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Global Finite-Energy Solutions of the Maxwell–Schrödinger System

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Abstract: The existence of global finite-energy solutions is proved for the initial value problem for the Maxwell–Schrödinger system in the Coulomb, Lorentz and temporal gauges

1. Introduction

We consider the coupled Maxwell–Schrödinger system in three space dimensions for a nonrelativistic charged particle in an electromagnetic field [4]. This system occurs as a model in laser physics [1]. Although, of course, the system is not Lorentz invariant, it is rotationally invariant and gauge invariant. In this paper we prove the existence of global finite-energy solutions of the initial value problem for the Maxwell–Schrödinger system in the Coulomb gauge.

K. Nakamitsu and M. Tsutsumi in [3 and 5] proved that the initial value problem for the Maxwell–Schrödinger system in the Lorentz gauge is globally well-posed in a space of smooth functions in dimensions one and two, and locally well-posed in dimension three. Y. Tsutsumi in [6] proved, by constructing the modified wave operator, that there exist global smooth solutions in the Coulomb gauge for a certain class of scattered data as $t \to +\infty$. However, the problem in three dimensions with initial condition at a finite time has remained open. The present paper resolves this existence question, but we have not succeeded in proving uniqueness.

In Sect. 2 we write the equations and introduce the approximate system to be used to make the construction. Essentially it is to replace the imaginary i in the Schrödinger equation by $i + \varepsilon$ with a small dissipation constant ε. In Sect. 3 we construct global solutions of the approximate system in the Coulomb gauge. In Sect. 4 we pass to the limit $\varepsilon \to 0$, thereby obtaining global weak solutions. In Sect. 5 we prove the analogous result in the Lorentz gauge and in the temporal gauge.

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We will denote by $\|\cdot\|_p$ the norm in $L^p = L^p(\mathbb{R}^3)$, and by $\|\cdot\|_{k,p}$ the norm in the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R}^3)$. We write H^k for $W^{k,2}$ as usual, and $\mathbb{R}^+$ for the interval $[0,\infty)$. We denote the space of all weakly continuous mapping of $\mathbb{R}^+$ to a Banach space X by $C_w(\mathbb{R}^+;X)$, and the space of all bounded weakly continuous mappings of $\mathbb{R}^+$ to X by $BC_w(\mathbb{R}^+;X)$.

2. The Maxwell–Schrödinger System

We will use Greek indices μ, ν, ... to run integers from 0 to 3, Latin indices j, k, ... to run integers from 1 to 3, and the summation convention for repeated indices. We write ∂_0 for $\partial/\partial t$, ∂_j for $\partial/\partial x^j$, and define ∂^μ by $\partial^0 = \partial_0$ and $\partial^j = -\partial_j$.

A classical Maxwell–Schrödinger (MS) field in $3+1$ dimensional space-time consists of a vector potential with four real components A_μ and a complex scalar field ψ. After a suitable rescaling, we may write the equations of motion as

$$\partial^\mu F_{\mu\nu} + J_\nu = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.1}$$

$$iD_0\psi + D_jD_j\psi = 0, \quad D_\mu = \partial_\mu - iA_\mu\,, \tag{2.2}$$

together with a gauge condition on the potential. Here the J_ν are the components of the charge-current density, given by

$$J_0 = |\psi|^2\,, \quad J_j = i(\overline{\psi}D_j\psi - \psi\overline{D_j\psi})\,.$$

We introduce a viscosity parameter $\varepsilon > 0$, and regard the MS system (2.1)–(2.2) as the limit $\varepsilon \to 0$ of the regularized system

$$\partial^\mu F_{\mu\nu} + J_\nu + \varepsilon R_\nu = 0\,, \tag{2.3}$$

$$D_0\psi - (i+\varepsilon)D_jD_j\psi = 0\,, \tag{2.4}$$

where

$$R_0 = 0, \quad R_j = \partial_j\{|\,\psi\,|^2 - 2\Delta^{-1}(\overline{D_k\psi}D_k\psi)\}\,, \tag{2.5}$$

with $\Delta^{-1}f = -(4\pi r)^{-1} * f$. The terms R_ν are defined so that Eqs. (2.3) and (2.4) are compatible. In fact, $\partial^\mu\partial^\nu F_{\mu\nu} = 0$ since $F_{\mu\nu} = -F_{\nu\mu}$. Thus (2.3) requires the equation of continuity

$$\partial^\nu(J_\nu + \varepsilon R_\nu) = 0\,. \tag{2.6}$$

But by (2.4),

$$\begin{aligned}\partial^\nu J_\nu &= \partial_0\,|\,\psi\,|^2 - 2\mathrm{Re}\,\{\mathrm{i}\partial_j(\overline{\psi}D_j\psi)\}\\ &= 2\mathrm{Re}\,\{\overline{\psi}(D_0\psi - \mathrm{i}D_jD_j\psi)\} = 2\varepsilon\mathrm{Re}\,\{\overline{\psi}D_jD_j\psi\}\,.\end{aligned}$$

Thus (2.6) implies $\partial^\nu R_\nu = -2\mathrm{Re}\,\{\overline{\psi}D_jD_j\psi\}$. This is the compatibility condition for R_ν. It has the solution (2.5).

We define the charge Q and the energy E by

$$\begin{aligned}Q &= \int |\,\psi\,|^2\,dx\,,\\ E &= \int\{\overline{D_j\psi}D_j\psi + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\}\,dx\,.\end{aligned}$$

Lemma 2.1. *Let ψ and A_μ be smooth functions on* $[0, T] \times \mathbb{R}^3$ *satisfying* (2.3) *and* (2.4). *Then we have*

$$Q(T) + 2\varepsilon \int_0^T dt \int \overline{D_j\psi} D_j\psi \, dx = Q(0) \,, \tag{2.7}$$

$$E(T) + 2\varepsilon \int_0^T dt \int \{\overline{D_jD_j\psi} D_kD_k\psi \, dx + V \mid \psi \mid^2\} \, dx = E(0) \,, \tag{2.8}$$

where V is the nonnegative function defined by

$$V = \tfrac{1}{2} \mid \psi \mid^2 - \Delta^{-1}(\overline{D_j\psi} D_j\psi) \,.$$

Proof. From (2.4) we have

$$\frac{dQ}{dt} = 2\mathrm{Re} \int \overline{\psi} D_0\psi \, dx = 2\varepsilon \mathrm{Re} \int \overline{\psi} D_jD_j\psi \, dx = -2\varepsilon \int \overline{D_j\psi} D_j\psi \, dx \,.$$

This implies (2.7). Similarly, from (2.4),

$$\begin{aligned}
\frac{d}{dt}\int \overline{D_j\psi} D_j\psi \, dx &= 2\mathrm{Re} \int \overline{D_j\psi} D_0D_j\psi \, dx \\
&= 2\mathrm{Re} \int \overline{D_j\psi}(D_jD_0 + iF_{j\,0})\psi \, dx \\
&= 2\mathrm{Re} \int \{-\overline{D_jD_j\psi} D_0\psi + iF_{j\,0}\overline{D_j\psi}\psi\} \, dx \\
&= -\int \{2\varepsilon \overline{D_jD_j\psi} D_kD_k\psi \, dx + F_{j\,0} J_j\} \, dx \,.
\end{aligned}$$

