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Wave Equations on q-Minkowski Space

U. Meyer

D.A.M.T.P., University of Cambridge, Silver Street, Cambridge CB39EW, United Kingdom
E-mail: um102@amtp.cam.ac.uk

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Abstract: We give a systematic account of a "component approach" to the algebra of forms on q-Minkowski space, introducing the corresponding exterior derivative, Hodge star operator, coderivative, Laplace-Beltrami operator and Lie-derivative. Using this (braided) differential geometry, we then give a detailed exposition of the q-d'Alembert and q-Maxwell equation and discuss some of their non-trivial properties, such as for instance, plane wave solutions. For the q-Maxwell field, we also give a q-spinor analysis of the q-field strength tensor.

1. Introduction

This paper develops some elements of braided differential geometry on quantum Minkowski space and uses these new tools to define and analyse the two simplest wave equations on this non-commutative spacetime, namely the q-d'Alembert and the q-Maxwell equation. In order to distinguish our approach from other related work it might be useful to emphasize that we are constructing generalisations of *classical* wave equations in position space and not a deformation of quantum theory, as for instance in [13], where wave equations in momentum space were constructed by using irreducible representations of the q-Poincaré group. At present, there is no q-Fourier transform in this case, and it does not seem to be possible to compare the results of the two approaches.

In our exposition of braided differential geometry we present forms in a slightly different way than in some earlier papers by other authors. We use what one might call a *component approach* to forms, but will show in Proposition 2.7 that the two possible approaches are equivalent. This different approach to forms has the consequence that the q-exterior derivative d is constructed in terms of braided differential operators ∂^a and not vice versa, as for instance in [12]. The additional ingredient needed for this construction is a q-Lorentz covariant antisymmetrisation operation, which we introduce. In a similar fashion, we also define the q-Hodge star operator, q-coderivative, q-Lie derivative, and q-Laplace-Beltrami operator. The advantage of the component approach is that for instance the q-electromagnetic field is given terms of components and admits a very simple $SL_q(2,\mathbb{C})$-spinor decomposition into self-dual and anti-self-dual parts, which will be discussed in the last section of this paper.

Preliminaries. In a previous paper [11], we gave a detailed account of the q-deformation of spacetime and its symmetry group (see [10] for a comparison with the approach of [2, 1]). The key idea was that q-Minkowski space should be given by 2×2 braided Hermitian matrices, which were introduced by S. Majid in [3] as a non-comutative deformation of the algebra of complex-valued polynomial functions on the space of ordinary Hermitian matrices. Braided matrices have a central and grouplike element, the so-called *braided determinant*, which plays the role of a q-norm and which determines a q-deformed Minkowski metric.

As given in [3], the braided matrices did not generalise the additive group structure of Minkowski space, which should be reflected in our dual and q-deformed setting by a *braided coaddition* as introduced in [6]. A braided coaddition is a braided coproduct of the from $\underline{\Delta} x = x \underline{\otimes} 1 + 1 \underline{\otimes} x$ which extends as an algebra map with respect to a *braided tensor product* $\underline{\otimes}$ and not the ordinary tensor product $\otimes$. Braided tensor products are like the super tensor products encountered in the theory of superspaces, but with the ± 1 factors replaced by braid statistics. The tensor product algebra is given by $(a \underline{\otimes} b)(c \underline{\otimes} d) = a \Psi(b \underline{\otimes} c) d$, where Ψ is the so-called *braiding* which measures how two independent copies of a system fail to commute. In the commutative case, Ψ is simply given by the twist map $\Psi(a \otimes b) = b \otimes a$. In the non-commutative case the braiding is determined by a background quantum group, which acts as the symmetry group of the system. A good introduction to the theory of braided matrices and braided groups is in [4].

The braiding and background quantum group which allows for quantum Minkowski space to have a braided coaddition was found in [11]. This construction gives rise to a natural quantum Lorentz group which preserves both the braided coaddition and the non-commutative algebra structure of quantum Minkowski space. The final result is given in terms of two solutions of the four dimensional QYBE:

$$R_{M}{}^{cd}{}_{ab} = {R^{-1}}^{LC}{}_{BI} R^{B'I}{}_{JA} R^{A'J}{}_{KD'} \tilde{R}^{KD}{}_{C'L} \qquad R_{L}{}^{ab}{}_{cd} = R^{CI}{}_{JB} R^{B'J}{}_{KA} R^{A'K}{}_{LD'} \tilde{R}^{LD}{}_{C'I} \,,$$

where P denotes the permutation map and

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad q \in \mathbb{R} \tag{1}$$

is the standard $SU_q(2)$ R-matrix. The matrix $\tilde{R}$ is defined as $((R^{t_2})^{-1})^{t_2}$, where t_2 denotes transposition in the second tensor component. We also use multi-indices $a = (AA') = (11), (12), (21), (22)$. These two matrices obey the relation

$$0 = (PR_L + 1)(PR_M - 1) \,, \tag{2}$$

which is needed to show the existence of a braided coaddition. In terms of these data, q-Minkowski space M_q is given as the algebra of quantum covectors $M_q = V^*(R_M)$ in the notation of [8]. It has generators x_a and a star structure $x_a^* = x_{\bar{a}}$, where $\bar{a} = (A'A)$ dentoes the twisted multi-index. The quantum Lorentz group $\mathscr{L}_q$ is defined as a quotient of the FRT algebra $A(R_L)$ with generators λ^a_b by the metric relation $\lambda^a{}_c\lambda^b{}_d g^{cd} = g^{ab}$, where the q-metric is given by $g^{ab} = (q+q^{-1})^{-1}\varepsilon_{AC}R^{A'C}_{DB}$ $\varepsilon^{DB'}$ in terms of the $SL_q(2,\mathbb{C})$-spinor metric

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1/\sqrt{q} \\ -\sqrt{q} & 0 \end{pmatrix} . \tag{3}$$

We are working in a "spinorial basis," where the metric has two negative and two positive eigenvalues. There is also a $*$-Hopf algebra morphism $\mathscr{L}_q \to SL_q(2,\mathbb{C})$ given by $\lambda^a{}_b \mapsto t^{\dagger B}{}_A t^{B'}{}_{A'}$ where t are the generators of $SL_q(2,\mathbb{C})$. This map induces a push forward of $\mathscr{L}_q$-comodules.

If we now consider the coaction of $\mathscr{L}_q$ on M_q, one problem arises: In order to obtain full covariance under the coaction by the q-Lorentz group, we have to adjoin $\mathscr{L}_q$ by a single invertible central and grouplike element ς [6]. The extended q-Lorentz group is denoted by $\tilde{\mathscr{L}}_q$, and its covariant right coaction on q-Minkowski space is given by

$$\beta_{M_q} : x_a \mapsto x_b \otimes \lambda^b{}_d \varsigma$$

Similarly, there is a covariant coaction of the extended algebra $\widetilde{SL}_q(2,\mathbb{C})$ given by $x^A{}_{A'} \mapsto x^B{}_{B'} \otimes t^{\dagger A}_B t^{B'}_{A'}\varsigma$. Since the element ς measures the degree of elements of M_q, it is often called the *dilaton element* [16, 6].

2. Differential Forms on Quantum Minkowski Space

Differential operators ∂^a on quantum Minkowski space were first presented by O. Ogievetski *et al.* in [12], where suitable commutation relations between these operators were introduced by hand. A general theory of braided differential operators was developed only subsequently by S. Majid in [7], and allows for a more systematic presentation of the algebra from [12]. This general construction works for any algebra of quantum vectors which can be equipped with a braided coaddition. The action of braided differential operators ∂^a on quantum vectors is then defined by formally "differentiating" the braided coaddition. Applied to q-Minowski space (equipped with the braided coaddition from [11]) this construction yields an algebra of braided differential operators $\mathscr{D}$ with generators ∂^a which obey the $V(R_M)$-relations $\partial^a\partial^b = R_M{}^{ab}{}_{cd}\partial^d\partial^c$ [7, Propsoition 2.2]. After changing q for q^{-1} to match out conventions, it is easily seen that this is just the algebra from [12. Eq. (5.2)] written in a compact form. This algebra $\mathscr{D}$ acts on quantum Minkowski space with an action $\alpha: \mathscr{D}\underline{\otimes}M_q \to M_q$ such that a *braided Leibniz rule* holds [7, Lemma 2.2]:

$$\partial^a fg = (\partial^a f)g + \circ\Psi_L^{-1}(\partial^a \otimes f)g . \tag{4}$$

Again, these are just the corresponding relations from [12, Sect. 5] in a compact form. Relation (4) also explains why we insist on calling the operators ∂^a *braided* differential operators. For unlike in the commutative case, M_q is not a $\mathscr{D}$-module algebra, but what one might call a *braided* $\mathscr{D}$-module algebra. By repeating the construction from [11] one can easily show that $\mathscr{D}$ can be equipped with a braided coaddition $\underline{\Delta}\partial^a = \partial^a\underline{\otimes}1 + 1\underline{\otimes}\partial^a$, making it into a braided Hopf algebra. The braided Leibniz rule is then seen to be nothing but the statement

$\alpha(h \otimes ab) = \cdot \circ (\alpha \otimes \alpha) \circ (id \otimes \Psi \otimes id) \circ (\underline{\Delta} \otimes id)(h \otimes a \otimes b)$, which is a *braided* generalisation of the classical notion of a module algebra.

