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Local Borel Summability of Euclidean Φ_4^4: A Simple Proof via Differential Flow Equations

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Abstract: It is shown how the differential flow equation (or, equivalently, the continuous renormalization group) method can be employed to give an astonishingly easy proof of the local Borel summability of the renormalized perturbative Euclidean massive Φ_4^4.

1. Introduction

Understanding rigorously (sometimes even only part of) the large order behaviour of perturbation theory in quantum field theory has proved to be quite a challenge (see e.g. chapter II.6 in [R] for a general review and references). For instance, let us consider the perturbative Euclidean massive Φ_d^4.

For $d \in \{2, 3\}$ it is known since a few years that the renormalized perturbation series for the connected Green functions is Borel summable and that the Borel transform exhibits an instanton singularity precisely as predicted by the Lipatov argument; the proof is based on constructive field theory techniques.

However, when $d = 4$, even the most sophisticated methods do not seem to be sufficient to go beyond a proof of the local existence of the Borel transform. In more detail, the combinatorially involved machinery of either elaborate BPHZ techniques [dCR] or discrete renormalization group/GN tree expansion methods [GN] proved adequate to establish local Borel summability, but without a good estimate of the minimal radius of convergence of the Borel transform. It required the introduction of multi-scale phase-space cluster expansion methods [MNRS, DFR] to obtain largely improved estimates (in fact, the suspected best possible estimates) on the radius of convergence. All attempts to prove the existence (or, less expected, the absence) of instanton or renormalon singularities failed, so far.

The purpose of this paper is to demonstrate that, somewhat unexpectedly, there is an easy and rather short proof of the local existence of the Borel transform for $d = 4$

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– yet, up to now, without a particularly good control over the radius of convergence (at the moment it is not clear to what extent this control can be improved). The method which is employed is the *continuous* renormalization group/differential flow equation technique [P, KK1] which is elementary and in particular free of any combinatorial troubles.

Outlining the contents of this paper, Sect. 2 begins with a brief summary of some basic facts on the differential flow equation method; for more details the reader is referred to [KK1]. Then we establish an improved version of the boundedness theorem of [KK1]; accordingly, each connected amputated renormalized Green function at scale Λ (i.e. when all (internal) momenta in the range $[\Lambda, \Lambda_0]$ have been integrated out, where Λ_0 is the momentum space UV cutoff) can be bounded by a power counting factor times a polynomial in $\log(\Lambda)$; also, and this is the important point for our purposes here, one can give a rather accurate upper bound on the degrees (of these polynomials) and one can compute recursive relations for the coefficients (of these polynomials).

The main result of Sect. 3 is an upper bound on the coefficients of these polynomials which implies local Borel summability. The technically very simple proof of this bound is based on the recursive relations and on the knowledge of the upper bounds on the degrees mentioned before. The proof (being recursive) also relies on an educated guess of what kind of bounds might be compatible with the recursive relations. No effort is made to produce optimal bounds; instead we will concentrate on brevity and simplicity.

There is every reason to believe that the methods presented in this paper extend to massless Φ_4^4 (see [KK2]) or to theories including fermions, for instance to QED (see [KK3]). In view of the existing rigorous construction of (weakly coupled) infrared Φ_4^4, perturbative massless Φ_4^4 is hardly more than a testing ground for new methods. The situation for QED is quite different; note, however, that local Borel summability of QED has already been proven in [FHRW] using the GN tree expansion technique.

It is hoped that the differential flow equation method can be utilized to shed some new light on the long standing problem concerning instantons/renormalons, for instance in Φ_4^4.

2. Perturbative Renormalizability Revisited

Let $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})$ be the momentum space connected amputated n-point Green function ($n \geqq 1$), at perturbative order $r \geqq 1$, of the perturbative Euclidean massive, and for the sake of simplicity even, Φ_4^4 theory with UV-cutoff Λ_0, as introduced in [KK1]; $p_1,\ldots,p_{n-1}, p_n (\equiv -p_1 - \cdots - p_{n-1})$ are the external momenta, and Λ is a scale parameter which varies continuously over the interval $[0, \Lambda_0]$. The indices Λ, Λ_0 indicate that the internal momenta have been integrated out approximately over $[\Lambda, \Lambda_0]$. For more details concerning the definition of $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}$ see [KK1].

Because $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}$ is a connected Green function one readily checks that [KK1]

$$\mathcal{L}_{r,n}^{\Lambda,\Lambda_0} \equiv 0, \quad \text{unless} \quad n \in \{2, 4, \ldots, 2r+2\} . \tag{2.1}$$

The $\mathcal{L}$'s obey, by construction [KK1], the general renormalization conditions (r.c.)

a) $\mathcal{L}_{r,2}^{0,\Lambda_0}(0) = a_r^R$,

b) $\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{r,2}^{0,\Lambda_0}(0) = \delta_{\mu,\nu} \cdot b_r^R$,

c) $\mathcal{L}_{r,4}^{0,\Lambda_0}(0) = c_r^R$, (2.2)

where the renormalization constants $\{a_r^R, b_r^R, c_r^R : r \geqq 1\}$ are a set of arbitrarily chosen finite numbers; for reasons of simplicity we will assume that a_r^R, b_r^R, c_r^R do not depend on Λ_0. (The renormalization conditions (2.2) could have been imposed as well at arbitrary nonzero external momenta [KKS].) Next, Euclidean invariance of our theory tells us that e.g.