On the other hand, using the identity $\partial_0 F_{j\,k} + \partial_j F_{k\,0} + \partial_k F_{0\,j} = 0$ and (2.3), we have

$$\begin{aligned}
\frac{d}{dt}\int \tfrac{1}{4} F_{\mu\,\nu} F_{\mu\,\nu} dx &= \int \{F_{j\,0}\partial_0 F_{j\,0} + \tfrac{1}{2} F_{j\,k}\partial_0 F_{j\,k}\} \, dx \\
&= \int \{F_{j\,0}\partial_0 F_{j\,0} - \tfrac{1}{2} F_{j\,k}(\partial_j F_{k\,0} + \partial_k F_{0\,j})\} \, dx \\
&= -\int F_{j\,0}\partial^\mu F_{\mu\,j} \, dx = \int F_{j\,0}(J_j + \varepsilon R_j) \, dx \\
&= \int \{F_{j\,0} J_j - 2\varepsilon \partial_j F_{j\,0} V\} \, dx \\
&= \int \{F_{j\,0} J_j + 2\varepsilon \partial^\mu F_{\mu\,0} V\} \, dx \\
&= \int \{F_{j\,0} J_j - 2\varepsilon V \mid \psi \mid^2\} \, dx \,.
\end{aligned}$$

Adding the above two results, we deduce (2.8).

3. Regularized MS Field in the Coulomb Gauge

We begin by constructing global solutions of the initial value problem for the regularized system (2.3)–(2.4) in the Coulomb gauge $\partial_j A_j = 0$. In this gauge, Eq. (2.3) for $\nu = 0$ and $\nu = j \neq 0$ reduce to the equations

$$\Delta A_0 = J_0 \,, \tag{3.1}$$

$$\Box A_j = \partial_j \partial_0 A_0 - J_j - \varepsilon R_j \,, \tag{3.2}$$

respectively. These equations may be written as follows. By (3.1), we may set $A_0 = \Delta^{-1}J_0$ in (2.4). Furthermore, using (2.6) and (2.5), we have

$$\begin{aligned}\partial_j\partial_0 A_0 = \partial_j\partial_0\Delta^{-1}J_0 &= \partial_j\partial_k\Delta^{-1}(J_k + \varepsilon R_k)\\ &= \partial_j\partial_k\Delta^{-1}J_k + \varepsilon R_j\,.\end{aligned}$$

Thus the system (2.3)–(2.4) in the Coulomb gauge reduces to the following system for the fields $A = (A_1, A_2, A_3)$ and ψ :

$$\Box A = -PJ\,, \quad \nabla\cdot A = 0\,, \tag{3.3}$$

$$\partial_0\psi - (i+\varepsilon)\Delta\psi = K_\varepsilon\,, \tag{3.4}$$

where $J = (J_1, J_2, J_3)$, $PJ = J - \nabla(\nabla\cdot\Delta^{-1}J)$ is the projection of J on the divergence-free vector fields, and

$$K_\varepsilon = i(\Delta^{-1}J_0)\psi - (i+\varepsilon)(2iA\cdot\nabla\psi + A\cdot A\psi)\,.$$

We consider the system (3.3)–(3.4) with the initial condition

$$(A, \partial_0 A, \psi)|_{t=0} = (a, b, \phi)\,. \tag{3.5}$$

The Coulomb gauge condition imposes the initial constraints

$$\nabla\cdot a = \nabla\cdot b = 0\,. \tag{3.6}$$

The main result of this section is the following.

Lemma 3.1. *Let $\varepsilon > 0$ and $k \geqq 1$. Assume that $(a, b, \phi) \in H^k \times H^{k-1} \times H^k$ and a, b satisfy* (3.6). *Then there is a unique solution (A, ψ) of* (3.3)–(3.5) *such that*

$$(A, \partial_0 A, \psi) \in C(\mathbb{R}^+; H^k \times H^{k-1} \times H^k)\,, \qquad \nabla\psi \in L^2(\mathbb{R}^+; H^k)\,.$$

Moreover, we have

$$\|A(t)\|_2 < C\{1+t\}\,, \qquad \sum_{\mu=0}^{3}\|\partial_\mu A(t)\|_2 < C\,, \qquad \|\psi(t)\|_{1,2} < C\,, \tag{3.7}$$

for some constants $C = C(\|a\|_{1,2}, \|b\|_2, \|\phi\|_{1,2})$ independent of t and ε.

To prove the lemma, we write the wave equation in (3.3) as the pair of equations

$$\partial_0 A = B, \qquad \partial_0 B - \Delta A = -PJ\,.$$

We convert the system consisting of this pair, of (3.4), and of the initial condition (3.5) to the integral equation

$$u = G_\Phi N(u)\,.$$

Here $\Phi = (a, b, \phi)$, $u = (A, B, \psi)$, $N(u) = (0, -PJ, K_\varepsilon)$, and G_Φ is the linear mapping $(A, B, \psi) \to (A^G, B^G, \psi^G)$ defined as

$$\begin{pmatrix} A^G(t) \\ B^G(t)\end{pmatrix} = M(t)\begin{pmatrix} a \\ b\end{pmatrix} + \int_0^t M(t-s)\begin{pmatrix} A(s) \\ B(s)\end{pmatrix} ds\,, \tag{3.9}$$

$$\psi^G(t) = S(t)\phi + \int_0^t S(t-s)\psi(s)\,ds\,, \tag{3.10}$$

where

$$M(t) = \exp\left\{ t \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \right\} , \qquad S(t) = \exp\{(i+\varepsilon)t\Delta\} .$$

The operator P is the projection of the space of real 3-vectors in H^k onto its subspace consisting of divergence-free vectors. We denote this subspace as H^k_σ. We define X^k_0 to be the space of all triples $\Phi = (a, b, \phi)$ in $H^k_\sigma \times H^{k-1}_\sigma \times H^k$, with the norm

$$\|\Phi\|_{X^k_0} = \|a\|_{k,2} + \|b\|_{k-1,2} + \|\phi\|_{k,2} .$$

Let $T > 0$, let $I = [0, T]$, and let $\|\cdot\|_{k,p,q}$ be the norm in $L^q(I; W^{k,p})$. As a solution space for the integral equation (3.8), we take the space $X^k(T)$ of all triples $u = (A, B, \psi)$ in $C(I; X^k_0)$ such that $\nabla\psi$ is in $L^2(I; H^k)$, with the norm defined by

$$\|u\|_{X^k(T)} = \|A\|_{k,2,\infty} + \|B\|_{k-1,2,\infty} + \|\psi\|_{k,2,\infty} + \|\nabla\psi\|_{k,2,2} .$$

We also introduce the space $Y^k(T)$ of all $u = (A, B, \psi)$ in $L^2(I; H^k_\sigma \times H^{k-1}_\sigma \times W^{k,3/2})$, with the norm

$$\|u\|_{Y^k(T)} = \|A\|_{k,2,2} + \|B\|_{k-1,2,2} + \|\psi\|_{k,3/2,2} .$$

We write $X^k(T)$, $Y^k(T)$ simply as X^k, Y^k when there is no danger of confusion.