By writing everything in the compact notation, it is also easily seen that q-differentiation is covariant in the sense that α is an $\tilde{\mathscr{L}}_q$-comodule morphism. Due to the covariance of the braided tensor product (see [5] for a detailed discussion), $\mathscr{D}\underline{\otimes}M_q$ is $\tilde{\mathscr{L}}_q$-covariant.

An open problem in this context is the question of the $*$-structure on $\mathscr{D}$. It is possible to equip $\mathscr{D}$ with a suitable $*$-structure making it into a braided $*$-Hopf algebra in the sense of [9], but the obvious choice for "$*$" does not commute with the action α. It has been speculated that the very notion of a $*$-structure needs to be q-deformed, but we will not attempt to solve this problem here.

2.1. q-Antisymmetrisers. In our component approach to forms, the exterior derivative is defined in terms of the braided differential operators ∂^a and not vice versa. An essential ingredient in this approach is an $\tilde{\mathscr{L}}_q$-covariant q-antisymmetrisation operation, which we now introduce. For this purpose, we define a q-deformed notion of antisymmetry, and call an $\tilde{\mathscr{L}}_q$-tensor $T_{\dots ab\dots}$ *q-antisymmetric* in adjacent indices a and b if

$$T_{\dots ab \dots} = T_{\dots cd \dots} R_{L\ ab}^{\ dc} \,. \tag{5}$$

Here T_{ab} and T^{ab}, etc. denote any elements of right $\tilde{\mathscr{L}}_q$-comodules, which transform as $T_{ab} \mapsto T_{cd} \otimes \lambda^c{}_a \lambda^d{}_b \varsigma^n$ and $T^{ab} \mapsto T^{cd} \otimes S\lambda^a{}_c S\lambda^b{}_d \varsigma^m$, respectively, where S denotes the antipode in $\mathscr{L}_q$. We do not require a tensor to have a specific ς-scaling property, and therefore n and m can be any integers. If a tensor is q-antisymmetric in any two adjacent indices, it is called *totally q-antisymmetric*.

At first sight, the definition (5) seem to cover only the case of lower indices. However, it is known from [11], that one can use the q-metric g^{ab} and its inverse to raise and lower indices in a q-Lorentz covariant fashion. The key ingredient in the proof is the relation $R_{L\ ef}^{\ kl} = g_{pf} g_{qe} R_{L\ ab}^{\ qp} g^{ak} g^{bl}$ between the R-matrix and the q-metric. Thus if for example a tensor $T_{\dots ab \dots}$ is q-antisymmetric in a and b then the tensor with upper indices $T^{\cdots ab \cdots} = T_{\dots ij \dots} \cdots g^{ia} g^{jb} \cdots$ obeys $T^{\cdots ab \cdots} = -R_{L\ dc}^{\ ab} T^{\cdots cd \cdots}$. Hence it is sufficient to define q-antisymmetry for either upper or lower indices.

Similarly to the definition of q-antisymmetry, we call a tensor *q-symmetric* in a and b if $T_{\dots ab \dots} = T_{\dots cd \dots} R_{M\ ab}^{\ dc}$. Again, this translates into a corresponding formula for upper indices: this time by virtue of an analogous relation between the matrix R_M and g. It is crucial to note that the two R-matrices used in the definition of q-symmetry and q-antisymmetry are genuinely different and not identical up to scaling as in the Hecke case.

We would like to define a q-antisymmetrisation operation which assigns to any q-Lorentz tensor a q-antisymmetric one in a covariant fashion. Since the two R-matrices R_L and R_M obey the relation (2), one might suspect that $(PR_M - 1)$ would be a good candidate for a q-antisymmetriser. However, this operator is not a projector, and it is also not quite clear how to obtain higher antisymmetrisers. We shall therefore take a different approach.

In the classical case, the space of totally antisymmetric tensors of valence four is one-dimensional, and one can choose a basis vector ε_{abcd} with $\varepsilon_{1234} = 1$, which then defines a projector (antisymmetriser) $\frac{1}{4!}\varepsilon^{dcba}\varepsilon_{efgh}$ onto this one-dimensional space. By successively contracting indices, one obtains lower antisymmetrisers. This construction turns out to be applicable also in the q-deformed case, where a q-epsilon tensor is also uniquely determined:

Lemma 2.1. *Up to a factor, there is exactly one complex valued tensor ε_{abcd} which is totally q-antisymmetric.*

Proof. It suffices to verify explicitly that the system of linear equations $\varepsilon_{abcd} = -\varepsilon_{ijcd}R_{L}{}^{ji}{}_{ab} = -\varepsilon_{aijd}R_{L}{}^{ji}{}_{bc} = -\varepsilon_{abij}R_{L}{}^{ji}{}_{cd}$ has a one-dimensional solution space. □

The non-zero entries of ε_{abcd} in the normalisation $\varepsilon_{1234} = 1$ are:

$\varepsilon_{1234} = 1$	$\varepsilon_{1243} = -q^{-2}$	$\varepsilon_{1324} = -1$	$\varepsilon_{1342} = q^2$
$\varepsilon_{1414} = 1 - q^2$	$\varepsilon_{1423} = 1$	$\varepsilon_{1432} = -1$	$\varepsilon_{1444} = 1 - q^{-2}$
$\varepsilon_{2134} = -1$	$\varepsilon_{2143} = q^{-2}$	$\varepsilon_{2314} = 1$	$\varepsilon_{2341} = -1$
$\varepsilon_{2413} = -q^{-2}$	$\varepsilon_{2431} = q^{-2}$	$\varepsilon_{2434} = q^{-2} - 1$	$\varepsilon_{3124} = 1$
$\varepsilon_{3142} = -q^2$	$\varepsilon_{3214} = -1$	$\varepsilon_{3241} = 1$	$\varepsilon_{3412} = q^2$
$\varepsilon_{3421} = -q^2$	$\varepsilon_{3424} = 1 - q^2$	$\varepsilon_{4123} = -1$	$\varepsilon_{4132} = 1$
$\varepsilon_{4141} = q^2 - 1$	$\varepsilon_{4144} = q^{-2} - 1$	$\varepsilon_{4213} = 1$	$\varepsilon_{4231} = -1$
$\varepsilon_{4243} = 1 - q^{-2}$	$\varepsilon_{4312} = -1$	$\varepsilon_{4321} = 1$	$\varepsilon_{4342} = q^2 - 1$
$\varepsilon_{4414} = 1 - q^{-2}$	$\varepsilon_{4441} = q^{-2} - 1$		

Using a different approach, a q-epsilon tensor for q-Minkowski space was also presented by A. Schirrmacher in [14]. But this q-epsilon tensor is not q-antisymmetric in the sense of our definition and does not coincide with the one given above.

After using the q-metric to obtain the corresponding q-epsilon tensor with upper indices, we can define q-antisymmetrisers by successively contracting indices of these two q-epsilon tensors:

Definition 2.2. *The q-antisymmetrisation of an $\tilde{\mathscr{L}}_q$-tensor $T_{\cdots a_1 \cdots a_n \cdots}$ in adjacent indices $a_1 \cdots a_n$ is defined as*

$$T_{\cdots [a_1 \cdots a_n] \cdots} = T_{\cdots c_1 \cdots c_n \cdots} \mathscr{A}_{\{n\}}{}^{c_1 \cdots c_n}{}_{a_1 \cdots a_n},$$

where the q-antisymmetrisers $\mathscr{A}_{\{k\}}$ are given by

$$\mathscr{A}_{\{k\}}{}^{a_1 \cdots a_k}{}_{b_1 \cdots b_k} = n_k^{-1} \varepsilon^{c_{4-k} \cdots c_1 a_k \cdots a_1} \varepsilon_{c_1 \cdots c_{4-k} b_1 \cdots b_k}$$

for $n < 5$ and are zero otherwise. The normalisation factors $n_1 = n_3 = 2(1 + q^2 + q^4)$, $n_2 = (1 + q^2)(1 + q^2)$, $n_4 = q^{-2}2(1 + q^2 + q^4)(1 + q^2)(1 + q^2)$ are a q-deformation of $(4 - k)!k!$.

It is evident that the q-antisymmetrisation of a tensor is q-antisymmetric in the sense of (5) and with some more effort one can show that the operation of q-antisymetrisation also has all the other relevant projector properties:

Proposition 2.3. *By explicit calculation, one can show:*

1. *The antisymmetrisers $\mathscr{A}_{\{k\}}$ are projectors:*

$$\mathscr{A}_{\{k\}}^2 = \mathscr{A}_{\{k\}}, \quad \text{i.e.} \quad T_{\cdots [[a_1 \cdots a_n]] \cdots} = T_{\cdots [a_1 \cdots a_n] \cdots}. \tag{6}$$

2. *Lower dimensional q-antisymmetrisers cancel on higher dimensional ones*:

$$T_{\dots[a_1\cdots[a_k\cdots a_1]\cdots a_n]} = T_{\dots[a_1\cdots a_n]\cdots} \,. \tag{7}$$

3. *The one-dimensional projector is trivial*:

$$\mathscr{A}_{\{1\}} = 1, \quad i.e. \quad T_{\dots[a]\dots} = T_{\dots a\dots} \,.$$

4. *The two-dimensional antisymmetriser $\mathscr{A}_{\{2\}}$ factors through $(PR_M - 1)$*:

$$\mathscr{A}_{\{2\}} = (PR_M - 1)B = B'(PR_M - 1)\,, \tag{8}$$

where B and B' are invertible matrices.

