin{aligned} \psi(\alpha) &= \sum_{i=0}^{d-1}\Big(\sum_{j=i}^{d-1} \beta_{d-j} c_{j-i}\Big)\alpha^i, \\ \varphi(\alpha) &= \Phi'(\alpha) = \sum_{i=0}^{d-1} c_i (d-i)\alpha^{d-i-1}, \\ t &= t(N,k,l) = -\frac{4^{k-1}N(N+1)}{3(k-l-2)\big((k-3)!\big)}, \\ \beta_i &= \sum_{r=0}^{[(i-1)/2]} \Big(\tbinom{i-1}{r} - \tbinom{i-1}{r-1}\Big) p^{r(k-1)} c(p^{i-2r-1}, h_0^*), \\ \Phi(x) &= \sum_{i=0}^{d} c_i x^{d-i} = \Phi^{[2,+]}(x)\cdot\Phi^{[1]}(x)\cdot\Phi^{[0]}(x). \end{aligned}$$

Moreover, $c(m,h_0^*)$ is the m-th Fourier coefficient of h_0^*; $\Phi^{[2,+]}(x)$, $\Phi^{[1]}(x)$, and $\Phi^{[0]}$ are the characteristic *polynomials of the Hecke operator $T(p)$ in $S_k^0(N^2)^+$, $S_k^0(N)$, and $S_k^0(1)$, respectively. In this situation we assume that $\Phi(x)$ has no multiple roots, and we take $\binom{i-1}{-1}$ as 0.*

Remark. The number x_1 in the proof of Lemma 4.1 is equal to $\psi(\alpha)/\varphi(\alpha)$ in this proposition.

We emphasize that d is almost a half of $\dim S_k(N^2)$.

5. NUMERICAL EXAMPLES

We assume $F = \mathbb{Q}(\sqrt{5})$. In this section we give two numerical examples of L-values $L(1, \mathrm{Ad}(f) \otimes \chi)$, where $f \in S_k(1)$ is a primitive form and $\chi = \left(\frac{5}{\cdot}\right)$. By [Shimura 1971, Propositions 3.64 and 3.65], we remark that if $f \in S_k(1)$ then $f^\chi \in S_k^0(N^2)^+$. In the following examples, we use the same notation as in Lemma 4.1 and Proposition 4.2.

$$\boxed{\text{We fix } p = 2.}$$

Example 1. We let $k = 22$, $l = 4$, and f be a primitive form belonging to $S_{22}(1)$. Then f^χ is a primitive form belonging to $S_{22}^0(5^2)^+$. We shall obtain the special value $D(20; f^\chi, E_{4,1}^*)/\big(\pi^{22}\langle f^\chi, f^\chi\rangle\big)$.

Now, in Proposition 4.2, we have

$$t = t(5, 22, 4) = -\frac{2^{23}}{3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19}.$$

The characteristic polynomials of $T(2)$ for each subspaces are given as follows:

$$\begin{aligned}
\Phi^{[2,+]}(x) = {} & (x - 288) \\
& \cdot (x^3 - 1312x^2 - 2780624x + 2939762688) \\
& \cdot (x^4 + 2910x^3 - 4542888x^2 \\
& \quad - 15642931840x - 4309053579264) \\
& \cdot (x^7 + 737x^6 - 10943402x^5 - 8245654024x^4 \\
& \quad + 30199145968768x^3 + 18103551357526016x^2 \\
& \quad - 19599947376572104704x \\
& \quad - 12132460755606042574848), \\
\Phi^{[1]}(x) = {} & (x^3 + 1312x^2 - 2780624x - 2939762688) \\
& \cdot (x^4 - 2910x^3 - 4542888x^2 \\
& \quad + 15642931840x - 4309053579264), \\
\Phi^{[0]}(x) = {} & x + 288.
\end{aligned}$$