Lemma 3.2. *Let $\varepsilon > 0$. Let $k \geqq 1$ and $\Phi \in X^k_0$. Then G_Φ maps Y^k to X^k, with*

$$\|G_\Phi u\|_{X^k} \leqq C_\varepsilon \|\Phi\|_{X^k_0} + C_\varepsilon\{T^{1/4} + T^{1/2}\}\|u\|_{Y^k} .$$

Proof. Let $u = (A, B, \psi) \in Y^k$, and let (A^G, B^G, ψ^G) be defined by (3.9)–(3.10). Note that $M(t)$ is an isometry on $H^k_\sigma \times H^{k-1}_\sigma$. Therefore, from (3.9) we have $(A^G, B^G) \in C(I; H^k_\sigma \times H^{k-1}_\sigma)$, with

$$\|A^G\|_{k,2,\infty} + \|B^G\|_{k-1,2,\infty} \leqq C\{\|a\|_{k,2} + \|b\|_{k-1,2}\} + CT^{1/2}\{\|A\|_{k,2,2} + \|B\|_{k-1,2,2}\} .$$

Thus to complete the proof, it is sufficient to show that ψ^G is in $C(I; H^k)$ and satisfies

$$\|\psi^G\|_{k,2,\infty} + \|\nabla\psi^G\|_{k,2,2} \leqq C_\varepsilon\{\|\phi\|_{k,2} + T^{1/4}\|\psi\|_{k,3/2,2}\} . \tag{3.11}$$

To see this, we may assume $k = 0$. Now $\psi_0 = S(t)\phi$ is the solution of the initial value problem

$$\partial_0\psi_0 - (i+\varepsilon)\Delta\psi_0 = 0 , \qquad \psi_0|_{t=0} = \phi .$$

The standard energy-type estimate shows

$$\|S(t)\phi\|_2^2 + 2\varepsilon \int_0^t \|\nabla S(t)\phi\|_2^2 ds = \|\phi\|_2^2 . \tag{3.12}$$

On the other hand, the standard $L^p - L^q$ estimate for the heat evolution operator $e^{\varepsilon t\Delta}$ implies

$$\|S(t)f\|_2 \leqq C_\varepsilon t^{-1/4}\|f\|_{3/2} , \tag{3.13}$$

$$\|\nabla S(t)f\|_2 \leqq C_\varepsilon t^{-3/4}\|f\|_{3/2} . \tag{3.14}$$

By (3.13), we have

$$\left\| \int_0^t S(t-s)\psi(s)ds \right\|_2 \leqq C_\varepsilon \int_0^t (t-s)^{-1/4} \|\psi(s)\|_{3/2} ds$$
$$\leqq C_\varepsilon t^{1/4} \left\{ \int_0^t \|\psi(s)\|_{3/2}^2 ds \right\}^{1/2}, \tag{3.15}$$

and it follows that $\psi^G \in C(I; L^2)$. By (3.14) and the singular integral inequality, we also have

$$\int_0^t \left\| \nabla \int_0^\tau S(\tau - s)\psi(s)ds \right\|_2^2 d\tau \leqq C_\varepsilon \int_0^t \left\{ \int_0^\tau (\tau - s)^{-3/4} \|\psi(s)\|_{3/2} ds \right\}^2 d\tau$$
$$\leqq C_\varepsilon \left\{ \int_0^t \|\psi(s)\|_{3/2}^{4/3} ds \right\}^{3/2}$$
$$\leqq C_\varepsilon t^{1/2} \int_0^t \|\psi(s)\|_{3/2}^2 ds . \tag{3.16}$$

From (3.12), (3.15) and (3.16), we deduce (3.11) for $k = 0$.

We next show that the nonlinear term $N(u)$ is locally Lipschitz as a mapping from X^k to Y^k for $k \geqq 1$. To this end, we recall the Gagliardo–Nirenberg inequality

$$\|d^j f\|_p \leqq C \|d^k f\|_q^s \|f\|_r^{1-s},$$

where $1/p - j/3 = s(1/q - k/3) + (1-s)/r$, $j/k \leqq s \leqq 1$ (if $k - j - 3/q$ is a non-negative integer, only $s < 1$ is allowed), and where

$$\|d^j f\|_p = \left\{ \sum_{|\alpha|=j} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_p \right\}^{1/p}. \tag{3.17}$$

We will also use the estimate

$$\|\Delta^{-1} f\|_6 \leqq C \|f\|_{6/5}, \tag{3.18}$$

which follows for instance from applying the generalized Young's inequality to the expression $\Delta^{-1} f = (4\pi r)^{-1} * f$.

Lemma 3.3. *Let $k \geqq 1$. Then $N(u)$ maps X^k to Y^k. We have*

$$\|N(u)\|_{Y^k} \leqq (1 + T^{1/2}) Z_{\varepsilon,k}(\|u\|_{X^1}) \|u\|_{X^k}, \tag{3.19}$$

$$\|N(u) - N(u')\|_{Y^k} \leqq (1 + T^{1/2}) Z_{\varepsilon,k}(\|u\|_{X^k} + \|u'\|_{X^k}) \|u - u'\|_{X^k}, \tag{3.20}$$

where $Z_{\varepsilon,k}(r) = C_{\varepsilon,k}\{r + r^2\}$.

Proof. Let $u = (A, B, \psi) \in X^k$. We recall the definition $N(u) = (0, -PJ, K_\varepsilon)$. We will estimate the components of $N(u)$ using (3.17) and (3.18). We recall that $J_j = 2\mathrm{Re}\{i\overline{\psi} D_j \psi\}$. We first show that

$$\|J\|_{k-1,2,2} \leqq C\{1 + T^{1/2}\}\{\|u\|_{X^1} + \|u\|_{X^1}^2\}\|u\|_{X^k}. \tag{3.21}$$

Indeed, let $f_1 = \overline{\psi}\nabla\psi$ and $f_2 = A\|\psi\|^2$. The L^2-norms of f_1 and f_2 are estimated as

$$\|f_1\|_2 \leqq \|\psi\|_6\|\nabla\psi\|_3 \leqq C\|\psi\|_{1,2}\|\nabla\psi\|_{1,2}\,,$$
$$\|f_2\|_2 \leqq \|A\|_6\|\psi\|_6^2 \leqq C\|A\|_{1,2}\|\psi\|_{1,2}^2\,.$$

These give

$$\begin{aligned}\|J\|_{0,2,2} &\leqq \|\psi\|_{1,2,\infty}\|\nabla\psi\|_{1,2,2} + T^{1/2}\|A\|_{1,2,\infty}\|\psi\|_{1,2,\infty}^2\\ &\leqq C\{1+T^{1/2}\}\{\|u\|_{X^1}^2 + \|u\|_{X^1}^3\}\,.\end{aligned}\tag{3.22}$$

We also have the estimates

$$\begin{aligned}\|d^k f_1\|_2 &\leqq C\sum_{a+b=k}\|d^a\psi\|_4\|d^b\nabla\psi\|_4\\ &\leqq C\sum_{a+b=k}\|d^k\psi\|_4^{a/k}\|\psi\|_4^{1-a/k}\|d^k\nabla\psi\|_4^{b/k}\|\nabla\psi\|_4^{1-b/k}\\ &\leqq C\sum_{a+b=k}\|\psi\|_{k+1,2}^{a/k}\|\psi\|_{1,2}^{1-a/k}\|\nabla\psi\|_{k+1,2}^{b/k}\|\nabla\psi\|_{1,2}^{1-b/k}\,,\end{aligned}$$

and

$$\begin{aligned}\|d^k f_2\|_2 &\leqq C\sum_{a+b+c=k}\|d^a A\|_6\|d^b\psi\|_6\|d^c\psi\|_6\\ &\leqq C\sum_{a+b=k}\|d^k A\|_6^{a/k}\|A\|_6^{1-a/k}\|d^k\psi\|_6^{b/k}\|\psi\|_6^{2-b/k}\\ &\leqq C\sum_{a+b=k}\|A\|_{k+1,2}^{a/k}\|A\|_{1,2}^{1-a/k}\|\psi\|_{k+1,2}^{b/k}\|\psi\|_{1,2}^{2-b/k}\\ &\leqq C\{\|A\|_{1,2}+\|\psi\|_{1,2}\}^2\{\|A\|_{k+1,2}+\|\psi\|_{k+1,2}\}\,.\end{aligned}$$

