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$N = 2$ Topological Yang-Mills Theory on Compact Kähler Surfaces

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Abstract: We study a topological Yang-Mills theory with $N = 2$ fermionic symmetry. Our formalism is a field theoretical interpretation of the Donaldson polynomial invariants on compact Kähler surfaces. We also study an analogous theory on compact oriented Riemann surfaces and briefly discuss a possible application of Witten's non-Abelian localization formula to the problems in the case of compact Kähler surfaces.

1. Introduction

Several years ago, Witten introduced the topological Yang-Mills theory (TYMT) [1] on general 4-manifolds to provide a quantum field theoretical interpretation of the Donaldson polynomial invariants [2]. The basic property of the TYMT is that there is a fermionic symmetry which localizes the path integral to an integral over the moduli space $\mathscr{M}$ of anti-self-dual (ASD) connections. Geometrically, the fermionic operator δ_W acts on $\mathscr{M}$ as the exterior derivative. The action functional of the TYMT can be written as an δ_W-exact form,

$$S_W = \delta_W V\,. \tag{1.1}$$

In the TYMT, correlation functions of physical observables correspond to the Donaldson polynomial invariants.

The moduli space $\mathscr{M}$ of ASD connections on a compact Kähler surface has natural complex and Kähler structures [3], which implies that the TYMT has actually $N = 2$ fermionic symmetry generated by the holomorphic and the anti-holomorphic parts of δ_W, i.e. $\delta_W = \mathbf{s} + \bar{\mathbf{s}}$. Geometrically, we can interpret $\bar{\mathbf{s}}$ as the Dolbeault cohomology operator on $\mathscr{M}$. Then, the $N = 2$ version of the topological action may be written as

$$S = \mathbf{s}\bar{\mathbf{s}}\mathbf{B}_{\mathbf{T}}\,. \tag{1.2}$$

In the first part of this paper, we study TYMT on compact Kähler surfaces with $N = 2$ fermionic symmetry. In Sect. 2, we briefly sketch Donaldson theory and the TYMT of Witten in order to make this paper reasonably self-contained and to set up notations for the later sections. In Sect. 3, we construct the $N = 2$ TYMT on compact Kähler surfaces. After a short discussion on the anti-self-duality relative to Kähler form, we show that the theory has $N = 2$ fermionic symmetry thereby deriving the corresponding algebra. We construct topological actions and study zero-modes of fermionic fields which naturally represent the cohomology structures of $\mathscr{M}$. In Sect. 4, we study topological observables and their correlation functions which can be interpreted as the Donaldson invariants on compact Kähler surfaces. As an example, we show that one of the correlation functions can be identified with the symplectic volume of the moduli space under some favorable conditions. This may be compared with the differential geometrical approach of Donaldson (sketched in p. 294–295 of [2]). Note that the main application of Donaldson's original work [2] was to the differential topology of complex algebraic surfaces. This is because, over compact Kähler surfaces, one can use algebro-geometric techniques which enhance the computability of the invariants. Although it is not clear that our treatment will lead to some explicit expressions for the invariants, we may hope to illuminate our understanding of Donaldson theory in the field theoretical interpretation.

Recently, Witten obtained a general expression for the two-dimensional analogue of the Donaldson invariants [4, 5]. In the second part of this paper, we discuss some analogies between the $N = 2$ TYMT on compact Kähler surfaces and the TYMT on compact Riemann surfaces (Sect. 3 of [5]). In Sect. 5.1, we study an $N = 2$ version of the TYMT on compact oriented Riemann surface as a straightforward application of the techniques developed in the first part of this paper. In Sect. 5.2, we give a basic description of Donaldson's algebro-geometrical approach to his invariants [2]. In Sect. 5.3, we suggest that the non-Abelian localization formula of Witten [5] can be applied to Donaldson theory on compact Kähler surfaces.

We understand that Galperin and Ogievetsky studied a TYMT on compact Kähler surfaces, as well as the hyperKähler case, with extended fermionic symmetry [6]. Their construction is based on the twisting of the $N = 2$ super-Yang-Mills theory [1].

2. Preliminary

In this section, we will briefly sketch the Donaldson invariants [2] and the TYMT of Witten [1]. A comprehensive exposition of Donaldson theory can be found in [7]. It will be also useful to consult with [8]. A detailed introduction of topological field theories in general can be found in [9].

2.1 Sketch of the Donaldson Invariants

Let M be a smooth oriented compact Riemann four-manifold and let E be a vector bundle over M with a reduction of the structure group to $SU(2)$. Let Ad(E) be the Lie algebra bundle associated to E by the adjoint representation. The bundle E is classified by the instanton number $k = \langle c_2(E), [M]\rangle$. Throughout this paper, we will assume that k is strictly positive. Let $\mathscr{A}$ be the space of all connections on E and $\mathscr{G}$ be the group of gauge transformations. Since any two connections differ by an element of Ad(E)-valued one form, $\mathscr{A}$ is an affine space and its tangent space $T\mathscr{A}$ consists of Ad(E)-valued one form on M. We define a quotient space $\mathscr{B} = \mathscr{A}/\mathscr{G}$, the set of gauge equivalence classes of connections. We also introduce $\mathscr{B}^* = \mathscr{A}^*/\mathscr{G}$, where $\mathscr{A}^*$ denotes the space of irreducible connections. Due to Atiyah and Singer

[10], there exists an universal bundle $\mathbb{E}$ over the product space $M \times \mathscr{B}^*$ as an adjoint bundle. Let $A^{m,n}(M \times \mathscr{B}^*)$ denote the space of Ad($\mathbb{E}$)-valued m-forms on M and n-forms on $\mathscr{B}^*$ respectively. Let $\mathscr{F}$ be the total curvature two-form, which can be decomposed into components

$$\begin{aligned} \mathscr{F}^{2,0} &= dA + A \wedge A \,, \\ \mathscr{F}^{1,1} &= \delta_H A \,, \\ \mathscr{F}^{0,2} &= -G_A[\delta_H A, *\delta_H A] \,, \end{aligned} \tag{2.1}$$

where $G_A = (d_A * d_A)^{-1}$ and $\delta_H A$ represents tangent vectors on $\mathscr{B}^*$, i.e. $\delta_H A$ is the horizontal projection of the Ad(E)-valued one-form δA ($d_A^* \delta_H A = 0$).

One can interpret the operator δ_H as the exterior covariant derivative (the horizontal part of exterior derivative δ on $\mathscr{A}$) on $\mathscr{B}^*$ [11],

$$\begin{aligned} d_A &: A^{m,n}(M \times \mathscr{B}^*) \to A^{m+1,n}(M \times \mathscr{B}^*) \,, \\ \delta_H &: A^{m,n}(M \times \mathscr{B}^*) \to A^{m,n+1}(M \times \mathscr{B}^*) \,. \end{aligned} \tag{2.2}$$

Then, one can find that

$$\delta_H A = \mathscr{F}^{1,1} \,, \qquad \delta_H \mathscr{F}^{1,1} = -d_A \mathscr{F}^{0,2} \,, \qquad \delta_H \mathscr{F}^{0,2} = 0 \,. \tag{2.3}$$

Note that δ_H is nilpotent up to a gauge transformation generated by $\mathscr{F}^{0,2}$, i.e. $\delta_H^2 A = -d_A \mathscr{F}^{0,2}$. Provided that δ_H acts on gauge invariant functionals, δ_H can be interpreted as the de Rham cohomology operator on $\mathscr{B}^*$. That is, δ_H is an operator of the equivariant cohomology. The de Rham cohomology classes on $\mathscr{B}^*$ can be easily constructed from the characteristic class $c_2(\mathbb{E}) = \frac{1}{8\pi^2}\mathrm{Tr}\mathscr{F}^2$ which is a closed form of degree 2 on $M \times \mathscr{B}^*$,

$$(d + \delta_H)\frac{1}{8\pi^2}\mathrm{Tr}\mathscr{F}^2 = 0 \,. \tag{2.4}$$

We expand $c_2(\mathbb{E})$ as follows;

$$\frac{1}{8\pi^2}\mathrm{Tr}\mathscr{F}^2 = \sum_{r=0}^{4} \mathscr{W}_r^{4-r} \qquad \text{where} \quad \mathscr{W}_r^{4-r} \in A^{r,4-r}(M \times \mathscr{B}^*) \,, \tag{2.5}$$

and define $W_r{}^{4-r} \equiv \int_M \mathscr{W}_r^{4-r} \wedge O_{4-r}$ for a harmonic $4-r$ form O_{4-r} on M. Equation (2.4) shows that $W_r{}^{4-r}$ is an element of de Rham cohomology classes on $\mathscr{B}^*$,

$$\delta_H W_r{}^{4-r} = 0 \,, \qquad W_r{}^{4-r} \in H^{4-r}(\mathscr{B}^*) \,, \tag{2.6}$$

which depends only on the cohomology class of $O_{4-r} \in H^{4-r}(M)$. In the Poincaré dual picture, one can define $W_r{}^{4-r} \equiv \int_{Y_r} \mathscr{W}_r^{4-r}$ in terms of a r-dimensional homology cycle Y_r which is Poincaré dual to O_{4-r}.

Let $\mathscr{M}$ be the moduli space of ASD connections – the set of gauge equivalence classes of ASD connections. For a given ASD connection A, a neighborhood of the point $[A]$ in the moduli space $\mathscr{M}$ of ASD connections should satisfy the following equations;

$$\begin{aligned} &F^+(A+\delta A) = d_A^+\delta A + (\delta A \wedge \delta A)^+ = 0\,,\\ &d_A^*\delta A = 0\,, \end{aligned} \tag{2.7}$$

where F^+ denotes the self-dual part of the curvature and $*$ is the Hodge star operator. The second equation in (2.7) restricts the infinitesimal variation δA to the direction orthogonal to the pure gauge variation. The linearisation of Eq. (2.7) gives rise to an elliptic operator $\delta_A = d_A^+ \oplus d_A^*$ which leads to the instanton complex of Atiyah-Hitchin-Singer [12];

$$0 \longrightarrow A^0(\mathrm{Ad}(E)) \xrightarrow{d_A} A^1(\mathrm{Ad}(E)) \xrightarrow{d_A^+} A_+^2(\mathrm{Ad}(E)) \longrightarrow 0\,. \tag{2.8}$$

The cohomology groups of this elliptic complex are

$$\begin{aligned} &H_A^0 = \mathrm{Ker}\,\triangle^0\,, \qquad \triangle^0 = d_A^* d_A\,,\\ &H_A^1 = \mathrm{Ker}\,\triangle^1\,, \qquad \triangle^1 = d_A d_A^* + d_A^{+*} d_A^+\,,\\ &H_A^2 = \mathrm{Ker}\,\triangle^2\,, \qquad \triangle^2 = d_A^+ d_A^{+*}\,, \end{aligned} \tag{2.9}$$

where $h_A^i = \dim H_A^i$ denotes the *i-th* Betti number. The zeroth cohomology is trivial when A is irreducible and the first cohomology group can be identified with the tangent space at $[A]$ on $\mathscr{M}$. The singularities in $\mathscr{M}$ arise if $h_A^0 \neq 0$ or $h_A^2 \neq 0$. The formal dimension of $\mathscr{M}$, for a simply connected M, is given by the index of the instanton complex,

$$\mathrm{ind}(\delta_A) = h_A^1 - h_A^0 - h_A^2 = 8k - 3(1 + b^+(M))\,, \tag{2.10}$$

where $b^+(M)$ is the dimension of the self-dual harmonic two-forms on M.