Since $\alpha = c(2, f^\chi) = 288$,

$$\begin{aligned}
x_1 &= \psi(288)/\varphi(288) \\
&= -\frac{2^{13} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 17 \cdot 13963 \cdot 219169}{13 \cdot 89 \cdot 173 \cdot 313 \cdot 2939 \cdot 3617 \cdot 11489}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{D(20; f^\chi, E_{4,1}^*)}{\pi^{22}\langle f^\chi, f^\chi\rangle} \\
&\quad = \frac{2^{36} \cdot 7 \cdot 13963 \cdot 219169}{3^4 \cdot 5 \cdot 11 \cdot 13^2 \cdot 19 \cdot 89 \cdot 173 \cdot 313 \cdot 2939 \cdot 3617 \cdot 11489}.
\end{aligned}$$

Next we show the L-value

$$L(1, \mathrm{Ad}(f) \otimes \chi)/\big(\pi^{24}\langle f, f\rangle\big).$$

Firstly, by an easy computation we have

$$\frac{D(20; f, E_{4,1}^*)}{\pi^{22}\langle f, f\rangle} = \frac{2^{32}}{3^6 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 3617}. \tag{5–1}$$

On the other hand, in the same way as in [Doi and Ishii 1994] we obtain

$$\begin{aligned}
&\frac{D(20; \widehat{f}, \widehat{E_{4,1}^*})}{5^{1/2}\pi^{44}\langle \widehat{f}, \widehat{f}\rangle} \\
&\quad = \frac{2^{51} \cdot 13963 \cdot 219169}{3^{15} \cdot 5^8 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 71 \cdot 457 \cdot 3617 \cdot 33092833}.
\end{aligned} \tag{5–2}$$

Now we put $g = 2^{-1}\zeta(-3)E_{4,1}^*$. Then g is a primitive form belonging to $G_4(1)$ and $\widehat{g} = 2^{-2}\zeta_F(-3)\widehat{E_{4,1}^*}$. In this case, the constant c defined in Theorem 2.1 is given as

$$c = \frac{2^{36} \cdot 7 \cdot 13 \cdot 89 \cdot 173 \cdot 313 \cdot 2939 \cdot 3617 \cdot 11489}{3^8 \cdot 5^{26} \cdot 7^3 \cdot 11 \cdot 17 \cdot 19 \cdot 457 \cdot 33092833},$$

where $a(5) = 21640950$. Therefore, by Theorem 2.1, we obtain

$$\frac{L(1, \mathrm{Ad}(f) \otimes \chi)}{\pi^{24}\langle f, f\rangle} = \frac{2^{49} \cdot 71}{3^4 \cdot 5^{21} \cdot 7 \cdot 11 \cdot 19}.$$

We note that the discriminant of the field generated by the Fourier coefficients of "proper" Hilbert cusp form of weight $(22, 22)$ with respect to $\mathrm{GL}(2)_{/F}$ is

$$5 \cdot 71 \cdot 2867327.$$

For the primes 5 and 2867327, one finds these primes appear in the numerator of $L(1, \mathrm{Ad}(f) \otimes \chi)$ (see [Doi et al. 1998]). Here f is the "Neben" primitive cusp form with the character $\chi = \left(\frac{5}{\cdot}\right)$, level 5 and weight 22.

Example 2. We take $k = 24$, $l = 4$ and let f be a primitive form belonging to $S_{24}(1)$. Here we show the special value $D(22; f^\chi, E_{4,1}^*)/\big(\pi^{24}\langle f^\chi, f^\chi\rangle\big)$. In Proposition 4.2, we have

$$t = t(5, 24, 4) = -\frac{2^{28}}{3^{11} \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19},$$

The characteristic polynomials of $T(2)$ for each subspace are:

$$\begin{aligned}\Phi^{[2,+]}(x)=&(x^2+1080x-20468736)\\&\cdot(x^3+666x^2-7619376x-5728572416)\\&\cdot(x^4-780x^3-22815912x^2\\&\quad-16729054720x+38396524609536)\\&\cdot(x^8-3015x^7-46537314x^6+118994796720x^5\\&\quad+643319168247936x^4\\&\quad-1262140353147755520x^3\\&\quad-248868226804133193318 4x^2\\&\quad+2550805516444321122877440x\\&\quad+1009109905157137608984231936),\\ \Phi^{[1]}(x)=&(x^3-666x^2-7619376x+5728572416)\\&\cdot(x^4+780x^3-22815912x^2\\&\quad+16729054720x+38396524609536),\\ \Phi^{[0]}(x)=&x^2-1080x-20468736.\end{aligned}$$