These imply

$$\begin{aligned}\|d^k f_1\|_{0,2,2} &\leqq C\sum_{a+b=k}\|\psi\|_{k+1,2,\infty}^{a/k}\|\psi\|_{1,2,\infty}^{1-a/k}\|\nabla\psi\|_{k+1,2,2}^{b/k}\|\nabla\psi\|_{1,2,2}^{1-b/k}\\ &\leqq C\{\|\psi\|_{1,2,\infty}+\|\nabla\psi\|_{1,2,2}\}\{\|\psi\|_{k+1,2,\infty}+\|\nabla\psi\|_{k+1,2,2}\}\,,\end{aligned}$$

and

$$\|d^k f_2\|_{0,2,2} \leqq CT^{1/2}\{\|A\|_{1,2,\infty}+\|\psi\|_{1,2,\infty}\}^2\{\|A\|_{k+1,2,\infty}+\|\psi\|_{k+1,2,\infty}\}\,,$$

respectively. Thus

$$\|d^k J\|_{0,2,2} \leqq C\{1+T^{1/2}\}\{\|u\|_{X^1}+\|u\|_{X^1}^2\}\|u\|_{X^{k+1}}\,.$$

This estimate, together with (3.22), proves (3.21) for all $k \geqq 1$.

We will next show that

$$\|K_\varepsilon\|_{k,3/2,2} \leqq C\{1+T^{1/2}\}\{\|u\|_{X^1}+\|u\|_{X^1}^2\}\|u\|_{X^k}\,.\tag{3.23}$$

We recall the definition $K_\varepsilon = ig_1 - (i+\varepsilon)(2ig_2+g_3)$, where $g_1 = (\Delta^{-1}J_0)\psi$, $g_2 = A\cdot\nabla\psi$, and $g_3 = A\cdot A\psi$. Using (3.18), we have

$$\begin{aligned}\|g_1\|_{3/2} &\leqq \|\Delta^{-1}J_0\|_6\|\psi\|_2 \leqq C\|J_0\|_{6/5}\|\psi\|_2\\ &\leqq C\|\psi\|_3\|\psi\|_2^2 \leqq C\|\psi\|_{1,2}^3\,,\end{aligned}$$

$$\|g_2\|_{3/2} \leqq \|A\|_6 \|\nabla\psi\|_2 \leqq C\|\nabla A\|_2 \|\nabla\psi\|_2 ,$$

$$\|g_3\|_{3/2} \leqq \|A\|_6 \|A\|_6 \|\nabla\psi\|_3 \leqq C\|\nabla A\|_2^2 \|\psi\|_{1,2} .$$

These estimates yield

$$\begin{aligned} \|K_\varepsilon\|_{0,3/2,2} &\leqq CT^{1/2}\{\|\psi\|_{1,2,\infty}^2 + \|A\|_{1,2,\infty} + \|A\|_{1,2,\infty}^2\}\|\psi\|_{1,2,\infty} \\ &\leqq CT^{1/2}\{\|u\|_{X^1}^2 + \|u\|_{X^1}^3\} . \end{aligned} \tag{3.24}$$

We also have the following estimates for the derivatives,

$$\begin{aligned} \|d^k g_1\|_{3/2} &\leqq \sum_{a+b=k} \|d^a \Delta^{-1} J_0\|_6 \|d^b\psi\|_2 \\ &\leqq C\|J_0\|_{6/5}\|d^k\psi\|_2 + \sum_{\substack{a+b=k \\ a\neq 0}} \|d^a J_0\|_{6/5}\|d^b\psi\|_2 \\ &\leqq C\|\psi\|_{1,2}^2\|\psi\|_{k,2} + C\sum_{\substack{a+b=k \\ a\neq 0}} \left\{ \sum_{i+j=a} \|d^i\psi\|_{6a/(2a+i)}\|d^j\psi\|_{6a/(2a+j)} \right\} \|d^b\psi\|_2 \\ &\leqq C\|\psi\|_{1,2}^2\|\psi\|_{k,2} + C\sum_{\substack{a+b=k \\ a\neq 0}} \|d^a\psi\|_2\|\psi\|_3\|d^b\psi\|_2 \\ &\leqq C\|\psi\|_{1,2}^2\|\psi\|_{k,2} , \\ \|d^k g_2\|_{3/2} &\leqq C\sum_{a+b=k} \|d^a A\|_{6k/(k+2a)}\|d^b\nabla\psi\|_{6k/(k+2b)} \\ &\leqq C\sum_{a+b=k} \|d^k A\|_2^{a/k}\|A\|_6^{1-a/k}\|d^k\nabla\psi\|_2^{b/k}\|\nabla\psi\|_6^{1-b/k} \\ &\leqq C\sum_{a+b=k} \|A\|_{k,2}^{a/k}\|A\|_{1,2}^{1-a/k}\|\nabla\psi\|_{k,2}^{b/k}\|\nabla\psi\|_{1,2}^{1-b/k} , \\ \|d^k g_3\|_{3/2} &\leqq C\sum_{a+b+c=k} \|d^a A\|_{6k/(k+2a)}\|d^b A\|_{6k/(k+2b)}\|d^c\psi\|_{3k/c} \\ &\leqq C\sum_{a+b=k} \|d^k A\|_2^{a/k}\|A\|_6^{2-a/k}\|d^k\psi\|_3^{b/k}\|\psi\|_\infty^{1-b/k} \\ &\leqq C\sum_{a+b=k} \|A\|_{k,2}^{a/k}\|A\|_{1,2}^{2-a/k}\|\nabla\psi\|_{k,2}^{b/k}\|\nabla\psi\|_{1,2}^{1-b/k} . \end{aligned}$$

When we take the L^2-norm in time, we obtain

$$\begin{aligned} \|d^k g_1\|_{0,3/2,2} &\leqq CT^{1/2}\|\psi\|_{k,2,\infty}\|\psi\|_{0,2,\infty}^2 , \\ \|d^k g_2\|_{0,3/2,2} &\leqq C\sum_{a+b=k} \|A\|_{k,2,\infty}^{a/k}\|A\|_{1,2,\infty}^{1-a/k}\|\nabla\psi\|_{k,2,2}^{b/k}\|\nabla\psi\|_{1,2,2}^{1-b/k} \\ &\leqq C\{\|A\|_{k,2,\infty} + \|\nabla\psi\|_{k,2,2}\}\{\|A\|_{1,2,\infty} + \|\nabla\psi\|_{1,2,2}\} , \\ \|d^k g_3\|_{0,3/2,2} &\leqq C\sum_{a+b=k} \|A\|_{k,2,\infty}^{a/k}\|A\|_{1,2,\infty}^{2-a/k}\|\nabla\psi\|_{k,2,2}^{b/k}\|\nabla\psi\|_{1,2,2}^{1-b/k} \\ &\leqq C\{\|A\|_{k,2,\infty} + \|\nabla\psi\|_{k,2,2}\}\{\|A\|_{1,2,\infty} + \|\nabla\psi\|_{1,2,2}\}^2 , \end{aligned}$$

respectively. Thus

$$\|d^k K_\varepsilon\|_{0,3/2,2} \leqq C\{1+\varepsilon\}\{1+T^{1/2}\}\{\|u\|_{X^1} + \|u\|_{X^1}^2\}\|u\|_{X^k} .$$

This estimate, together with (3.24), proves (3.23). From (3.21) and (3.23), we have (3.19). The proof of (3.20) is similar.

