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ASYMPTOTIC HIGH-ORDER SCHEMES FOR INTEGRO-DIFFERENTIAL PROBLEMS ARISING IN MARKETS WITH JUMPS*

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Abstract. In this paper we deal with the numerical approximation of integro-differential equations arising in financial applications in which jump processes act as the underlying stochastic processes. Our aim is to find finite differences schemes which are high-order accurate for large time regimes. Therefore, we study the asymptotic time behavior of such equations and we define as *asymptotic high-order schemes* those schemes that are consistent with this behavior. Numerical tests are presented to investigate the efficiency and the accuracy of such approximations.

Key words. Integro-differential equations, jump processes, finite differences methods, asymptotic behavior

AMS subject classifications. 65M15, 65M06, 47G20, 35L45

1. Introduction

It is common sense that the choice of approximation schemes of a differential problem depends on the qualitative analysis that we make of the problem, however rudimentary. If we expect, for example, to have solutions which are uniformly bounded in time, we will try to find methods which preserve this property. This is not always easy, also due to the problem of obtaining a fine "qualitative analysis" of the solutions, in particular for nonlinear problems.

In this paper we are concerned with time dependent integro-differential problems which arise in some option pricing problems described by pure jump markets. For these problems, we shall introduce some schemes which are increasingly accurate for large times, with respect to the asymptotic behavior of solutions. This property of accuracy is required in order to get better results for large time simulations when computing perturbations of nonconstant stable states.

Given a family of stable asymptotic states for a given evolutionary problem, we say that a numerical scheme is *Asymptotic High Order*. In the following we simply write AHO, if it is high-order accurate, with respect to the local truncation error, when restricted to every element of this family.

A similar approach has been first introduced by Roe [17] for hyperbolic conservation laws with source term. The author proposed the upwinding of the source term, giving a first example of a first order monotone scheme, which is second order on all steady states. In the same context, the difficulties on the numerical treatment of the source term gave origin to a large series of works dealing with the derivation and properties of the so-called "well-balanced" solvers. Well-balanced schemes have been introduced by Bermudez and Vazquez [2], Greenberg and LeRoux [9], and then further developed by Gosse and LeRoux [8], LeVeque [13], Jin [10], Botchorishvili, Perthame and Vasseur [3], and Perthame and Simeoni [15]. See the book [4] and references therein for more references and comments.

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Here we focus our attention on the numerical solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = \gamma(K*u-u), \\ \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where the convolution kernel verifies

$$\text{[K1]} \qquad K \geq 0, \quad \int_{-\infty}^{+\infty} K(x)dx = 1, \quad K(-x) = K(x),$$

and $u_0(x)$ is an integrable function.

The interest in such models can be motivated by their applications in *mathematical finance* in which the pure jump process acts as the underlying stochastic process, see for instance [16, 5, 7]. Another interesting application is when considering Asian options with jumps, where at least one space direction has no diffusion term, and so we are faced again with a pure diffusion process at least in that direction. It is also worth mentioning that similar equations arise as simplified models for *radiating gases*, see [18, 12].

In the context of mathematical finance, direct numerical issues, for the general case of jump-diffusion processes, were already considered in [1], [14] and in the book by R. Cont [7]. A first convergence theory for monotone explicit schemes was given for a general class of integro-differential Cauchy problems in [5]. In [6] the authors proposed to apply Implicit-Explicit (IMEX) Runge-Kutta methods for the time integration of those equations, to solve the integral term explicitly, giving schemes with high-order time-accuracy, and stable under weak time-step restrictions.

In this paper, we focus our attention on semi-discrete finite difference approximations for the space variable. Our objective is to use the analysis of the time-asymptotic behavior of the solutions to the Cauchy problem (1.1) to design high-order numerical schemes for large times. Here we only consider the case where the asymptotic states are described by stationary solutions. Therefore, we can find schemes that are consistent and monotone, but such that they are high-order accurate when computed on every stationary solution. In that way, for every given initial condition, the scheme selects automatically its right asymptotic state, then improving for large times the accuracy of the scheme for a fixed space mesh.

The paper is organized as follows. Section 2 is devoted to some background. In Section 3 we consider AHO schemes for a simpler model problem, a linear hyperbolic purely differential equation, obtaining a fourth-order scheme in the steady state. In Section 4 we investigate the asymptotic behavior of the solution to the Cauchy problem (1.1). The asymptotic behavior for this problem has been studied in [11] and in [18] as a reduced case of a more general theory for a hyperbolic-elliptic coupled system. Starting from these results we obtain the decay rate of the solution and its derivatives for suitable initial data. Applying the results of Section 4, in Section 5 we give a rigorous result about the accuracy of AHO schemes, and in Section 6 we present two such schemes. Section 7 is devoted to numerical tests.

2. Notations and Background

In the following, we shall consider problems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = L(u) \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where the operator $L(\cdot)$ involves the partial derivatives of u in the space variable and the convolution term; L does not involve t explicitly and we shall assume that L is an homogeneous linear operator.

Our finite difference approximation will be defined on a uniform mesh, with the time interval Δt constant. The discrete region $\Omega_h \subset \mathbb{R}$ is covered by a mesh which we shall assume has uniform spacing h. Individual values at mesh points will be denoted by U_j^n, with $j \in \mathcal{J}_h = \{1,...,J\}$, and values of U at all such points on time level n will be denoted by $U^n := \{U_j^n, j \in \mathcal{J}_h\}$. We shall denote by u_j^n the mesh values of the function $u(x,t)$ which will usually be the point values $u(x_j,t_n)$. Then, as for the mesh point values U_j^n above, we define $u^n := \{u_j^n, j \in \mathcal{J}_h\}$. We shall consider the discrete l^2 norm, defined as

$$\|U_j^n\|_{l^2} := \Big(h \sum_{j\in\mathcal{J}_h} |U_j^n|^2\Big)^{1/2}. \tag{2.2}$$