$$\partial_{p_\mu} \mathcal{L}_{r,2}^{0,\Lambda_0}(0) = 0 . \tag{2.3}$$

Hence, the r.c. (2.2) and Eq. (2.3) control all the quantities $\partial_p^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}$ of dimension $n + |w| \leqq 4$ at $\Lambda = 0$ (and $p = 0$). On the other side, since the bare interaction of the Φ_4^4 theory contains only dimension $\leqq 4$ vertices, all the objects $\partial_p^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}$ of dimension $n + |w| > 4$ are well under control at $\Lambda = \Lambda_0$:

$$\partial_p^w \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0} \equiv 0, \quad \text{if} \quad n + |w| > 4 . \tag{2.4}$$

It is convenient to measure the $\mathcal{L}$'s by the norm $\|(\cdot)\|_{(2\Lambda,\eta)}$, where [KK1]

$$\|\partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} := \max_{\substack{p_1,\dots,p_{n-1} : |p_j| \leqq \max\{2\Lambda,\eta\}, 1 \leqq j \leqq n-1 \\ w : |w| = z}} |\partial_p^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n-1})| ; \tag{2.5}$$

here, η with $0 \leqq \eta < \infty$ is arbitrary *but fixed once and for all.*

The $\mathcal{L}$'s obey an infinite set of coupled differential flow equations [P, KK1]. Upon estimating these differential equations using the norm (2.5), one arrives at (cf. (2.27), (2.28) in [KK1])

$$\begin{aligned} \|\partial_\Lambda \partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \leqq C_1 \cdot \Bigg\{ & \binom{n+2}{2} \cdot \|\partial^z \mathcal{L}_{r,n+2}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \\ & + \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z'' \leqq z}} n' \cdot n'' \cdot \|\partial^{z'} \mathcal{L}_{r',n'}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \\ & \cdot \|\partial^{z''} \mathcal{L}_{r'',n''}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \Bigg\} \\ & (0 \leqq \Lambda \leqq 1, \ 0 \leqq z \leqq 3) \end{aligned} \tag{2.6}$$

and at

$$\begin{aligned} \|\partial_\Lambda \partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \leqq C_1 \cdot \Bigg\{ & \binom{n+2}{2} \cdot \Lambda \cdot \|\partial^z \mathcal{L}_{r,n+2}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \\ & + \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z''+z'''=z}} n' \cdot n'' \cdot \Lambda^{-3-z'''} \\ & \cdot \prod_{\# = ',''} \|\partial^{z^\#} \mathcal{L}_{r^\#,n^\#}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \Bigg\} . \\ & (1 \leqq \Lambda \leqq \Lambda_0, \ 0 \leqq z \leqq 3) \end{aligned} \tag{2.7}$$

The constant $C_1 \geqq 1$ in (2.6), (2.7) is *independent* of z (because we restricted z to the range $0 \leqq z \leqq 3$) and of r, n, Λ, Λ_0, η. (Notice that in order to simplify the formulae in the present paper as much as possible, the scale parameter Λ_1 in [KK1] has now been fixed to $\Lambda_1 := 1$.)

Define the set S by

$$S := \{(n, z) : (n = 2 \wedge 0 \leqq z \leqq 3) \vee (n = 4 \wedge 1 \leqq z \leqq 3)\}\,, \tag{2.8a}$$

and define, for $n \in \{2, 4, \ldots\}$ and $0 \leqq z \leqq 3$,

$$\varepsilon_{n,z} := \begin{cases} 0, & (n, z) \in S \\ 1, & \text{else}\,. \end{cases} \tag{2.8b}$$

The following is a more precise version of Theorem 3 in [KK1] and of Proposition 4′ in [KKS].

Theorem 1. *For any* r.c. (2.2), *for any fixed* η $(0 \leqq \eta < \infty)$, *and for* $0 \leqq z \leqq 3$ *the connected amputated Green functions of the perturbative Euclidean massive even* Φ_4^4 *theory satisfy the bounds*

$$\|\partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}\|_{(2\Lambda,\eta)} \leqq (C_1 \cdot e^6)^{2r-n/2} \cdot \begin{cases} e^{(1-\Lambda)n} \cdot B_{r,n,z}\,, & 0 \leqq \Lambda \leqq 1 \\ \Lambda^{4-n-z} \cdot P_{r,n,z}\log(\Lambda)\,, & 1 \leqq \Lambda \leqq \Lambda_0\,, \end{cases} \tag{2.9}$$

where: a) $P_{r,n,z}\log(\Lambda)$ *is a polynomial in* $\log(\Lambda)$ *of the form*

$$P_{r,n,z}\log(\Lambda) = \sum_{j \geqq 0} A_{r,n,z,j} \cdot (\log(\Lambda))^j \tag{2.10}$$

with

$$A_{r,n,z,j} = 0, \quad \textit{for } j > (r + \varepsilon_{n,z} - n/2)\,. \tag{2.11}$$

b) $A_{r,n,z,j}$ $(\geqq 0)$ *and* $B_{r,n,z}$ $(\geqq 0)$ *are independent of* Λ, Λ_0. c) *The* A*'s and* B*'s obey the recursive equations* (2.12)–(2.18) *below.*