Let M be a simply connected, oriented 4-manifold. Then, the essential cohomological data are contained in $H^2(M)$. Thus, we have a de Rham cohomology on $\mathscr{B}^*$,

$$W_2{}^2 = \frac{1}{4\pi^2}\int\limits_M \mathrm{Tr}\left(\mathscr{F}^{0,2}\mathscr{F}^{2,0} + \tfrac{1}{2}\mathscr{F}^{1,1}\wedge\mathscr{F}^{1,1}\right)\wedge O_2\,, \tag{2.11}$$

which depends only on the cohomology class of $O_2 \in H^2(M)$ or on the homology class of $\Sigma \in H_2(M)$ which is Poincaré dual to O_2. Equivalently, $W_2{}^2$ defines the Donaldson μ-map, $\mu(\Sigma)$. The basic idea of Donaldson is to use the moduli space $\mathscr{M}^*$ of irreducible ASD connections as a fundamental homology cycle in order to evaluate a cup product of the cohomology classes $\mu(\Sigma_i)$ for $\Sigma_i \in H_2(M)$,

$$\langle \mu(\Sigma_1) \cup \cdots \cup \mu(\Sigma_d), [\mathscr{M}^*]\rangle\ . \tag{2.12}$$

In order to have a differential topological interpretation of the above pairings, further specifications are required. That is, the moduli space $\mathscr{M}^*$ should be a smooth oriented $2d$-dimensional manifold and should carry fundamental homology classes. According to a theorem of Freed and Uhlenbeck [13], the moduli space $\mathscr{M}^*$ of irreducible connections is a smooth manifold with the actual dimension being equal to the formal dimension for a generic choice of Riemann metric on M. It is also known that there is no reducible instanton for $b^+(M) > 1$ [2]. For an odd $b^+(M) = 1 + 2a$, the dimension of the moduli space is even, $\dim\mathscr{M}^* = 2d = 4k - 3(1 + a)$. Donaldson also proved the orientability of the moduli space [14]. The next and the most important condition is to understand the compactness properties of $\mathscr{M}^*$ so that one may define the fundamental homology classes. In practice, the moduli space is rarely compact and this is the main subtlety in defining the Donaldson invariants. However, there is a natural compactification $\overline{\mathscr{M}}^*$ of $\mathscr{M}^*$ [15, 16]. One may extend the cohomology classes $\mu(\Sigma_i)$ to the compactified space and try to evaluate their cup products on $\overline{\mathscr{M}}^*$. For a large enough k (or for the stable range $4k > 3b^+(M)+3$), the compactified space carries a fundamental homology class. In this way, we have a well defined pairing. It is more convenient to work in the Poincaré dual picture of (2.12). Donaldson showed that one can arrange codimension 2 cycles V_{Σ_i} – which represent the cohomology classes $\mu(\Sigma_i)$, on $\mathscr{M}^*$ so that the intersection

$$\mathscr{M}^* \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \tag{2.13}$$

is compact in the stable range. The Donaldson invariant is defined by the intersection number of (2.13);

$$q_{k,M}\left(\Sigma_1, \ldots, \Sigma_d\right) = \#(\mathscr{M}^* \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}), \tag{2.14}$$

which is an invariant of the oriented diffeomorphism type of M.

2.2 *Witten's Interpretation and Generalization*

The TYMT in a special limit can be viewed as an integral representation of the Donaldson invariants. If we denote $\widehat{W}_r^{4-r}$ to be the restriction of W_r^{4-r} on $\mathscr{B}^*$ to $\mathscr{M}^*$, the Donaldson invariants can be represented by an integral of wedge products of the cohomology classes $\widehat{W}_r^{4-r_i} \in H^{4-r_i}(\mathscr{M}^*)$ over $\mathscr{M}^*$,

$$\int_{\mathscr{M}^*} \widehat{W}_{r_1}^{4-r_1} \wedge \cdots \wedge \widehat{W}_{r_k}^{4-r_k}\,, \tag{2.15}$$

which vanishes unless $\sum_{i=1}^{k}(4-r_i) = \dim(\mathscr{M}^*)$. Of course, we should have some concrete procedure to obtain $\widehat{W}_r^{4-r}$ from W_r^{4-r}.

The basic idea of the TYMT is to introduce a fermionic symmetry δ_W and basic multiplet (A, Ψ, Φ) with the transformation law analogous to (2.3),

$$\delta_W A = \Psi\,, \qquad \delta_W \Psi = -d_A \Phi\,, \qquad \delta_W \Phi = 0\,. \tag{2.16}$$

Geometrically, Witten's fermionic operator δ_W corresponds to δ_H. One can construct a δ_W-invariant action after introducing additional fields such that the fixed point locus of the fermionic symmetry (δ_W-invariant configuration) is the moduli space of ASD connections. An appropriate action may be written as

$$S_W = \delta_W\left(\frac{1}{h^2}\int_M \mathrm{Tr}\left[-\mathscr{X} \wedge (H + F^+) + \bar{\Phi}(d_A * \Psi + \alpha\mathscr{Y})\right]\right)\,, \tag{2.17}$$

where the self-dual two-forms $(\mathscr{X}, H)$ (as well as the zero-forms $(\bar{\Phi}, \mathscr{Y})$) are analogous to the antighost multiplet for the anti-self-duality constraint, $F^+ = 0$, (and those for the Coulomb gauge condition $d_A * \Psi = 0$ respectively) in the usual BRST quantization,

$$\begin{aligned} \delta_W \bar{\Phi} &= \mathscr{Y} , & \delta_W \mathscr{Y} &= [\Phi, \bar{\Phi}] , \\ \delta_W \mathscr{X} &= H , & \delta_W H &= [\Phi, \mathscr{X}] . \end{aligned} \tag{2.18}$$

This transformation law can be deduced from the property that δ_W is nilpotent up to a gauge transformation generated by Φ in the basic multiplet, i.e. $\delta_W^2 A = -d_A \Phi$. These properties allow one to interpret the TYMT as a BRST quantized version of an underlying theory with topological symmetry [17–21]. We introduce a quantum number U, analogous to the ghost number of BRST quantization, which assigns the value 1 to δ_W and $(0, 1, 2, -1, 0, -2, -1)$ to $(A, \Psi, \Phi, \mathscr{X}, H, \bar{\Phi}, \mathscr{Y})$. Note that the U numbers of the basic multiplet (A, Ψ, Φ) can be interpreted as the form degrees on $\mathscr{B}^*$ in the universal bundle formalism.

For $\alpha = 0$, we find

$$\begin{aligned} S_W = & \frac{1}{h^2} \int_M \operatorname{Tr} \Big[-H \wedge (H + F^+) - \mathscr{X} \wedge (d_A \Psi)^+ + \mathscr{X}[\Phi, \mathscr{X}] \\ & + \mathscr{Y} d_A * \Psi - \bar{\Phi}(d_A * d_A \Phi + [\Psi, *\Psi]) \Big] . \end{aligned} \tag{2.19}$$

We can integrate H out by the Gaussian integral or by setting $H = -\frac{1}{2} F^+$. The resulting action is δ_W-invariant if the $\mathscr{X}$ transformation is modified to $\delta \mathscr{X} = -\frac{1}{2} F^+$. In Witten's interpretation, the instanton cohomology groups are realized by the zero-modes of the fermionic variables $(\mathscr{Y}, \Psi, \mathscr{X})$,

$$d_A \mathscr{Y} = 0 , \qquad d_A^* \Psi = d_A^+ \Psi = 0 , \qquad d_A^{+*} \mathscr{X} = 0 . \tag{2.20}$$

For an example, the $\mathscr{X}$ and $\mathscr{Y}$ equations of motion

$$(d_A \Psi)^+ = 0 , \qquad d_A * \Psi = 0 , \tag{2.21}$$

show that a zero-mode of Ψ represents a tangent vector on the smooth part of instanton moduli space. The number of the non-trivial solutions of (2.20) for $(\mathscr{Y}, \Psi, \mathscr{X})$ are equal to (h_A^0, h_A^1, h_A^2). Note that the U numbers of $\mathscr{Y}$, Ψ and $\mathscr{X}$ are -1, 1 and -1 respectively. Thus, the formal dimension of $\mathscr{M}$ is just the number of fermionic zero-modes carrying $U = 1$ minus the number of fermionic zero-modes carrying $U = -1$. If there are no $\mathscr{Y}$ and $\mathscr{X}$ zero-modes, $\mathscr{M}$ is a smooth manifold with the actual dimension being equal to the number of Ψ zero-modes. The $\bar{\Phi}$ integral leads to a delta function constraint,

$$\Phi = -G_A [\Psi, *\Psi] , \tag{2.22}$$

which coincides with the universal bundle formalism (2.1).

Geometrically, observables of the TYMT correspond to the de Rham cohomology classes ${W_r}^{4-r}$ on $\mathscr{B}^*$ [11]. Correlation functions of observables can be formally written as

$$\begin{aligned} & \left\langle {W_{r_1}}^{4-r_1} \ldots {W_{r_k}}^{4-r_k} \right\rangle \\ & = \frac{1}{\operatorname{vol}(\mathscr{G})} \int (\mathscr{D}X) \exp(-S_W) {W_{r_1}}^{4-r_1} \ldots {W_{r_k}}^{4-r_k} . \end{aligned} \tag{2.23}$$

Due to the fermionic symmetry, the path integral of the TYMT is localized to an integral over the fixed point locus,

$$\delta_W \mathscr{X} = 0 \,, \qquad \delta_W \Psi = 0 \,, \tag{2.24}$$

which is the instanton moduli space and the space of Φ zero-modes, modulo gauge symmetry. The localization of the path integral to an integral over the fixed point locus is a general property of any cohomological field theory [22]. Equivalently, the semiclassical limit, which can be shown to be exact, of the path integral coincides with the above localization. If there are no reducible connections, the path integral reduces to an integral over the moduli space $\mathscr{M}^*$ of irreducible ASD connections. Provided that there are no $\mathscr{Y}$ and $\mathscr{X}$ zero-modes (that is, h^0_A and h^2_A vanish everywhere), the path integral (after integrating out non-zero modes) reduces to an integral of wedge products of closed differential forms on $\mathscr{M}^*$,

$$\langle W_{r_1}{}^{4-r_1} \cdots W_{r_k}{}^{4-r_k} \rangle = \int_{\mathscr{M}^*} \widehat{W}_{r_1}{}^{4-r_1} \wedge \cdots \wedge \widehat{W}_{r_k}{}^{4-r_k} \,. \tag{2.25}$$

This is the integral representation (2.15) of the Donaldson invariants [1]. The cohomology classes $\widehat{W}_{r_i}{}^{4-r_i}$ on $\mathscr{M}^*$ can be obtained from $W_{r_i}{}^{4-r_i}$ after replacing F by its instanton value, Ψ by its zero-modes and Φ by the zero-mode parts of its expectation value,

$$\langle \Phi \rangle = -\int_M G_A[\Psi, *\Psi] \,. \tag{2.26}$$

Clearly, the integral (2.25) will vanish unless the integrand is a top form on $\mathscr{M}^*$, $\sum_{i=1}^{k}(4 - r_i) = \dim(\mathscr{M}^*)$. Or, because the path integral measure has ghost number anomaly due to the Ψ zero-modes, Eq. (2.25) will vanish unless we insert an appropriate set of observables with net ghost number being equal to the number of Ψ zero-modes.

One can see that the noncompactness of the moduli space may make the topological interpretation of Eq. (2.25) problematic [2]. It is also unclear how the compactification procedure can be implanted in the field theoretical interpretation. However, we should emphasis here that the viewpoint adopted in this subsection is just a recipe to evaluate the correlation functions under favorable situations. It is important to note that the energy-momentum tensor of the TYMT is δ_W-exact,

$$T_{\alpha\beta} = \delta_W \lambda_{\alpha\beta} \,. \tag{2.27}$$

It follows that the correlation functions of the TYMT give the Donaldson invariants, which are valid regardless of whether the instanton moduli space exists and what properties it has [1]. In order to appreciate the real virtues of the TYMT, however, considerable progress in our understanding of quantum field theory may be required.

[1] See the four-steps (p. 381–383 in [1]) for the precise meanings

[2] This is briefly discussed in [23]

3. $N=2$ Topological Yang-Mills Theory

Let M be an n complex dimensional compact Kähler manifold endowed with a Kähler metric. Picking a complex structure on M together with the Kähler metric, one can determine the Kähler form ω. Let Ω^2 be the space of (real) two-forms on M. Using the complex structure and the Kähler form on M, one can decompose Ω^2 as

$$\Omega^2 = \Omega^2_+ \oplus \Omega^2_- \,, \tag{3.1}$$

where

$$\begin{aligned} \Omega^2_+ &= \Omega^{2,0} + \Omega^{0,2} + \Omega^{1,1}_\omega \,, \\ \Omega^2_- &= \Omega^{1,1}_\perp \,. \end{aligned} \tag{3.2}$$

We denote $\Omega^{1,1}_\omega$ to be the space of $(1,1)$-forms which is parallel to ω and $\Omega^{1,1}_\perp$ to be the orthogonal complement of $\Omega^{1,1}_\omega$. For an example, an element α of Ω^2_+ can be written as $\alpha = \alpha^{2,0} + \alpha^{0,2} + \alpha^0\omega$ where $\alpha^{0,2} = \overline{\alpha^{2,0}}$ and α^0 is a real zero-form. For $n=2$, the decompositions (3.1) and (3.2) are identical to the decompositions of the space of two-forms into the spaces of self-dual and anti-self two-forms. Let E be a vector bundle over M with reduction of the structure group to $SU(2)$. Let $\mathscr{A}$ be the space of all connections on E and $\mathscr{G}$ be the group of gauge transformations on E. We can introduce a moduli space $\mathscr{M}$ as the subspace of $\mathscr{B} = \mathscr{A}/\mathscr{G}$ cut by the following equations;

$$F^{2,0}(A) = 0\,, \qquad F^{0,2}(A) = 0\,, \qquad \Lambda F^{1,1}(A) = 0\,, \tag{3.3}$$

where Λ is the adjoint of the wedge multiplication by ω

$$\Lambda : \Omega^{p,q} \to \Omega^{p-1,q-1}\,. \tag{3.4}$$

In particular, the action of Λ on a $(1,1)$-form measures the component parallel to ω.