The general form of two times level difference scheme we shall consider will be written as

$$S_1 U^{n+1} = S_0 U^n, \tag{2.3}$$

where the difference operators S_0, S_1 are independent of n. Since we are most interested in the space approximation, we shall often use a simple forward-time approximation,

$$S_1 = \frac{1}{\Delta t}, \quad S_0 = \frac{1}{\Delta t} - \tilde{S}_0, \quad \frac{U^{n+1} - U^n}{\Delta t} + \tilde{S}_0 U^n = 0, \tag{2.4}$$

where at each point $j \in \mathcal{J}_h$, the linear difference operator $\tilde{S}_0$ is written in the form of a sum over near neighbors $\mathcal{N}(j)$ of j,

$$(\tilde{S}_0 U^n)_j = \sum_{l\in\mathcal{N}(j)} \alpha_{j,l} U_l^n, \quad \forall j \in \mathcal{J}_h. \tag{2.5}$$

The notation $\alpha_{j,l}$ denotes the fact that the coefficients may depend on j as well as l, to enable us to incorporate the numerical boundary conditions in (2.5), so that values of U at all points outside Ω_h can be eliminated. The truncation error of the whole scheme (2.3) is defined in terms of the exact solution u as

$$\mathcal{T}^n(\cdot,u) := S_1 u^{n+1} - S_0 u^n, \tag{2.6}$$

and consistency of the difference scheme (2.3) with the problem (2.1) as

$$\mathcal{T}_j^n \to 0 \text{ as } \Delta t(h) \to 0 \quad \forall j \in \mathcal{J}_h$$

for all sufficiently smooth solutions u of (2.1). We mean by $\Delta t(h) \to 0$ that Δt may tend to zero at a rate determined by h. Moreover, if p and q are the largest integers for which

$$|\mathcal{T}_j^n| \le C[h^p + (\Delta t)^q], \text{ as } \Delta t(h) \to 0 \quad \forall j \in \mathcal{J}_h,$$

for sufficiently smooth u, the scheme is said to have an order of accuracy p in x and q in t.

We shall assume that operator (2.5) represents the differential operator $L(\cdot)$ in the limit $h \to 0$, and we define the truncation error for the space approximation in terms of the steady state solution w of $L(w)=0$, as

$$\tilde{\mathcal{T}}(\cdot,w) := \tilde{S}_0 w. \tag{2.7}$$

Dealing with the evaluation of the integral term, we shall first consider a finite integration interval instead of the whole real line, choosing only those points for which the density K has a significant non zero values. This choice would not introduce significant errors and we can then neglect its contribution on subsequent error evaluations (see [5] for more details).
We shall use standard quadrature rules and, by the symmetry of K, we shall write

$$(K*u)(x_j,t_n) \approx (\mathbf{K}*U^n)_j := h \sum_{i=-N}^{N} \beta_i K_i U^n_{i+j}, \quad \forall j \in \mathcal{J}_h \tag{2.8}$$

for some coefficients β_i such that $\sum_{i=-N}^{N} \beta_i = 2N$, $K_i := K(x_i)$ and $\mathbf{K} := \{\beta_i K_i, i = -N,...,N\}$. In the numerical test we shall use the following classical fourth-order rule

$$\{\beta_{-N},\beta_{-N+1},...,\beta_{N-1},\beta_N\} = \{3/8,7/6,23/24,1,...,1,23/24,7/6,3/8\}. \tag{2.9}$$

3. An Asymptotic 4-Order scheme for linear hyperbolic equations with a source term

We start our discussion on AHO schemes by first considering a simple model, namely problem (1.1) with $K \equiv 0$. In other terms, we look at the following linear hyperbolic conservation law with the source term,

$$\begin{cases} u_t + a u_x = -u, \ \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \ \forall x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where constant $a>0$. By using the fact that the solution is explicitly given by

$$u(x,t) = e^{-t} u_0(x-at),$$

or just using Fourier transform methods, it is easy to show that, if $u_0 \in L^2(\mathbb{R})$, then the solution $u(t)$ tends to zero in the L^2-norm as t tends to infinity, with a rate e^{-t}, that is:

$$\|u\|_2 = e^{-t} \|u_0\|_2. \tag{3.2}$$

Assuming enough regularity on the initial data, the same result holds true for higher derivatives $\partial^r u / \partial t^r$. Moreover, for every steady state w of (3.1), such that $u_0 - w \in L^2(\mathbb{R})$, it holds trivially

$$\|u-w\|_2 = e^{-t} \|u_0 - w\|_2.$$

Now, we would use this result to construct AHO schemes. Then, we focus on the numerical approximation of (3.1) coupled with its steady state equation

$$a w_x = -w. \tag{3.3}$$

Starting from a general three-point scheme for (3.1)

$$\frac{dU_j}{dt} + \alpha_{-1} U_{j-1} + \alpha_0 U_j + \alpha_1 U_{j+1} = 0, \quad j \in \mathbb{Z}, \tag{3.4}$$

we shall define the coefficients α_i to get a AHOp scheme, that is a scheme of p-order when computed on the solution w of (3.3). We remark, that in this simple case, this is equivalent to constructing a two-step p-order finite difference scheme for the ODE (3.3).
First of all, notice that the scheme (3.4) is consistent with equation (3.1) if, for every h small enough,

$$\alpha_{-1}+\alpha_0+\alpha_1=1+c_0h,\ -\alpha_{-1}+\alpha_1=\frac{a}{h}+c_1, \tag{3.5}$$

for some constants or infinitesimal functions of h, $c_0(h)$ and $c_1(h)$.
Let now $\mathcal{T}_h(x,u(x,t))$ be the local truncation error introduced by the semi-discretization (3.4),

$$\mathcal{T}_h(x,u(x,t))=\frac{du(x,t)}{dt}+\alpha_{-1}u(x-h,t)+\alpha_0u(x,t)+\alpha_1u(x+h,t),\quad (x,t)\in\mathbb{R}\times\mathbb{R}^+. \tag{3.6}$$

To get an AHOp method, we need, for all $x\in\mathbb{R}$,

$$\tilde{\mathcal{T}}_h(x,w)=\alpha_{-1}w(x-h)+\alpha_0w(x)+\alpha_1w(x+h)=\mathcal{O}(h^p),\ \text{as}\ h\to 0, \tag{3.7}$$

where w is a smooth solution of equation (3.3). By Taylor expansion and by using equation (3.3), the AHO condition (3.7) becomes, for all $x\in\mathbb{R}$

$$\Big[\alpha_{-1}+\alpha_0+\alpha_1+\sum_{n=1}^{p}\frac{1}{n!}\big(\frac{h}{a}\big)^n\big((-1)^n\alpha_{-1}+\alpha_1\big)\Big]w(x)=\mathcal{O}(h^p),\ \text{as}\ h\to 0. \tag{3.8}$$