The q-antisymmetrisers are also "Hermitian" with respect to the q-deformed metric. For instance, $\mathscr{A}_{\{2\}}$ obeys the relation $\mathscr{A}_{\{2\}cd}^{ab} = g_{cj}g_{di}\mathscr{A}_{\{2\}kl}^{ij}g^{al}g^{bk}$, and similar relations hold for the other q-antisymmetrisers.

Of particular importance for the following is relation (8), which ensures that q-symmetric tensors are in the kernel of the q-antisymmetrisers. Relations (7) and (8) imply:

Corollary 2.4. *If an $\tilde{\mathscr{L}}_q$-tensor $T_{\dots a_1\cdots a_k\dots}$ is q-symmetric in any two adjacent indices a_i and a_{i+1}, then $T_{\dots[a_1\cdots a_ia_{i+1}\cdots a_k]\dots} = 0$.*

Thus although q-symmetry and q-antisymmetry are defined in terms of two different R-matrices, the two notions are compatible in this sense. This corollary will be used later when we introduce the external derivative d and show that $d^2 = 0$.

Up to now we have used the index notation also for q-antisymmetrised tensors, thus implying certain transformation properties of these objects. However, *a priori* it does not seem to be obvious that the q-antisymmetrisation operation is $\tilde{\mathscr{L}}_q$-covariant.

Proposition 2.5. *The coaction by the q-Lorentz group commutes with q-antisymmetrisation.*

Proof. We need to show that monomials of generators of $\mathscr{L}_q$ commute with the q-antisymmetriser $\mathscr{A}_{\{n\}}$. For this purpose, note that the uniqueness of ε from Lemma 2.1 can be used to establish that the generators of the q-Lorentz group obey the relation

$$\lambda^a{}_e\lambda^b{}_f\lambda^c{}_g\lambda^d{}_h\varepsilon^{hgfe} = n_4^{-1}\varepsilon^{dcba}\varepsilon_{mnop}\lambda^m{}_e\lambda^n{}_f\lambda^o{}_g\lambda^p{}_h\varepsilon^{hgfe}$$

Hence we find for the q-antisymmetriser $\mathscr{A}_{\{2\}}$:

$$\begin{aligned}
\lambda^a{}_e\lambda^b{}_f\varepsilon^{dcfe}\varepsilon_{cdkl} &= \lambda^a{}_e\lambda^b{}_f\delta^c{}_g\delta^d{}_h\varepsilon^{hgfe}\varepsilon_{cdkl} \\
&= \lambda^a{}_e\lambda^b{}_f\lambda^n{}_g\lambda^m{}_h\varepsilon^{hgfe}S^{-1}\lambda^d{}_mS^{-1}\lambda^c{}_n\varepsilon_{cdkl} \\
&= n_4^{-1}\varepsilon^{mnba}\varepsilon_{opqr}\lambda^o{}_e\lambda^p{}_f\lambda^q{}_g\lambda^r{}_h\varepsilon^{hgfe}S^{-1}\lambda^d{}_mS^{-1}\lambda^c{}_n\varepsilon_{cdkl} \\
&= n_4^{-1}\varepsilon^{mnba}S^{-1}\lambda^d{}_mS^{-1}\lambda^c{}_n\varepsilon_{opqr}\lambda^o{}_e\lambda^p{}_f\lambda^q{}_g\lambda^r h : \varepsilon^{hgfe}\varepsilon_{cdkl} \\
&= \varepsilon^{mnba}S^{-1}\lambda^d{}_mS^{-1}\lambda^c{}_n\varepsilon_{opqr}\lambda^o{}_c\lambda^p{}_d\lambda^q{}_k\lambda^r{}_l \\
&= \varepsilon^{mnba}\varepsilon_{nmqr}\lambda^q{}_k\lambda^r{}_l\,,
\end{aligned}$$

and similar for $\mathscr{A}_{\{3\}}$ and $\mathscr{A}_{\{4\}}$. The case of $\mathscr{A}_{\{1\}}$ is trivial. □

2.2. The q-Exterior Algebra. After these remarks on q-antisymmetrisation, we will now introduce the notion of forms on quantum Minkowski space. Similar to [17], we first define an algebra Λ as the associative $\mathbb{C}$-algebra generated by 1 and four elements dx_a with relations $dx_a dx_b = -dx_m dx_n R_L{}^{nm}{}_{ab}$. This algebra is an $\tilde{\mathscr{L}}_q$-comodule algebra with coaction $dx_a \mapsto dx_b \otimes \lambda^b{}_a \varsigma$. As a linear space, it is a direct sum $\Lambda = \bigoplus_{k=0}^4 \Lambda_k$, where Λ_k is the $\mathbb{C}$-linear space spanned by 1 and the elements $dx_{a_1} \cdots dx_{a_k}$. In particular, we have $\Lambda_0 \cong \mathbb{C}$. We also define the dual algebra Λ^* with generators dx^a, relations $dx^a dx^b = -R_L{}^{ab}{}_{cd} dx^d dx^c$ and dual pairing given by $\langle dx^a, dx_b\rangle = \delta^a{}_b$. There is a q-metric induced isomorphism between these algebras defined by $dx^a \mapsto dx_b g^{ab}$, which is also a $\tilde{\mathscr{L}}_q$-comodule morphism. The proof is just like the proof of $V(R_M) \cong V^*(R_M)$ from [11].

For the construction of forms on quantum Minkowski space there are now two possibilities. On the one hand, one could consider the subalgebra of $\tilde{\mathscr{L}}_q$-scalars in $\Lambda^* \underline{\otimes} M_q$ and define k-forms as $\tilde{\mathscr{L}}_q$-scalars in $\Lambda_k^* \underline{\otimes} M_q$, as suggested by the construction in [17]. On the other hand, however, one could take a "component" approach, which is the one we shall use in this paper:

Definition 2.6. *A k-form* $\mathbf{w}$ *on q-Minkowski space is an* $\tilde{\mathscr{L}}_q$*-comodule morphism* $\mathbf{w} : \Lambda_k \longrightarrow M_q$.

Over the ring of all $\tilde{\mathscr{L}}_q$-scalars in M_q, k-forms form a linear space, which is denoted by Ω_k. In terms of this space, we can show that the two approaches to forms on quantum Minkowski space are equivalent:

Proposition 2.7. *The linear space* Ω_k *is isomorphic to the space of* $\tilde{\mathscr{L}}_q$*-scalars in* $\Lambda_k^* \underline{\otimes} M_q$.

Proof. All k-forms are of the form $\mathbf{w} : dx_{a_1} \cdots dx_{a_k} \mapsto f_{a_1 \cdots a_k}$ for some element $f_{a_1 \cdots a_k}$ of M_q. This means that we can define a linear map $\phi : \mathbf{w} \mapsto dx_{a_1} \cdots dx_{a_k} \underline{\otimes} f_{b_k \cdots b_1}\, g^{a_k b_k} \cdots g^{a_1 b_1}$ into the space of $\tilde{\mathscr{L}}_q$-scalars in $\Lambda_k^* \underline{\otimes} M_q$. On the other hand, any $\tilde{\mathscr{L}}_q$-scalar $h \in \Lambda_k^* \underline{\otimes} M_q$ is of the form $h = dx^{a_k} \cdots dx^{a_1} \underline{\otimes} h_{a_1 \cdots a_k}$ for some $h_{a_1 \cdots a_k} \in M_q$, and we can define a map $\psi : h \mapsto \langle dx^{a_k} \cdots dx^{a_1} \cdots\rangle h_{a_1 \cdots a_k}$ in Ω_k. It is easy to see that $\phi \circ \psi = id$ and $\psi \circ \phi = id$, and hence the two spaces are isomorphic. $\square$

For any k-form $\mathbf{w}$, the element $\mathbf{w}(dx_{a_1} \cdots dx_{a_k})$ is a completely q-antisymmetric tensor in M_q. All such tensors in M_q are in the image of the q-antisymmetrisers from Definition 2.2 and we find:

Proposition 2.8. *All k-forms on quantum Minkowski space are of the form* $\mathbf{w}(dx_{a_1} \cdots dx_{a_k}) = w_{[a_1 \cdots a_k]}$ *for some* $w_{a_1 \cdots a_k} \in M_q$.

Proof. The proposition follows from the observation that the dimension of totally q-antisymmetric tensors over the ring of $\tilde{\mathscr{L}}_q$-scalars in M_q coincides with the ranks of the q-antisymmetrisers $\mathscr{A}_{\{k\}}$. As in the classical case, these spaces have dimensions $1, 4, 6, 4, 1$ for $p = 0, 1, 2, 3, 4$, and dimension 0 for $p > 4$. $\square$

The one-dimensional space Ω_4 is spanned by the *top form* $\varepsilon : dx_{a_1} \cdots dx_{a_4} \mapsto \varepsilon_{a_1 \cdots a_4}$, and Ω_0 by the form $\mathbf{1} : \xi \mapsto \xi \cdot 1$ for $\xi \in \Lambda_0 = \mathbb{C}$. As a corollary of relation (7) from Proposition 2.3 one finds that the $\mathbb{C}$-linear space $\Omega = \bigoplus_{k=0}^4 \Omega_k$ can be equipped with an algebra structure.