Recursive equations for the A*'s and* B*'s*:

Case 1) If $n + z > 4$:

$$\begin{aligned} A_{r,n,z,j} = &\binom{n+2}{2} \sum_{k \geqq j} A_{r,n+2,z,k} \cdot \frac{k!}{j!}(n+z-4)^{j-k-1} \\ &+ \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z''\leqq z}} n' \cdot n'' \cdot \sum_{\substack{k'\geqq 0,\, k''\geqq 0 \\ k'+k''\geqq j}} A_{r',n',z',k'} \cdot A_{r'',n'',z'',k''} \\ &\cdot \frac{(k'+k'')!}{j!} \cdot (n+z-4)^{j-k'-k''-1}\,, \end{aligned} \tag{2.12}$$

$$B_{r,n,z} = (\text{r.h.s.}(2.12))|_{j=0} + f_{r,n,z}\,, \tag{2.13}$$

where

$$\begin{aligned} f_{r,n,z} := \frac{1}{n+2}\Bigg(&\binom{n+2}{2} B_{r,n+2,z} \\ &+ \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z''\leqq z}} n' \cdot n'' \cdot B_{r',n',z'} \cdot B_{r'',n'',z''}\Bigg)\,. \end{aligned} \tag{2.14}$$

Case 2) If $n+z=4$: (i.e. $n=4,\ z=0$ or $n=2,\ z=2$)

$$B_{r,n,z}=\delta_{n,2}|b_r^R|+\delta_{n,4}|c_r^R|+f_{r,n,z}+4\cdot\max\{2,\eta\}\cdot B_{r,n,z+1}\,, \tag{2.15}$$

$$\begin{aligned}A_{r,n,z,j}&=\delta_{j,0}(\delta_{n,2}|b_r^R|+\delta_{n,4}|c_r^R|+f_{r,n,z})\\&\quad+\delta_{j,\geqq 1}\left(\binom{n+2}{2}A_{r,n+2,z,j-1}\cdot\frac{1}{j}\right.\\&\quad\left.+\sum_{\substack{r'+r''=r\\ n'+n''=n+2\\ z'+z''\leqq z}}n'n''\cdot\sum_{\substack{k'\geqq 0,\,k''\geqq 0\\ k'+k''=j-1}}A_{r',n',z',k'}\cdot A_{r'',n'',z'',k''}\cdot\frac{1}{j}\right)\\&\quad+4\cdot\max\{2,\eta\}\cdot A_{r,n,z+1,j}\,.\end{aligned} \tag{2.16}$$

Case 3) If $n+z<4$: (i.e. $n=2,\ z=0,1$)

$$B_{r,n,z}=\delta_{z,0}|a_r^R|+f_{r,n,z}+4\cdot\max\{2,\eta\}\cdot B_{r,n,z+1}\,, \tag{2.17}$$

$$\begin{aligned}A_{r,n,z,j}&=\delta_{j,0}(\delta_{z,0}|a_r^R|+f_{r,n,z})+\binom{n+2}{2}A_{r,n+2,z,j}\cdot(4-n-z)^{-1}\\&\quad+\sum_{\substack{r'+r''=r\\ n'+n''=n+2\\ z'+z''\leqq z}}n'n''\cdot\sum_{\substack{k'\geqq 0,\,k''\geqq 0\\ k'+k''=j}}A_{r',n',z',k'}\cdot A_{r'',n'',z'',k''}\cdot(4-n-z)^{-1}\\&\quad+4\cdot\max\{2,\eta\}\cdot A_{r,n,z+1,j}\,.\end{aligned} \tag{2.18}$$

Proof of Theorem 1. The simple but somewhat lengthy proof follows the induction scheme which is standard in the framework of the flow equation method [KK1]. Working out the details is left to the reader. Let me merely point out that one makes use of (2.1)–(2.8) and, if $(n+z)>4$, of

$$x^{-\mu}(\log x)^k=-\frac{d}{dx}\left[x^{-\mu+1}\sum_{j=0}^{k}(\mu-1)^{j-k-1}\frac{k!}{j!}(\log x)^j\right],\quad \mu\neq 1\,,$$

which shows that for $1\leqq\Lambda\leqq\Lambda_0$ and $\mu>1$:

$$\int_{\Lambda}^{\Lambda_0}dx\,x^{-\mu}(\log x)^k\leqq\Lambda^{-\mu+1}\sum_{j=0}^{k}(\mu-1)^{j-k-1}\frac{k!}{j!}(\log(\Lambda))^j\,.$$

If $(n+z)=4$ one employs $x^{-1}(\log x)^k=(k+1)^{-1}\frac{d}{dx}[(\log x)^{k+1}]$, whereas for $(n+z)<4$ one estimates (for $\mu\in\mathbb{R}$)

$$\int_1^{\Lambda}dx\,x^{\mu}(\log x)^k\leqq(\log(\Lambda))^k\cdot\int_1^{\Lambda}dx\,x^{\mu}\,. \qquad\clubsuit$$