Let us fix $p=4$. We shall take into account consistency conditions (3.5), by also using the fact that the coefficient α_0 can be written as the sum of a singular part, which comes from the derivative approximation, and a bounded part, namely

$$\alpha_0=\frac{\alpha_0^s}{h}+\alpha_0^c,\quad \alpha_0^s,\ \alpha_0^c\in\mathbb{R}. \tag{3.9}$$

Therefore, relation (3.8) becomes, for all x

$$\begin{aligned}&\Big[h\big(c_0-\frac{c_1}{a}-\frac{\alpha_0^s}{2a^2}\big)+\frac{h^2}{a^2}\big(\frac{1}{3}-\frac{\alpha_0^c}{2}\big)+\frac{h^3}{a^2}\big(\frac{c_0}{2}-\frac{c_1}{6a}-\frac{\alpha_0^s}{4!a^2}\big)\\ &+\frac{h^4}{4!a^4}\big(\frac{4}{5}-\alpha_0^c\big)+\frac{h^5}{4!a^4}\big(c_0-\frac{c_1}{5a}\big)\Big]w(x)=\mathcal{O}(h^p),\ \text{as}\ h\to 0.\end{aligned} \tag{3.10}$$

We can choose the values of the constants α_0^s, α_0^c, c_0 and c_1 in such a way that all of the expression between round parentheses vanishes. It is clear that in this way we can get at least a fourth order approximation. In that case, the solution is given, for every fixed value of the parameter c_1, by

$$\alpha_0^c=\frac{2}{3},\ c_0=\frac{c_1}{5a},\ \alpha_0^s=-\frac{8a}{5}c_1, \tag{3.11}$$

while, using (3.5), the coefficients α_1 and α_{-1} are given by

$$\alpha_1=\frac{1}{2}(1-a_0-c_0h+\frac{a}{h}+c_1),\ \alpha_{-1}=\frac{1}{2}(1-a_0-c_0h-\frac{a}{h}-c_1). \tag{3.12}$$

Then, we have constructed a class, depending on the parameter c_1, of semi-discrete schemes for (3.1), of fourth-order approximations to the equation $aw_x + w = 0$, namely

$$\begin{aligned}\frac{dU_j}{dt} &+ \frac{a}{2h}(U_{j+1}-U_{j-1}) + \frac{1}{6}(U_{j+1}+4U_j+U_{j-1}) + \frac{4a}{5h}c_1(U_{j+1}-2U_j+U_{j-1}) \\ &+ \frac{c_1}{2}(U_{j+1}-U_{j-1}) + \frac{h}{10a}c_1(U_{j+1}+U_{j-1}) = 0. \end{aligned} \tag{3.13}$$

Remark 3.1. *Notice that a scheme which is a second-order approximation to the steady equation (3.3), was introduced by Roe in [17]. It corresponds to the special choice*

$$c_0 = 0, \quad c_1 = -\frac{1}{2}, \quad \alpha_0^s = a, \quad \alpha_0^c = \frac{1}{2}. \tag{3.14}$$

3.1. Monotonicity. Let us now consider a forward-time approximation to (3.4). To preserve the monotonicity, we should have

$$\alpha_{-1} \le 0, \ \alpha_1 \le 0 \text{ and } (1-\Delta t\alpha_0) \ge 0. \tag{3.15}$$

Thanks to (3.11) and (3.12), we conclude that we have monotonicity if and only if

$$c_1 \le -5/8.$$

This yields a sub-class of forward-time monotone AHO4 schemes approximating (3.1). As a special case, for $c_1 = -5/8$ and $\alpha_0^s = a$, it gives the upwinding of the space derivative,

$$\begin{aligned}\frac{U_j^{n+1}-U_j^n}{\Delta t} &+ \frac{a}{h}\left(U_j^n - U_{j-1}^n\right) + \frac{1}{6}\left(U_{j+1}^n + 4U_j + U_{j-1}^n\right) \\ &- \frac{5}{16}\left(U_{j+1}^n - U_{j-1}^n\right) - \frac{h}{16a}\left(U_{j+1}^n + U_{j-1}^n\right) = 0,\end{aligned} \tag{3.16}$$

under the CFL condition

$$\Delta t \le \frac{h}{(a+2/3h)}.$$

3.2. Asymptotic accuracy. As is well-known, monotonicity, joint to consistency, yields a result of convergence for the scheme (3.16). We emphasize that this scheme is globally accurate for order 1, but it has the property of improving its accuracy as time increases. For the truncation error (3.6), we can actually write

$$\begin{aligned}\mathcal{T}_h(x,u(x,t)) = \frac{du(x,t)}{dt} &+ \alpha_{-1}[u(x-h,t)-w(x-h)] + \alpha_0[u(x,t)-w(x)] \\ &+ \alpha_1[u(x+h,t)-w(x+h)] + \tilde{\mathcal{T}}_h(x,w),\end{aligned} \tag{3.17}$$

where w is the steady state and $\tilde{\mathcal{T}}_h(x,w)$ has been defined in (3.7). Computing the local $L^2(\Omega_h)$-norm and using monotonicity, we get

$$\begin{aligned}\|\mathcal{T}_h(u)\|_{L^2(\Omega_h)} \le \|u_t\|_{L^2(\Omega_h)} &+ (|\alpha_{-1}|+|\alpha_0|+|\alpha_1|)\,\|u-w\|_{L^2(\Omega_h)} \\ &+ \|\tilde{\mathcal{T}}_h(w)\|_{L^2(\Omega_h)}.\end{aligned} \tag{3.18}$$

By construction of the scheme (3.8)-(3.11),

$$\left\|\tilde{\mathcal{T}}_h(x_j,w_j)\right\|_{L^2(\Omega_h)} = \frac{h^4}{180a^4}\|w\|_{L^2(\Omega_h)} + \mathcal{O}(h^5),$$

and for small values of h, $(|\alpha_{-1}|+|\alpha_0|+|\alpha_1|) = \mathcal{O}(1/h)$.
We now apply relation (3.2) and subsequent remarks to get

$$\|\mathcal{T}_h(x_j,u(x_j,t))\|_{L^2(\Omega_h)} \le C_1 e^{-t} + C_2\frac{e^{-t}}{h} + \frac{h^4}{180a^4}\|w\|_{L^2(\Omega_h)},$$

where C_1 and C_2 are positive constants. Therefore, for fixed h, as $t \ge \ln Ch^{-5}$, for some constant C, the accuracy of the scheme increases up to the order of consistency with the stationary solution, here for example to the order 4, since the term $\frac{e^{-t}}{h}$ is dominated by a $\mathcal{O}(h^5)$. Numerical evidence of this improved accuracy will be discussed in Section 7.1.