Behind the simple equation (3.3), there is a beautiful theorem of Donaldson-Uhlenbeck-Yau [24, 25], which generalize a theorem of Narasimhan-Seshadri [26] (see also Atiyah-Bott [27]), on the stable bundles: An irreducible holomorphic bundle $\mathscr{E}$ over a compact Kähler manifold is stable if and only if there is an unique unitary irreducible (Einstein-Hermitian) connection [28] on $\mathscr{E}$ with $\Lambda F^{1,1} = c \times I_{\mathscr{E}}$, where c is a topological invariant depending only on the cohomology classes of ω and $c_1(\mathscr{E})$ and $I_{\mathscr{E}}$ is the identity endomorphism. Note that each solution of Eq. (3.3) lies in the subspace $\mathscr{A}^{1,1}$, consisting of connections whose curvatures are type $(1,1)$, of $\mathscr{A}$. Thus, one can associate a holomorphic vector bundle $\mathscr{E}_A$ for each solution A. We can introduce the moduli space of holomorphic vector bundles as a set of isomorphism classes of holomorphic vector bundles. The moduli space can be identified with the quotient space $\mathscr{A}^{1,1}/\mathscr{G}^{\mathbb{C}}$, where $\mathscr{G}^{\mathbb{C}}$ denotes the complexification of $\mathscr{G}$. By the theorem of Donaldson-Uhlenbeck-Yau, one can identify $\mathscr{M}^*$ with the moduli space $\mathscr{M}^s_M \subset \mathscr{A}^{1,1}/\mathscr{G}^{\mathbb{C}}$ of stable holomorphic vector bundles [3]. For instance, the moduli space of irreducible ASD $SU(2)$ connections is isomorphic to the moduli space of stable $SL(2,\mathbb{C})$ bundles. The moduli space of irreducible flat $SU(2)$ connections over a compact oriented Riemann surface Σ is also isomorphic to the moduli space of stable $SL(2,\mathbb{C})$ bundles over Σ.

Though the material in Sect. 3.1 and in the later sections are generally valid (after slight modifications) on an arbitrary dimensional compact Kähler manifold, we restrict our attention to $n = 1, 2$ cases because we do not know the full details of the moduli spaces otherwise.

[3] See chapter 6 in ref. [7] for details

3.1 $N=2$ Algebra

It is straightforward to obtain the $N=2$ algebra from the basic δ_W algebra using the natural complex structure on $\mathscr{B}$ induced from M. Picking a complex structure J on M, one can introduce a complex structure $J_{\mathscr{A}}$ on $\mathscr{A}$,

$$J_{\mathscr{A}}\delta A = J\delta A\ , \qquad \delta A \in T\mathscr{A}\ , \tag{3.5}$$

by identifying $T^{1,0}\mathscr{A}$ and $T^{0,1}\mathscr{A}$ in $T\mathscr{A} = T^{1,0}\mathscr{A} \oplus T^{0,1}\mathscr{A}$ with the Ad(E) valued $(1,0)$-forms and $(0,1)$-forms on M respectively. One can also introduce a Kähler structure $\tilde{\omega}$ on $\mathscr{A}$ given by

$$\tilde{\omega} = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\,(\delta A \wedge \delta A) \wedge \omega^{n-1}\ . \tag{3.6}$$

Note that the complex structure $J_{\mathscr{A}}$ can descend to $\mathscr{A}/\mathscr{G}$ while the Kähler structure $\tilde{\omega}$ does not in general.

Now we turn back to the universal bundle in Sect. 2.1. Using the complex structures J and $J_{\mathscr{A}}$, we can decompose $A^{m,\ell}(M \times \mathscr{B}^*)$ into $A^{p,q,r,s}(M \times \mathscr{B}^*)$, for $m = p+q$, $\ell = r+s$. In terms of the form degree, one can decompose $\mathscr{F}^{1,1}$ as $\mathscr{F}^{1,1} = \mathscr{F}^{1,0,1,0} + \mathscr{F}^{0,1,1,0} + \mathscr{F}^{1,0,1,0} + \mathscr{F}^{1,0,0,1}$. The fact that $\mathscr{F}^{1,1} = \delta_H A$ (2.1) and the specific choice of the complex structure (3.5) lead to $\mathscr{F}^{1,1} = \mathscr{F}^{1,0,1,0} + \mathscr{F}^{0,1,0,1}$. To see this more explicitly, we introduce the following decompositions of the exterior covariant derivatives d_A and δ_H;

$$d_A = \partial_A + \bar{\partial}_A\ , \qquad \delta_H = \delta'_H + \delta''_H\ , \tag{3.7}$$

where

$$\begin{aligned}
\partial_A &: A^{p,q,r,s}(M \times \mathscr{B}^*) \to A^{p+1,q,r,s}(M \times \mathscr{B}^*)\ ,\\
\bar{\partial}_A &: A^{p,q,r,s}(M \times \mathscr{B}^*) \to A^{p,q+1,r,s}(M \times \mathscr{B}^*)\ ,\\
\delta'_H &: A^{p,q,r,s}(M \times \mathscr{B}^*) \to A^{p,q,r+1,s}(M \times \mathscr{B}^*)\ ,\\
\delta''_H &: A^{p,q,r,s}(M \times \mathscr{B}^*) \to A^{p,q,r,s+1}(M \times \mathscr{B}^*)\ .
\end{aligned} \tag{3.8}$$

Let $A = A' + A''$ be the decomposition of A into the $(1,0)$ part A' and the $(0,1)$ part A'', then we have $\delta_H A = \delta'_H A' + \delta''_H A''$. The curvature two-form $\mathscr{F}^{0,2}$ on $\mathscr{B}^*$ can be decomposed into its $(2,0)$, $(1,1)$, and $(0,2)$ components ϕ, φ and $\bar{\phi}$. According to our choice of complex structure [4], one can see that the only non-vanishing component is φ.

Now we can deduce the $N=2$ algebra from the decompositions of the basic δ_W algebra, Eq. (2.3), according to the above structures. Introducing the decompositions

[4] The Hodge star operator $*$ maps a r-form on M into $2n-r$-form

$$* : \Omega^r(M) \to \Omega^{2n-r}(M)\ ,$$

and $*^2 = 1$. Note that $*$ maps a (p,q)-form on M into $(n-q, n-p)$ form

$$* : \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)\ .$$

From the definition of $\mathscr{F}^{0,2}$ in Eq. (2.1), it follows that $\mathscr{F}^{0,2}$ is type $(1,1)$ on $\mathscr{B}^*$.

$\delta_W = \mathbf{s} + \bar{\mathbf{s}}$, $\Psi = \psi + \bar{\psi}$ and $\Phi = \varphi$, one can find that

$$\begin{aligned} &\mathbf{s}A' = \psi\,, && \mathbf{s}\psi = 0\,, && \\ &\bar{\mathbf{s}}A' = 0\,, && \bar{\mathbf{s}}\psi = -\partial_A\varphi\,, && \bar{\mathbf{s}}\varphi = 0\,, \\ &\mathbf{s}A'' = 0\,, && \mathbf{s}\bar{\psi} = -\bar{\partial}_A\varphi\,, && \mathbf{s}\varphi = 0\,. \\ &\bar{\mathbf{s}}A'' = \bar{\psi}\,, && \bar{\mathbf{s}}\bar{\psi} = 0\,. && \end{aligned} \tag{3.9}$$

We introduce another quantum number R which assigns values 1 and -1 to $\mathbf{s}$ and to $\bar{\mathbf{s}}$ respectively. Note that both $\mathbf{s}$ and $\bar{\mathbf{s}}$ increase the U number by 1 since they are the decompositions of δ_W. Geometrically, the fermionic operators $\mathbf{s}$ and $\bar{\mathbf{s}}$ correspond to δ'_H and δ''_H respectively. Then, a (p,q,r,s)-form on $M \times \mathscr{B}^*$ has the U number $r+s$ and the R number $r-s$.

One finds that $\mathbf{s}$ and $\bar{\mathbf{s}}$ are nilpotent and anti-commutative with each other up to a gauge transformation generated by φ

$$\mathbf{s}^2 = 0\,, \qquad (\mathbf{s}\bar{\mathbf{s}} + \bar{\mathbf{s}}\mathbf{s})A = -d_A\varphi\,, \qquad \bar{\mathbf{s}}^2 = 0\,. \tag{3.10}$$

Geometrically, the operators $\mathbf{s}$ and $\bar{\mathbf{s}}$ can be interpreted as the Dolbeault cohomology generators on $\mathscr{B}^*$, provided that they act on a gauge invariant quantity. An arbitrary gauge invariant quantity α in the form $\alpha = \mathbf{s}\bar{\mathbf{s}}\beta$ is invariant under the transformations generated by both $\mathbf{s}$ and $\bar{\mathbf{s}}$. An $\mathbf{s}$ and $\bar{\mathbf{s}}$-closed form α is automatically δ_W-closed;

$$\mathbf{s}\alpha = \bar{\mathbf{s}}\alpha = 0 \to (\mathbf{s} + \bar{\mathbf{s}})\alpha = \delta_W\alpha = 0\,. \tag{3.11}$$

On the other hand, the converse of the above relation,

$$\delta_W\alpha = 0 \longrightarrow \mathbf{s}\alpha = \bar{\mathbf{s}}\alpha = 0, \;\; or \;\; \mathbf{s}\alpha = 0 \longleftrightarrow \bar{\mathbf{s}}\alpha = 0\,, \tag{3.12}$$

is not generally valid, because the Kähler structure (3.6) on $\mathscr{A}$ does not descend to $\mathscr{B}$ in general. However, this is not a real problem because the configuration space of the TYMT is, due to the localization of the path integral, the moduli space of the ASD connection which has Kähler structure. Furthermore, the failure of Eq. (3.12) simply means the failure of the Hodge decomposition theorem [29];

$$H^r(\mathscr{B}^*) \neq \sum_{p+q=r} H^{p,q}(\mathscr{B}^*)\,. \tag{3.13}$$

Anyway, we can define the Dolbeault cohomology of $\mathscr{B}^*$ by the $\bar{\mathbf{s}}$ operator, which satisfies the Hodge decomposition after restriction to $\mathscr{M}^*$. In Sect. 4.1, we will see that there is an important set of elements of the Dolbeault cohomology of $\mathscr{B}^*$ which are both $\mathbf{s}$ and $\bar{\mathbf{s}}$ closed.

To construct an action, we need to introduce more fields to impose the anti-self-duality relative to the Kähler form,

$$F^{2,0} = 0\,, \qquad F^{0,2} = 0\,, \qquad \Lambda F^{1,1} = 0\,. \tag{3.14}$$

From the relations (3.10), one can establish the following general transformation rules for an anti-ghost $\mathscr{J}$ and its auxiliary fields $\mathscr{K}, \bar{\mathscr{K}}$;

$$\begin{aligned} &\mathbf{s}\mathscr{J} = \mathscr{K}\,, && \mathbf{s}\mathscr{K} = \mathbf{s}^2\mathscr{J} = 0\,, \\ &\bar{\mathbf{s}}\mathscr{J} = \bar{\mathscr{K}}\,, && \bar{\mathbf{s}}\bar{\mathscr{K}} = \bar{\mathbf{s}}^2\mathscr{J} = 0\,, \\ &(\mathbf{s}\bar{\mathbf{s}} + \bar{\mathbf{s}}\mathbf{s})\mathscr{J} = \mathbf{s}\bar{\mathscr{K}} + \bar{\mathbf{s}}\mathscr{K} = [\varphi, \mathscr{J}]\,. && \end{aligned} \tag{3.15}$$

We introduce a self-dual two-form B with $(U, R) = (-2, 0)$ which is the anti-ghost for the constraint (3.1). We also introduce the auxiliary fields $-\chi$ and $\bar{\chi}$ – self-dual two forms carrying $(U, R) = (-1, 1)$ and $(U, R) = (-1, -1)$ respectively. From (3.15), we have

$$\begin{aligned} &\mathbf{s}B = -\chi , \qquad \mathbf{s}\chi = 0 , \\ &\bar{\mathbf{s}}B = \bar{\chi} , \qquad \bar{\mathbf{s}}\bar{\chi} = 0 , \\ &\mathbf{s}\bar{\chi} - \bar{\mathbf{s}}\chi = [\varphi, B] . \end{aligned} \tag{3.16}$$