Lemma 3.4. *Let $k \geqq 1$ and $\Phi \in X_0^k$. Then there is a $T_{\max} > 0$ such that the equation $u = G_\Phi N(u)$ has a unique solution u in $X^k(T)$ for all $T \in (0, T_{\max})$, and such that if $T_{\max} < \infty$, then*

$$\overline{\lim_{t \uparrow T_{\max}}} \|u(t)\|_{X_0^1} = \infty . \tag{3.25}$$

Moreover, if $\Phi_n \in X_0^k$ with $\Phi_n \to \Phi$ in X_0^k, and if $T \in (0, T_{\max})$, then the solution u_n of (3.8) *with Φ replaced by Φ_n belongs to $X^k(T)$ for sufficiently large n, and $u_n \to u$ in $X^k(T)$.*

Proof. By Lemmas 3.2 and 3.3, we have

$$\|G_\Phi N(u)\|_{X^k} \leqq \alpha \|\Phi\|_{X_0^k} + \beta(T, \|u\|_{X^1}) \|u\|_{X^k} , \tag{3.26}$$

$$\|G_\Phi N(u) - G_\Phi N(u')\|_{X^k} \leqq \beta(T, \|u\|_{X^k} + \|u'\|_{X^k}) \|u - u'\|_{X^k} , \tag{3.27}$$

for some $\alpha > 0$, and $\beta(T, r) = C\{T^{1/4} + T\}\{r + r^2\}$. We choose $R > 0$ and $T_k = T_k(R) > 0$ so that

$$\|\Phi\|_{X_0^k} < R, \tag{3.28}$$

$$\beta(T_k, 4\alpha R) < 1/2 . \tag{3.29}$$

Let $B^k(T_k)$ be the closed ball of center 0 and radius $2\alpha R$ in the space $X^k(T_k)$. Then by (3.26) and (3.27), $G_\Phi N(\,\cdot\,)$ is a contraction mapping of $B^k(T_k)$ into itself. Therefore the equation $u = G_\Phi N(u)$ has a unique solution u in $B^k(T_k)$. Since T_k depends only on k and R, a standard continuation argument shows that there exists a $T_{\max} > 0$ such that u extends to a unique solution in $X^k(T)$ for all $T \in (0, T_{\max})$, and furthermore that if $T_{\max} < \infty$, then

$$\overline{\lim_{t \uparrow T_{\max}}} \|u(t)\|_{X_0^k} = \infty . \tag{3.30}$$

We claim that (3.25) holds where the smaller norm in X_0^1 is used. For, otherwise there would be a constant $L > 0$ such that

$$\|u(t)\|_{X_0^1} < L$$

for $0 \leqq t < T_{\max}$. By (3.26) and (3.27) with $k = 1$, there exists $T_1 > 0$ depending only on L such that the equation $v = G_v N(v)$ has a unique solution $v \in B^1(T_1)$ for all $\Psi \in X_0^1$ with X_0^1-norm $< L$. Let $0 < \delta < T_1$ and $T_* = T_{\max} - \delta$. Let u_* be the unique solution of $u_* = G_{u(T_*)} N(u_*)$ in $B^1(T_1)$. Then u_* coincides with $u(\,\cdot\, + T_*)$ on $[0, \delta)$, so that $u_* \in X^k(T)$ for all T in $(0, \delta)$. Moreover, (3.26) with Φ replaced by $u(T_*)$ and u by u_* is valid for $T \in (0, \delta)$. It implies that

$$\begin{aligned} \|u_*\|_{X^k(T)} &= \|G_{u(T_*)} N(u_*)\|_{X^k(T)} \\ &\leqq \alpha \|u(T_*)\|_{X_0^k} + \beta(T, \|u_*\|_{X^1(T)}) \|u_*\|_{X^k(T)} \\ &\leqq \alpha \|u(T_*)\|_{X_0^k} + \beta(\delta, 2\alpha L) \|u_*\|_{X^k(T)} \,. \end{aligned}$$

If we choose δ so small that $\beta(\delta, 2\alpha L) < 1/2$, this inequality yields

$$\|u_*\|_{X^k(T)} < 2\alpha\|u(T_*)\|_{X_0^k}$$

for $0 < T < \delta$. But this implies

$$\overline{\lim_{t\uparrow T_{\max}}}\|u(t)\|_{X_0^k} \leqq 2\alpha\|u(T_*)\|_{X_0^k},$$

which contradicts (3.30). This proves (3.25).

To prove the last part of the lemma, it is sufficient to show that $u_n \to u$ in $X^k(T_k)$, with T_k as above. Let n be sufficiently large so that (3.28) with Φ replaced by Φ_n holds. The above existence argument shows that both u, u_n belong to $B^k(T_k)$. Then using Lemma 3.2 and (3.20), we have

$$\begin{aligned}\|u - u_n\|_{X^k(T_k)} &= \|G_\Phi N(u) - G_{\Phi_n}N(u_n)\|_{X^k(T_k)}\\ &\leqq \alpha\|\Phi - \Phi_n\|_{X_0^k} + \beta(T_k, \|u\|_{X^k} + \|u_n\|_{X^k})\|u - u_n\|_{X^k(T_k)}\\ &\leqq \alpha\|\Phi - \Phi_n\|_{X_0^k} + \beta(T_k, 4\alpha R)\|u - u_n\|_{X^k(T_k)}.\end{aligned}$$

Together with (3.29), this implies that $u_n \to u$ in $X^k(T_k)$.

Proof of Lemma 3.1. By Lemma 3.4, there exists a unique solution (A, ψ) of the approximate system (3.3)–(3.5) such that

$$(A, \partial_t A, \psi) \in C([0, T); H^k \times H^{k-1} \times H^k), \quad \nabla\psi \in L^2([0, T); H^k),$$

for some $T > 0$. Moreover, $T = \infty$ if $(A, \partial_t A, \psi)$ does not blow up in the $H^1 \times L^2 \times H^1$-norm in finite time. Thus to complete the proof, it is sufficient to show that (3.7) holds *a priori*. In view of the last part of Lemma 3.4, we may assume that A and ψ are smooth.