Corollary 2.9. *The q-wedge product $\wedge : \Omega_k \times \Omega_r \to \Omega_{k+r}$ defined by*

$$\mathbf{w} \wedge \mathbf{v} : dx_{a_1} \cdots dx_{a_{k+r}} \mapsto w_{[a_1 \cdots a_k} v_{a_{k+1} \cdots a_{k+r}]}$$

is an associative $\mathbb{C}$-algebra structure on Ω, with identity $\mathbf{1} \in \Omega_0$.

Finally, we define the notion of a real form. By virtue of the relation $R_L{}^{ab}{}_{cd} = R_L{}^{\bar{b}\bar{a}}{}_{\bar{d}\bar{c}}$ from [11, Proposition 2.3], Λ can be equipped with a $*$-structure $dx_a^* = dx_{\bar{a}}$. A k-form $\mathbf{w}$ is then called *real* if $\mathbf{w}(dx_{a_1} \cdots dx_{a_k})^* = \mathbf{w}((dx_{a_1} \cdots dx_{a_k})^*)$.

As already mentioned before, this paper defines the exterior derivative d in terms of the braided differential operators ∂^a and the q-antisymmetrisers introduced in the last section:

Definition 2.10. *The q-exterior derivative $d : \Omega_k \mapsto \Omega_{k+1}$ is defined by $d\mathbf{w} : dx_{a_1} \cdots dx_{a_{k+1}} \mapsto \partial_{[a_1} w_{a_2 \cdots a_{k+1}]}$.*

Forms whose q-exterior derivative vanishes are called *closed* and forms which are themselves q-exterior derivatives are said to be *exact*. The crucial test for a definition of a "q-deformed exterior derivative" is whether exact forms are closed.

Proposition 2.11. *Exact forms on q-Minkowski space are closed*: $d^2 = 0$.

Proof. Relation (7) implies:

$$d^2\mathbf{w}(dx_{a_1} \cdots dx_{a_{k+2}}) = \partial_{[a_1} \partial_{a_2} w_{a_3 \cdots a_{k+1}]} = \partial_{[[a_1} \partial_{a_2]} w_{a_3 \cdots a_{k+1}]} = 0.$$

Here we used that braided differential operators ∂_a obey the relations of $V'(R_M)$ [7], i.e. $\partial_{a_1} \partial_{a_2}$ is q-symmetric and hence $\partial_{[a_1} \partial_{a_2]} = 0$ by virtue of Corollary 2.4. □

As a consequence of the braided Leibniz rule (4), we find for the action of the q-exterior derivative d on wedge products of forms:

Corollary 2.12. *The q-exterior derivative acts as*

$$d\mathbf{w} \wedge \mathbf{v} = (d\mathbf{w}) \wedge \mathbf{v} + (-1)^k \mathbf{w} \wedge d\mathbf{v}$$

on wedge products $\mathbf{w} \wedge \mathbf{v}$, where $\mathbf{w}$ is a k-form.

Proof. The crucial point is that the inverse braiding brings up R-matrices, which cancel on the q-antisymmetriser because of the symmetry property (5). We prove the corollary only for 1-forms $\mathbf{w}$, the general case follows immediately by using the hexagon identity for the braiding Ψ. Thus let $\mathbf{w}$ be a 1-form and $\mathbf{v}$ a k-form. On Λ_{k+2} we have by virtue of (4) and the q-antisymmetry of ε^{abcd}:

$$\begin{aligned}
\partial_{[a_1} w_{a_2} v_{a_3 \cdots a_{k+2}]} &= (\partial_{[a_1} w_{a_2}) v_{a_3 \cdots a_{k+2}]} + \cdot \circ \Psi^{-1}(\partial_{[a_1} \otimes w_{a_2}) v_{a_3 \cdots a_{k+2}]} \\
&= (\partial_{[a_1} w_{a_2}) v_{a_3 \cdots a_{k+2}]} + w_c \otimes \partial_d R^{-1}_L{}^{cd}{}_{[a_2 a_1} v_{a_3 \cdots a_{k+2}]} \\
&= (\partial_{[a_1} w_{a_2}) v_{a_3 \cdots a_{k+2}]} + w_{[a_2} \otimes \partial_{a_1} v_{a_3 \cdots a_{k+2}]} \,.
\end{aligned}$$

Here we used the inverse braiding $\Psi^{-1}(\partial_{a_1} \otimes w_{a_2}) = w_c \otimes \partial_d R^{-1}_L{}^{cd}{}_{a_2 a_1}$ (see [8, Prop. 3.2] for a useful list of braidings between various standard algebras). □

The second operator on Ω one can define with the tools at hand is the q-Hodge star operator. It is defined in terms of the metric g^{ab} and the tensor ε_{abcd}.

Definition 2.13. *The q-Hodge star operator* $* : \Omega_k \longrightarrow \Omega_{4-k}$ *is defined by*

$$^{*}\mathbf{w}(dx_{a_1} \cdots dx_{a_{4-k}}) = w_{[c_1 \cdots c_k]} H_{\{k\}}{}^{c_1 \cdots c_k}{}_{a_1 \cdots a_{4-k}} ,$$

where $H_{\{k\}}{}^{c_1 \cdots c_k}{}_{a_1 \cdots a_{4-k}} = n_k^{-1/2} \varepsilon_{a_1 \cdots a_{4-k} b_1 \cdots b_k} g^{b_1 c_k} \cdots g^{b_k c_1}$ *and* $n_0 = n_4$.

This q-Hodge star operator generalises the characteristic properties of the classical Hodge star operation. For example, the top form ε and the identity $\mathbf{1}$ are conjugate with respect to the q-Hodge star: $^{*}\varepsilon = \mathbf{1}$. Also, we can show for $*^2 : \Omega_k \longrightarrow \Omega_k$:

Proposition 2.14. $*^2|_{\Omega_k} = (-1)^{k(4-k)}$.

Proof. By explicit calculation, one can verify the following relations between the q-antisymmetrisers $\mathscr{A}_{\{k\}}$ and the matrices $H_{\{k\}}$, which implement the q-Hodge star operation:

$$\begin{aligned} &H_{\{0\}} H_{\{4\}} = 1, && -H_{\{1\}} H_{\{3\}} = \mathscr{A}_{\{1\}} , \\ &H_{\{2\}} H_{\{2\}} = \mathscr{A}_{\{2\}}, && -H_{\{3\}} H_{\{1\}} = \mathscr{A}_{\{3\}} , \\ &H_{\{4\}} H_{\{0\}} = \mathscr{A}_{\{5\}} . && \end{aligned} \tag{9}$$

Together with (6), these relations imply the proposition. Since we are working in a "spinorial basis" we do not obtain an additional (-1)-factor in Proposition 2.14, as in the case of an "x, y, z, t-basis." □

It is also possible to "shift" the q-Hodge star operator in the q-wedge product of two k-forms:

Lemma 2.15. *If* $\mathbf{w}$ *and* $\mathbf{v}$ *are both k-forms, then* $^{*}\mathbf{w} \wedge \mathbf{v} = (-1)^k \mathbf{w} \wedge {}^{*}\mathbf{v}$

Proof. Verify by explicit calculation

$$\begin{aligned} H_{\{1\}}{}^{a}{}_{bcd} \mathscr{A}_{\{4\}}{}^{bcde}{}_{klmn} &= -H_{\{1\}}{}^{e}{}_{bcd} \mathscr{A}_{\{4\}}{}^{abcd}{}_{klmn} , \\ H_{\{2\}}{}^{ab}{}_{cd} \mathscr{A}_{\{4\}}{}^{cdef}{}_{klmn} &= H_{\{2\}}{}^{ef}{}_{cd} \mathscr{A}_{\{4\}}{}^{abcd}{}_{klmn} , \\ H_{\{3\}}{}^{abc}{}_{d} \mathscr{A}_{\{4\}}{}^{defg}{}_{klmn} &= -H_{\{3\}}{}^{efg}{}_{d} \mathscr{A}_{\{4\}}{}^{abcd}{}_{klmn} . \end{aligned}$$

This proves the lemma. □

Now that we are given both a well-behaved exterior derivative and a q-Hodge star operator, it is straightforward to define a q-coderivative, q-Laplace–Beltrami operator and q-Lie derivative

Definition 2.16. *The q-coderivative* $\delta : \Omega_k \longrightarrow \Omega_{k-1}$ *and the q-Laplace–Beltrami operator* $\Delta : \Omega_k \longrightarrow \Omega_k$ *on k-forms on quantum Minkowski space are defined as* $\delta = {}^{*}d^{*}$ *and* $\Delta = \delta d + d\delta$*, respectively.*

Forms $\mathbf{w}$ on quantum Minkowski space which satisfy $\delta\mathbf{w} = 0$ are called *co-closed*, and forms which are themselves q-coderivatives are called *co-exact.* As a corollary of Proposition 2.14 and Proposition 2.11, one finds:

Corollary 2.17. *Co-exact forms on quantum Minkowski space are co-closed:* $\delta^2 = 0$.