4. L^2-decay properties for the integro-differential problem

In this section we shall investigate the asymptotic behavior of the solution to the Cauchy problem (1.1). The asymptotic behavior for this problem has been studied in [11] and in [18] as a reduced case of a more general theory of hyperbolic-elliptic coupled systems. Starting from these results we shall focus on the decay rate of the solution and its derivatives, in the case $u_0 \in L^1 \cap L^2(\mathbb{R})$. Notice that, all the following results also hold true in the case of non-constant stationary solution w such that $u_0 - w \in L^1 \cap L^2(\mathbb{R})$, and we shall use them to study the order properties of numerical schemes for problems where the initial data u_0 is a perturbation of a stationary solution.

We shall denote by $\mathcal{K}$ the operator $\mathcal{K}u = K * u - u$ and, for the Fourier transform $\widehat{\mathcal{K}} := -m(\xi)\hat{u}$, where $m(\xi) = 1 - \hat{K}(\xi)$, we shall assume

[K2] $m(\xi)$ is real, non-negative, even and increasing on $\mathbb{R}^+$, and $1 \ge m(\xi) \ge \omega\min(1,\xi^2)$ for some positive constant ω.

Then, the following results for the decay rate of the solution u and its space derivatives hold.

PROPOSITION 4.1. [18] *Under the assumptions* [K2], *given an initial data $u_0 \in L^1 \cap L^2(\mathbb{R})$, the weak solution of (1.1) decays in L^2 with*

$$\frac{\|u(t)\|_2}{\|u_0\|_1} \le Ct^{-1/4} + \mathcal{O}(t^{-1/2}), \tag{4.1}$$

where the constant C depends on the kernel K.

PROPOSITION 4.2. [11] *Under the assumptions* [K2], *given an initial data $u_0 \in L^1 \cap H^1(\mathbb{R})$, we have the estimate*

$$\|\partial_x u\|_2 \le Ce^{-ct}\|\partial_x u_0\|_2 + C(1+t)^{-3/4}\|u_0\|_1, \tag{4.2}$$

where the constant C depends on the kernel K. Now, we would like to study the decay rate of $\partial_t u$. Notice that, assuming strong regularity conditions on the initial data u_0, it is easy to see that all subsequent time-derivatives of u decay at worst as u.

PROPOSITION 4.3. *Assuming conditions* [K2], *given an initial data $u_0 \in L^1 \cap H^1(\mathbb{R})$, it holds that*

$$\|u_t\|_2 \le Ct^{-1/2}. \tag{4.3}$$

Proof. First, we show that this result holds true for $v(y,t)=u(x,t)$ with $y=x-at$. The function v solves the following ordinary differential equation

$$v_t(y,t)=K*v(y,t)-v(y,t). \tag{4.4}$$

By Fourier transform,

$$\hat{v}(\xi,t)=\hat{v}_0(\xi)e^{-m(\xi)t}, \tag{4.5}$$

and, for $0\leq m(\xi)\leq 1$,

$$\|v_t\|_2\leq\|v\|_2. \tag{4.6}$$

Let us set $g(t):=\|\hat{v}_t\|_2^2$. Then, for every $t>0$, $g(t)$ is a non negative and continuous function of t. Moreover, we can prove that $g(t)$ is an integrable function. In fact, multiplying equation (4.4) by the complex conjugate $\bar{\hat{v}}$, it holds

$$\partial_t\|\hat{v}\|_2^2+2\|\hat{v}_t\|_2^2\leq 0,$$

and integrating in t yields

$$\int_0^{+\infty}\|\hat{v}_t\|_2^2dt<+\infty.$$

From (4.5) we also get that $g(t)$ is monotone decreasing, since multiplying by the complex conjugate $\bar{\hat{v}}_t$, and deriving in time, we obtain

$$\partial_t|\hat{v}_t|^2=-2m(\xi)^3|\hat{v}_0|^2e^{-2m(\xi)t}.$$

This yields

$$\partial_t\int|\hat{v}_t|^2d\xi=-2\int m(\xi)^3|\hat{v}_0|^2e^{-2m(\xi)t}d\xi\leq 0.$$

Then, for the function $g(t)$ it holds

$$g(t)t\leq\int_0^t g(s)ds\leq\int_0^{+\infty}g(s)ds:=C<+\infty,$$

which gives

$$\|v_t\|_2\leq Ct^{-1/2}.$$

Now, coming back to the function u, we have,

$$v_t=u_t+au_x,$$

then

$$\|u_t\|_2\leq\|v_t\|_2+|a|\|u_x\|_2. \tag{4.7}$$

By Proposition 4.2, the slowest term in (4.7) is $\|v_t\|_2$ and so we can conclude the proof. ∎

5. Asymptotic high-order schemes. consistency and accuracy

Starting from the asymptotic analysis given in the previous section, we would like to generalize the accuracy result obtained in Subsection 3.2 for the hyperbolic equation. Let us consider the general problem (2.1) coupled with its steady equation $L(w)=0$. By relations (2.3)-(2.7), the local truncation error (2.6) may be written as

$$\mathcal{T}(x_j,u_j^n) = \frac{u_j^{n+1}-u_j^n}{\Delta t} + \tilde{S}_0(u_j^n - w_j) + \tilde{\mathcal{T}}(x_j,w_j), \quad \forall j \in \mathcal{J}_h, \forall n. \tag{5.1}$$