Remark. Equations (2.12)–(2.18) are but *one* (but, within the flow equation scheme, a quite accurate) possibility to define recursively the coefficients A and B. Clearly, less accurate recursion relations could be obtained from (2.12)–(2.18) by increasing (some of) the coefficients on the r.h.s. of (2.12)–(2.18), e.g. by replacing $(n+z-4)^{j-k} \mapsto 1$ in (2.12), etc. In fact, in order to prove the local Borel summability we will not have to rely on the full structure of the r.h.s. of (2.12)–(2.18); rather, our proof will be unaffected by e.g. replacing $(n+z-4)^{j-k} \mapsto 1$ and $(n+z-4)^{j-k'-k''} \mapsto 1$ in (2.12), $j^{-1} \mapsto 1$ in (2.16), and by multiplying the r.h.s of (2.12)–(2.18) by an r, n, z, j-independent constant.

3. Local Borel Summability

In this section we are going to prove $r!$-type bounds, uniformly in Λ_0, on the connected amputated Green functions $\mathcal{L}_{r,n}^{0,\Lambda_0}$.

Obviously, such bounds could not be proven if the renormalization constants a_r^R, b_r^R and c_r^R would grow too rapidly with r. Hence we make the necessary

Assumption. There exists $C_2 \geqq 1$ such that

$$|a_r^R|,\ |b_r^R|,\ |c_r^R| \leqq r! \cdot (C_2)^r . \tag{3.1}$$

Recall (from (2.1) and (2.11)) that, because the $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}$ are nontrivial (i.e. $r \geqq 1$) connected Green functions of the even Φ_4^4 theory, $A_{r,n,z,j}$ and $B_{r,n,z}$ can only be nonzero if the conditions $r \geqq 1$, $n \in \{2, 4, \ldots, 2r+2\}$ and $0 \leqq j \leqq r + \varepsilon_{n,z} - n/2$ are met. On these potentially nontrivial A's and B's we now have the following bounds.

Theorem 2. *Assume* (3.1). *Fix arbitrary real* $\alpha > 1$, $\beta > 2$, $\gamma \geqq 0$ *and* $\kappa > 1$. *Then, for all* $r \geqq 1$, $n \in \{2, 4, \ldots, 2r+2\}$, $0 \leqq z \leqq 3$ *and* $0 \leqq j \leqq r + \varepsilon_{n,z} - n/2$, *we find*

$$\begin{aligned} A_{r,n,z,j} &\leqq K^{4-z}(K^4)^{3r-n/2-1} \cdot r^{-\alpha} n^{-\beta} \kappa^{-j} \\ &\quad \cdot (\max\{r+2-n/2,\ n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma} , \end{aligned} \tag{3.2}$$

$$B_{r,n,z} \leqq \text{r.h.s.}(3.2)|_{j=0} . \tag{3.3}$$

Here, K is an r, n, z, j-independent constant which has to be chosen large enough; for example, (3.2) *and* (3.3) *hold if*

$$K \geqq e^{4(3+\alpha+\beta+\gamma)} \frac{1}{1-1/\kappa} \left(C_2 + \max\{2, \eta\} + \kappa \left(1 + \frac{1}{(\alpha-1)(\beta-2)} \right) \right) . \tag{3.4}$$

Remark. The bounds (3.2), (3.3) and (3.4) are by no means the optimal results which could be achieved by the methods used in this paper. Even with a small effort one could come up with much better bounds. However, since we don't attempt to find the best possible estimates anyway, I chose to prove the bounds (3.2)–(3.4) because of their attractively simple form.

Proof. The proof of Theorem 2 is, as usual, based on induction; and the induction scheme is (again as usual) the standard one [KK1].

It is convenient to introduce the notation, for $t \in \{12, 13, 15, 16, 17, 18\}$, $(2.t)_{\text{lin}}$ resp. $(2.t)_{\text{quad}}$ for that part on the r.h.s. of Eq. $(2.t)$ which is 0^{th} and 1^{st} order resp. 2^{nd} order in A, B.

For given r, n, z, j in the range specified in Theorem 2 we know, according to the induction hypothesis (within our standard induction scheme), that the potentially nontrivial A's and B's on the r.h.s. of Eq. (2.t) (where $t \in \{12, 13\}$ if $n+z>4$, etc.) do obey the bounds (3.2), (3.3); and the induction step consists in proving (3.2), (3.3) for $A_{r,n,z,j}$ and $B_{r,n,z}$ by bounding the r.h.s. of the relevant (2.t).

Before getting into the details of the induction step, let me try to explain the bound (3.2), (3.3) by outlining the role played by the various factors on the r.h.s. of (3.2), (3.3) while bounding the r.h.s. of Eq. (2.t). First of all, the factor $(\max\{(\cdot),(\cdot)\})^{(\cdots)}$ is, on the one hand, used to annihilate the dangerous factor $\sim \frac{1}{n}\binom{n+2}{2} \simeq n$ in $(2.t)_{\text{lin}}$, and on the other hand it nicely conspires with $k!/j!$ (in $(2.t)_{\text{lin}}$) and with $(k'+k'')!/j!$ (in $(2.t)_{\text{quad}}$) to terms which can be bounded by const $\cdot (\max\{r+2-n/2, n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma}$ where const depends at most on γ. Therefore, at this point, what remains to be done to complete the induction step is to control sums of the type $\sum_{k\geqq j}\kappa^{-k}$, $\sum_{r'+r''=r}(r')^{-\alpha}(r'')^{-\alpha}$ and $\frac{1}{n}\sum_{n'+n''=n+2} n'n''(n')^{-\beta}(n'')^{-\beta}$ in a way which is consistent with the induction hypothesis; the functions which are involved (i.e. κ^{-k}, etc.) precisely do the job.