Due to the redundancy in the last relation, we can introduce one more auxiliary field which can not be uniquely determined. We choose $H \equiv \mathbf{s}\bar{\chi} - \frac{1}{2}[\varphi, B]$, a self-dual two form with $(U, R) = (0, 0)$. Then, one can find that

$$\begin{aligned} \mathbf{s}\bar{\chi} &= H + \tfrac{1}{2}[\varphi, B] , \qquad & \mathbf{s}H &= \tfrac{1}{2}[\varphi, \chi] , \\ \bar{\mathbf{s}}\chi &= H - \tfrac{1}{2}[\varphi, B] , \qquad & \bar{\mathbf{s}}H &= \tfrac{1}{2}[\varphi, \bar{\chi}] . \end{aligned} \tag{3.17}$$

If we interpret χ and $\bar{\chi}$ as the decompositions of $\mathscr{X}$ according to the R number, $\mathscr{X} \equiv (\chi + \bar{\chi})/2$, our choice leads to

$$(\mathbf{s} + \bar{\mathbf{s}})\mathscr{X} = H , \qquad (\mathbf{s} + \bar{\mathbf{s}})H = [\varphi, \mathscr{X}] , \tag{3.18}$$

which coincides with the $N = 1$ algebra (2.18). But the anti-ghost B and the commutator $\frac{1}{2}[\varphi, B]$ in (3.17) have no counterparts in the $N = 1$ algebra. The U and the R numbers of the all fields introduced in this subsection can be summarized by

$$\begin{array}{cccccccccc} \text{Fields} & A' & A'' & \psi & \bar{\psi} & \varphi & B & \chi & \bar{\chi} & H \\ U \text{ Number} & 0 & 0 & 1 & 1 & 2 & -2 & -1 & -1 & 0 \\ R \text{ Number} & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \end{array} . \tag{3.19}$$

3.2 $N = 2$ Actions

Let M be a compact Kähler surface with Kähler form ω. The action for the $N = 2$ TYM on M can be written as

$$S = \mathbf{s}\bar{\mathbf{s}}\, \mathbf{B_T} , \tag{3.20}$$

which is a natural $N = 2$ version of Witten's action (2.17). The unique choice of $\mathbf{B_T}$ with $(U, R) = (-2, 0)$, so that the action has $(U, R) = (0, 0)$, is

$$\mathbf{B_T} = \int_M \mathscr{B}_{\mathbf{T}} = -\frac{1}{h^2} \int_M \operatorname{Tr} \left[B \wedge *F + \chi \wedge *\bar{\chi} \right] . \tag{3.21}$$

Note that the self-dual two-forms $B, \chi, \bar{\chi}, H$ can be written as

$$\begin{aligned} B &= B^{2,0} + B^{0,2} + B^0\omega , \qquad & \chi &= \chi^{2,0} + \chi^{0,2} + \chi^0\omega , \\ H &= H^{2,0} + H^{0,2} + H^0\omega , \qquad & \bar{\chi} &= \bar{\chi}^{2,0} + \bar{\chi}^{0,2} + \bar{\chi}^0\omega , \end{aligned} \tag{3.22}$$

which give rise to

$$\begin{aligned} \mathbf{B_T} = -\frac{1}{h^2} \int_M \operatorname{Tr} \Big[& B^{2,0} \wedge *F^{0,2} + B^{0,2} \wedge *F^{2,0} + \chi^{2,0} \wedge *\bar{\chi}^{0,2} \\ & + \chi^{0,2} \wedge *\bar{\chi}^{2,0} + \left(B^0 f + \chi^0 \bar{\chi}^0 \right) \omega^2 \Big] , \end{aligned} \tag{3.23}$$

where $f \equiv \frac{1}{2}\hat{F} = \frac{1}{2}\Lambda F^{1,1}$. From (3.9), (3.16), (3.17), (3.20) and (3.23), we find that

$$\begin{aligned} S = &\frac{1}{h^2}\int_M \mathrm{Tr}\Big[-H^{2,0}\wedge *\left(H^{0,2}+F^{0,2}\right) - H^{0,2}\wedge *\left(H^{2,0}+F^{2,0}\right) - \chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} \\ &- \bar{\chi}^{0,2}\wedge *\partial_A\psi - [\varphi,\chi^{2,0}]\wedge *\bar{\chi}^{0,2} - [\varphi,\chi^{0,2}]\wedge *\bar{\chi}^{2,0} + \tfrac{1}{2}[\varphi,B^{2,0}]\wedge *F^{0,2} \\ &- \tfrac{1}{2}[\varphi,B^{0,2}]\wedge *F^{2,0} + \tfrac{1}{2}[\varphi,B^{2,0}]\wedge *[\varphi,B^{0,2}] - \Big(H^0\left(H^0+f\right) \\ &+ \tfrac{1}{2}\bar{\chi}^0\Lambda\bar{\partial}_A\psi + \tfrac{1}{2}\chi^0\Lambda\partial_A\bar{\psi} + [\varphi,\chi^0]\bar{\chi}^0 + \tfrac{1}{2}[\varphi,B^0]f - \tfrac{1}{4}[\varphi,B^0]^2 \\ &- \tfrac{1}{2}B^0\Lambda\left(\partial_A\bar{\partial}_A\varphi + [\psi,\bar{\psi}]\right)\Big)\omega^2\Big]\,, \end{aligned} \tag{3.24}$$

where we have used $\bar{\partial}_A\bar{\partial}_A\varphi = [F^{0,2},\varphi]$. We can integrate out $H^{2,0}$, $H^{0,2}$ and H^0 from the action by setting $H = -\frac{1}{2}F^+$ or by the Gaussian integral, which leads to a modified transformation law

$$\begin{aligned} &\mathbf{s}\bar{\chi}^{2,0} = -\tfrac{1}{2}F^{2,0} + \tfrac{1}{2}[\varphi,B^{2,0}]\,, \qquad &&\bar{\mathbf{s}}\chi^{2,0} = -\tfrac{1}{2}F^{2,0} - \tfrac{1}{2}[\varphi,B^{2,0}]\,, \\ &\mathbf{s}\bar{\chi}^{0,2} = -\tfrac{1}{2}F^{0,2} + \tfrac{1}{2}[\varphi,B^{0,2}]\,, &&\bar{\mathbf{s}}\chi^{0,2} = -\tfrac{1}{2}F^{0,2} - \tfrac{1}{2}[\varphi,B^{0,2}]\,, \\ &\mathbf{s}\bar{\chi}^{0} = -\tfrac{1}{2}f + \tfrac{1}{2}[\varphi,B^{0}]\,, &&\bar{\mathbf{s}}\chi^{0} = -\tfrac{1}{2}f - \tfrac{1}{2}[\varphi,B^{0}]\,. \end{aligned} \tag{3.25}$$

One can see that the locus of $\mathbf{s}$ and $\bar{\mathbf{s}}$ fixed points in the above transformations is precisely the space of ASD connections. Due to the localization principle of the cohomological field theories [22], the path integral reduces to the integral over the locus of the $\mathbf{s}$ and $\bar{\mathbf{s}}$ fixed points, $\mathbf{s}\chi = \bar{\mathbf{s}}\bar{\chi} = 0$, $\mathbf{s}\bar{\psi} = \bar{\mathbf{s}}\psi = 0$, which is the instanton moduli space with the space of φ zero-modes ($d_A\varphi = 0$).

Note that the terms $\mathrm{Tr}\left(-\frac{1}{2}[\varphi,B]\wedge *F + \frac{1}{4}[\varphi,B]\wedge *[\varphi,B]\right)$ in the action can be dropped without changing the theory, since: i) they do not change the fixed point locus; ii) the first term vanishes at the fixed point locus; iii) the $B^{2,0}$ and $B^{0,2}$ equations of motion are trivial algebraic ones; iv) they do not contribute to the B^0 equation of motion, provided that the theory localizes to the fixed point locus. The terms containing the commutator, $[\varphi,B]$, are trivial because the commutator is originated from the gauge degree of freedom in the anti-ghost multiplet, as we can see from Eq. (3.16). However, another type of gauge degree of freedom appearing in the basic transformation law (3.9) should be maintained since we are dealing with the equivariant cohomology. In our action functional, it gives a very important term, $\frac{1}{2}\mathrm{Tr}\,B^0\Lambda\left(\partial_A\bar{\partial}_A\varphi + [\psi,\bar{\psi}]\right)\omega^2$, which leads to the B^0 equation of motion,

$$i\bar{\partial}_A^*\bar{\partial}_A\varphi + \Lambda[\psi,\bar{\psi}] = 0\,, \tag{3.26}$$

where we have used the Kähler identities;

$$\partial_A^* = i[\Lambda,\bar{\partial}_A]\,, \qquad \bar{\partial}_A^* = -i[\Lambda,\partial_A]\,. \tag{3.27}$$

Now we can choose a simpler transformation law for the anti-ghost multiplet by dropping the $[\varphi, B]$ term in Eq. (3.16),

$$\begin{array}{llll} \mathbf{s}B = -\chi\,, & \mathbf{s}\chi = 0\,, & \mathbf{s}\bar{\chi} = H\,, & \mathbf{s}H = 0\,, \\ \bar{\mathbf{s}}B = \bar{\chi}\,, & \bar{\mathbf{s}}\bar{\chi} = 0\,, & \bar{\mathbf{s}}\chi = H\,, & \bar{\mathbf{s}}H = 0\,. \end{array} \tag{3.28}$$

The corresponding action is

$$\begin{aligned} S = & \frac{1}{h^2}\int\limits_M \mathrm{Tr}\Big[-H^{2,0}\wedge *\left(H^{0,2}+F^{0,2}\right) - H^{0,2}\wedge *\left(H^{2,0}+F^{2,0}\right) \\ & + B^{2,0}\wedge *[F^{2,0},\varphi] - \chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} - \bar{\chi}^{0,2}\wedge *\partial_A\psi \\ & - \Big(H^0\left(H^0+f\right) + \tfrac{1}{2}\bar{\chi}^0\Lambda\bar{\partial}_A\psi + \tfrac{1}{2}\chi^0\Lambda\partial_A\bar{\psi} \\ & - \tfrac{1}{2}B^0\Lambda\left(\partial_A\bar{\partial}_A\varphi + [\psi,\bar{\psi}]\right)\Big)\omega^2\Big]\,. \end{aligned} \tag{3.29}$$

One can easily check that this action is equivalent to the original one. We can also consider the simplest possible action of the $N = 2$ TYMT by discarding the $\mathrm{Tr}\,\chi\wedge *\bar{\chi}$ term from Eq. (3.21) and using the transformation law (3.28),

$$\begin{aligned} S = & \frac{1}{h^2}\int\limits_M \mathrm{Tr}\Big[-H^{2,0}\wedge *F^{0,2} - H^{0,2}\wedge *F^{2,0} + B^{2,0}\wedge *[F^{2,0},\varphi] - \chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} \\ & - \bar{\chi}^{0,2}\wedge *\partial_A\psi - \Big(H^0 f + \tfrac{1}{2}\bar{\chi}^0\Lambda\bar{\partial}_A\psi + \tfrac{1}{2}\chi^0\Lambda\partial_A\bar{\psi} \\ & - \tfrac{1}{2}B^0\Lambda\left(\partial_A\bar{\partial}_A\varphi + [\psi,\bar{\psi}]\right)\Big)\omega^2\Big]\,. \end{aligned} \tag{3.30}$$

The localization of this action to the moduli space of ASD connections is provided by the delta function gauges instead of the fixed point argument. It is a typical property of general cohomological field theory that there can be different realizations of the same theory.

3.3 Fermionic Zero-Modes

Now we will discuss the geometrical meaning of the zero-modes of the various fermionic fields $(\psi, \bar{\psi}, \chi^{2,0}, \bar{\chi}^{0,2}, \chi, \bar{\chi})$. Let $\mathscr{E}_A$ be a holomorphic structure induced by an ASD connection A. Let $\mathrm{End}^0(\mathscr{E}_A)$ be the trace-free endomorphism bundle of $\mathscr{E}_A$.

