Commun. Math. Phys. 178, 563 – 570 (1996)

Communications in
Mathematical
Physics

Volumes of Restricted Minkowski Sums and the Free Analogue of the Entropy Power Inequality

Stanislaw J. Szarek[1, ★], **Dan Voiculescu**[2, ★]

[1] Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106-7058, USA

[2] Department of Mathematics, University of California, Berkeley, California 94720-3840, USA

Received: 28 August 1995 / Accepted: 27 September 1995

Abstract: In noncommutative probability theory independence can be based on free products instead of tensor products. This yields a highly noncommutative theory: free probability theory (for an introduction see [9]). The analogue of entropy in the free context was introduced by the second named author in [8]. Here we show that Shannon's entropy power inequality ([6, 1]) has an analogue for the free entropy $\chi(X)$ (Theorem 2.1).

The free entropy, consistent with Boltzmann's formula $S = k \log W$, was defined via volumes of matricial microstates. Proving the free entropy power inequality naturally becomes a geometric question.

Restricting the Minkowski sum of two sets means to specify the set of pairs of points which will be added. The relevant inequality, which holds when the set of addable points is sufficiently large, differs from the Brunn–Minkowski inequality by having the exponent $1/n$ replaced by $2/n$. Its proof uses the rearrangement inequality of Brascamp–Lieb–Lüttinger ([2]). Besides the free entropy power inequality, note that the inequality for restricted Minkowski sums may also underlie the classical Shannon entropy power inequality (see 3.2 below).

1. The Inequality for Restricted Minkowski Sums

If $A, B \subset \mathbb{R}^n$ (or any vector space), the Minkowski sum of A and B is defined by

$$A + B = \{x + y : (x, y) \in A \times B\} .$$

An important property of the Minkowski sum in $\mathbb{R}^n$ is the Brunn–Minkowski inequality ([4, 5])

$$\lambda(A + B)^{1/n} \geqq \lambda(A)^{1/n} + \lambda(B)^{1/n} ,$$

where λ denotes n-dimensional Lebesgue measures. We introduce a modified concept of a sum.

★ Research supported in part by grants from the National Science Foundation.

1.1 Definition. *Let A, B be subsets of a vector space and $\Theta \subset A \times B$. We will call*

$$A +_{\Theta} B = \{x + y : (x, y) \in \Theta\}$$

the restricted (to Θ) sum of A and B.

We then have the following inequality (in what follows, all sets and functions are assumed to be measurable; λ denotes the Lebesgue measure in the appropriate dimension that may vary from place to place).

1.2 Theorem. *Let $\rho \in (0, 1)$, $n \in \mathbb{N}$ and let $A, B \subset \mathbb{R}^n$ be such that*

$$\rho \leqq \left(\frac{\lambda(B)}{\lambda(A)}\right)^{\frac{1}{n}} \leqq \rho^{-1} .$$

Furthermore, let $\Theta \subset A \times B \subset \mathbb{R}^{2n}$ be such that

$$\lambda(\Theta) \geqq (1 - c\min\{\rho\sqrt{n}, 1\})\lambda(A)\lambda(B) .$$

Then

$$\lambda(A +_{\Theta} B)^{2/n} \geqq \lambda(A)^{2/n} + \lambda(B)^{2/n} . \tag{1.1}$$

($c > 0$ is a numerical constant, independent of ε, n, A, B and Θ).

The following simple but illuminating example shows that, in general, one cannot expect a significantly stronger assertion: let B^n be the Euclidean ball in $\mathbb{R}^n$, $A = B^n$, $B = \rho B^n$ and $\Theta = \{(x, y) \in A \times B : \langle x, y\rangle \leqq 0\}$. Then

(1) $\lambda(\Theta) = \frac{1}{2}\lambda(A)\lambda(B)$,

(2) $A +_{\Theta} B = (1 + \rho^2)^{\frac{1}{2}} B^n$,

and we have equality in (1.1). We now state a lemma which is an elaboration of this example

1.3 Lemma. *Let ρ, n be as in Theorem 1.2 and let*

$$\Theta = \{(x, y) : x, y \in \mathbb{R}^n, |x| \leqq 1, \ |y| \leqq \rho, \ |x + y| \leqq (1 + \rho^2)^{\frac{1}{2}}\} .$$

Then

$$\lambda(\Theta) \leqq (1 - c\min\{\rho\sqrt{n}, 1\})\lambda(B^n)\lambda(\rho B^n) ,$$

where $c > 0$ is a universal constant.