Carrying out the induction step is, once one knows how to proceed, almost a triviality. However, due to the length of the detailed argument I decided to divide it into several easy Lemmas (Lemmas 3–9 below).

Lemma 3. *For all* $r\geqq 1$, $n\in\{2,4,\ldots,2r\}$, $\varepsilon\in\{0,1\}$ *and* $0\leqq j\leqq r+\varepsilon-n/2$ *one has*

$$\left(\frac{\max\{r+1-n/2,\, n+2\}}{\max\{r+2-n/2,\, n\}}\right)^{r+\varepsilon-n/2-j} \leqq 2e^2 . \tag{3.5}$$

Let $\delta\in\mathbb{R}$; *then, for all r and n as above*

$$\left(\frac{\max\{r+1-n/2,\, n+2\}}{\max\{r+2-n/2,\, n\}}\right)^{\delta} \leqq 2^{|\delta|} . \tag{3.6}$$

Proof. Let us first check (3.5). a) if $r+1-n/2\geqq n+2$: Then $r+2-n/2\geqq n$ and thus l.h.s.(3.5)$\leqq 1$. b) if $r+1-n/2<n+2$: Since $r+\varepsilon-n/2-j$ is always $\geqq 0$ and $n\leqq \max\{(\cdot), n\}$, we have l.h.s.(3.5)$\leqq \left(\frac{n+2}{n}\right)^{r+\varepsilon-n/2-j} \leqq \left(\frac{n+2}{n}\right)^{r+1-n/2}$ $\leqq (1+2/n)^{n+1}\leqq 2e^2$. – Inequality (3.6) is proven similarly by distinguishing the 3 cases $r+1-n/2\geqq n+2$; $r+1-n/2<n+2$ and $\delta\geqq 0$; $r+1-n/2<n+2$ and $\delta\leqq 0$. ♣

Lemma 4. *Assume* (3.4) (*and hence we know that e.g.* $K^8\geqq C_2\cdot e^{(3+\alpha+\gamma)}$). *Then, for all* $r\geqq 1$, $n\in\{2,4\}$, $\varepsilon\in\{0,1\}$ *and* $0\leqq z\leqq 3$:

$$\begin{aligned} r!\cdot(C_2)^r \leqq\; & K^{4-z-1}(K^4)^{3r-n/2-1}\cdot r^{-\alpha}n^{-\beta} \\ & \cdot(\max\{r+2-n/2,\, n\})^{r+\varepsilon-n/2-1-\gamma} \\ & \cdot(C_2\cdot 4^{\beta}\cdot e^{4(3+\alpha+\gamma)}) . \end{aligned} \tag{3.7}$$

Proof. It is evident that for the r, n, ε's under consideration

$$r!(C_2)^r(K^4)^{-3r+n/2+1}\cdot r^\alpha n^\beta(\max\{r+2-n/2,\,n\})^{-r-\varepsilon+n/2+1+\gamma}$$

$$\begin{aligned}&\leqq\frac{r!}{(r+2-n/2)^{r+2-n/2}}\cdot(\max\{r+1,\,4\})^{3+\gamma}\cdot(C_2/K^8)^{r-1}\cdot C_2r^\alpha 4^\beta\\&\leqq(r+3)^{3+\alpha+\gamma}\cdot(C_2/K^8)^{r-1}\cdot C_2 4^\beta\\&\leqq e^{(r-1)(3+\alpha+\gamma)}(C_2/K^8)^{r-1}\cdot e^{4(3+\alpha+\gamma)}C_2 4^\beta\,.\end{aligned}$$ ♣

Lemmas 3 and 4 permit us now to prove

Lemma 5. *For all t*

$$\begin{aligned}(2.t)_{\mathrm{lin}}\leqq K^{4-z-1}&(K^4)^{3r-n/2-1}\cdot r^{-\alpha}n^{-\beta}\kappa^{-j}\\&\cdot(\max\{r+2-n/2,\,n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma}\\&\cdot\left(e^{4(3+\alpha+\beta+\gamma)}\frac{1}{1-1/\kappa}(C_2+\max\{2,\,\eta\}+\kappa)\right)\,,\end{aligned}\tag{3.8}$$

where $j:=0$ *if* $t\in\{13,\,15,\,17\}$.