From that, we have the following estimate

$$\|\mathcal{T}(x_j,u_j^n)\|_{l^2(\Omega_h)} \le \left\|\frac{u_j^{n+1}-u_j^n}{\Delta t}\right\|_{l^2(\Omega_h)} + \left\|\tilde{S}_0(u_j^n-w_j)\right\|_{l^2(\Omega_h)} + \left\|\tilde{\mathcal{T}}(x_j,w_j)\right\|_{l^2(\Omega_h)}. \tag{5.2}$$

Using (2.5) and the Cauchy inequality $(\sum a_k b_k)^2 \le (\sum a_k^2)(\sum b_k^2)$, the second term on the right may be estimated by

$$\left|\tilde{S}_0(u_j^n-w_j)\right|^2 = \Big|\sum_{l\in\mathcal{N}(j)} \alpha_{j,l}(u_l^n - w_l)\Big|^2 \le \Big(\sum_{l\in\mathcal{N}(j)} \alpha_{j,l}^2\Big)\Big(\sum_{l\in\mathcal{N}(j)} (u_l^n-w_l)^2\Big).$$

If A is the $J\times J$ coefficients matrix, $A=(\alpha_{j,l})_{j=1,\dots,J,l=1,\dots,J}$, where $\alpha_{j,l}=0\ \forall l\notin\mathcal{N}(j)$, it is

$$\sum_{l\in\mathcal{N}(j)} \alpha_{j,l}^2 \le \Big(\sum_{l\in\mathcal{N}(j)} |\alpha_{j,l}|\Big)^2 \le \Big(\max_{j\in\mathcal{J}_h}\sum_{l\in\mathcal{N}(j)} |\alpha_{j,l}|\Big)^2 = \|A\|_\infty^2.$$

Moreover, since $\forall j \ \ \mathrm{card}(\mathcal{N}(j)) \le \ \mathrm{card}(\mathcal{J}_h)$,

$$\sum_{l\in\mathcal{N}(j)} (u_l^n - w_l)^2 \le \sum_{l\in\mathcal{J}_h} (u_l^n-w_l)^2 = \|u^n - w\|_{l^2(\Omega_h)}^2.$$

Then, we obtain

$$\begin{aligned}\|\mathcal{T}(x_j,u_j^n)\|_{l^2(\Omega_h)} \le & \left\|\frac{u_j^{n+1}-u_j^n}{\Delta t}\right\|_{l^2(\Omega_h)} + \ \mathrm{card}(\mathcal{J}_h)\|A\|_\infty \|u_l^n - w_l\|_{l^2(\Omega_h)} \\ & + \left\|\tilde{\mathcal{T}}(x_j,w_j)\right\|_{l^2(\Omega_h)},\end{aligned} \tag{5.3}$$

where $\mathrm{card}(\mathcal{J}_h) = J = |\Omega_h|/h$.

PROPOSITION 5.1. *Assume that w is a steady state of (2.1) and consider the solution u associated to an initial data u_0 which differs from w by an integral disturbance, $(u_0 - w) \in L^1\cap H^1(\mathbb{R})$. Then, for the truncation error generated by a consistent scheme of the form (2.3)-(2.5), it holds that*

$$\|\mathcal{T}(x,u)\|_{l^2(\Omega_h)} = C_1 t^{-1/2} + C_2\frac{t^{-1/4}}{h^2} + \|\tilde{\mathcal{T}}(x,w)\|_{l^2(\Omega_h)}, \tag{5.4}$$

Proof. For the consistency assumption on scheme (2.3)-(2.5), the operator (2.5) represents the integro-differential operator $L(\cdot)$ in the limit $h\to 0$ and hence $\|A\|_\infty = \mathcal{O}(1/h)$. Moreover,

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} \approx [u_t(\xi)]_j^n.$$

Since the discrete l^2-norm approximates the integral L^2-norm, applying estimates (4.3) and (4.1) to the first and the second term on (5.3) respectively, we obtain the required relation (5.4), where $t=n\Delta t$. □

This proposition allows us to conclude that increasing the time of the simulation, the order of the scheme may increase up to the order of the space approximation. In fact, for the pure differential case, for fixed h we can choose t large enough to have $t \geq (\|\tilde{\mathcal{T}}(x,w)\| h^2)^{-4}$, and so obtain

$$\lim_{t\to+\infty} \|\mathcal{T}(x,u)\|_{l^2(\Omega_h)}(t) = \|\tilde{\mathcal{T}}(x,w)\|_{l^2(\Omega_h)}.$$

6. Some asymptotic p-order schemes for the linear integro-differential equation

Let us now deal with our objective problem,

$$u_t + a u_x = \gamma(K * u - u). \tag{6.1}$$

We should construct a scheme of high order on the steady state w solution of

$$a w_x = \gamma(K * w - w). \tag{6.2}$$

We write the general scheme (2.4) as

$$\begin{aligned}\frac{U_j^{n+1} - U_j^n}{\Delta t} &+ \alpha_{-1} U_{j-1} + \alpha_0 U_j + \alpha_1 U_{j+1} - \gamma(\mathbf{K} * U)_j \\ &+ h(\boldsymbol{\alpha'} * U)_j + h^2(\boldsymbol{\alpha''} * U)_j + \ldots = 0,\end{aligned} \tag{6.3}$$

where the added terms $\boldsymbol{\alpha'}^{('')} = (\alpha'^{('')}_{-N}, \ldots, \alpha'^{('')}_{N})^T$ and $(\boldsymbol{\alpha'}^{('')} * U)_j = \sum_{i=-N}^{N} \alpha'^{('')}_i U_{i+j}$, may help us to get high-order accuracy.
First of all, the consistency property is assured if coefficients α_{-1}, α_0, α_1 verify

$$\alpha_{-1} + \alpha_0 + \alpha_1 = \gamma + c_0 h, \; -\alpha_{-1} + \alpha_1 = \frac{a}{h} + c_1, \tag{6.4}$$

for some constants or infinitesimal function of h, $c_0(h)$ and $c_1(h)$.
Then, to get an AHO scheme of order p we need, as $h \to 0$,

$$\alpha_{-1} w_{j-1} + \alpha_0 w_j + \alpha_1 w_{j+1} - \gamma(\mathbf{K} * w)_j + h(\boldsymbol{\alpha'} * w)_j + h^2(\boldsymbol{\alpha''} * w)_j + \ldots = \mathcal{O}(h^p). \tag{6.5}$$