We postpone the proof of the lemma (which depends on a careful, but completely elementary computation) and show how it implies the theorem. We observe first that Lemma 1.3 yields the following special case of the theorem:

$$A = \rho_1 B^n , \qquad B = \rho_2 B^n , \qquad \Theta = \{(x, y) \in A \times B : x + y \in RB^n\} , \tag{1.2}$$

where $\rho_1, \rho_2, R > 0$ are arbitrary constants. The case $\rho_1 = 1, \rho_2 = \rho < 1$ follows directly and the general one by symmetry and homogeneity.

The strategy for the rest of the proof is now as follows: if $A_0, B_0 \subset \mathbb{R}^n$ and $\Theta_0 \subset A_0 \times B_0$, we will show that there are A, B, Θ of the form (1.2) verifying

(i) $\lambda(A_0) = \lambda(A)$, $\lambda(B_0) = \lambda(B)$,

(ii) $\lambda(\Theta) \geqq \lambda(\Theta_0)$,

(iii) $\lambda(A +_{\Theta} B) \leqq \lambda(A_0 +_{\Theta_0} B_0)$.

Now if the original A_0, B_0, Θ_0 had yielded a counterexample to the theorem, the corresponding A, B, Θ would have, *a fortiori*, worked as such, contrary to the remark following Lemma 1.3. Accordingly it remains to realize (i)–(iii) for given A_0, B_0, Θ_0.

Step 1. Set $C = A_0 +_{\Theta_0} B_0$ and

$$\Theta_1 = \{(x,y) \in A_0 \times B_0 : x + y \in C\},$$

then $A_0 +_{\Theta_0} B_0 = A_0 +_{\Theta_1} B_0$, while clearly $\lambda(\Theta_1) \geqq \lambda(\Theta_0)$.

Step 2. Define $\rho_1, \rho_2, R > 0$ via

$$\lambda(A_0) = \lambda(\rho_1 B^n), \qquad \lambda(B_0) = \lambda(\rho_2 B^n), \qquad \lambda(C) = \lambda(RB^n).$$

We then have

$$\begin{aligned}\lambda(\Theta_1) &= \lambda(\{(x,y) \in A_0 \times B_0 : x + y \in C\}) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{A_0}(x)\chi_{B_0}(y)\chi_C(x+y)dxdy \\ &\leqq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\rho_1 B^n}(x)\chi_{\rho_2 B^n}(y)\chi_{RB^n}(x+y)dxdy \\ &= \lambda(\{(x,y) \in \rho_1 B^n \times \rho_2 B^n : x + y \in RB^n\}), \end{aligned} \tag{1.3}$$

as required for (i)–(iii) (and concluding the derivation of Theorem 1.2 from Lemma 1.3). The inequality in (1.3) is a special case of [2, Theorem 3.4], which, in a much more general setting, estimates an integral of a product of nonnegative functions by that of their spherical (or Schwartz) symmetrizations; we thank Alain Pajor for pointing out the paper [2] to us. □

Proof of Lemma 1.3 (Sketch). We will show that, for an appropriate choice of $c_1 > 0$ and with $\tau = \frac{1}{2}\min\{\rho\sqrt{n}, 1\}$, one has

$$1 \geqq |x_0| \geqq 1 - \tau/n \Rightarrow \lambda(\{y : |y| \leqq \rho,\ |x_0 + y| > (1+\rho^2)^{\frac{1}{2}}\}) \geqq c_1\lambda(\rho B^n). \tag{1.4}$$