Proof. a) We begin with $t=12$. If $n=2r+2$ then $(2.12)_{\mathrm{lin}}=0$ and hence does satisfy (3.8); therefore we will assume from now on that $n\leqq 2r$. Due to the induction hypothesis, and using $K^{4-z}\leqq K^4$ and $(n+z-4)^{j-k}\leqq 1$, we see that

$$\begin{aligned}(2.12)_{\mathrm{lin}}\leqq&(K^4)^{3r-n/2-1}\cdot r^{-\alpha}(n+2)^{-\beta}\cdot\frac{1}{n+z-4}\binom{n+2}{2}\\&\cdot\sum_{k=j}^{r+\varepsilon_{n+2,z}-n/2-1}\frac{k!}{j!}(\max\{r+1-n/2,\,n+2\})^{r+\varepsilon_{n+2,z}-n/2-k-1-\gamma-1}\kappa^{-k}\,.\end{aligned}\tag{3.9}$$

a1) $n\geqq 6$: In this case $\varepsilon_{n+2,z}=\varepsilon_{n,z}$, and we continue by rewriting (3.9) as

$$\begin{aligned}\text{r.h.s.(3.9)}=&(K^4)^{3r-n/2-1}\cdot r^{-\alpha}n^{-\beta}\kappa^{-j}\cdot(\max\{r+2-n/2,\,n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma}\\&\cdot\left(\frac{n}{n+2}\right)^\beta\cdot\left[\frac{(\max\{r+1-n/2,\,n+2\})^{-1}}{n+z-4}\binom{n+2}{2}\right]\\&\cdot\sum_{k=j}^{r+\varepsilon_{n+2,z}-n/2-1}\left\{\left[\frac{k!}{j!}(\max\{r+1-n/2,\,n+2\})^{-k+j}\right]\right.\\&\cdot\left.\left(\frac{\max\{r+1-n/2,\,n+2\}}{\max\{r+2-n/2,\,n\}}\right)^{r+\varepsilon_{n,z}-n/2-j-1-\gamma}\kappa^{j-k}\right\}\,.\end{aligned}\tag{3.10}$$

Now apply Lemma 3, use the fact that $\beta\geqq 0$, $\gamma\geqq 0$, $\kappa>1$, the bound (valid for all k with $j\leqq k\leqq r+1-n/2$ and so in particular for all k satisfying $j\leqq k\leqq r+\varepsilon_{n+2,z}-n/2-1$)

$$\frac{k!}{j!}(\max\{r+1-n/2,\,n+2\})^{-k+j}\leqq\frac{k!}{j!(r+1-n/2)^{k-j}}\leqq\frac{k!}{j!k^{k-j}}\leqq 1\,,$$

and the estimate (valid for all $n\geqq 6$)

$$\frac{(\max\{r+1-n/2,\, n+2\})^{-1}}{n+z-4}\binom{n+2}{2} \leqq \frac{1}{2}\,\frac{(n+2)(n+1)}{(n+2)(n+z-4)} < 2 \tag{3.11}$$

in order to arrive at (3.8).

a2) $n \leqq 4$: Then $\varepsilon_{n+2,z} \leqq \varepsilon_{n,z}+1$ and we bound (3.9) accordingly by

$$\text{r.h.s.}(3.9) \leqq (K^4)^{3r-n/2-1}\cdot r^{-\alpha}(n+2)^{-\beta}\cdot\frac{1}{n+z-4}\binom{n+2}{2}$$

$$\cdot\sum_{k=j}^{r+\varepsilon_{n+2,z}-n/2-1}\frac{k!}{j!}(\max\{r+1-n/2,\, n+2\})^{r+\varepsilon_{n,z}-n/2-k-1-\gamma}\kappa^{-k};$$

and now we proceed as before, the only difference being that instead of (3.11) we cannot do better than bounding $\frac{1}{n+z-4}\binom{n+2}{2} \leqq 15$ (because the n-suppressing factor $(\max\{r+1-n/2,\, n+2\})^{-1}$ isn't at our disposal any longer).

b) For all other t's the proof of (3.8) works similarly. The renormalization constants have already been bounded in a useful way in Lemma 4; and one observes that, for $t \in \{15, 16\}$, we have $\varepsilon_{n+2,z} = \varepsilon_{n,z}$ and $\varepsilon_{n,z+1} \leqq \varepsilon_{n,z}$, whereas for $t \in \{17, 18\}$ we have $\varepsilon_{n+2,z} \leqq \varepsilon_{n,z}+1$ and $\varepsilon_{n,z+1} = \varepsilon_{n,z}$. ♣

As a result, Lemma 5 has shown that the 0th and 1st order part on the r.h.s. of Eqs. (2.t) can be bounded in a way which is consistent with the induction hypothesis. So we may turn our attention now towards the $(2.t)_{\text{quad}}$'s.

Lemma 6. *Let $\delta \in \mathbb{R}$, $\delta > 1$, and $m', m'' \in \mathbb{N}$. Then*

$$\sum_{m'+m''=m}(m')^{-\delta}(m'')^{-\delta} \leqq \frac{2^{\delta+1}\cdot\delta}{\delta-1}\cdot m^{-\delta}\,. \tag{3.12}$$