Thus, although d, δ and Δ are defined in terms of deformed antisymmetrisers and differential operators on a non-commutative space, their abstract properties resemble very much the classical case. It is straightforward to verify that Δ commutes with d, δ and $*$, and that

$$\delta^* = (-1)^{k*}d, \qquad {}^*\delta = (-1)^{k+1}d^*, \qquad d\delta^* = {}^*\delta d\,.$$

Some further properties of these operators will be given in the context of the following sections. In particular we will analyse the explicit action of these operators on zero and 1-forms, which is of interest to physical applications.

The q-Hodge star operator also enables us to generalise the idea of a Lie derivative. For this purpose, we introduce a *q-inner product* on the q-exterior algebra Ω as a bilinear map $(\ ,\ ) : \Omega_k \times \Omega_r \to \Omega_{k-r}$ defined by $(\mathbf{w}, \mathbf{v}) = i_{\mathbf{v}}\mathbf{w} = {}^*(\mathbf{v} \wedge^{*-1}(\mathbf{w}))$. The q-inner product is "transposed" to the q-wedge product in the sense that

$$(\mathbf{v} \wedge \mathbf{w}, \mathbf{u}) = i_{\mathbf{v}\wedge\mathbf{w}}\mathbf{u} = i_{\mathbf{v}}(i_{\mathbf{w}}\mathbf{u}) = (\mathbf{w}, i_{\mathbf{v}}\mathbf{u})\,,$$

and it also obeys ${}^*\mathbf{w} = i_{\mathbf{w}}\varepsilon$ and $\delta i_{\mathbf{v}}\mathbf{w} = i_{\mathbf{v}}\delta\mathbf{w} + (-1)^k i_{d\mathbf{v}}\mathbf{w}$, where $\mathbf{v}$ is a k-form. Furthermore, Lemma 2.15 implies $(\mathbf{v}, \mathbf{w}) = i_{\mathbf{v}}\mathbf{w} = i_{*\mathbf{v}}{}^*\mathbf{w} = ({}^*\mathbf{v}, {}^*\mathbf{w})$ for any two k-forms $\mathbf{v}$ and $\mathbf{w}$. In terms of this q-inner product, we now introduce:

Definition 2.18. *Let $\mathbf{v}$ be a 1-form. The q-Lie derivative $L_{\mathbf{v}} : \Omega_k \to \Omega_k$ with respect to $\mathbf{v}$ is defined as $L_{\mathbf{v}} = i_{\mathbf{v}} \circ d + d \circ i_{\mathbf{v}}$.*

The q-Lie derivative commutes with the q-exterior derivative and we also have $L_{\mathbf{f}\wedge\mathbf{v}}\mathbf{w} = \mathbf{f} \wedge L_{\mathbf{v}}\mathbf{w} + d\mathbf{f} \wedge i_{\mathbf{v}}\mathbf{w}$ for zero forms $\mathbf{f}$ and 1-forms $\mathbf{v}$. For the action of the q-Lie derivative on zero forms on q-Minkowski space, we find:

Proposition 2.19. *The action of $L_{\mathbf{v}}$ on $\mathbf{f} \in \Omega_0$ is given by $L_{\mathbf{v}}\mathbf{f} : 1 \mapsto v^a\partial_a f$.*

Proof. First note that $L_{\mathbf{v}}\mathbf{f} = i_{\mathbf{v}}d\mathbf{f}$, since $\mathbf{f}$ is a zero form. Then show by explicit calculation that

$$H^a_{\{1\}cde}H^{bcde}_{\{4\}} = -g^{ba}\,. \tag{10}$$

Hence $L_{\mathbf{v}}\mathbf{f}(1) = -v_b\partial_a f H^a_{\{1\}cde}\mathscr{A}^{bcde}_{\{4\}klmn}H^{klmn}_{\{4\}} = -v_b\partial_a f H^a_{\{1\}cde}H^{bcde}_{\{4\}} = v_b\partial_a g^{ba} f$. $\square$

3. The q-d'Alembert Equation

The simplest case of a wave equation on q-Minkowski space is the q-d'Alembert equation, where fields are 0-forms $\boldsymbol{\varphi}$ and the wave equation is given by the q-Laplace–Beltrami operator.

Definition 3.1. *A solution of the q-d'Alembert equation is a 0-form $\boldsymbol{\varphi}$ such that $\Delta\boldsymbol{\varphi} = 0$.*

This equation can be written less abstractly, in terms of the braided differential operators and the value φ on $1 \in \Lambda_0$ of $\boldsymbol{\varphi}$.

Proposition 3.2. *The q-d'Alembert equation is equivalent to*

$$\Box\varphi = 0\,, \tag{11}$$

where $\Box = \partial_a \partial_b g^{ab}$ *is the q-d'Alembert operator.*

Proof. Since ${}^*\boldsymbol{\varphi}$ is a 4-form, $d\delta\boldsymbol{\varphi}$ vanishes and thus $\Delta\boldsymbol{\varphi} = \delta d\boldsymbol{\varphi}$. With relation (10), we find:

$$\begin{aligned} 0 = \delta d\boldsymbol{\varphi}(1) &= \partial_f \partial_a \varphi H^{a}_{\{1\}cde} \mathscr{A}^{fcde}_{\{4\}klmn} H^{klmn}_{\{4\}} \\ &= \partial_f \partial_a \varphi H^{a}_{\{1\}cde} H^{fcde}_{\{4\}} = \partial_f \partial_a g^{fa} \varphi\,, \end{aligned}$$

which proves the equivalence of the q-d'Alembert equation and (11). □

Equation (11) is indeed the obvious choice for a q-d'Alembert equation, as remarked by many authors before. This form is also convenient for proving that the q-d'Alembert equation is $\tilde{\mathscr{L}}_q$-covariant. Keeping in mind the various transformation properties, one can show that the action α of the operator $\Box$ commutes with the coaction by $\tilde{\mathscr{L}}_q$, i.e. $\beta_{M_q} \circ \alpha \circ (\Box \otimes \varphi) = \alpha \circ \beta_{\mathscr{D}\otimes M_q} \circ (\Box \otimes \varphi)$. One could also write down a *q-Klein Gordon equation* of the form $(\Box + m^2)\varphi = 0$, but this equation would only be $\tilde{\mathscr{L}}_q$-covariant if the "mass" m transformed as $m \mapsto m \otimes \varsigma$, i.e. not as an $\tilde{\mathscr{L}}_q$-scalar. One might argue that this transformation property in itself is not necessarily harmful, but the results of the next section on plane wave solutions seem to suggest to us that $\tilde{\mathscr{L}}_q$-covariant wave equations on M_q are inherently massless.

A solution of the q-d'Alembert equation determines a conserved current:

Proposition 3.3. *Let* φ *be a solution of the q-d'Alembert equation. Then the current* 1*-form* $\mathbf{j}$,

$$\mathbf{j} = \overline{\boldsymbol{\varphi}} \wedge i d\boldsymbol{\varphi} - q^{-2} i d\overline{\boldsymbol{\varphi}} \wedge \boldsymbol{\varphi}$$

is conserved: $\delta\mathbf{j} = 0$.

Proof. Equation (10), Corollary 2.12, and the relation $q^{-2} R_L^{-1}{}^{cd}_{ab} g^{ba} = g^{cd}$ imply:

$$\begin{aligned} \delta\mathbf{j}(dx_a) &= i\partial_b(\overline{\varphi}\partial_a\varphi - q^{-2}(\partial_a\overline{\varphi})\varphi) H^{a}_{\{1\}cde} \mathscr{A}^{bcde}_{\{4\}klmn} H^{klmn}_{\{4\}} \\ &= i\partial_b(\overline{\varphi}\partial_a\varphi - q^{-2}(\partial_a\overline{\varphi})\varphi) H^{a}_{\{1\}cde} H^{bcde}_{\{4\}} H_{\{0\}klmn} H^{klmn}_{\{4\}} \\ &= i\partial_b(\overline{\varphi}\partial_a\varphi - q^{-2}(\partial_a\overline{\varphi})\varphi) g^{ba} \\ &= i((\partial_b\overline{\varphi})(\partial_a\varphi) - (\partial_c\overline{\varphi})(\partial_d\varphi) q^{-2} R_L^{-1}{}^{cd}_{ab}) g^{ba} \\ &= 0\,. \end{aligned}$$

Here we used repeatedly the relations (9). □

Since the question of the $*$-structure on the braided differential operators is still unsolved, it is not quite clear whether this current is real or not. However, for the plane wave solutions which we will discuss next, one can establish that the corresponding current is indeed real.