It then follows that

$$\lambda(B^n \times \rho B^n \backslash \Theta) \geqq (1 - \tau/n)^n c_1 \lambda(B^n) \cdot \lambda(\rho B^n)$$

and that clearly implies the lemma. To show (1.4), we denote $r_0 = |x_0|$ and assume, as we may, that $x_0 = (r_0, 0, \dots, 0)$ and $n \geqq 2$. Then (the reader is advised to draw a picture)

$$\begin{aligned}&|\{y : |y| \leqq \rho,\ |x_0 + y| \leqq (1+\rho^2)^{\frac{1}{2}}\}| \\ &\quad = |B^{n-1}| \cdot \left(\int_{-\rho}^{s} (\rho^2 - u^2)^{\frac{n-1}{2}} du + \int_{s}^{t} (1 + \rho^2 - (r_0 + u)^2)^{\frac{n-1}{2}} du\right),\end{aligned}$$

where $s = (1 - r_0^2)/2r_0$ and $t = (1+\rho^2)^{\frac{1}{2}} - r_0$. Since $s \leqq (\tau/n) \cdot (1 + \frac{\tau/n}{2r_0}) \leqq (\rho/\sqrt{n})(1 + O(n^{-1}))$, the contribution of the first integral constitutes a proportion of $\lambda(\rho B^n)$ that is strictly smaller than 1 (uniformly in n) and asymptotically, as

$n \to \infty$, is of order $\Phi(1) \cdot \lambda(\rho B^n)$, where Φ is the c.d.f. of a standard $N(0,1)$ Gaussian random variable. Similarly, the contribution of the second integral is shown to be $o(1) \cdot \lambda(\rho B^n)$ as $n \to \infty$ (or, more exactly, less than $(\rho/\sqrt{n}) \cdot \lambda(\rho B^n)$ for all $n \geqq 2$); we omit the rather routine details. Combining the two estimates yields (1.4), hence Lemma 1.3.

1.4 Remark. Theorem 1.2 is optimal in the following sense: there exist constants $\alpha, A > 0$ such that, for any $n \in \mathbb{N}$ (resp. for any $n \in \mathbb{N}$, $\rho \in (0,1)$), there exist $A, B \in \mathbb{R}^n$ (resp. with $\rho \leqq (\lambda(B)/\lambda(A))^{1/n} \leqq \rho^{-1}$) and $\Theta \subset A \times B$ with $\lambda(\Theta) > \alpha\lambda(A)\lambda(B)$ (resp. $\lambda(\Theta) > (1 - A\rho n^{\frac{1}{2}})\lambda(a)\lambda(B)$) such that the assertion of the theorem does not hold.

1.5 Corollary. *There exist $c, C > 0$ such that, for any $\delta \in [0,c]$, $n \in \mathbb{N}$, any $A, B \subset \mathbb{R}^n$ and any $\Theta \subset A \times B$ with $\lambda(\Theta) \geqq (1-\delta)\lambda(A)\lambda(B)$ one has*

$$\lambda(A +_\Theta B)^{2/n} \geqq \left(1 - \frac{C\delta}{n}\right) (\lambda(A)^{2/n} + \lambda(B)^{2/n}) . \tag{1.5}$$

Proof. We may assume that $\lambda(A) = 1 \geqq \lambda(B) = \rho^n$. Let $c > 0$ be one given by Theorem 1.2; we may clearly assume that $c \leqq 1/2$. If $\rho \geqq \delta/(c\sqrt{n})$, we may apply Theorem 1.2 and get the assertion, in fact without the factor $(1 - \frac{C\delta}{n})$. On the other hand, regardless of the size of ρ one has (just by Fubini's theorem),

$$\lambda(A +_\Theta B) \geqq (1-\delta)\lambda(A) = 1 - \delta ,$$

hence

$$\lambda(A +_\Theta B)^{2/n} \geqq 1 - \frac{3\delta}{n} ,$$

and it is easy to check that, for an appropriate choice of C, the right-hand side of (1.5) does not exceed the latter quantity if $\rho < \delta/(c\sqrt{n})$.

1.6 Remark. Redoing the argument of Theorem 2.1 in the context of Corollary 1.5 (rather than formally applying the assertion of the theorem) does not produce a sharper result. However, it is possible to obtain an assertion similar to that of Corollary 1.5 under much weaker assumptions, namely, in the notation of Theorem 1.2, if $\gamma \in (0,1)$ then the condition $\lambda(\Theta) \geqq \gamma\lambda(A)\lambda(B)$ implies a version of (1.5) with $(1 - C\delta/n)$ replaced by $(1 - C\rho(\log(1 + 1/\gamma)/n)^{\frac{1}{2}})$.