Proof. For $x \in \mathbb{R}$ we denote by $\{x\}$ the largest integer which is smaller than or equal to x. We have

$$\text{l.h.s.}(3.12) = m^{-2\delta}\sum_{m'=1}^{m-1}\left(\frac{m'}{m}\right)^{-\delta}\left(1-\frac{m'}{m}\right)^{-\delta} \leqq m^{-2\delta}\cdot 2\cdot\sum_{m'=1}^{\{m/2\}}\left(\frac{m'}{m}\right)^{-\delta}\cdot 2^{\delta}$$

$$\leqq 2^{\delta+1}\cdot m^{-\delta}\sum_{m'=1}^{\infty}(m')^{-\delta} \leqq 2^{\delta+1}\cdot m^{-\delta}\left(1+\int_1^\infty dx\,x^{-\delta}\right)\,. \quad ♣$$

As a Corollary to Lemma 6 we see that for $\alpha > 1$, $\beta > 2$,

$$\frac{1}{n+2}\sum_{\substack{r'+r''=r\\ n'+n''=n+2}}(r')^{-\alpha}(r'')^{-\alpha}(n')^{1-\beta}(n'')^{1-\beta} \leqq r^{-\alpha}n^{-\beta}\cdot\frac{\alpha\beta\cdot 2^{1+\alpha+\beta}}{(\alpha-1)(\beta-2)}\,. \tag{3.13}$$

Lemma 7. *Let $r \geqq 1$, $n \in \{2, 4, \ldots, 2r+2\}$, $0 \leqq z \leqq 3$ and $0 \leqq j \leqq r+\varepsilon_{n,z}-n/2$; and similarly for $r^\#, n^\#, z^\#, j^\#$ with $\# \in \{', ''\}$. Moreover, assume that $r'+r''=r$, $n'+n''=n+2$, $z'+z'' \leqq z$. Then, for $n+z \neq 4$ and $j'+j'' \geqq j$:*

$$\frac{(j'+j'')!}{j!}\cdot(\max\{r'+2-n'/2,\ n'\})^{r'+\varepsilon_{n',z'}-n'/2-j'}$$

$$\cdot(\max\{r''+2-n''/2,\ n''\})^{r''+\varepsilon_{n'',z''}-n''/2-j''}$$

$$\leqq(\max\{r+2-n/2,\ n\})^{r+\varepsilon_{n,z}-n/2-j}\ ; \tag{3.14}$$

but for $n+z=4$ *and* $j'+j''=j-1$:

$$(\max\{r'+2-n'/2,\ n'\})^{r'+\varepsilon_{n',z'}-n'/2-j'}$$

$$\cdot(\max\{r''+2-n''/2,\ n''\})^{r''+\varepsilon_{n'',z''}-n''/2-j''}$$

$$\leqq(\max\{r+2-n/2,\ n\})^{r+\varepsilon_{n,z}-n/2-j}\ . \tag{3.15}$$

Proof. Because $(r^{\#}+2-n^{\#}/2)\geqq 1$ and $(r'+2-n'/2)+(r''+2-n''/2)=(r+2-n/2)+1$, we see that $(r^{\#}+2-n^{\#}/2)\leqq(r+2-n/2)$; also, $n^{\#}\geqq 2$ and $n'+n''=n+2$ imply $n^{\#}\leqq n$; hence $\max\{r^{\#}+2-n^{\#}/2,\ n^{\#}\}\leqq\max\{r+2-n/2,\ n\}$. Therefore, since $r^{\#}+\varepsilon_{n^{\#},z^{\#}}-n^{\#}/2-j^{\#}\geqq 0$, it is now obvious that

$$(\max\{r'+2-n'/2,\ n'\})^{r'+\varepsilon_{n',z'}-n'/2-j'}(\max\{r''+2-n''/2,\ n''\})^{r''+\varepsilon_{n'',z''}-n''/2-j''}$$

$$\leqq(\max\{r+2-n/2,\ n\})^{r+\varepsilon_{n',z'}+\varepsilon_{n'',z''}-n/2-1-j'-j''}\ .$$

This yields (3.15); in order to prove (3.14) we need the additional observation that $j'+j''\leqq r+1-n/2\leqq\max\{r+2-n/2,\ n\}$. ♣

Lemma 8. (*Notation as in Lemma* 7) *For* $k\in\mathbb{N}_0$ *define*

$$\mu(k):=\sum_{k'=0}^{r'+1-n'/2}\ \sum_{k''=0}^{r''+1-n''/2}\delta_{k'+k'',k}\ . \tag{3.16}$$

Then, for all $\gamma\geqq 0$, *all* r, n, $r^{\#}$, $n^{\#}$, k :

$$\mu(k)\cdot\left(\frac{\max\{r+2-n/2,\ n\}}{\max\{r'+2-n'/2,\ n'\}\cdot\max\{r''+2-n''/2,\ n''\}}\right)^{1+\gamma}\leqq 2^{1+\gamma}\ . \tag{3.17}$$

Proof. Define $m^{\#}:=\max\{r^{\#}+2-n^{\#}/2,\ n^{\#}\}$, and $m_+:=\max\{m',\ m''\}$, $m_-:=\min\{m',\ m''\}$. Notice that (3.16) implies that $\mu(k)\leqq r^{\#}+2-n^{\#}/2\leqq m_-$. Furthermore, considering separately the cases $r+2-n/2\geqq n$ and $<n$, one readily verifies that $\max\{r+2-n/2,\ n\}\leqq m'+m''\leqq 2m_+$.