As we are working in an algebraic framework, we will present our deformed exponentials as "formal power series." Strictly speaking, they are not elements of our algebra, but of a suitable completion. These plane wave solutions are indexed by a copy of $V(R_M)$ regarded as momentum space with generators p^a. This algebra of q-momenta is an $\tilde{\mathscr{L}}_q$-comodule algebra with coaction $p^a \mapsto p^b \otimes S\lambda^a_b \varsigma^{-1}$, i.e. has the ς-scaling property as appropriate for momenta. The relations between the p's are described in terms of R_M, but on the *q-deformed light cone* P_0 defined as the quotient of $V(R_M)$ by the relation $g_{ab}\, p^a p^b = 0$, one also has:

Lemma 3.4. *On the quotient P_0, the generators of $V(R_M)$ obey the $V(q^{-2}R_L)$ relations $p^a p^b = q^{-2} R_L{}^{ab}_{cd}\, p^d\, p^c$.*

Proof. Let $p = (a, b, c, d)$ be the vector of generators. The algebra $V(q^{-2}R_L)$ has the same relations as $V(R_M)$ except for $cb = q^2 bc + (1 - q^2)dd$, which differs from the corresponding relation $cb = bc - (1 - q^2)ad - (1 - q^{-2})dd$. However, in the quotients $V(q^{-2}R_L)/(g_{ab}\, p^a p^b = 0)$ and P_0, the generators obey $ad - q^{-2}cd = 0$ and we can rewrite both relations as $ad = bc - (1 - q^{-2})dd$. $\square$

The q-light cone is invariant under the coaction by $\tilde{\mathscr{L}}_q$ in the sense that the coaction β by the q-Lorentz group on $V(R_M)$ descends to a covariant coaction $\beta : P_0 \to P_0 \otimes \tilde{\mathscr{L}}_q$. Using P_0 as an "index set" we define a family of *q-deformed plane waves*:

$$\exp(ix \cdot p) = \sum_{n=0}^{\infty} \frac{i^n}{[n]!} x_{a_1} \cdots x_{a_n} \underline{\otimes} p^{a_n} \cdots p^{a_1} \tag{12}$$

as a formal power series in $M_q \underline{\otimes} P_0$, where $[n] = 1 + q^2 + \cdots + q^{2(n-1)}$ and $[n]! = [1] \cdots [n]$.

Proposition 3.5. *The family of P_0-indexed complex valued plane waves $\varphi(p)$ given by $1 \mapsto \exp(\pm ix \cdot p)$ are solutions of the q-d'Alembert equation.*

Proof. The elements $\exp(\pm ix \cdot p)$ transform as scalars under the coaction by $\tilde{\mathscr{L}}_q$, since the dilaton terms always cancel, and hence $\varphi(p)$ is an $\tilde{\mathscr{L}}_q$-comodule morphism. It remains to show $\Box \exp(ix \cdot p) = 0$. By virtue of Lemma 3.4, we find:

$$\begin{aligned}
\partial^c \exp(ix \cdot p) &= \sum_n i^n \frac{1}{[n]!} \partial^c x_{a_1} \cdots x_{a_n} p^{a_n} \cdots p^{a_1} \\
&= \sum_n i^n \frac{1}{[n]!} \delta^c_{d_1} x_{d_2} \cdots x_{d_n} (\delta^{d_1 \cdots d_n}_{e_1 \cdots e_n} + PR_L{}^{d_1 d_2}_{e_1 e_2} \delta^{d_3 \cdots d_n}_{e_3 \cdots e_n} + PR_L{}^{d_1 d_2}_{e_1 g_1} PR_L{}^{g_1 d_3}_{e_2 e_3} \\
&\quad \times \delta^{d_4 \cdots d_n}_{e_4 \cdots e_n} + \cdots + PR_L{}^{d_1 d_2}_{e_1 g_1} \cdots PR_L{}^{g_{n-2} d_n}_{e_{n-1} e_n}) p^{e_n} \cdots p^{e_1} \\
&= \sum_n i^n \frac{1 + q^2 + \cdots + q^{2(n-1)}}{[n]!} x_{a_2} \cdots x_{a_n} p^{a_n} \cdots p^{a_2}\, p^c \\
&= \exp(x \cdot p) i p^c\, ,
\end{aligned}$$

and hence $\Box \exp(ix \cdot p) = 0$. We used the braided Leibniz rule to evaluate $\partial^c x_{a_1} \cdots x_{a_n}$. $\square$

These plane wave type solutions exist *only* on the q-light cone, giving further support to our claim that wave equations on quantum Minkowski space should be massless. In general we do not know whether the conserved current associated to a solution of the q-d'Alembert equation is real, but in the case of plane wave solutions, we can show:

Lemma 3.6. *Any two monomials in the formal power series* $\exp(-ix \cdot p)$ *and* $\exp(ix \cdot p)$ *commute.*

Proof. Using the statistics relations $p^a x_b = x_c {R_L^{-1}}^{ac}{}_{db} p^d$ and Lemma 3.4, which implies ${R_L^{-1}}^{ab}{}_{cd} p^c p^d = q^{-2} p^b p^a$ we can show:

$$
\begin{aligned}
p^c x_{a_1} \cdots x_{a_m} \underline{\otimes} p^{a_m} \cdots p^{a_1} &= x_{a_1} \cdots x_{a_m} \underline{\otimes} {R_L^{-1}}^{a_1 c}{}_{d_1 b_1} \cdots {R_L^{-1}}^{a_m d_{m-1}}{}_{d_m b_m} p^{d_m} p^{b_m} \cdots p^{b_1} \\
&= q^{-2m} x_{a_1} \cdots x_{a_m} \underline{\otimes} p^{a_m} \cdots p^{a_1} p^c ,
\end{aligned}
$$

and hence with $q^{-2} R_M{}^{ab}{}_{cd} p^d p^c = R_L{}^{ab}{}_{cd} p^d p^c$, which also follows from Lemma 3.4:

$$
\begin{aligned}
&(x_{a_1} \cdots x_{a_n} p^{a_n} \cdots p^{a_1})(x_{b_1} \cdots x_{b_m} p^{b_m} \cdots p^{b_1}) \\
&= q^{-2nm} x_{a_1} \cdots x_{a_n} x_{b_1} \cdots x_{b_m} p^{b_m} \cdots p^{b_1} p^{a_n} \cdots p^{a_1} \\
&= q^{-2nm} x_{c_1} \cdots x_{c_m} x_{d_1} \cdots x_{d_n} \\
&\qquad \begin{matrix} R_M{}^{d_1 c_m}{}_{f_1 e_1} & R_M{}^{d_2 e_1}{}_{h_1 e_2} & \cdots & R_M{}^{d_n e_{n-1}}{}_{k_1 b_m} \\ R_M{}^{f_1 c_{m-1}}{}_{f_2 g_1} & R_M{}^{h_1 g_1}{}_{h_2 g_2} & \cdots & R_M{}^{k_1 g_{n-1}}{}_{k_2 b_{m-1}} \\ \cdots & \cdots & \cdots & \cdots \\ R_M{}^{f_{m-1} c_1}{}_{a_1 l_1} & R_M{}^{h_{m-1} l_1}{}_{a_2 l_2} & \cdots & R_M{}^{k_{m-1} l_{n-1}}{}_{a_n b_1} \end{matrix} \\
&\qquad p^{b_m} \cdots p^{b_1} p^{a_n} \cdots p^{a_1} \\
&= x_{c_1} \cdots x_{c_m} x_{d_1} \cdots x_{d_n} \\
&\qquad \begin{matrix} R_L{}^{d_1 c_m}{}_{f_1 e_1} & R_L{}^{d_2 e_1}{}_{h_1 e_2} & \cdots & R_L{}^{d_n e_{n-1}}{}_{k_1 b_m} \\ R_L{}^{f_1 c_{m-1}}{}_{f_2 g_1} & R_L{}^{h_1 g_1}{}_{h_2 g_2} & \cdots & R_L{}^{k_1 g_{n-1}}{}_{k_2 b_{m-1}} \\ \cdots & \cdots & \cdots & \cdots \\ R_L{}^{f_{m-1} c_1}{}_{a_1 l_1} & R_L{}^{h_{m-1} l_1}{}_{a_2 l_2} & \cdots & R_L{}^{k_{m-1} l_{n-1}}{}_{a_n b_1} \end{matrix} \\
&\qquad p^{b_m} \cdots p^{b_1} p^{a_n} \cdots p^{a_1} \\
&= (x_{b_1} \cdots x_{b_m} p^{b_m} \cdots p^{b_1})(x_{a_1} \cdots x_{a_n} p^{a_n} \cdots p^{a_1}) . \quad \square
\end{aligned}
$$

This lemma suggests that in a suitable completion of our algebra, we can reorder terms and verify that $\exp(ix \cdot p)\exp(-ix \cdot p) = 1$. In this case, we would find $\mathbf{j}(dx_a^*) = \mathbf{j}(dx_{\bar{a}}) = \exp(-ix \cdot p)\exp(ixp)p_{\bar{a}} - q^{-2}\exp(-ix \cdot p)p_{\bar{a}}\exp(ixp) = p_{\bar{a}} - q^{-2}\exp(-ix \cdot p)p_{\bar{a}}\exp(ixp) = \mathbf{j}(dx_a)^*$ and the conserved current $\mathbf{j}$ associated to the plane wave solutions $\varphi(p)$ would be seen to be real.

4. The q-Maxwell Equation

For q-Maxwell equations, we apply a similar strategy as for the q-d'Alembert equation: we first give a more abstract definition in terms of δ and d and then show how this equation looks in terms of the maybe more familiar braided differential operators ∂.

Definition 4.1. *A solution of the q-Maxwell equation is a* 1*-form* $\mathbf{A}$ *such that* $\delta d\mathbf{A} = 0$.