2. The Free Entropy Power Inequality

The free entropy $\chi(X_1, \ldots, X_n)$ for an n-tuple of selfadjoint elements $X_j \in M$, M a von Neumann alegbra with a normal faithful trace state τ, was defined in [8] part II. The definition involves sets of matricial microstates $\Gamma_R(X_1, \ldots, X_n; m, k, \varepsilon)$ (see Sect. 2 in [8] part II). The microstates are points in $(\mathcal{M}_k^{sa})^n$, where $\mathcal{M}_k^{sa}$ denotes the selfadjoint $k \times k$ matrices. λ will denote the Lebesgue measure on $(\mathcal{M}_k^{sa})^n$ corresponding to the euclidean norm

$$\|(A_1, \ldots, A_n)\|_{HS}^2 = \mathrm{Tr}(A_1^2 + \cdots + A_n^2) .$$

For one random variable we have (Prop. 4.5 in [8] part II) that:

$$\chi(X) = \int\int \log|s-t|d\mu(s)d\mu(t) + \frac{3}{4} + \frac{1}{2}\log 2\pi\,, \tag{2.1}$$

where μ is the distribution of X (see 2.3 in [9]) or equivalently the measure on $\mathbb{R}$ obtained by applying the trace τ to the spectral measure of X.

2.1 Theorem. *Let $X, Y \in M$, $X = X^*$, $Y = Y^*$ and assume X, Y are free. Then*

$$\exp(2\chi(X)) + \exp(2\chi(Y)) \leqq \exp(2\chi(X+Y))\,. \tag{2.2}$$

Using the explicit formula for $\chi(X)$ and the fact that the distribution of the sum of two free random variables is obtained via the free convolution $\boxplus$ (see 3.1 in [9]) there is an equivalent form of the preceding theorem.

2.1′ Theorem. *Let α, β be compactly supported probability measures on $\mathbb{R}$. Then*

$$\begin{aligned}&\exp(2\int\int \log|s-t|d\alpha(s)d\alpha(t)) + \exp(2\int\int \log|s-t|d\beta(s)d\beta(t))\\ &\quad \leqq \exp(2\int\int \log|s-t|d(\alpha\boxplus\beta)(s)d(\alpha\boxplus\beta)(t)\,.\end{aligned} \tag{2.3}$$

Proof of Theorem 2.1. The proof will be technically similar to Sects. 4 and 5 of [8] part II. Let $Z \in M$, $Z = Z^*$ distributed according to Lebesgue measure on $[0,1]$ and let U_1, U_2 be unitaries with Haar distributions in (M,τ) and assume Z, U_1, U_2 are $*$-free. Let further $h_1, h_2 : [0,1] \to \mathbb{R}$ be C^1-functions with $h_1'(t) > 0$, $h_2'(t) > 0$ for all $t \in [0,1]$. Remark that it suffices to prove the theorem in the case $X = U_1 h_1(Z) U_1^*$, $Y = U_2 h_2(Z) U_2^*$ (i.e., the distributions of X and Y are the push-forwards by h_1 and h_2 of Lebesgue measure on $[0,1]$). Indeed see 2° in the proof of Proposition 4.5 in [8] part II) there are sequences $h_{j,n}$ of functions as above, such that

$$\lim_{n\to\infty} \chi(U_1 h_{1,n}(Z)U_1^*) = \chi(X)\,, \qquad \lim_{n\to\infty} \chi(U_2 h_{2,n}(Z)U_2^*) = \chi(Y)\,,$$

$h_{1,n}(Z), h_{2,n}(Z)$ converging in distribution to X, Y and $\|h_{j,n}\|_\infty < R$ for some fixed constant R. Then

$$\|U_1 h_{1,n}(Z)U_1^* + U_2 h_{2,n}(Z)U_2^*\| \leqq 2R\,,$$

and $U_1 h_{1,n}(Z)U_1^* + U_2 h_{2,n}(Z)U_2^*$ converges in distribution to $X + Y$ because of our freeness assumptions. By 2.6 in [8] part II we have

$$\limsup_{n\to\infty} \chi(U_1 h_{1,n}(Z)U_1^* + U_2 h_{2,n}(Z)U_2^*) \leqq \chi(X+Y)\,,$$

and hence it suffices to prove Theorem 2.1, in case $X = U_1 h_1(Z)U_1^*$, $Y = U_2 h_2(Z)U_2^*$.