As done in the hyperbolic case, we define $\alpha_0 = \alpha_0^s/h + \alpha_0^c$ and we write the Taylor expansion for w_j,

$$w_{j\pm1} = w_j \pm h w_x(x_j) + \frac{h^2}{2} w_{xx}(x_j) \pm \frac{h^3}{6} w_{xxx}(x_j) + \mathcal{O}(h^4) \tag{6.6}$$

where derivatives may be computed by using equation (6.2),

$$a w_x = \gamma(K * w - w), \; a w_{xx} = \gamma\Big(K_x * w - \frac{\gamma}{a}(K * w - w)\Big), \ldots \tag{6.7}$$

Substituting expressions (3.5), (3.9), (6.6) and (6.7) in (6.5) and, for simplicity, stopping up to order $p=3$, we get for all j

$$\begin{aligned} h(\boldsymbol{\alpha'} * w)_j + h^2(\boldsymbol{\alpha''} * w)_j + \frac{h}{2}\frac{\gamma}{a}\alpha_0^s(\mathbf{K}_x * w)_j + \frac{h^2}{6}\gamma(\mathbf{K}_{xx} * w)_j \\ + h\big(c_1 + \frac{\gamma}{2a}\alpha_0^s\big)\frac{\gamma}{a}(\mathbf{K} * w)_j + h\big(c_0 - c_1\frac{\gamma}{a} - \frac{\gamma^2}{2a^2}\alpha_0^s\big)w_j \\ + h^2\frac{\gamma}{2a}\big(-\alpha_0^c + \frac{2\gamma}{3}\big)(\mathbf{K}_x * w)_j - h^2\frac{\gamma^2}{2a^2}\big(-\alpha_0^c + \frac{2\gamma}{3}\big)(\mathbf{K} * w)_j \\ + \frac{h^2}{2}\frac{\gamma^2}{a^2}\big(-\alpha_0^c + \frac{2\gamma}{3}\big)w_j = \mathcal{O}(h^3). \end{aligned} \tag{6.8}$$

Notice that in the algebraic manipulation we used the relation $(K_x * w)(\cdot) \approx -(\mathbf{K}_x * w)$, obtained from expression (2.8).
Solving the round brackets in (6.8) in a suitable way, we obtain that second-order asymptotic schemes may be defined by

$$c_0 = 0, \quad c_1 = -\frac{\gamma}{2a}\alpha_0^s, \quad \alpha'_j = -\frac{\gamma}{2a}\alpha_0^s(\mathbf{K}_x)_j, \tag{6.9}$$

while third-order ones by

$$c_0 = 0, \quad c_1 = -\frac{\gamma}{2a}\alpha_0^s, \quad \alpha_0^c = \frac{2}{3}\gamma, \quad \alpha'_j = -\frac{\gamma}{2a}\alpha_0^s(\mathbf{K}_x)_j, \quad \alpha''_j = -\frac{\gamma}{6}(\mathbf{K}_{xx})_j. \tag{6.10}$$

A similar procedure may be followed to get higher orders.
In both relations (6.9) and (6.10) is left a degree of freedom, that we may define to preserve the monotonicity. As a special case we shall consider the following monotone asymptotic second-order scheme (AHO2),

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{h} + \gamma\frac{U_j^n + U_{j-1}^n}{2} - \gamma(\mathbf{K} * U^n)_j - \frac{h\gamma}{2}(\mathbf{K}_x * U^n)_j = 0, \tag{6.11}$$

under the CFL condition $\Delta t \leq h/(a + h/2)$, $h \leq 2a$, and the following monotone asymptotic third-order scheme (AHO3),

$$\begin{aligned} \frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{h} + \frac{\gamma}{12}\big(5U_{j-1}^n + 8U_j^n - U_{j+1}^n\big) - \gamma(\mathbf{K} * U^n)_j \\ - \frac{h\gamma}{2}(\mathbf{K}_x * U^n)_j - h^2\frac{\gamma}{6a}(\mathbf{K}_{xx} * U^n)_j = 0, \end{aligned} \tag{6.12}$$

under the CFL condition $\Delta t \leq h/(a + 2/3h)$, $h \leq 12/5a$. Notice that for (6.11) we get the upwinding of the source term u as in the Roe scheme, see Remark 3.1.

7. Numerical tests

Now, our aim is to show how, for large time simulations, AHO schemes give better numerical results than standard approximations. Therefore, we shall focus our attention on the numerical error as a function of time: for all tests, we shall fix the grid steps h and Δt and we will plot the difference

$$e(n\Delta t) = \|U^n - u^n\|_{l^2}, \tag{7.1}$$

increasing time $t = n\Delta t$. For tests on the hyperbolic problem, the function u will be the exact solution, while for tests on the integro-differential model, it will be the asymptotic function w.