The $\chi^{2,0}$, $\bar{\chi}^{0,2}$, χ^0, and $\bar{\chi}^0$ equations of motion (up to gauge transformations) are

$$\bar{\partial}_A\bar{\psi} = 0\,, \qquad \partial_A\psi = 0\,, \qquad \Lambda\bar{\partial}_A\psi = 0\,, \qquad \Lambda\partial_A\bar{\psi} = 0\,. \tag{3.31}$$

Using the Kähler identities (3.27), one finds that a zero-mode of $\bar{\psi}$ becomes an element of the $(0,1)$-*th* twisted Dolbeault cohomology group of $\mathrm{End}^0(\mathscr{E}_A)$,

$$\mathbf{H}^{0,1} = \{\bar{\partial}_A\bar{\psi} = 0 \quad \text{and} \quad \bar{\partial}_A^*\bar{\psi} = 0\}\,. \tag{3.32}$$

The geometrical meaning of the zero-modes of ψ and $\bar{\psi}$, which are the non-trivial solutions of Eq. (3.31), can be easily understood as follows ([3] and Chap. 7.2 of [28]); for a given ASD connection $A = A' + A''$,

$$F^{2,0}(A') = 0\,, \qquad F^{0,2}(A'') = 0\,, \qquad \Lambda F^{1,1}(A', A'') = 0\,, \tag{3.33}$$

an infinitely close instanton solution $A + \delta A$ (after linearizations) should satisfy

$$\begin{aligned} F^{2,0}(A' + \delta A') &\sim \partial_A \delta A' = 0\,, \\ F^{0,2}(A'' + \delta A'') &\sim \bar{\partial}_A \delta A'' = 0\,, \\ \Lambda F^{1,1}(A' + \delta A', A'' + \delta A'') &\sim \Lambda(\bar{\partial}_A \delta A' + \partial_A \delta A'') = 0\,. \end{aligned} \tag{3.34}$$

In addition, we require δA to be orthogonal to the pure gauge variation,

$$d_A^* \delta A = 0\,, \tag{3.35}$$

so that a δA satisfying (3.34) and (3.35) represents a tangent vector at A on $\mathscr{M}$. Using the Kähler identities (3.6), one can rewrite (3.35) as

$$\Lambda(\bar{\partial}_A \delta A' - \partial_A \delta A'') = 0\,. \tag{3.36}$$

Combining (3.34) and (3.36), we can see that the zero-modes of ψ and $\bar{\psi}$ are holomorphic and anti-holomorphic tangent vectors, respectively, on the smooth part of $\mathscr{M}$. Thus, we have

$$H_A^1 \approx \mathbf{H}^{0,1}\,, \qquad h_A^1 = 2\mathbf{h}^{0,1}\,, \tag{3.37}$$

where $\mathbf{h}^{0,q} \equiv \dim_{\mathbb{C}} \mathbf{H}^{0,q}(\mathrm{End}^0(\mathscr{E}_A))$.

One can also find that the zero-modes of $\chi^{2,0}, \bar{\chi}^{0,2}, \chi^0$ and $\bar{\chi}^0$ satisfy the following equations (the ψ and $\bar{\psi}$ equations of motion);

$$\begin{aligned} \partial_A^* \chi^{2,0} + i\partial_A \chi^0 &= 0\,, \\ \bar{\partial}_A^* \bar{\chi}^{0,2} - i\bar{\partial}_A \bar{\chi}^0 &= 0\,, \end{aligned} \tag{3.38}$$

with obvious relations $\partial_A \chi^{2,0} = \bar{\partial}_A \bar{\chi}^{0,2} = 0$ coming from the dimensional reasoning. Note that a self-dual two form α satisfies [3]

$$d_A^{*+} \alpha = \partial_A^* \alpha^{2,0} + \bar{\partial}_A^* \alpha^{0,2} + i(\partial_A \alpha^0 - \bar{\partial}_A \alpha^0)\,. \tag{3.39}$$

If A is an ASD connection [5], we have

$$\left|d_A^{*+}\alpha\right|^2 = \left|\partial_A^* \alpha^{2,0}\right|^2 + \left|\bar{\partial}_A^* \alpha^{0,2}\right|^2 + \left|d_A \alpha^0\right|^2\,. \tag{3.40}$$

Thus, Eq. (3.38) reduces to

$$\partial_A^* \chi^{2,0} = \partial_A \chi^0 = 0\,, \qquad \bar{\partial}_A^* \bar{\chi}^{0,2} = \bar{\partial}_A \bar{\chi}^0 = 0\,. \tag{3.41}$$

In particular, a zero-mode of $\bar{\chi}^{0,2}$ is an element of the $(0,2)$-*th* twisted Dolbeault cohomology

$$\mathbf{H}^{0,2} = \{\bar{\partial}_A \bar{\chi}^{0,2} = 0 \quad \text{and} \quad \bar{\partial}_A^* \bar{\chi}^{0,2} = 0\}\,, \tag{3.42}$$

and a zero-mode of $\bar{\chi}^0$ is an element of H_A^0.

An important point is that the zero-mode of the fermionic field ψ and those of χ are always accompanied by their counterparts, i.e. the zero-modes of $\bar{\psi}$ and $\bar{\chi}$ respectively. Thus, the net violation of the R number by the fermionic zero-modes is

[5] Similar equation is also valid for an EH connection on a compact Kähler manifold

always zero, while the net violation of the U number by the fermionic zero-modes is equal to $2(\mathbf{h}^{0,1} - h^0_A - \mathbf{h}^{0,2})$. That is, the half of the net U number violation is identical to the number of $\bar{\psi}$ zero-modes minus the number of $\bar{\chi}^0$ zero-modes minus the number of $\bar{\chi}^{0,2}$ zero-modes.

The various fermionic zero-modes of the $N = 2$ TYMT naturally realize the cohomology group of the Atiyah-Hitchin-Singer and Itoh's instanton complex [3, 12][6],

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0(\mathrm{Ad}(E)) & \xrightarrow{d_A} & \Omega^1(\mathrm{Ad}(E)) & \xrightarrow{d_A^+} & \Omega^2_+(\mathrm{Ad}(E)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^{0,0}(\mathrm{End}^0(\mathscr{E}_A)) & \xrightarrow{\bar{\partial}_A} & \Omega^{0,1}(\mathrm{End}^0(\mathscr{E}_A)) & \xrightarrow{\bar{\partial}_A} & \Omega^{0,2}(\mathrm{End}^0(\mathscr{E}_A)) & \longrightarrow & 0\,. \end{array} \tag{3.43}$$

The first elliptic complex is the original Atiyah-Hitchin-Singer instanton complex (Eq. 2.8) and its cohomology H^q_A (with $h^q_A \equiv \dim_{\mathbb{R}} H^q_A)$) was given by Eq. (2.9). The second one is the twisted Dolbeault complex of $\mathrm{End}^0(\mathscr{E}_A)$ and its cohomology group was denoted by $\mathbf{H}^{0,q}$ (with $\mathbf{h}^{0,q} \equiv \dim_{\mathbb{C}} \mathbf{H}^{0,2})$). Then (Theorem 7.2.21 and Eq. (7.2.29) in [28]), we have

$$\begin{aligned} &H^0_A \otimes C \approx \mathbf{H}^{0,0}\,, \qquad && h^0_A = \mathbf{h}^{0,0}\,, \\ &H^1_A \approx \mathbf{H}^{0,1}\,, && h^1_A = 2\mathbf{h}^{0,1}\,, \\ &H^2_A \approx \mathbf{H}^{0,2} \oplus H^0_A\,, && h^2_A = 2\mathbf{h}^{0,2} + h^0_A\,. \end{aligned} \tag{3.44}$$

In particular,

$$\dim_{\mathbb{R}}(\mathscr{M}) = h^1_A - h^0_A - h^2_A = 2(\mathbf{h}^{0,1} - h^0_A - \mathbf{h}^{0,2})\,. \tag{3.45}$$

Thus the formal (real) dimension of the moduli space is identical to the net violation of the U number by the fermionic zero-modes, i.e. the U number anomaly. Note that there are no zero-modes except ψ and $\bar{\psi}$ pairs if and only if the cohomology group H^2_A is trivial. Then, the moduli space is a smooth Kähler manifold with complex dimension being equal to the number of $\bar{\psi}$ zero-modes. For example, if we consider a simply connected compact Kähler surface with a strictly positive geometric genus $p_g(M) > 0$. From the the formula $b^+(M) = 1+2p_g(M)$, we have $b^+(M) \geq 3$ and, then, it is known that there are no reducible instantons for a generic choice of metric. It is also known that for a large enough integer k, the second cohomology group vanishes at least over a dense subset of $\mathscr{M}^*$, or for a dense set of stable bundles $\mathscr{E}_A$ the cohomology group $\mathbf{H}^{0,2}$ vanishes [2]. Then, the generic part in the instanton moduli space is a smooth Kähler manifold with complex dimension, $\dim_{\mathbb{C}}(\mathscr{M}) = 4k - 3(1 + p_g(M))$.

Finally, we note that the zero-modes of bosonic variables A, φ, B^0 do not contribute to the U number violation in the path integral measure. This is because the numbers of φ and B^0 zero-modes are the same, each arising from a reducible connection, while they carry the same U numbers up to sign. Thus, the U number anomaly is an obstruction to having a well defined path integral measure, which should be absorbed by inserting appropriate observables [1].

6 The generalization to the moduli space of EH connections on a compact Kähler manifold was studied by Kim [30]. See also Chap. 7 of [28].

4. Observables and Correlation Functions

4.1 Observables

From an obvious s and $\bar{\mathrm{s}}$ invariant $\mathscr{W}_{0,0}{}^{2,2} = \frac{1}{2}\mathrm{Tr}\,\varphi^2$, which are gauge invariant and metric independent[7], one can find the following topological descent equations;

$$\mathrm{s}\mathscr{W}_{p,q}{}^{r,s} + \bar{\mathrm{s}}\mathscr{W}_{p,q}{}^{r+1,s-1} + \partial\mathscr{W}_{p-1,q}{}^{r+1,s} + \bar{\partial}\mathscr{W}_{p,q-1}{}^{r+1,s} = 0\,, \tag{4.1}$$

where we generally denote $\mathscr{W}_{\alpha,\beta}{}^{\rho,\delta}$ as an α,β-form on M and (ρ,δ) form of $\mathscr{B}$ and all the indices are positive (that is, $\mathscr{W}_{\alpha,\beta}{}^{\rho,\delta}$ with negative indices vanish in the above equation), $\alpha+\beta+\rho+\delta = 4$ and $\alpha,\beta,\rho,\delta = 0,1,2$. Explicitly,

$$\begin{aligned}
&\mathscr{W}_{0,0}{}^{2,2} = \tfrac{1}{2}\mathrm{Tr}\,(\varphi^2)\,, && \mathscr{W}_{1,1}{}^{1,1} = \mathrm{Tr}\,(\varphi F^{1,1} + \psi\wedge\bar{\psi})\,,\\
&\mathscr{W}_{1,0}{}^{2,1} = \mathrm{Tr}\,(\psi\varphi)\,, && \mathscr{W}_{2,1}{}^{0,1} = \mathrm{Tr}\,(\bar{\psi}\wedge F^{2,0})\,,\\
&\mathscr{W}_{0,1}{}^{1,2} = \mathrm{Tr}\,(\bar{\psi}\varphi)\,, && \mathscr{W}_{2,1}{}^{1,0} = \mathrm{Tr}\,(\psi\wedge F^{1,1})\,,\\
&\mathscr{W}_{2,0}{}^{2,0} = \tfrac{1}{2}\mathrm{Tr}\,(\psi\wedge\psi)\,, && \mathscr{W}_{1,2}{}^{0,1} = \mathrm{Tr}\,(\bar{\psi}\wedge F^{1,1})\,,\\
&\mathscr{W}_{2,0}{}^{1,1} = \mathrm{Tr}\,(\varphi F^{2,0})\,, && \mathscr{W}_{1,2}{}^{1,0} = \mathrm{Tr}\,(\psi\wedge F^{0,2})\,,\\
&\mathscr{W}_{0,2}{}^{0,2} = \tfrac{1}{2}\mathrm{Tr}\,(\bar{\psi}\wedge\bar{\psi})\,, && \mathscr{W}_{2,2}{}^{0,0} = \mathrm{Tr}\,(F^{0,2}\wedge F^{2,0} + \tfrac{1}{2}F^{1,1}\wedge F^{1,1})\,,\\
&\mathscr{W}_{0,2}{}^{1,1} = \mathrm{Tr}\,(\varphi F^{0,2})\,,
\end{aligned} \tag{4.2}$$

and all the other components vanish. Note that Eqs. (4.1) and (4.2) can be combined into a single identity,

$$(\partial + \bar{\partial} + \mathrm{s} + \bar{\mathrm{s}})\frac{1}{2}\mathrm{Tr}\,(F^{2,0} + F^{0,2} + F^{1,1} + \psi + \bar{\psi} + \varphi)^2 = 0\,. \tag{4.3}$$

Let

$$W_{\alpha,\beta}{}^{\rho,\delta} \equiv \frac{1}{4\pi^2}\int_M \mathscr{W}_{\alpha,\beta}{}^{\rho,\delta}\wedge O_{2-\alpha,2-\beta}\,, \tag{4.4}$$

where $O_{2-\alpha,2-\beta} \in H^{2-\alpha,2-\beta}(M)$ denotes a harmonic $(2-\alpha,2-\beta)$-form on M. Equation (4.1) gives

$$\mathrm{s}W_{p,q}{}^{r,s} + \bar{\mathrm{s}}W_{p,q}{}^{r+1,s-1} = 0\,, \tag{4.5}$$

where we have used the condition that M is a compact Kähler surface. Because $\mathscr{B}$, in general, do not have Kähler structure, Eq. (4.5) does not imply that every $W_{\alpha,\beta}{}^{\rho,\delta}$ is both s and $\bar{\mathrm{s}}$ closed. For an example,

$$\mathrm{s}W_{2,1}{}^{0,1} = -\frac{1}{4\pi^2}\int_M \mathrm{Tr}\,(\bar{\partial}_A\varphi\wedge F^{2,0} + \bar{\psi}\wedge\partial_A\psi)\wedge O_{0,1} \neq 0\,. \tag{4.6}$$

If we restrict $W_{p,q}{}^{r,s}$ to $\mathscr{M}^*$, all of them are both s and $\bar{\mathrm{s}}$ closed due to the Kähler structure on $\mathscr{M}^*$. In quantum field theory, however, it is unnatural to use a quantity which is invariant only on-shell as an observable, though it is not entirely impossible.