Let A_0 be defined by (3.1) and let Q and E be the charge and the energy defined in Sect. 2. Thus $Q(0) = \|\phi\|_2^2$, and $E(0)$ is estimated as

$$\begin{aligned}E(0) &= \|(\nabla - ia)\phi\|_2^2 + \tfrac{1}{2}\|\nabla\Delta^{-1}|\phi|^2 - b\|_2^2 + \tfrac{1}{4}\|\nabla \times a\|_2^2\\ &\leqq \{\|\nabla\phi\|_2 + \|a\|_4\|\phi\|_4\}^2 + C\{\|\phi\|_3\|\phi\|_2 + \tfrac{1}{2}\|b\|_2\}^2 + \tfrac{1}{4}\|\nabla \times a\|_2^2\\ &\leqq C(\|a\|_{1,2}, \|b\|_2, \|\phi\|_{1,2}),\end{aligned}$$

where we have used the boundedness of $\nabla\Delta^{-1}$ as an operator from $L^{6/5}$ to L^2. Since A_μ and ψ satisfy the regularized system (2.3)–(2.4), Lemma 2.1 implies that $E(t) \leqq E(0)$ and $Q(t) \leqq Q(0)$. Using the Coulomb gauge condition, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int A_jA_j\,dx &= \int \partial_0 A_j A_j\,dx = \int\{F_{0j} + \partial_j A_0\}A_j\,dx\\ &= \int\{F_{0j}A_j - A_0\partial_j A_j\}\,dx = \int F_{0j}A_j\,dx\\ &\leqq 2^{1/2}E(0)^{1/2}\|A\|_2.\end{aligned}$$

By integration, this implies

$$\|A(t)\|_2 \leqq \|a\|_2 + 2^{1/2}E(0)^{1/2}\,t, \tag{3.31}$$

which proves the first estimate in (3.7). In the Coulomb gauge, the energy for the field A may be rewritten as

$$\begin{aligned}\tfrac{1}{2}\int F_{\mu\nu}F_{\mu\nu}\,dx &= \int\{\partial_\mu A_\nu\partial_\mu A_\nu - \partial_\mu A_\nu\partial_\nu A_\mu\}\,dx \\ &= \int\{\partial_\mu A_\nu\partial_\mu A_\nu - \partial_0 A_0\partial_0 A_0 - 2\partial_j A_0\partial_0 A_j - \partial_j A_k\partial_k A_j\}\,dx \\ &= \int\{\partial_\mu A_j\partial_\mu A_j + \partial_j A_0\partial_j A_0 + 2A_0\partial_0\partial_j A_j + A_k\partial_k\partial_j A_j\}\,dx \\ &= \int\{\partial_\mu A_j\partial_\mu A_j + \partial_j A_0\partial_j A_0\}\,dx\,.\end{aligned}$$

It follows that

$$\sum_\mu \|\partial_\mu A(t)\|_2^2 + \sum_j \|\partial_j A_0(t)\|^2 \leqq E(0)\,, \tag{3.32}$$

so that the second estimate in (3.7) holds. We also have

$$\begin{aligned}\|\partial_j\psi\|_2 &\leqq \|D_j\psi\|_2 + \|A_j\|_6\|\psi\|_3 \\ &\leqq \|D_j\psi\|_2 + C\sum_{k=1}^3 \|\partial_k A_j\|_2 \sum_{k=1}^3 \|\partial_k\psi\|_2^{1/2}\|\psi\|_2^{1/2} \\ &\leqq E(0)^{1/2} + CE(0)^{1/2}Q(0)^{1/4}\sum_{k=1}^3 \|\partial_k\psi\|_2^{1/2}\,,\end{aligned}$$

where we have used (3.32). Thus

$$\sum_j \|\partial_j\psi(t)\|_2 \leqq C(Q(0),E(0))\,,$$

which together with $\|\psi(t)\|_2^2 \leqq Q(0)$ proves the last estimate in (3.7).

4. Global MS Field in the Coulomb Gauge

We now prove the existence of global finite energy solutions of the initial value problem for the exact MS system in the Coulomb gauge, Eqs. (3.3)–(3.4) with $\varepsilon = 0$, by making use of Lemma 3.1 and a compactness method.

Theorem 4.1. *Assume that* $(a, b, \phi) \in H^1 \times L^2 \times H^1$ *and* a, b *satisfy* (3.6). *Then there exists at least one solution* (A, ψ) *of* (3.3)–(3.5) *with* $\varepsilon = 0$, *such that*

$$A \in C_w(\mathbb{R}^+; L^2), \quad \partial_\mu A \in BC_w(\mathbb{R}^+; L^2), \quad \psi \in BC_w(\mathbb{R}^+; H^1)\,.$$

Theorem 4.2. *Let* A, ψ *be as in Theorem* 4.1. *If we define* A_0 *by* (3.1), *then* A_0, A, ψ *satisfy the MS system in its original form* (2.1)–(2.2).

Proof of Theorem 4.1. Let $\varepsilon_n > 0$ with $\varepsilon_n \to 0$. By Lemma 3.1, for each n there is a solution (ψ_n, A_n) of the regularized equations

$$\Box A_n = -PJ(A_n, \psi_n), \quad \nabla\cdot A_n = 0\,, \tag{4.2}$$

$$\partial_0\psi_n - (i + \varepsilon_n)\Delta\psi_n = K_{\varepsilon_n}(A_n, \psi_n)\,, \tag{4.3}$$

such that $(A_n, \partial_0 A_n, \psi_n)$ belongs to $C(\mathbb{R}^+ : H^1 \times L^2 \times H^1)$ and assumes the initial value (a, b, ϕ) at $t = 0$. Moreover, we have

$$\|A_n(t)\|_2 < C\{1 + t\}, \quad \|\partial_\mu A_n(t)\|_{1,2} < C\,, \tag{4.4}$$

$$\|\psi_n(t)\|_{1,2} < C\,, \tag{4.5}$$

where in the following discussion C denotes various constants independent of t and n. Now from the definition of $J = (J_1, J_2, J_3)$ in Sect. 2,

$$\begin{aligned}\|J(\psi_n, A_n)\|_{3/2} &\leqq 2\{\|\psi_n\|_6\|\nabla\psi_n\|_2 + \|A_n\|_6\|\psi_n\|_6^2\} \\ &\leqq C\{1 + \|\nabla A_n\|_2\}\|\psi_n\|_{1,2}^2 .\end{aligned}$$

By (4.4) and (4.5), this implies

$$\|J(\psi_n, A_n)(t)\|_{3/2} < C . \tag{4.6}$$

Also, the three estimates above (3.24) give

$$\|K_{\varepsilon_n}(\psi_n, A_n)\|_{3/2} \leqq C\{\|\psi_n\|_{1,2}^2 + \|\nabla A_n\|_2 + \|\nabla A_n\|_2^2\}\|\psi_n\|_{1,2} ,$$

which shows that

$$\|K_{\varepsilon_n}(\psi_n, A_n)(t)\|_{3/2} < C . \tag{4.7}$$

By (4.4)–(4.7), we can extract a subsequence of $\{(A_n, \psi_n)\}$, which we denote again by $\{(A_n, \psi_n)\}$, such that

$$A_n \to A \text{ weakly}^* \text{ in } L^\infty([0, T]; H^1), \quad 0 < T < \infty , \tag{4.8}$$

$$\partial_0 A_n \to \partial_0 A \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^+; L^2) , \tag{4.9}$$

$$\psi_n \to \psi \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^+; H^1) , \tag{4.10}$$

$$J(\psi_n, A_n) \to \beta \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^+; L^{3/2}) , \tag{4.11}$$

$$K_{\varepsilon_n}(\psi_n, A_n) \to \alpha \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^+; L^{3/2}) , \tag{4.12}$$

for some A, ψ satisfying (4.1) and $\alpha, \beta \in L^\infty(\mathbb{R}^+; L^{3/2})$. Passing to the limit $n \to \infty$ in (4.2) and (4.3), and using (4.8) and (4.10)–(4.12), we have

$$\Box A = -P\beta, \quad \nabla \cdot A = 0 , \tag{4.13}$$

$$\partial_0\psi - i\Delta\psi = \alpha , \tag{4.14}$$

in $\mathscr{D}'(\mathbb{R}^+; H^{-1})$.