Using the results from the preceding sections, we can rewrite this rather abstract relation to resemble the classical equation $\partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu A_\mu = \partial^\mu\partial_{[\mu}A_{\nu]} = 0$:

Proposition 4.2. *The q-Maxwell equation is equivalent to the set of four equations*

$$\partial^c\partial_{[c}A_{z]} = 0\,, \tag{13}$$

or alternatively

$$\Box A_z - \partial_z\partial^c A_c = 0\,. \tag{14}$$

Here ∂ *denote the braided differential operators on* M_q, $\Box$ *the q-d'Alembert operator and* "[ ]" *the q-antisymmetriser.*

Proof. First verify by explicit calculation that

$$H_{\{2\}}{}^{ab}{}_{cd}H_{\{3\}}{}^{xcd}{}_{z} = \frac{(1+q^2)^2}{q(2(1+q^2+q^4))^{1/2}}\, g^{xc}\mathcal{A}_{\{2\}}{}^{ab}{}_{cz}\,. \tag{15}$$

By virtue of this relation, we obtain

$$\begin{aligned}
0 = \delta d\mathbf{A}(dx_z) &= \partial_x\partial_a A_a \mathcal{A}_{\{2\}}{}^{ab}{}_{cd}H_{\{2\}}{}^{cd}{}_{ef}\mathcal{A}_{\{3\}}{}^{xef}{}_{klm}H_{\{3\}}{}^{klm}{}_{z} \\
&= \partial_x\partial_a A_b H_{\{2\}}{}^{ab}{}_{cd}\mathcal{A}_{\{2\}}{}^{cd}{}_{ef}\mathcal{A}_{\{3\}}{}^{xef}{}_{klm}H_{\{3\}}{}^{klm}{}_{z} = \partial_x\partial_a A_b H_{\{2\}}{}^{ab}{}_{ef}\mathcal{A}_{\{3\}}{}^{xef}{}_{klm}H_{\{3\}}{}^{klm}{}_{z} \\
&= \partial_x\partial_a A_b H_{\{2\}}{}^{ab}{}_{ef}H_{\{3\}}{}^{xef}{}_{j}H_{\{1\}}{}^{j}{}_{klm}H_{\{3\}}{}^{klm}{}_{z} = \partial_x\partial_a A_b H_{\{2\}}{}^{ab}{}_{ef}H_{\{3\}}{}^{xef}{}_{z} \\
&= \partial_x\partial_a A_b g^{xc}\mathcal{A}_{\{2\}}{}^{ab}{}_{cz} = \partial^a\partial_{[c}A_{z]}\,,
\end{aligned}$$

where we used the definitions of δ and d, the relations (9) and (7), and finally (15). This establishes the equivalence of the q-Maxwell equation and (13). In order to prove the second part, note that the generators of $V(R_M)$ obey

$$x_a x^b = x^c x_d R_M{}^{bd}{}_{ca}\,. \tag{16}$$

Hence with (8), we find:

$$0 = \partial^c \partial_{[c} A_{z]} = \partial^c (\partial_c A_z - \partial_i A_j R_M{}^{ji}{}_{cz})$$
$$= \partial^c \partial_c A_z - \partial^c \partial_i A_j R_M{}^{ji}{}_{cz} = \Box A_z - \partial_z \partial^c A_c \,.$$

This proves the equivalence of (13) and (14). $\Box$

In the form (13), the $\tilde{\mathscr{L}}_q$-covariance of the q-Maxwell equations can be easily established. Again, a massive field equation, i.e. a *q-Proca equation* $\partial^c \partial_{[c} A_{z]} = m^2 A_z$ would be $\tilde{\mathscr{L}}_q$-covariant only if m transformed as $m \mapsto m \otimes \varsigma^{-1}$, and again we shall find q-deformed plane wave solutions only on the q-light cone.

As in the undeformed case, solutions to the q-Maxwell equation have a *gauge freedom*. If $\mathbf{A}$ is a solution of the q-Maxwell equation and φ a 0-form then by virtue of theorem 2.11 the 1-form $\mathbf{A} + d\varphi$ is also a solution. Provided it is possible to solve the inhomogeneous equation $\Delta\varphi = -\delta\mathbf{A}$, we can use this gauge freedom to arrange for $\mathbf{A}$ to satisfy the *q-Lorentz gauge* condition $\delta\mathbf{A} = 0$. Using an argument similar to the proof of proposition 3.2, one can show that the q-Lorentz gauge condition is satisfied if and only if $\partial^c A_c = 0$. Proposition 4.2 implies that in this case $\mathbf{A}$ obeys $\Delta\mathbf{A} = 0$ or equivalently $\Box A_z = 0$. As in the classical case, a field $\mathbf{A}$ satisfying the q-Lorentz gauge has a *residual gauge freedom* $\mathbf{A} \mapsto \mathbf{A} + d\varphi$, where φ is a solution of the q-d'Alembert equation.

The q-Maxwell equation also has a family of plane wave solutions. However, in this case the solutions are indexed by the q-momentum p^a (the generators of the q-light cone P_0) and the "q-amplitude" A_z, which are generators of a copy of M_q. We define the algebra Y as the quotient of $P_0 \underline{\otimes} M_q$ by the relation $p^c \underline{\otimes} A_c = 0$. This algebra Y labels plane wave solutions to the q-Maxwell equation:

Proposition 4.3. *The family of Y-indexed* 1*-forms* $\mathbf{A}$ *given by* $dx_z \mapsto \exp(ix \cdot p) \underline{\otimes} A_z$ *are solutions of the q-Maxwell equation and satisfy the q-Lorentz gauge condition* $\delta\mathbf{A} = 0$.

Proof. Using the q-Maxwell equations in the form (13), one finds:

$$\partial^c \partial[c \exp(ix \cdot p) \underline{\otimes} A_z] = \partial^c (\partial_c \exp(ix \cdot p) \underline{\otimes} A_z - \partial_m \exp(ix \cdot p) \underline{\otimes} A_n R_M{}^{nm}{}_{cz})$$
$$= \exp(ix \cdot p) p^c p_m \underline{\otimes} A_n R_M{}^{nm}{}_{cz})$$
$$= \exp(ix \cdot p) p_z p^c \underline{\otimes} A_c = 0 \,.$$

Here we used (8), Proposition 3.5 and relation (16). These solutions obviously satisfy the q-Lorentz gauge condition. $\Box$

A solution $\mathbf{A}$ of the q-Maxwell equation defines a 2-form $\mathbf{F} = d\mathbf{A}$, *the q-field strength tensor* which obeys the two equations

$$d\mathbf{F} = 0, \qquad \delta\mathbf{F} = 0 \,. \tag{17}$$

Proposition 4.2 implies that the second relation is equivalent to

$$\partial^c F_{cd} = 0 \,. \tag{18}$$

and we also find

$$\Box F_{ab} = \Box \partial_{[a} A_{b]} = \partial_{[a} \Box A_{b]} = \partial_{[a} \partial_{b]} \partial^c A = 0 \,,$$

using the fact that $\Box$ is central in $\mathscr{D}$ and the q-Maxwell equation for **A**. Since at present we do not have a q-Poincaré lemma, we only know that (17) is implied by the q-Maxwell equations, but we cannot prove that they are equivalent.

5. q-Spinor Analysis of the q-Field Strength Tensor

In this section we give a $SL_q(2,\mathbb{C})$-spinor description of the q-field strength tensor **F** similar to the classical case. For this purpose, we need a few elements of the $SL_q(2,\mathbb{C})$-spinor calculus, some aspects of which were already discussed in [15]. This case is very simple since the R-matrix (1) is of *Hecke-type* and obeys $0=(PR+q^{-1})(PR-q)$. This means that one can take either $(PR-q)$ or $(PR^{-1}-q^{-1})$ as a q-antisymmetriser for $SL_q(2,\mathbb{C})$-spinors. The Hecke relation ensures that after a suitable normalisation these operators are projectors. Furthermore, one does not have any problems with higher q-antisymmetrisers, since they are all zero. One could also define a q-antisymmetriser by first identifying a q-antisymmetric ε_{AB}, similar to the procedure in Sect. 2.1, but this approach gives the same result. The q-antisymmetric spinor ε_{AB} is simply the $SL_q(2,\mathbb{C})$-spinor metric (3), which obeys $q\varepsilon_{AB}=-\varepsilon_{CD}R^{-1}{}^{DC}_{\ AB}$. One can easily verify that then $A^{AB}_{\ \ CD}=(q+q^{-1})^{-1}\varepsilon^{BA}\varepsilon_{CD}=(q+q^{-1})^{-1}(PR^{-1}-q^{-1})^{AB}_{\ \ CD}$ obeys $A^2=A$ by virtue of the Hecke relation. We also define a *q-symmetriser* $S=1/2(1-A)=(q+q^{-1})^{-1}(PR^{-1}+q)$, and the *$q$-symmetrisation* "( )" and *q-antisymmetrisation* "[ ]" of a multivalent q-spinor $T_{\dots AB\dots}$ with two adjacent lower indices A and B as $T_{\dots(AB)\dots}=T_{\dots CD\dots}S^{CD}_{\ \ AB}$ and $T_{\dots[AB]\dots}=T_{\dots CD\dots}A^{CD}_{\ \ AB}$, respectively. Again, one obtains similar relations for upper indices. Due to the Hecke relation, the q-(anti)-symmetrisation of a q-spinor is q-(anti)-symmetric:

$$T_{\dots[(AB)]\dots}=0, \qquad T_{\dots([AB])\dots}=0\,, \tag{19}$$

and we also have a decomposition

$$T_{\dots AB\dots}=T_{\dots(AB)\dots}+T_{\dots[AB]\dots}\,. \tag{20}$$

The q-(anti)-symmetrisation is $SL_q(2,\mathbb{C})$-covariant in the sense that both operations commute with the coaction by $SL_q(2,\mathbb{C})$. Furthermore, if $T_{\dots CD\dots}$ is a multivalent q-spinor then

$$T_{\dots[CD]\dots}=\frac{1}{q^{-1}+q}\,\varepsilon_{CD}T_{\dots B}{}^{B}{}_{\dots}\,. \tag{21}$$

In this formula we do not violate the index notation by writing ε_{CD} on the left since C and D are adjacent indices and the generators of $SL_q(2,\mathbb{C})$ preserve the spinor metric.