[7] Our particular choice is that our main interest in this paper is the $SU(2)$ invariants [1]. We can consider a general invariant polynomial in φ obeying all the criterions for a topological observable. And, the basic results discussed below can be applied without modifications

We can find a set of well defined topological observables,

$$
\begin{aligned}
W_{0,0}{}^{2,2} &= \frac{1}{8\pi^2}\int_M \operatorname{Tr}(\varphi^2) O_{2,2}\,, \\
W_{1,0}{}^{2,1} &= \frac{1}{4\pi^2}\int_M \operatorname{Tr}(\psi\varphi)\wedge O_{1,2}\,, \\
W_{0,1}{}^{1,2} &= \frac{1}{4\pi^2}\int_M \operatorname{Tr}(\bar{\psi}\varphi)\wedge O_{2,1}\,, \\
W_{1,1}{}^{1,1} &= \frac{1}{4\pi^2}\int_M \operatorname{Tr}(\varphi F^{1,1} + \psi\wedge\bar{\psi})\wedge O_{1,1}\,, \\
W_{2,2}{}^{0,0} &= \frac{1}{4\pi^2}\int_M \operatorname{Tr}(F^{0,2}\wedge F^{2,0} + \tfrac{1}{2}F^{1,1}\wedge F^{1,1})\,.
\end{aligned}
\tag{4.7}
$$

Using the Bianchi identity, $d_A F_A = 0$, and integration by parts, one can see that they are both $\mathbf{s}$ and $\bar{\mathbf{s}}$ closed. Geometrically, an observable $W_{p,q}{}^{r,s}$ is an element of the (r,s)-*th* Dolbeault cohomology group on $\mathscr{B}^*$, $W_{p,q}{}^{r,s} \in H^{r,s}(\mathscr{B}^*)$, which depends only on the cohomology class of $O_{2-p,2-q}$ in M. Finally, we note interesting descent equations among $\mathscr{W}_{0,0}{}^{2,2}, \mathscr{W}_{1,1}{}^{1,1}$ and $\mathscr{W}_{2,2}{}^{0,0}$,

$$
\begin{aligned}
\mathbf{s}\bar{\mathbf{s}}\operatorname{Tr}(\varphi F^{1,1} + \psi\wedge\bar{\psi}) &= -\partial\bar{\partial}\,\tfrac{1}{2}tr(\varphi^2)\,, \\
\mathbf{s}\bar{\mathbf{s}}\operatorname{Tr}(F^{0,2}\wedge F^{2,0} + \tfrac{1}{2}F^{1,1}\wedge F^{1,1}) &= -\partial\bar{\partial}\operatorname{Tr}(\varphi F^{1,1} + \psi\wedge\bar{\psi})\,.
\end{aligned}
\tag{4.8}
$$

4.2 Correlation Functions

In principle, we can give formulas for the Donaldson invariants on an arbitrary compact Kähler surface for an arbitrary compact group. Following the manipulations developed by Witten, we can formally show that the correlation functions of the $N = 2$ TYMT are topological invariants in general circumstances. It is crucial to prove that the energy-momentum tensor $T_{\alpha\bar{\beta}}$ defined by the variation of the action under an infinitesimal change of the Kähler metric $g^{\alpha\bar{\beta}} \to g^{\alpha\bar{\beta}} + \delta g^{\alpha\bar{\beta}}$,

$$
\delta S = \frac{1}{2}\int_M \sqrt{g}\,\delta g^{\alpha\bar{\beta}}\, T_{\alpha\bar{\beta}}\,,
\tag{4.9}
$$

is a $\mathbf{s}$ and $\bar{\mathbf{s}}$ exact form

$$
T_{\alpha\bar{\beta}} = \mathbf{s}\bar{\mathbf{s}}\lambda_{\alpha\bar{\beta}}\,.
\tag{4.10}
$$

The above relation is an immediate consequence of (3.20), if the variation operator $\delta/\delta g^{\alpha\bar{\beta}}$ commutes to both $\mathbf{s}$ and $\bar{\mathbf{s}}$ off shell. The only subtlety here is that the various anti-ghost and auxiliary fields $B, \chi, \bar{\chi}, H$ are subject to the self-duality constraint (3.22) which must be preserved after the deformation of the metric [1]. Clearly, the variations of $B, \chi, \bar{\chi}, H$ come only from their $(1,1)$ components, which should remain parallel to the new Kähler form associated with the deformed Kähler metric [6]. Thus, an arbitrary change $\delta g^{\alpha\bar{\beta}}$ in the Kähler metric must be accompanied by

$$
g^{\alpha\bar{\beta}}\delta H_{\alpha\bar{\beta}} = -H_{\alpha\bar{\beta}}\delta g^{\alpha\bar{\beta}}\,.
\tag{4.11}
$$

We have the same relations for $\chi_{\alpha\bar{\beta}}$, $\bar{\chi}_{\alpha\bar{\beta}}$ and $B_{\alpha\bar{\beta}}$. One can easily see that $\delta/\delta g^{\alpha\bar{\beta}}$ commutes to $\mathbf{s}$ and $\bar{\mathbf{s}}$ off shell. Consequently, we have

$$T^{\alpha\bar{\beta}} = \mathbf{s}\bar{\mathbf{s}}\lambda_{\alpha\bar{\beta}}\,, \qquad \lambda_{\alpha\bar{\beta}} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\alpha\bar{\beta}}}\mathscr{B}_{\mathrm{T}}\,. \tag{4.12}$$

Now it is an easy exercise to show that the correlation functions of the observables are topological invariants by following essentially the same manipulation as given in Sect. 3 of [1]. Here, we will only note the ghost number anomaly. Recall that the action (3.24) has two global symmetries generated by U and R charges at the classical level. If the formal dimension of the moduli space is non-zero, we have an U number anomaly because the net U ghost number violation $\triangle U$ coincides with the formal dimension. Let $\left\{W_{p_i,q_i}{}^{r_i,s_i}\right\}$ be the general topological observables which can be constructed from $\mathscr{W}_{0,0}{}^{2k,2k} = \frac{1}{2}\mathrm{Tr}\,\varphi^{2k}$. Then, a correlation function

$$\left\langle\prod_{i=1}^{\ell} W_{p_i,q_i}{}^{r_i,s_i}\right\rangle = \frac{1}{\mathrm{vol}(\mathscr{G})}\int(\mathscr{D}X)\exp(-S)\cdot\prod_{i=1}^{\ell} W_{p_i,q_i}{}^{r_i,s_i}\,, \tag{4.13}$$

vanishes unless $\prod_{i=1}^{\ell} W_{p_i,q_i}{}^{r_i,s_i}$ carries the U number $\triangle U$ and the R number zero. Thus, we have the following superselection rule;

$$\dim_{\mathbb{R}}(\mathscr{M}) = \triangle U = \sum_{i=1}^{\ell}(r_i + s_i)\,, \qquad \sum_{i=1}^{\ell}(r_i - s_i) = 0\,, \tag{4.14}$$

which is equivalent to the condition $\dim_{\mathbb{C}}(\mathscr{M}) = \sum_{i=1}^{\ell} r_i = \sum_{i=1}^{\ell} s_i$.

4.3 Differential Forms on Moduli Space

Throughout this sub-section, we assume that there are the zero-modes of $(\psi,\bar{\psi})$ pairs only. In this case, the path integral localizes to the moduli space $\mathscr{M}^*$ of irreducible ASD connections which is a finite dimensional smooth Kähler manifold. Let $\dim_{\mathbb{C}}(\mathscr{M}^*) \equiv d$, the number of the $(\psi,\bar{\psi})$ zero-modes pairs. The basic result of the $N=1$ TYMT is that the expectation value of some products of observables reduces to an integral of wedge products of the de Rham cohomology classes on $\mathscr{M}^*$ over $\mathscr{M}^*$ [1]. In the $N=2$ TYMT theory, the path integral will reduce to an integral of wedge products of the Dolbeault cohomology classes on $\mathscr{M}^*$.

We can obtain an element $\widehat{W}_{p,q}{}^{r,s} \in H^{r,s}(\mathscr{M}^*)$ of Dolbeault cohomology group on $\mathscr{M}^*$ from $W_{p,q}{}^{r,s}$ after restricting to $\mathscr{M}^*$. Clearly, this restriction is provided by the localization of the path integral. In the exact semiclassical limit, a correlation function

$$\left\langle\prod_{i=1}^{n} W_{p_i,q_i}{}^{r_i,s_i}\right\rangle = \frac{1}{\mathrm{vol}(\mathscr{G})}\int(\mathscr{D}X)\exp(-S)\cdot\prod_{i=1}^{n} W_{p_i,q_i}{}^{r_i,s_i} \tag{4.15}$$

reduces to an integral of wedge products of the elements of $H^{r_i,s_i}(\mathscr{M}^*)$,

$$\left\langle\prod_{i=1}^{n} W_{p_i,q_i}{}^{r_i,s_i}\right\rangle = \int_{\mathscr{M}^*}\widehat{W}_{p_1,q_1}{}^{r_1,s_1}\wedge\cdots\wedge\widehat{W}_{p_n,q_n}{}^{r_n,s_n}\,, \tag{4.16}$$

after integrating all non-zero modes out. Clearly, the correlation function (4.16) vanishes unless the integrand is a top form, a (d, d)-form, on $\mathscr{M}^*$,

$$\sum_{i=1}^{n} (r_i, s_i) = (d, d) . \tag{4.17}$$

This condition coincides with the superselection rule due to the ghost number anomalies

$$\triangle R = \sum_{i=1}^{n} (r_i + s_i) = 2d , \qquad \triangle U = \sum_{i=1}^{n} (r_i - s_i) = 0 . \tag{4.18}$$

Explicitly, $\widehat{W_{p,q}}^{r,s}$ can be obtained from $W_{p,q}{}^{r,s}$ by replacing

$$\begin{aligned} &\bullet\ F \text{ with its instanton value ,} \\ &\bullet\ \psi \text{ and } \bar{\psi} \text{ with their zero-modes ,} \\ &\bullet\ \varphi \text{ with the zero-mode parts of } <\varphi>= \int_M d\mu \, \frac{i}{\delta_A^* \delta_A} \Lambda[\psi, \bar{\psi}] , \end{aligned} \tag{4.19}$$

where the last relation resulted from the B^0 equation of motion (3.26).