It has been known from the time of Hecke that

$$K_f=K_{f^\chi}=K_{\hat{f}}=\mathbb{Q}(\sqrt{144169}).$$

For $\alpha\in K_f$, we put $N(\alpha)=N_{K_f/\mathbb{Q}}(\alpha)$. By Proposition 4.2, we obtain

$$N\left(\frac{D(22;f^\chi,E^*_{4,1})}{\pi^{24}\langle f^\chi,f^\chi\rangle}\right)=\frac{2^{78}\cdot p_1\cdot p_2}{3^{11}\cdot5^3\cdot7^3\cdot11^2\cdot13^2\cdot19\cdot31^3\cdot829^2\cdot5167^2\cdot q_0^2\cdot q_1\cdot q_2\cdot q_3},$$

where we have given names to several primes:

$$\begin{aligned}p_1&=23820607970513,\\p_2&=5324628462248993,\\q_0&=43867,\\q_1&=144169,\\q_2&=80311577,\\q_3&=136379767.\end{aligned}$$

From this we can compute the special value

$$L(1,\mathrm{Ad}(f)\otimes\chi)/\big(\pi^{26}\langle f,f\rangle\big).$$

In the same fashion as for (5–1) we have

$$N\left(\frac{D(22;f,E^*_{4,1})}{\pi^{24}\langle f,f\rangle}\right)=\frac{2^{74}\cdot73}{3^{12}\cdot5^5\cdot7^4\cdot11^6\cdot13^2\cdot17\cdot19\cdot q_0^2\cdot q_1},$$

and just as for (5–2) we have

$$N\left(\frac{D(22;\widehat{f},\widehat{E^*_{4,1}})}{5^{1/2}\pi^{48}\langle\widehat{f},\widehat{f}\rangle}\right)=\frac{2^{143}\cdot73\cdot p_1\cdot p_2}{3^{30}\cdot5^{17}\cdot7^{10}\cdot11^9\cdot13^4\cdot17^2\cdot19^3\cdot41^2\cdot109\cdot p_0\cdot q_0^2\cdot q_1\cdot q_4^2},$$

where in addition we have set

$$\begin{aligned}p_0&=54449,\\q_4&=317680421579.\end{aligned}$$

In this case the constant c in Theorem 2.1 is

$$N(c)=\frac{2^{80}\cdot31^3\cdot829^2\cdot5167^2\cdot{q_0}^2\cdot q_2\cdot q_3}{3^{14}\cdot5^{52}\cdot7^5\cdot11^3\cdot13\cdot17^2\cdot19^2\cdot23^2\cdot41^2\cdot{q_4}^2},$$

where $a(5)=36534510\pm180480\sqrt{144169}$. Thus we obtain

$$N\left(\frac{L(1,\mathrm{Ad}(f)\otimes\chi)}{\pi^{26}\langle f,f\rangle}\right)=\frac{2^{81}\cdot109\cdot54449}{3^9\cdot5^{45}\cdot7^2\cdot11^2\cdot13\cdot17\cdot19\cdot23^2\cdot144169}.$$

By [Doi and Ishii 1994], we note that the discriminant of the field generated by the Fourier coefficients of "proper" Hilbert cusp form of weight $(24,24)$ with respect to $\mathrm{GL}(2)_{/F}$ is

$$5\cdot109\cdot54449\cdot15505829.$$

Here again the primes 5 and 15505829 appear in the numerator of $L(1,\mathrm{Ad}(f)\otimes\chi)$ for the Neben cusp form f with the character $\chi=\left(\frac{5}{\cdot}\right)$, level 5 and weight 24 (see [Doi et al. 1998]).

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