Like in 5.3 of [8] part II, let

$$\Omega(h_j;k) = \{A \in \mathscr{M}_k^{sa} \mid h_j(2s/2k) \leqq \lambda_{s+1}(A) \leqq h_j((2s+1)/2k), 0 \leqq s \leqq k+1\}\,,$$

where $\lambda_1(A) \leqq \cdots \leqq \lambda_k(A)$ are the eigenvalues of A. The last part of the proof of Proposition 4.5 in [8] part II shows that

$$\lim_{k\to\infty} (k^{-2}\log\lambda(\Omega(h_j;k)) + 2^{-1}\log k) = \chi(h_j(Z))\,, \tag{2.4}$$

where λ is the Lebesgue measure on $\mathscr{M}_k^{sa}$.

Let further $N \in \mathbb{N}$ and $\varepsilon > 0$ be given and

$$\Theta(k) = \{(A_1, A_2) \in \prod_{1 \leqq j \leqq 2} \Omega(h_j; k) \mid (A_1, A_2) \in \Gamma(U_1 h_1(Z) U_1^*, U_2 h_2(Z) U_2^*; N, k, \varepsilon)\} . \tag{2.5}$$

By Lemma 5.3 in [8] part II we have:

$$\lim_{k \to \infty} \frac{\lambda(\Theta(k))}{\lambda(\Omega(h_1; k) \times \Omega(h_2; k))} = 1 . \tag{2.6}$$

If $R > \|h_j\|_\infty$ then

$$\Theta(k) \subset \Gamma_R(U_1 h_1(Z) U_1^*, U_2 h_2(Z) U_2^*;\ N, k, \varepsilon) .$$

Further, given $N_1 \in \mathbb{N}$, $\varepsilon_1 > 0$ we may choose $N \in \mathbb{N}$, $\varepsilon > 0$ so that

$$(A_1, A_2) \in \Gamma_R(U_1 h_1(Z) U_1^*, U_2 h_2(Z) U_2^*;\ N, k, \varepsilon)$$

implies

$$A_1 + A_2 \in \Gamma_{2R}(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*;\ N, k, \varepsilon) .$$

In particular,

$$\Omega(h_1; k) +_{\Theta(k)} \Omega(h_2; k) \subset \Gamma_{2R}(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*;\ N_1, k, \varepsilon_1) . \tag{2.7}$$

Using Theorem 1.2 for $k \geqq k_0$ with k_0 sufficiently large, taking into account (2.6), we have

$$(\lambda(\Omega(h_1; k)))^{2/k^2} + (\lambda(\Omega(h_2; k)))^{2/k^2}$$

$$\leqq (\lambda(\Gamma_{2R}(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*;\ N_1, k, \varepsilon_1)))^{2/k^2} .$$

Given $\delta > 0$ we may choose k_0, N_1 large and ε_1 small, so that

$$k^{-2} \log \lambda(\Gamma_{2R}(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*;\ N_1, k, \varepsilon_1) + \frac{1}{2} \log k$$

$$\leqq \chi(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*) + \delta$$

for all $k \geqq k_0$.

We infer that for $k \geqq k_0$,

$$\exp(2k^{-2}(\log \lambda(\Omega(h_1; k)) + 2^{-1} \log k) + \exp(2k^{-2}(\log \lambda(\Omega(h_2; k)) + 2^{-1} \log k)$$

$$\leqq \exp(2(\chi(U_1 h_1(Z) U_1^* + U_2 h_2(Z) U_2^*) + \delta)) .$$

Letting $k \to \infty$ and taking into account that $\delta > 0$ was arbitrary, we get the desired inequality. $\square$