We note that the smooth part of the moduli space of ASD connections is a Kähler manifold. This was first proved by Itoh using a direct but difficult calculation [3]. His result is a generalization of a theorem of Atiyah and Bott on the Kähler structure of the moduli space of flat connections on Riemann surfaces [27]. It was futher generalized to the moduli space of EH connections on an arbitrary dimensional compact Kähler manifold by Kobayashi. Let M be an n complex dimensional compact Kähler manifold. The Kähler structure on the smooth part of the moduli space of EH connections can be proved very compactly by the symplectic (Mardsen-Weinstein) reduction (Chap. 7.6 in [28] and Chap. 6.5.1–3 in [7]). First, we restrict $\mathscr{A}$ to the subspace $\mathscr{A}^{1,1} \subset \mathscr{A}$ consisting of connections having curvature of type $(1,1)$. The subspace $\mathscr{A}^{1,1}$ is preserved by the action of $\mathscr{G}$ (as well as by the action of $\mathscr{G}^{\mathbb{C}}$) and its smooth part has the Kähler structure given by the restriction of (3.6). Let $Lie(\mathscr{G})$ be the Lie algebra of $\mathscr{G}$, which can be identified with the space of $\mathrm{Ad}(E)$-valued zero-forms. Then, we have a moment map $\mathfrak{m} : \mathscr{A}^{1,1} \to \Omega^0(\mathrm{Ad}(E))^*$

$$\mathfrak{m}(A) = -\frac{1}{4\pi^2} F_A^{1,1} \wedge \omega^{n-1} = -\frac{1}{4\pi^2} f \omega^n , \tag{4.20}$$

where $\Omega^0(\mathrm{Ad}(E))^* = \Omega^{2n}(\mathrm{Ad}(E))$ denotes the dual of $\Omega^0(\mathrm{Ad}(E))$. The reduced phase space (or the symplectic quotient) $\mathfrak{m}^{-1}(0)/\mathscr{G}$ can be identified with the moduli space $\mathscr{M}$ of EH connections. Then the smooth part of the reduced phase space has Kähler structure descended from $\mathscr{A}^{1,1}$ by the reduction theorem of Mardsden and Weinstein. We will also denote the Kähler structure on $\mathscr{M}$ by $\tilde{\omega}$. One can also consider the semi-stable set $\mathscr{A}^{1,1}_{ss} \subset \mathscr{A}^{1,1}$ and its ordinary (complex) quotient space $\mathscr{A}^{1,1}_{ss}/\mathscr{G}^{\mathbb{C}} \equiv \mathscr{M}^{ss}_M$ which is isomorphic to the symplectic quotient $\mathfrak{m}^{-1}(0)/\mathscr{G} = \mathscr{M}$. Then, the smooth part of $\mathscr{M}^{ss}_M$ (which is identical to the smooth part of $\mathscr{M}^{s}_M$) has the Kähler structure.

Let M be a compact Kähler surface. The second cohomology $H^2(M)$ is always non-trivial due to the cohomology class represented by the Kähler form ω. Thus, we have a non-trivial observable,

$$W_{1,1}{}^{1,1} = \frac{1}{4\pi^2} \int_M \mathrm{Tr}\,(\varphi F^{1,1} + \psi \wedge \bar{\psi}) \wedge \omega . \tag{4.21}$$

One can see that $\widehat{W}_{1,1}{}^{1,1}$ is the Kähler form $\tilde{\omega}$ on $\mathscr{M}^*$ descended symplectically from (3.6), since $F^{1,1} \wedge \omega$ vanishes for an instanton and the ψ and $\bar{\psi}$ zero-modes represent the holomorphic and the anti-holomorphic tangent vectors on $\mathscr{M}^*$. And, $\widehat{W}_{1,1}{}^{1,1}$ defines Donaldson's μ-map,

$$\begin{aligned} \mu(\omega) : H^{1,1}(M) &\to H^{1,1}(\mathscr{M}^*)\,, \\ \mu(\mathrm{PD}[\omega]) : H_{1,1}(M) &\to H^{1,1}(\mathscr{M}^*)\,, \end{aligned} \tag{4.22}$$

where PD$[\omega]$ denotes the homology class which is Poincaré dual to ω. Thus, we have

$$\left\langle \exp\left(\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\varphi F^{1,1} + \psi \wedge \bar{\psi}\right) \wedge \omega\right)\right\rangle = \int_{\mathscr{M}^*} \frac{\tilde{\omega}^d}{d!} = \mathrm{vol}(\mathscr{M}^*) > 0\,, \tag{4.23}$$

provided that $\mathscr{M}^*$ is a non-empty compact manifold. This argument is a field theoretical interpretation of Donaldson's differential geometric approach (Theorem 4.1 as explained in p. 294–295 of [2]). In practice, however, the moduli space is rarely compact and it is hard to avoid the singularities. These facts are both physically and mathematically the main obstacle to understanding Donaldson theory more fully [23]. For simply connected algebraic surfaces under some suitable conditions, Donaldson proved the positivity of the above invariant (Theorem 4.1 in [2]) using the algebro-geometrical approach. We will sketch his method in Sect. 5.2.

5. Other Approaches

We have studied a natural field theoretical interpretation of the Donaldson invariants on a compact Kähler surface. However, it is not clear whether our field theoretical method in general may give any new insight in calculating the invariants explicitly. We have seen that our formalism reduces to the differential-geometrical approach of Donaldson. On the other hand, Donaldson obtained his positivity theorem by an independent algebro-geometrical approach which is more suitable in the realistic problems [2]. His method is closely related to certain properties of the moduli space $\mathscr{M}^s_\Sigma$ of stable bundles over a compact Riemann surface Σ. In the language of stable bundles, it is very natural to consider an $N = 2$ TYMT on Σ which leads to a field theoretical interpretation of the intersection pairings on $\mathscr{M}^s_\Sigma$. Moreover, Witten obtained explicit expressions[8], which is general enough to include an arbitrary compact group and the reducible connections, of the general intersection pairings on the moduli space of flat connections on a Riemann surface [4, 5]. It should be emphasized that his solutions are based on field theoretical methods. On the other hand, the original field theoretical interpretation of the Donaldson's invariants has not produced any concrete result. The main purpose of this final section is to learn something useful to enhance the computability of the Donaldson invariants from Witten's solutions in two dimensions. Although the TYMT in two dimensions was completely solved by Witten, I will construct an $N = 2$ version of the theory to suggest a method, analogous to Witten's solutions in two dimensions, which may be useful in the calculation of the Donaldson invariants on compact Kähler surfaces.

[8] Some similar results were obtained by various mathematicians with different methods [31]

5.1 $N = 2$ TYMT on Compact Riemann Surfaces

It is straightforward to obtain a two dimensional version of the $N = 2$ TYMT or an $N = 2$ version of Witten's TYMT on Riemann surfaces (Sect. 3 of [5]). Let Σ be a compact oriented Riemann surface. Picking a complex structure J on Σ with an arbitrary metric, which is always a Kähler metric, we can determine a Kähler form ω. By the dimensional reasoning, every curvature two-form is type $(1,1)$ and parallel to ω,

$$F = f\omega \, . \tag{5.1}$$

Thus, the constraint (3.3) is satisfied if F is flat. In this sense a flat connection on Riemann surface can be regarded as the two dimensional analogue of the anti-self duality relative to Kähler form in compact Kähler surfaces. Similarly, the self-dual two-forms $B, \chi, \bar{\chi}, H$ reduce to

$$B = B^0\omega \, , \qquad \chi = \chi^0\omega \, , \qquad \bar{\chi} = \bar{\chi}^0\omega \, , \qquad H = H^0\omega \, . \tag{5.2}$$

The action of the two dimensional version of the $N = 2$ TYMT

$$S = \mathbf{s}\bar{\mathbf{s}}\mathbf{B}_{\mathbf{T}} \, , \tag{5.3}$$

can be obtained from the same form of $\mathbf{B}_{\mathbf{T}}$ given by Eq. (3.21)

$$\mathbf{B}_{\mathbf{T}} = -\frac{1}{h^2}\int_{\Sigma} \mathrm{Tr}\left[\left(B^0 f + \chi^0\bar{\chi}^0\right)\right]\omega \, . \tag{5.4}$$

The action is

$$\begin{aligned}\mathbf{S} = -\frac{1}{h^2}\int_{\Sigma} \mathrm{Tr}\Big[& H^0(H^0 + f) + [\varphi, \chi^0]\bar{\chi}^0 + \bar{\chi}^0 \Lambda \bar{\partial}_A \psi + \chi^0 \Lambda \partial_A \bar{\psi} \\ & - B^0 \Lambda \left(\partial_A \bar{\partial}_A \varphi + [\psi, \bar{\psi}]\right) + \tfrac{1}{2}[\varphi, B^0] f - \tfrac{1}{4}[\varphi, B^0][\varphi, B^0]) \Big]\omega \, . \end{aligned} \tag{5.5}$$

We can integrate H^0 out from the action by setting $H^0 = -\frac{1}{2}f$ or by the Gaussian integral, which leads to the modified transformation

$$\mathbf{s}\bar{\chi}^0 = -\tfrac{1}{2}f + \tfrac{1}{2}[\varphi, B^0] \, , \qquad \bar{\mathbf{s}}\chi^0 = -\tfrac{1}{2}f - \tfrac{1}{2}[\varphi, B^0] \, . \tag{5.6}$$

Thus, the fixed point locus of $\mathbf{s}$ and $\bar{\mathbf{s}}$ is the moduli space flat connections with the space of φ zero-modes, modulo gauge symmetry. The χ^0 and $\bar{\chi}^0$ equations of motion,

$$\Lambda \bar{\partial}_A \psi = 0 \, , \qquad \Lambda \partial_A \bar{\psi} = 0 \, , \tag{5.7}$$

together with the Kähler identities, show that the zero-modes of ψ and $\bar{\psi}$ are holomorphic and anti-holomorphic tangent vectors on the moduli space $\mathscr{M}_f$ of flat connections on Σ. If there is no reducible connection, the moduli space is a smooth Kähler manifold with complex dimension being equal to the number of the $\bar{\psi}$ zero-modes. And the path integral reduces an integral over the moduli space $\mathscr{M}_f^*$ of irreducible flat connections.

It is also straightforward to construct observables. Starting from $\mathscr{W}_{0,0}{}^{2,2} = \frac{1}{2}\mathrm{Tr}\,\varphi^2$, we can obtain essentially the same relations as in Sect. 4. The differences are that $\mathscr{W}_{p,q}{}^{r,s}$ is restricted to $p, q = 0, 1$, $r, s = 0, 1, 2$, and F should be replaced by zero in the reduction of ${W_{p,q}}^{r,s}$ to ${\widehat{W}_{p,q}}^{r,s}$. Note that the Hodge numbers on a compact oriented Riemann surface with genus g are given by

$$h^{0,0} = h^{1,1} = 1 , \qquad h^{1,0} = h^{0,1} = g . \tag{5.8}$$

In particular, the second cohomology group of Σ is represented by the Kähler form ω. Then, the complete set of relevant observables is

$$\begin{aligned}
{W_{0,0}}^{2,2} &= \frac{1}{8\pi^2} \int\limits_{\Sigma} \operatorname{Tr}(\varphi^2)\omega , \\
W_{1,0}^{(g)2,1} &= \frac{1}{4\pi^2} \int\limits_{\Sigma} \operatorname{Tr}(\psi\varphi) \wedge O_{0,1}^{(g)} , \\
W_{0,1}^{(g)1,2} &= \frac{1}{4\pi^2} \int\limits_{\Sigma} \operatorname{Tr}(\bar{\psi}\varphi) \wedge O_{1,0}^{(g)} , \\
{W_{1,1}}^{1,1} &= \frac{1}{4\pi^2} \int\limits_{\Sigma} \operatorname{Tr}(\varphi F + \psi \wedge \bar{\psi}) .
\end{aligned} \tag{5.9}$$

The observable

$$\frac{1}{4\pi^2} \int\limits_{\Sigma} \operatorname{Tr}\left((\varphi F + \psi \wedge \bar{\psi}\right) \tag{5.10}$$

reduces to the Kähler form $\tilde{\omega}_{\Sigma}$ on moduli space of flat connections [27], after replacing F by zero and $\psi, \bar{\psi}$ by their zero-modes. And, it gives the two-dimensional version of Donaldson's map

$$\begin{aligned}
\mu(1) &: H^{0,0}(\Sigma) \to H^{1,1}(\mathscr{M}_f^*) , \\
\mu(\mathrm{PD}[1]) &: H_{1,1}(\Sigma) \to H^{1,1}(\mathscr{M}_f^*) ,
\end{aligned} \tag{5.11}$$

Thus, Donaldson's map actually induces the Kähler structure $\tilde{\omega}_{\Sigma}$ on the moduli space $\mathscr{M}_f^*$. Consequently, the correlation function,

$$\left\langle \exp\left(\frac{1}{4\pi^2} \int\limits_{\Sigma} \operatorname{Tr}\left(\varphi F + \psi \wedge \bar{\psi}\right) \right) \right\rangle = \int\limits_{\mathscr{M}_f^*} \frac{\tilde{\omega}_{\Sigma}^d}{d!} = \operatorname{vol}(\mathscr{M}_f^*) , \tag{5.12}$$

reduces to the symplectic volume of $\mathscr{M}_f^*$, provided that the moduli space is compact. This is analogous to Eq. (4.23).

The intersection pairing corresponding to (5.12) was explicitly calculated by Witten [4] by adopting various methods: i) zero coupling limit of physical Yang-Mills theory; ii) Verlinde formula originated from the conformal field theory; iii) relating combinatorial treatment of physical Yang-Mills theory to the theory of Reidemeister-Ray-Singer torsion. Note that the first and third methods of Witten are related to physical Yang-Mills theory in two dimensions, which also leads to a complete formalism[9] for the general intersection pairings [5]. On the other hand, Verlinde's formula is an algebro-geometrical formalism although it has a physical origin [32].