We shall show that (ψ, A) is the desired solution. Now $\partial_0 A$ is bounded from $\mathbb{R}^+$ to L^2, and $\partial_0^2 A$, $\partial_0\psi$ are bounded from $\mathbb{R}^+$ to H^{-1} by (4.13) and (4.14). Thus $(A, \partial_0 A, \psi)$ is continuous as a mapping of $\mathbb{R}^+$ to $L^2 \times H^{-1} \times H^{-1}$. Also it is locally bounded from $\mathbb{R}^+$ to $H^1 \times L^2 \times H^1$. Thus it follows that $(A, \partial_0 A, \psi)$ is weakly continuous as a mapping of $\mathbb{R}^+$ to $H^1 \times L^2 \times H^1$.

We claim that the field (A, ψ) satisfies the initial condition (3.5). In fact, if $\langle \cdot , \cdot \rangle$ denotes the duality pairing, A_n satisfies

$$\int_0^t \langle A_n(s)\partial_0 f(s) + \partial_0 A_n(s) f(s), v\rangle ds = \langle a, v\rangle ,$$

and

$$\int_0^t \langle \partial_t A_n(s)\partial_0 f(s) + \{\Delta A_n(s) + PJ(A_n, \psi_n)(s)\} f(s), w\rangle ds = \langle b, w\rangle ,$$

for all $(v, w) \in L^2 \times H^1$ and all $f \in C^\infty(\mathbb{R}^+)$ with $f(0) = 1$ and $f(t) = 0$ for t large. Taking the limit $n \to \infty$ and using (4.8), (4.9), (4.11) and (4.13), we find that

$$\int_0^t \{A(s)\partial_0 h(s) + \partial_0 A(s) h(s)\} ds = a ,$$

$$\int_0^t \{\partial_0 A(s)\partial_0 h(s) + \partial_0^2 A(s)h(s)\}ds = b\,,$$

as integrals in L^2 and H^{-1}, respectively. This implies $(A, \partial_0 A)|_{t=0} = (a, b)$. Applying the same argument to ψ_n, we deduce that $\psi|_{t=0} = \phi$.

We also have $\alpha = K_0(A, \psi)$ and $\beta = J(A, \psi)$. To see this, let I be a bounded interval in $\mathbb{R}^+$, and Ω a bounded open set in $\mathbb{R}^3$. It is sufficient to show that α and β coincide with $K_0(A, \psi)$ and $J(A, \psi)$ on $I \times \Omega$, respectively. Now by (4.4), $\{(A_n, \partial_0 A_n)\}$ is a bounded sequence in $L^4(I; H^1(\Omega) \times L^2(\Omega))$. Since $H^1(\Omega) \subset L^4(\Omega) \subset L^2(\Omega)$, with the first imbedding compact and the second one continuous, a standard compactness lemma [2] then asserts that there is a subsequence of $\{A_n\}$, again denoted by $\{A_n\}$, such that

$$A_n \to A \text{ strongly in } L^4(I \times \Omega)\,. \tag{4.15}$$

Similarly, since $\{\partial_0 \psi_n\}$ is bounded in $L^\infty(\mathbb{R}^+; H^{-1})$ as indicated above, and so $\{(\psi_n, \partial_0 \psi_n)\}$ is a bounded sequence in $L^2(I; H^1(\Omega) \times H^{-1}(\Omega))$, we may assert that

$$\psi_n \to \psi \text{ strongly in } L^4(I \times \Omega)\,. \tag{4.16}$$

By (4.5), $|\psi_n|^2$ is bounded in $L^2(I; L^{6/5})$, while by (4.16) and the arbitrariness of Ω, we may assume $\psi_n \to \psi$ a.e. on $I \times \mathbb{R}^3$. It follows that $|\psi_n|^2 \to |\psi|^2$ weakly in $L^2(I; L^{6/5})$, and this implies

$$\Delta^{-1}|\psi_n|^2 \to \Delta^{-1}|\psi|^2 \text{ weakly in } L^2(I; L^6)\,, \tag{4.17}$$

since Δ^{-1} is bounded from $L^{6/5}$ to L^6. Therefore by (4.8), (4.10) and (4.15)–(4.17), we have

$$J(A_n, \psi_n) \to J(A, \psi) \text{ weakly in } L^{4/3}(I \times \Omega)\,,$$

$$K_{\varepsilon_n}(A_n, \psi_n) \to K_0(A, \psi) \text{ weakly in } L^{4/3}(I \times \Omega)\,.$$

Thus $\alpha = K_0(A, \psi)$ and $\beta = J(A, \psi)$ on $I \times \Omega$, which completes the proof.

Proof of Theorem 4.2. Let A_0 be defined by (3.1), that is, $\Delta A_0 = J_0$. Then A_0, A, ψ clearly satisfy the component of (2.1) with $\mu = 0$ and also (2.2). To see that (2.1) also holds for $\mu \neq 0$, we use the following fact. If $f \in H^1$ and $g \in H^{-1}$, and if $s > 3/2$, then the inequality

$$|\langle g, \zeta f\rangle| \leqq C\|g\|_{-1,2}\|f\|_{1,2}\|\zeta\|_{s,2}$$

holds for all $\zeta \in H^s$. Hence the product fg is well-defined as an element of H^{-s}. Since ψ is in $L^\infty(\mathbb{R}^+; H^1)$ and $D_0\psi = iD_jD_j\psi$ is in $L^\infty(\mathbb{R}^+; H^{-1})$, it follows that

$$\partial_0 J_0 = 2\mathrm{Re}\{\overline{\psi} D_0 \psi\} = 2\mathrm{Re}\{i\overline{\psi} D_j D_j \psi\} = \partial_j J_j\,,$$

by the definition of J_μ in Sect. 2. Since $A_0 = \Delta^{-1} J_0$, this implies

$$\partial_0 A_0 = \partial_k \Delta^{-1} J_k\,. \tag{4.18}$$

From (3.3), (4.18) and the definition of P, it follows that

$$\partial^\mu F_{\mu j} = \Box A_j - \partial_j \partial_0 A_0 = -(PJ)_j - \partial_j \partial_k \Delta^{-1} J_k = J_j\,,$$

as was to be shown.

5. Global Solutions in Other Gauges

In this section we consider the initial value problem for the MS system (2.1)–(2.2) in the Lorentz gauge $\partial^\mu A_\mu = 0$ and in the temporal gauge $A_0 = 0$.