3. Concluding Remarks and Open Problems

3.1. The Free Entropy Power Inequality for n-Tuples. To extend Theorem 2.1 to n-tuples of non-commutative random variables means to prove

$$\exp\left(\frac{2}{n}\chi(X_1,\ldots,X_n)\right) + \exp\left(\frac{2}{n}\chi(Y_1,\ldots,Y_n)\right) \leqq \exp\left(\frac{2}{n}\chi(X_1+Y_1,\ldots,X_n+Y_n)\right) \tag{3.1}$$

under the assumption that $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_n\}$ are free. The missing ingredient at this time is the generalization of Sect. 5 in [8] part II to n-tuples. The rest of the argument, i.e. the use of Theorem 1.2, would then be along the same lines as for $n = 1$. At present, partial generalizations of Theorem 2.1 can be obtained. The route to be followed is: first replace X and Y by n-tuples $(X_1,\ldots,X_n)$, $(Y_1,\ldots,Y_n)$ such that the $2n$ variables $X_1,\ldots,X_n$, $Y_1,\ldots,Y_n$ are free and note that in this situation the necessary facts about sets of matricial microstates can be obtained from Sect. 5 of [8] part II. Then the generalization of Theorem 2.1 will hold for n-tuples $(F_1(X_1,\ldots,X_n),\ldots,F_n(X_1,\ldots,X_n))$ and $(H_1(Y_1,\ldots,Y_n),\ldots,H_n(Y_1,\ldots,Y_n))$, where $X_1,\ldots,X_n$, $Y_1,\ldots,Y_n$ are free and $(F_1,\ldots,F_n)$,$(H_1,\ldots,H_n)$ are non-commutative functions satisfying suitable conditions, like the existence of an inverse of the same kind and extending to the matricial microstates. These kind of extensions have statements containing many technical conditions, the proof, except for some technicalities, being along the same lines as for $n = 1$. We don't pursue this here, hoping that better techniques will yield a proof of the free entropy power inequality in full generality.

3.2. Shannon's Classical Entropy Power Inequality and Restricted Minkowski Sums. We would like to signal that the inequality in Theorem 1.2 has the potential to provide a proof also of Shannon's classical entropy power inequality. The reason is that the classical entropy of an n-tuple of commutative random variables can be defined via microstates (using the diagonal subalgebra of the $n \times n$ matrix algebra instead of the full algebra) and the entropy power inequality would then correspond to the same kind of geometric problem at the level of microstates as in the free case. We are thinking of exploring this possibility in future work.

3.3. The Free Analogue of the Stam Inequality. It seems natural to look also for a free analogue of the Stam inequality ([7], see also [1,3]), of which the free entropy power inequality would be a consequence. With Φ denoting the free analogue of Fisher's information measure (see [8] part I) this problem amounts to proving that:

$$(\Phi(X+Y))^{-1} \leqq (\Phi(X))^{-1} + (\Phi(Y))^{-1}$$

if X, Y are free.

Acknowledgements. Part of this work was done while the first named author was visiting Université Paris VI and Université Marne la Vallee in May–July 1995. The second named author worked on this paper while visiting MIT and IHES in 1995. The authors express their gratitude to these institutions for their hospitality and support.

References

1. Blachman, N.M.: The convolution inequality for entropy powers. IEEE Trans. Inform. Theory **2**, 267–327 (1965)
2. Brascamp, H.J., Lieb, E.H., Luttinger, J.M.: A general rearrangement inequality for multiple integrals. J. Funct. Anal. **17**, 227–237 (1974)
3. Carlen, E.A.: Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. **101**, 194–211 (1991)
4. Pisier, G.: The Volume of Convex Bodies and Banach Space Geometry. Cambridge: Cambridge Univ. Press, 1989
5. Schneider, R.: Convex bodies: The Brunn–Minkowski theory. Encyclopedia of Mathematics and Its Applications **44**, Cambridge: Cambridge Univ. Press, 1993
6. Shannon, C.E., Weaver, W.: The Mathematical Theory of Communications. University of Illinois Press, 1963
7. Stam, A.J.: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Information and Control **2**, 101–112 (1959)
8. Voiculescu, D.: The analogues of entropy and of Fisher's information measure in free probability theory, I. Commun. Math. Phys. **155**, 71–92 (1993); ibidem II. Invent. Math. **118**, 411–440 (1994); *ibidem* III, IHES preprint, May 1995 (to appear in GAFA)
9. Voiculescu, D., Dykema, D., Nica, A.: Free Random Variables. CRM Monograph Series, Vol. 1, Providence, RI: American Mathematical Society, 1992

Communicated by A. Connes