We now apply these results to the field strength tensor **F**, or more generally, to any q-antisymmetric tensor $F_{ab}\in M_q$. Any such tensor defines an $SL_q(2,\mathbb{C})$ spinor

$$f^{AB}{}_{A'B'}=F_{AI'IB'}R^{I'B}_{\ \ A'I}$$

which is the object we will study in this section.

Proposition 5.1. *The tensor $f^{AB}{}_{A'B'}$ admits a decomposition*

$$f^{AB}{}_{A'B'}=\phi^{AB}\varepsilon_{A'B'}+\varepsilon^{AB}\psi_{A'B'}\,,$$

where ϕ^{AB} and $\psi_{A'B'}$ are q-symmetric $SL_q(2,\mathbb{C})$-spinors.

Proof. Since the tensor F_{ab} is q-antisymmetric, $f^{AB}{}_{A'B'}$ obeys:

$$\begin{aligned} f^{AB}{}_{A'B'} &= F_{AI'IB'} R^{I'B}_{A'I} = -F_{CC'DD'} R^{DD'CC'}_{L\ \ AI'IB'} R^{I'B}_{A'I} \\ &= -F_{CC'DD'} R^{C'K}_{LD} R^{D'L}_{MB'} R^{AN}{}_{KC} \tilde{R}^{MI}_{I'N} R^{I'B}_{A'I} = -f^{CD}{}_{C'D'} R^{D'C'}_{A'B'} R^{AB}{}_{DC} \,. \end{aligned} \tag{22}$$

Due to relation (20), we also have

$$f^{AB}{}_{A'B'} = f^{(AB)}{}_{(A'B')} + f^{[AB]}{}_{(A'B')} + f^{(AB)}{}_{[A'B']} + f^{[AB]}{}_{[A'B']} \,.$$

This implies with (22), (19) and the Hecke relation:

$$\begin{aligned} f^{AB}{}_{A'B'} &= -f^{CD}{}_{C'D'} R^{D'C'}_{A'B'} R^{AB}{}_{DC} = -q^2 f^{(AB)}{}_{(A'B')} + f^{[AB]}{}_{(A'B')} \\ &\quad + f^{(AB)}{}_{[A'B']} - q^{-2} f^{[AB]}{}_{[A'B']} \,, \end{aligned}$$

and also

$$\begin{aligned} f^{AB}{}_{A'B'} &= -f^{CD}{}_{C'D'} R^{-1C'D'}{}_{B'A'} R^{-1BA}{}_{CD} = -q^{-2} f^{(AB)}{}_{(A'B')} + f^{[AB]}{}_{(A'B')} \\ &\quad + f^{(AB)}{}_{[A'B']} - q^2 f^{[AB]}{}_{[A'B']} \,, \end{aligned}$$

and therefore $0 = f^{(AB)}{}_{(A'B')} + f^{[AB]}{}_{[A'B']}$. With relation (21), it follows

$$f^{AB}{}_{A'B'} = \phi^{AB} \varepsilon_{A'B'} + \varepsilon^{AB} \psi_{A'B'} \,,$$

where $\phi^{AB} = f^{(AB)C}{}_{C}$ and $\psi_{A'B'} = f_{C}{}^{C}{}_{(A'B')}$ are q-symmetric $SL_q(2,\mathbb{C})$-spinors. □

In the case of a real tensor, the two components ϕ and ψ are not independent, but are related by the star structure on M_q.

Proposition 5.2. *A q-antisymmetric tensor F_{ab} is real iff $\psi_{DC} = -\phi^{*CD}$ and can hence be written as*

$$f^{AB}{}_{A'B'} = \phi^{AB} \varepsilon_{A'B'} + \varepsilon_{BA} \phi^{*B'A'}$$

in terms of the q-symmetric $SL_q(2,\mathbb{C})$-spinor ϕ^{AB}.

Proof. If F_{ab} is real then

$$\begin{aligned} f^{*AB}{}_{A'B'} &= F^{*}_{AI'IB'} R^{I'B}_{A'I} = F_{B'II'A} R^{I'B}_{A'I} \\ &= F_{B'II'A} R^{A'I}_{BI'} = f^{B'A'}{}_{BA} \,, \end{aligned}$$

where we used the fact that R is of real type, i.e. obeys $R^{AB}_{\ \ CD} = R^{DC}_{\ \ BA}$. In components, this means that

$$\phi^{*AB}\varepsilon_{A'B'} + \varepsilon^{AB}\psi^{*}_{A'B'} = \phi^{B'A'}\varepsilon_{BA} + \varepsilon^{B'A'}\psi_{BA}$$

Due to the q-symmetry of ϕ and ψ and the q-antisymmetry of ε_{AB}, multiplication of this equation by $qR^{CD}_{\ \ BA}$ yields by virtue of (19):

$$q^2\phi^{*CD}\varepsilon_{A'B'} - \varepsilon^{CD}\psi^{*}_{A'B'} = -\phi^{B'A'}\varepsilon_{DC} + q^2\varepsilon^{B'A'}\psi_{DC}\,.$$

again using that R is of real type. Thus $\phi^{*CD}\varepsilon_{A'B'} = \varepsilon^{B'A'}\psi_{DC}$, which implies the proposition, since $\varepsilon^{B'A'} = -\varepsilon_{A'B'}$. □

Classically, this decomposition of the field strength tensor into spinors coincides with the decomposition into its self-dual and anti-self-dual part. The same result holds in the non-commutative case. By virtue of Proposition 2.14, any two-form $\mathbf{F}$ on quantum Minkowski space can be decomposed uniquely as $\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-$, where $\mathbf{F}^+ = \frac{1}{2}(\mathbf{F} + {}^*\mathbf{F})$ and $\mathbf{F}^- = \frac{1}{2}(\mathbf{F} - {}^*\mathbf{F})$ are self-dual and anti-self-dual, i.e. obey ${}^*\mathbf{F}^\pm = \pm\mathbf{F}^\pm$. The q-Maxwell equations (17) are then equivalent to either $d\mathbf{F}^+ = 0$ and $d\mathbf{F}^- = 0$ or the two equations

$$\delta\mathbf{F}^+ = 0, \qquad \delta\mathbf{F}^- = 0\,. \tag{23}$$

Proposition 5.3. *Let F_{AB} be a q-antisymmetric tensor. Then*

$$f^{+AB}_{\ \ \ \ \ A'B'} = \phi^{AB}\varepsilon_{A'B'}, \qquad f^{-AB}_{\ \ \ \ \ A'B'} = \varepsilon^{AB}\psi_{A'B'}$$

are the self-dual and anti-self dual parts of f.

Proof. It suffices to show that $f^\pm$ are selfdual and antiselfdual, respectively. On the tensor $f^{AB}_{\ \ \ A'B'}$, the q-Hodge star operation is implemented by the matrix

$$U^{ABA'B'}_{\{2\}CDC'D'} = \tilde{R}^{A'I}_{\ \ \ I'B}H^{\ AI'IB'}_{\{2\}CJ'JD'}R^{J'D}_{\ \ \ C'J}\,.$$

By explicit calculation, one verifies that this operator satisfies the relations

$$S^{AB}_{\ \ EF}\varepsilon_{A'B'}U^{ABA'B'}_{\{2\}CDC'D'} = S^{CD}_{\ \ EF}\varepsilon_{C'D'}\,,$$

$$\varepsilon^{AB}S^{E'F'}_{\ \ \ A'B'}U^{ABA'B'}_{\{2\}CDC'D'} = \varepsilon^{CD}S^{E'F'}_{\ \ \ C'D'}\,,$$

Since ϕ^{AB} and $\psi_{A'B'}$ are q-symmetric and thus eigenvectors of the q-symmetriser S, this implies that $\phi^{EF}\varepsilon_{A'B'}$ and $\varepsilon^{AB}\psi_{A'B'}$ are self-dual and anti-self-dual, respectively. □

If we are looking for real solutions of the q-Maxwell equations (17), it is thus sufficient to solve one of the two equations in (23). In terms of the $SL_q(2,\mathbb{C})$-spinor ψ this means:

Corollary 5.4. *For real* $\mathbf{F}$, *the q-Maxwell equation $\delta\mathbf{F}^-$ is equivalent to*

$$\nabla^{BI'}\psi_{I'B'} = 0\,, \tag{24}$$

where $\nabla^{CC'} = \tilde{R}^{C'C}_{\ \ \ A'A}\partial^{AA'}$.

Proof. Proposition 5.3 implies with (18):

$$0 = \partial^a F^-_{ab} = \tilde{R}^{I'B}_{A'I} \partial_A^{A'} \varepsilon^{AI} \psi_{I'B'} = \tilde{R}^{I'B}_{A'A} \partial^{AA'} \psi_{I'B'} = \nabla^{BI'} \psi_{I'B'}.$$

This proves the corollary. □

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