The problem in the Lorentz gauge reduces to the system

$$\Box A_v + J_v = 0, \quad \partial^\mu A_\mu = 0\,, \tag{5.1}$$

$$iD_0\psi + D_jD_j\psi = 0\,, \tag{5.2}$$

$$(A_\mu,\, \partial_0 A_\mu,\, \psi)|_{t=0} = (a_\mu,\, b_\mu,\, \phi)\,. \tag{5.3}$$

The equations require us to assume the initial constraints

$$b_0 = \partial_j a_j, \quad \Delta a_0 = \partial_j b_j + |\phi|^2\,. \tag{5.4}$$

Theorem 5.1. (*Lorentz gauge*) *Assume that* $(a_\mu, b_\mu, \phi) \in H^1 \times L^2 \times H^1$ *and the* a_μ, b_μ *satisfy* (5.4). *Then the initial value problem* (5.1)–(5.3) *has a solution* (ψ, A_μ) *such that*

$$(A_0,\, \partial_0 A_0) \in C(\mathbb{R}^+; H^1 \times L^2), \quad (A_j, \partial_0 A_j, \psi) \in C_w(\mathbb{R}^+; H^1 \times L^2 \times H^1)\,. \tag{5.5}$$

Proof. We will reduce the problem to the Coulomb gauge by a gauge transformation. Let a, b denote the 3-vectors with components a_j, b_j, respectively. Let $f = \Delta^{-1}\nabla \cdot a$. We define

$$(a^c,\, b^c, \phi^c) = (Pa,\, Pb,\, e^{-if}\phi)\,,$$

where $Pv = v - \nabla(\nabla \cdot \Delta^{-1}v)$ as before. Note that $\phi^c \in H^1$, since $\nabla f = (I-P)a$. By Theorem 4.1, the MS system in the Coulomb gauge, (3.3)–(3.4) with $\varepsilon = 0$, has a solution $(A^c,\, \psi^c)$, $A^c = (A_1^c,\, A_2^c,\, A_3^c)$, such that

$$A^c \in C_w(\mathbb{R}^+; L^2), \quad \partial_\mu A^c \in BC_w(\mathbb{R}^+; L^2), \quad \psi^c \in BC_w(\mathbb{R}^+; H^1)\,,$$

that satisfies the initial condition

$$(\psi^c,\, \partial_0 A^c,\, A^c)|_{t=0} = (a^c,\, b^c,\, \phi^c)\,.$$

Let

$$A_0^c = \Delta^{-1}|\psi^c|^2\,. \tag{5.6}$$

We define λ to be the solution of the initial value problem

$$\Box\lambda = -\partial_0 A_0^c, \quad (\lambda,\, \partial_0\lambda)|_{t=0} = (f,\, g)\,, \tag{5.7}$$

where f is as above and $g = \Delta^{-1}\nabla \cdot b$, so that $\nabla f \in H^1$ and $\nabla g \in L^2$. By (4.18),

$$\partial_0 A_0^c = \Delta^{-1}\nabla \cdot J^c\,,$$

where $J^c = J(A^c,\, \psi^c) \in L^\infty(\mathbb{R}^+; L^{6/5})$. Since $\Delta^{-1}\nabla$ is bounded from $L^{6/5}$ to L^2, this implies that $\partial_0 A_0^c$ is in $L^\infty(\mathbb{R}^+; L^2)$. Then the energy estimate for (5.7) shows that $\nabla\lambda \in C(\mathbb{R}^+; L^2)$ and $\partial_0\lambda \in C(\mathbb{R}^+; L^2_{loc})$. We define A_μ and ψ by the gauge transformation

$$A_\mu = A_\mu^c + \partial_\mu\lambda\,, \quad \psi = e^{i\lambda}\psi^c\,. \tag{5.8}$$

We shall show that (A_μ, ψ) is the desired solution of (5.1)–(5.3). Now from (5.6)–(5.8) and (5.4), we have

$$\begin{aligned} A_0|_{t=0} &= A_0^c|_{t=0} + g = \Delta^{-1}|\phi^c|^2 + \nabla \cdot \Delta^{-1} b = a_0 , \\ \partial_0 A_0|_{t=0} &= \partial_0 A_0^c|_{t=0} + \partial_0^2 \lambda|_{t=0} = \Delta\lambda|_{t=0} = \nabla \cdot a = b . \\ A_j|_{t=0} &= a_j^c + \partial_j f = (Pa)_j + ((I-P)a)_j = a_j , \\ \partial_0 A_j|_{t=0} &= b_j^c + \partial_j g = (Pb)_j + ((I-P)b)_j = b_j , \\ \psi|_{t=0} &= e^{if}\psi^c = \phi . \end{aligned}$$

Thus (A_μ, ψ) satisfies (5.3). Applying ∂^μ to the first equation in (5.8), and using (5.7) and $\partial_j A_j^c = 0$, we see that the A_μ satisfy the Lorentz gauge condition $\partial^\mu A_\mu = 0$.

We now prove (5.5) and (5.1)–(5.3). From the definition (5.8), A_μ and ψ belong to $C_w(\mathbb{R}^+; L^2)$. From (5.6)–(5.8), we have

$$\Box A_0 = -|\psi^c|^2 .$$

Since $(A_0, \partial_0 A_0)|_{t=0} = (a_0, b_0) \in H^1 \times L^2$, this equation implies that $A_0 \in C(\mathbb{R}^+; H^1)$ with $\partial_0 A_0 \in C(\mathbb{R}^+; L^2)$. Then $\partial_j A_j \in C(\mathbb{R}^+; L^2)$, since $\partial^\mu A_\mu = 0$. We also have

$$\partial_j A_k - \partial_k A_j = \partial_j A_k^c - \partial_k A_j^c \in C_w(\mathbb{R}^+; L^2) .$$

The last two facts imply that $\partial_k A_j \in C_w(\mathbb{R}^+; L^2)$ for all j and k. Similarly,

$$\partial_0 A_j = \partial_0 A_j^c - \partial_j A_0^c + \partial_j A_0 \in C_w(\mathbb{R}^+; L^2) .$$

Thus we have proved that $A_\mu, \partial_\nu A_\mu \in C_w(\mathbb{R}^+; L^2)$. In particular, $A_j \in C_w(\mathbb{R}^+; H^1)$. By (5.8), this implies $\partial_j \lambda \in C_w(\mathbb{R}^+; H^1)$. Then

$$\partial_j \psi = e^{i\lambda}\{\partial_j + i\partial_j \lambda\}\psi^c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2) .$$

Thus $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1)$. Now by Theorem 4.2, (A_μ^c, ψ^c) satisfies (2.1) and (2.2), which are invariant under the gauge transformation (5.8). Thus (A_μ, ψ) also satisfies (2.1)–(2.2) and hence (5.1)–(5.2) since $\partial^\mu A_\mu = 0$. The weak continuity of ψ in t now follows from (5.2). This completes the proof.

In the temporal gauge $A_0 = 0$, we may write the initial value problem for the MS system as

$$-\partial_0 \partial_j A_j = J_0, \quad A_0 = 0 , \tag{5.9}$$
$$\Box A_j + \partial_j \partial_k A_k = -J_j , \tag{5.10}$$
$$i\partial_0 \psi + D_j D_j \psi = 0 , \tag{5.11}$$
$$(A_j, \partial_0 A_j, \psi)|_{t=0} = (a_j, b_j, \phi) . \tag{5.12}$$

The initial constraint is

$$\partial_j b_j + |\phi|^2 = 0 . \tag{5.13}$$

Theorem 5.2. (*Temporal gauge*) *Assume that* $(a_j, b_j, \phi) \in H^1 \times L^2 \times H^1$ *and the* b_j *satisfy* (5.13). *Then the initial value problem* (5.9)–(5.13) *has a solution* (A, ψ) *such that*

$$A_j \in C_w(\mathbb{R}^+; H^1) , \quad \partial_0 A_j \in BC_w(\mathbb{R}^+; L^2) , \quad \psi \in BC_w(\mathbb{R}^+; H^1) .$$

Proof. We define f, a^c, b^c, ϕ^c, A_0^c, A^c and ψ^c as in the proof of Theorem 5.1. Using the solution λ of the simple equation

$$\partial_0 \lambda + A_0^c = 0 \,, \quad \lambda|_{t=0} = f \,,$$

we define ψ, A_μ by (5.8). Then (ψ, A_j) provides the solution of the theorem.

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