[9] Witten obtained a truly general expression which is valid regardless of what properties the moduli has, for an arbitrary Riemann surface

5.2 Algebro-Geometrical Approach

We have seen that the correlation functions (4.23) and (5.12) can be identified with the volumes of the moduli spaces $\mathscr{M}^*$ and $\mathscr{M}_f^*$ respectively, provided that the moduli spaces are compact. In this subsection, we will sketch an independent algebro-geometrical method of Donaldson on the intersection pairings [2], which is more suitable to deal with the non-compactness of the moduli spaces.

Due to Donaldson, a canonical identification of the correlation function can be obtained by a theorem of Gieseker [33] (and Proposition (5.4) in [2]). Let $E \to \Sigma$ be a complex vector bundle with reduction of structure group $SU(2)$ over Σ. From a fundamental result of Quillen [34], we have a determinant line bundle $\widetilde{\mathscr{L}_\Sigma} \to \mathscr{A}_\Sigma$ over the space of all connections on E. We also have a $\mathscr{G}$ invariant connection (Quillen metric) on $\widetilde{\mathscr{L}_\Sigma}$ with curvature two form given by $-2\pi i$ times the Kähler structure $\tilde{\omega}_\Sigma$ (Eq. (3.6) for $n = 1$) on $\mathscr{A}_\Sigma$. Then, we can obtain a holomorphic line bundle $\mathscr{L}_\Sigma$ over the moduli space $\mathscr{M}_f^*$ (or over the moduli space $\mathscr{M}_\Sigma^s$ of stable bundles on Σ) with curvature given by $-2\pi i\tilde{\omega}_\Sigma$. Thus, we have another representative of Donaldson's map $\mu(PD[1])$ by the first Chern class, $c_1(\mathscr{L}_\Sigma)$, of $\mathscr{L}_\Sigma$. We can also construct a line bundle $\mathscr{L}_\Sigma^{\otimes m}$. The theorem of Gieseker says that a section j of some power $\mathscr{L}_\Sigma^{\otimes m}$ embeds $\mathscr{M}_\Sigma^s$ as a quasi-projective variety in $\mathbb{CP}^N$. The degree of a quasi-projective variety Y can be defined by the degree of the closure $\overline{Y}$ in the projective space. The degree $\deg(\overline{j(\mathscr{M}_\Sigma^s)})$ of the d-dimensional projective variety $\overline{j(\mathscr{M}_\Sigma^s)}$ is defined by the number of points of intersections of $\overline{j(\mathscr{M}_\Sigma^s)}$ with a generic $(N - d)$ dimensional hyperplane $\mathbb{CP}^{N-d} \subset \mathbb{CP}^N$. Equivalently

$$\deg(\overline{j(\mathscr{M}_\Sigma^s)}) = \left\langle c_1\left(\mathscr{L}_\Sigma^{\otimes m}\right)^d, \overline{j(\mathscr{M}_\Sigma^s)}\right\rangle . \tag{5.13}$$

Thus we can identify the intersection pairing represented by the correlation function in (5.12) with $\deg(\overline{j(\mathscr{M}_\Sigma^s)})/(d!\, m^d)$.

Furthermore, Donaldson obtained his positivity theorem by constructing a quasi-projective embedding J of the moduli space of stable bundles on a simply connected algebraic surface using Gieseker's projective embedding and a theorem of Mehta and Ramanathan [35]. We will roughly describe Donaldson's proof. Let M be a simply connected algebraic surface with $p_g(M) > 0$ so that there is a Hodge metric compatible with a projective embedding. Then, the cohomology class of Kähler form ω is Poincaré dual to the hyperplane section class H. Let Σ be a Riemann surface representing H. Now consider a moduli space $\mathscr{M}_\Sigma^s$ of stable bundles over Σ. Using the theorem of Mehta and Ramanathan, Donaldson showed there is a restriction map $r : \mathscr{M}_M^s \to \mathscr{M}_\Sigma^s$, which is an embedding (for the precise statement, the reader should refer to Sect. 5 in [2]). Over $\mathscr{M}_\Sigma^s$ we have a holomorphic line bundle $\mathscr{L}_\Sigma$ as mentioned before. Then the Donaldson $\mu(\Sigma)$ map is the pull-back by r of the first Chern class of $\mathscr{L}_\Sigma$. Now for large m we have a quasi-projective embedding $j(\mathscr{M}_\Sigma^s)$. Thus the composition $J = j \circ r$ gives a quasi-projective embedding of $\mathscr{M}_M^s$. Then, for large k, Donaldson showed that his invariant $q_{k,M}(H, \ldots, H)$ is proportional to the $\deg(\overline{J(\mathscr{M}_M^s)})$. Since the degree of a non-empty variety is always positive, the positivity theorem is followed.

We will end this subsection by observing a relation between the Donaldson μ-maps, (4.22) and (5.11). Let M be a projective algebraic surface Kähler form ω and Σ represent the hyperplane section. Assume that we study the $N = 2$ TYMT's on M

and Σ. The Donaldson μ-maps on M and on Σ are represented by the topological observables,

$$\begin{aligned} \Omega &= \frac{1}{4\pi^2}\int_M \mathrm{Tr}\,(\varphi F + \psi \wedge \bar{\psi}) \wedge \omega\,, \\ \Omega_\Sigma &= \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\,(\varphi F + \psi \wedge \bar{\psi})\,. \end{aligned} \tag{5.14}$$

The Poincaré duality of ω to Σ means that we have the following identity for every $(1,1)$ form η on M (p. 230 in [28]);

$$\frac{i}{2\pi}\int_M (\log h)\partial\bar{\partial}\eta = \int_\Sigma \eta - \int_M \eta \wedge \omega\,, \tag{5.15}$$

where h is a globally defined smooth function on M which vanish exactly at Σ. If we choose $\eta = \frac{1}{4\pi^2}\mathrm{Tr}\,(\varphi F + \psi \wedge \bar{\psi})$, we have

$$\Omega - \Omega_\Sigma = \mathbf{s}\bar{\mathbf{s}}\int_M (\log h)\, c_2(E)\,, \tag{5.16}$$

where we have used Eq. (4.8) [10]. This is an interesting but not an accidental relation as explained in Sect. 6.5.4 in [7]. We want to construct a holomorphic line bundle $\mathscr{L}$ over $\mathscr{M}^s_M$ with curvature $-2\pi i$ times the Kähler form $\tilde{\omega} = \widehat{\Omega}$ on $\mathscr{M}^s_M$, so that the first Chern class $c_1(\mathscr{L})$ of $\mathscr{L}$ represents the Donaldson μ-map (4.22). The desired line bundle $\mathscr{L}$ can be obtained via pull back of the determinant line bundle $\mathscr{L}_\Sigma$ by the restriction map $r : \mathscr{M}^s_M \to \mathscr{M}^s_\Sigma$. Then, we get an induced connection on $\mathscr{L}$ with curvature form $-2\pi i\tilde{\omega}_\Sigma = -2\pi i\widehat{\Omega}_\Sigma$. Equation (5.16) implies that we can modify the pull back connection by the zero-form $\int_M (\log h)c_2(E)$ on $\mathscr{A}$ to get a new connection which gives the desired curvature $-2\pi i\tilde{\omega} = \widehat{\Omega}$. Since $\int_M (\log h)c_2(E)$ is gauge invariant, we have the desired result.

5.3 Non-Abelian Localization

One of great challenges of the TYMT is to overcome its formality and to provide some concrete computational methods. In this view, Witten's beautiful results on two dimensions are very encouraging [5]. The purpose of this subsection is to suggest that Witten's formalism can be applied to calculating the Donaldson invariants on a compact Kähler surface.

To begin with, we briefly summarize Witten's arguments [11]. Let Σ be a compact oriented Riemann surface with Kähler form ω. The partition function of Yang-Mills

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[10] Equation (4.8) also give a similar relation if choose $\eta = \frac{1}{8\pi^2}\mathrm{Tr}\,(\varphi^2)\omega$

[11] Note that the original arguments of Witten are slightly different to what follows, since the Kähler structure on a Riemann surface is not essential in his arguments. However, the Kähler structure is essential to deal with the higher dimensional case

theory on Σ can be written as

$$Z(\varepsilon) = \frac{1}{\mathrm{vol}(\mathscr{G})} \int_{\mathscr{A}_\Sigma} \mathscr{D}A\, \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, \mathscr{D}\varphi \exp\left[\frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(\varphi F + \psi \wedge \bar{\psi}\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma \omega\, \mathrm{Tr}\, \varphi^2\right] . \tag{5.17}$$

The decoupled fields $\psi, \bar{\psi}$ are introduced to give the correct path integral measure (the symplectic measure in the Kähler polarization). Witten's fundamental result is that the partition function $Z(\varepsilon)$ of the two dimensional Yang-Mills theory can be expressed as a sum of contributions of critical points S

$$Z(\varepsilon) = \sum_{\alpha \in S} Z_\alpha(\varepsilon) . \tag{5.18}$$

In the limit $\varepsilon = 0$, the only contribution of the critical points is the absolute minimum of the action and the partition function becomes an integral over the moduli space of flat connections. For $\varepsilon \neq 0$, the higher critical points will contribute, however, their contributions are exponentially small. More concretely,

$$\left\langle \exp\left(\frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\,\left(\varphi F + \psi \wedge \bar{\psi}\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma \omega\, \mathrm{Tr}\, \varphi^2\right)\right\rangle = Z(\varepsilon) + O(\exp(-c) , \tag{5.19}$$

where c is the smallest value of the Yang-Mills action, $\frac{1}{8\pi^2\varepsilon}\int_M \omega\, \mathrm{Tr}\, f^2$, for the higher critical points. Using the correspondence (5.19), Witten obtained general expressions for the intersection pairings on moduli space of flat connections.

The above example should be viewed as an application of Witten's non-Abelian localization formula, which is a non-Abelian generalization of the Duistermaat-Heckmann (DH) integral formula [36], that a partition function of a quantum field theory with an action functional given by the norm squared of a moment map can be expressed as a sum of contributions of critical points [5]. Now, we want to find a natural higher dimensional analogue of the Witten's solution in two dimensions. We can start this by observing some special properties of physical Yang-Mills theory in two dimensions: i) every oriented compact Riemann surface is a Kähler manifold; ii) every connection is of type $(1,1)$ which defines a holomorphic structure; iii) the action functional in the first order formalism, Eq. (5.17), has the $N = 2$ fermionic symmetry with the transformation law Eq. (3.9), or the action functional consists of the topological observables of the $N = 2$ TYMT; iv) the Yang-Mills action is given by the norm squared of the moment map. Then, our strategy is to design a variant of physical Yang-Mills theory on a compact Kähler surface M, which has all the properties (i)–(iv). An obvious candidate motivated from Eq. (5.17) is

$$Z(\varepsilon) = \frac{1}{\mathrm{vol}(\mathscr{G})} \int_{\mathscr{A}^{1,1}} \mathscr{D}A\, \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, \mathscr{D}\varphi \exp\left[\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\varphi F + \psi \wedge \bar{\psi}\right) \wedge \omega + \frac{\varepsilon}{8\pi^2}\int_M \frac{\omega^2}{2!}\mathrm{Tr}\, \varphi^2\right] , \tag{5.20}$$

where we have restricted $\mathscr{A}$ to $\mathscr{A}^{1,1}$. We further assume that ψ and $\bar{\psi}$ are tangent to $\mathscr{A}^{1,1}$, i.e. $\partial_A \psi = \bar{\partial}_A \bar{\psi} = 0$. Similarly to its two-dimensional ancestor, the decoupled fields $\psi, \bar{\psi}$ are introduced to give the symplectic measure and to ensure the $N = 2$ fermionic symmetry of the theory. The above partition function is identical to that of physical Yang-Mills theory on M restricted to $\mathscr{A}^{1,1}$, which can be called *holomorphic Yang-Mills theory* (HYMT), up to a topological term after integrating $\varphi, \psi, \bar{\psi}$ out. And the action functional of the HYMT is the norm squared of the moment map (4.20). Then, by the non-Abelian version of the DH integration formula of Witten, it is obvious that the four-dimensional counterpart of the mapping (5.19) exists between the intersection pairings on the moduli space of ASD connections and the partition function (5.20). The only technical difficulty is to define the restriction of $\mathscr{A}$ to $\mathscr{A}^{1,1}$ in a correct quantum theoretical way. This can be easily achieved because the constraints $F_A^{2,0} = F_A^{0,2} = \partial_A \psi = \bar{\partial}_A \bar{\psi} = 0$, which define the HYMT, are related by the $N = 2$ fermionic symmetry. It will be more suitable to define the HYMT as a deformation of the $N = 2$ TYMT, analogous to the simple mapping from the TYMT to physical Yang-Mills theory in two dimensions [5]. The details will be discussed elsewhere [37].

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