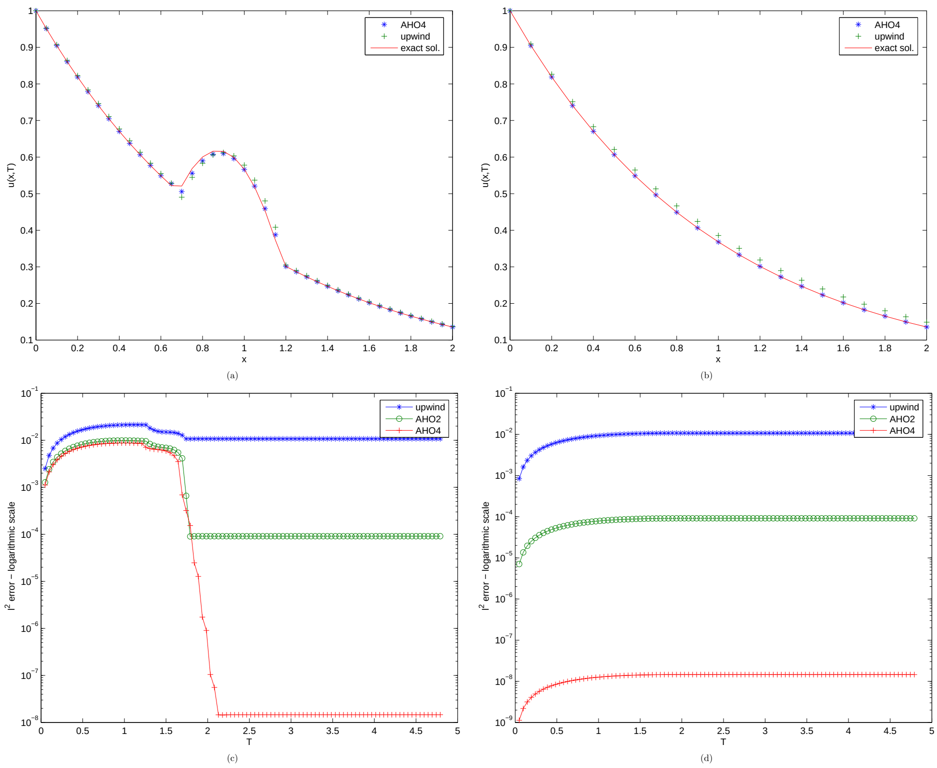

FIG. 7.1. *Linear hyperbolic example, subsection (7.1). In figure (a) and (b) we plot the solution to the upwind scheme (7.4) and to the AHO4 scheme (3.16) for problem (7.2)-(7.7) coupled with the exact solution (7.3). In (a) we show that after few time iterations the initial perturbation is still dominant and the schemes have the same accuracy, while in (b), increasing time, the initial perturbation vanishes and the numerical solution using upwind looses the steady state while not using AHO4. To make this more clear, in figure (c) we fix $h=0.05$ and we compare the l^2-error (7.1) using upwind, AHO2, AHO4 schemes, and while the increasing time. Then, it is evident that, once the steady state has been reached, the AHO2-error (o) decays rapidly to its order of accuracy, as well as AHO4-error ($*$) gains more and more accuracy. Moreover, to confirm the order of accuracy of the AHO4 scheme, we test the three schemes upwind, AHO2, AHO4 on the steady state solution (7.6) and in figure (d) we plot the resulting errors. As we expected, we gain accuracy using the AHO4 scheme, and we can affirm that, the scheme produces no time evolution on steady state data.*

7.1. A numerical test for the hyperbolic problem. Let us consider the scalar equation

$$u_t + a u_x = -u \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \tag{7.2}$$

with data $u(x,0) = u_0(x)$ and $u(0,t) = 1$, $\forall\ (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. The general solution is

$$u(x,t) = \begin{cases} u_0(x-at)e^{-t} & x \geq at \\ e^{-\frac{x}{a}} & x < at. \end{cases} \tag{7.3}$$

Then, the function $w(x) = e^{-x/a}$ is the time-asymptotic state of u and solves the steady equation $a w_x + w = 0$.
In Figure 7.1, we shall compare the AHO4 scheme (3.16) with the standard first-order

upwind scheme, that is actually an AHO1 scheme,

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{U_j-U_{j-1}}{h}+U_j=0, \tag{7.4}$$

and with the AHO2 scheme introduced by Roe [17],

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+a\frac{U_j-U_{j-1}}{h}+\frac{U_j+U_{j-1}}{2}=0. \tag{7.5}$$

We shall consider the solution to problem (7.2) with $a=1$ and with two different initial data,

$$u_0(x)=e^{-x}, \tag{7.6}$$

and

$$u_0(x)=e^{-x}-\Big[8\big(e^{-\frac{1}{4}}-\frac{1}{3}e^{-\frac{3}{4}}\big)x^2-\big(6e^{-\frac{1}{4}}-\frac{2}{3}e^{-\frac{3}{4}}\big)x\Big]\chi_{[1/4,3/4]}. \tag{7.7}$$

Datum (7.6) is interesting for testing the effectiveness of the theoretical accuracy obtained in previous sections, while the second one is interesting for testing the power of Asymptotic High-Order schemes for large time simulations. In Figure (7.1) we show that after a few time iterations, until the initial perturbation is still dominant, the three schemes (7.11)-(6.11)-(6.12), have the same accuracy. While increasing time, the initial perturbation vanishes and the numerical solution using upwind looses the steady state. To make this more clear, we compare the error (7.1) for the three approximations. Then, in figure 7.1-(c) it is evident that, once the steady state has been reached, the AHO2-error decays rapidly to its order of accuracy, while the AHO4-error gains more and more accuracy.

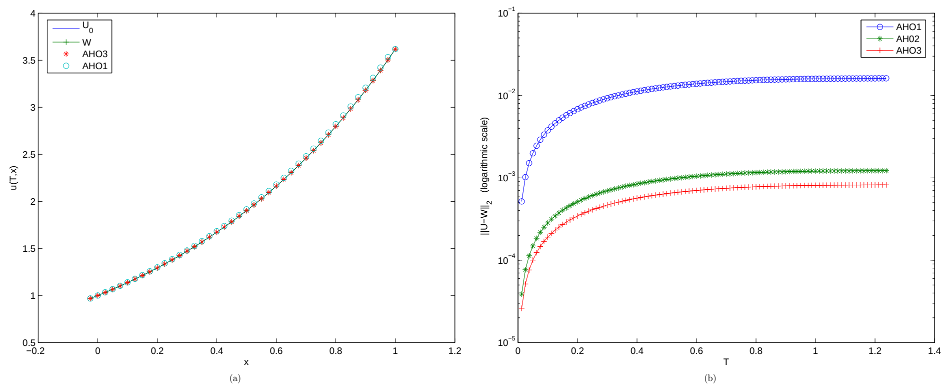

FIG. 7.2. *Integro-differential example, subsection (7.2). We consider schemes (7.11), (6.11), (6.12) to solve problem (7.8)-(7.12). In figure (a) we plot the solutions coupled with the steady state (7.12), while in figure (b) we compare the quantity (7.1) using the three schemes, for a large time simulation. As we expected, we gain accuracy using the AHO2 and AHO3 schemes.*

7.2. A numerical test for the integro-differential problem. Let us consider the integro-differential equation

$$u_t+u_x=(K*u-u). \tag{7.8}$$

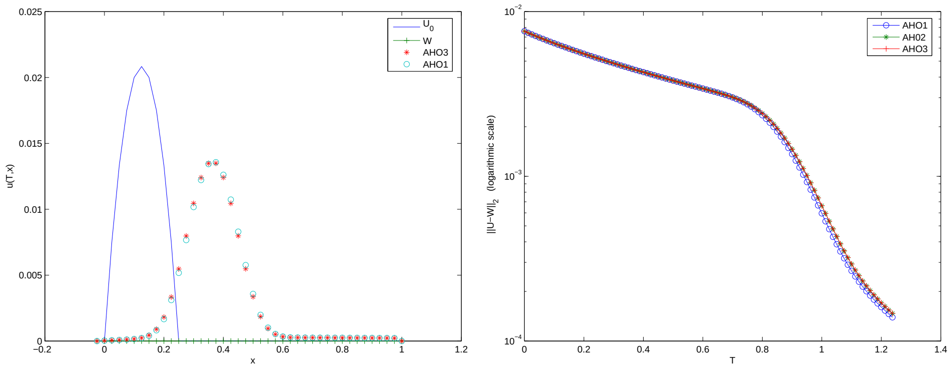

FIG. 7.3. *Integro-differential example, subsection (7.2). We consider schemes (7.11), (6.11), (6.12) to solve problem (7.8)-(7.13). In figure (a) we plot the three solutions coupled with the initial data (7.13), after a few time iterations, when the initial perturbation is still dominant, while in figure (b) we compare the quantity (7.1) using the three schemes, for a large time simulation. Both figures (a) and (b) show that the schemes have the same accuracy. This is exactly what we expect when the initial data is a perturbation of a constant state. In fact, by consistency assumptions, all schemes should be exact when computed on constants.*

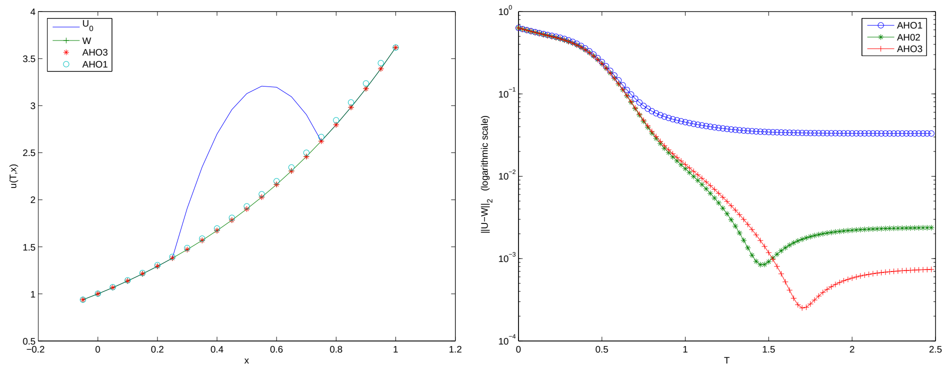

FIG. 7.4. *Integro-differential example, subsection (7.2). We consider schemes (7.11), (6.11), (6.12) to solve problem (7.8)-(7.14). In figure (a) we plot the solutions coupled with the initial data (7.14) after $M=100$ time iterations, while in figure (b) we compare the quantity (7.1) using the three schemes, for a large time simulation. It is clear that, once the steady state has been reached, the AHO2-error $(*)$ decays rapidly to its high-order of accuracy, while AHO3-error $(+)$ gains more and more accuracy.*

For the simplest model arising in finance, namely if we assume that the price of the underlying is just moving according to a Poisson process, we can take as a kernel K the Gaussian probability density

$$K(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{7.9}$$

Notice that, the Gaussian Kernel verifies assumptions [K2]. Then, our test problem (7.8)-(7.9) has the L^2-decay properties introduced in Section 4. With this choice of K, the steady states of (7.8) are

$$w(x)=c \quad \text{and} \quad w(x)=ce^{\alpha x}, \quad c\in\mathbb{R}, \tag{7.10}$$

where $\alpha \approx 1.28589$ solves the equation $\alpha = e^{\alpha^2/2} - 1$. In Figures 7.3-7.2-7.4, we shall compare the two Asymptotic High-Order schemes AHO2 (6.11) and AHO3 (6.12) with the following standard explicit approximation,

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{h} + U_j^n - (\mathbf{K} * U^n)_j = 0. \tag{7.11}$$

To be consistent with the previous notations we shall call this scheme AHO1.

We first check the accuracy of schemes AHO2 (6.11) and AHO3 (6.12) solving the equation (7.8) associated with the steady state as initial data,

$$u_0(x) = e^{\alpha x}. \tag{7.12}$$

Looking at Figure (7.2), it should be clear that we gain accuracy using the AHO2 and AHO3 schemes.
Secondly, we consider the solution to equation (7.8) associated with the following initial data

$$u_0(x) = \Big[-\frac{4}{3}x^2 + \frac{x}{3}\Big]\chi_{[0,\frac{1}{4}]}, \tag{7.13}$$

where $\chi_{[\cdot,\cdot]}$ is the characteristic function of interval $[\cdot,\cdot]$. We plot the numerical solution in Figure 7.3, to show that when the initial data is a perturbation of a constant state, all schemes have the same accuracy. This is actually true because, by the consistency, all schemes should be exact when computed on constants.
To end, in Figure 7.4 we consider problem (7.8) with initial data

$$u_0(x) = e^{\alpha x} + \big[ax^2 + bx + c\big]\chi_{[\frac{1}{4},\frac{3}{4}]}, \tag{7.14}$$

where $a = -b + 2(e^{3\alpha/4} - e^{\alpha/4})$, $b = 6e^{\alpha/4} - \frac{2}{3}e^{3\alpha/4} - \frac{16}{3}c$ and $c = -e^{3\alpha/4}$. We compare the quantity (7.1) using the three schemes, for a large time simulation. It is clear that, once the steady state has been reached, the AHO2-error decays rapidly to its high-order of accuracy, while AHO3-error gains more and more accuracy.

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