We may restrict our attention to the case $\mu(k)\geqq 1$; then

$$\text{l.h.s.(3.17)}\leqq\left(\mu(k)\cdot\frac{\max\{r+2-n/2,\ n\}}{m'\cdot m''}\right)^{1+\gamma}\leqq\left(\frac{m_-\cdot 2m_+}{m_-\cdot m_+}\right)^{1+\gamma}\ . \quad ♣$$

Combining Lemmas 6–8 we arrive at

Lemma 9. *For all t*

$$(2.t)_{\mathrm{quad}} \leqq K^{4-z-1}(K^4)^{3r-n/2-1} \cdot r^{-\alpha} n^{-\beta} \kappa^{-j} \cdot (\max\{r+2-n/2,\, n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma} \cdot \left(\frac{\alpha \cdot \beta \cdot \kappa}{(1-1/\kappa)(\alpha-1)(\beta-2)} \cdot 2^{10+\alpha+\beta+\gamma}\right), \tag{3.18}$$

where $j := 0$ *if* $t \in \{13, 15, 17\}$.

Proof. As an illustration we prove (3.18) for $t = 12$. With $K^{4-z} \leqq K^4$ and $(n+z-4)^{j-k'-k''} \leqq 1$ we have

$$(2.12)_{\mathrm{quad}} \leqq (K^4)^{3r-n/2-1} \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z''\leqq z}} (r')^{-\alpha}(r'')^{-\alpha}(n')^{1-\beta}(n'')^{1-\beta} \cdot \frac{1}{n+z-4} \cdot \sum_{k'=0}^{r'+\varepsilon_{n',z'}-n'/2} \; \sum_{k''=0}^{r''+\varepsilon_{n'',z''}-n''/2} \delta_{k'+k'',\geqq j} \cdot \kappa^{-k'-k''} \cdot \frac{(k'+k'')!}{j!} \cdot (\max\{r'+2-n'/2,\, n'\})^{r'+\varepsilon_{n',z'}-n'/2-k'-1-\gamma} \cdot (\max\{r''+2-n''/2,\, n''\})^{r''+\varepsilon_{n'',z''}-n''/2-k''-1-\gamma},$$

and upon making use of (3.14),

$$\leqq (K^4)^{3r-n/2-1}(\max\{r+2-n/2,\, n\})^{r+\varepsilon_{n,z}-n/2-j-1-\gamma} \cdot \kappa^{-j} \cdot \sum_{\substack{r'+r''=r \\ n'+n''=n+2 \\ z'+z''\leqq z}} (r')^{-\alpha}(r'')^{-\alpha}(n')^{1-\beta}(n'')^{1-\beta} \cdot \frac{1}{n+z-4} \cdot \sum_{k\geqq j}\left(\sum_{k'=0}^{r'+1-n'/2} \; \sum_{k''=0}^{r''+1-n''/2} \delta_{k'+k'',k}\right) \cdot \left(\frac{\max\{r+2-n/2,\, n\}}{\max\{r'+2-n'/2,\, n'\} \cdot \max\{r''+2-n''/2,\, n''\}}\right)^{1+\gamma} \cdot \kappa^{j-k}.$$

Applying (3.17) and (3.13) leads to (3.18). ♣

The proof of Theorem 2 is now completed by combining Lemmas 5 and 9. ♣

As pointed out earlier, we did not worry too much about the quality of our lower bound (3.4), and in fact it is far from being optimal. Nevertheless, given (3.4) one can ask what values one should choose for α, β, γ, κ; a short calculation reveals that in this case the values $\alpha = 5/4$, $\beta = 9/4, \gamma = 0$ and $\kappa = 2$ might be sensible.

It has been proved in [KK1] that

$$\mathcal{L}_{r,n} := \lim_{\Lambda_0 \to \infty} \mathcal{L}_{r,n}^{0,\Lambda_0}$$

exists, for all r, n. Since Theorems 1 and 2 imply that

$$\|\mathcal{L}_{r,n}^{0,\Lambda_0}\|_{(0,\eta)} \leqq (C_1 \cdot e^6)^{2r-n/2} e^n B_{r,n,0}$$

with (since $\max\{r+2-n/2, n\} \leqq 2(r+1)$)

$$B_{r,n,0} \leqq K^{12r} \cdot (2(r+1))^r ,$$

K given e.g. by (3.4), we infer the

Theorem 3. *For any renormalization conditions* (2.2) *subject to the bounds* (3.1), *and for any* η, $0 \leqq \eta < \infty$, *the renormalized connected amputated Green functions,* $\mathcal{L}_{r,n}$, *of the perturbative Euclidean massive even* Φ_4^4 *theory obey*

$$\|\mathcal{L}_{r,n}\|_{(0,\eta)} \leqq r! \cdot (\text{const})^r \cdot (\max\{2, \eta\})^{12r} , \tag{3.19}$$

where "const" *does not depend on* r, n, η.

As a conclusion, the deCalan–Rivasseau bound (3.19) proves local Borel summability: Define the Borel transform $\mathcal{B}_n$ of $\{\mathcal{L}_{r,n} : r \geqq 1\}$ by

$$\mathcal{B}_n(t)(p_1, \ldots, p_{n-1}) := \sum_{r \geqq 1} \frac{t^r}{r!} \mathcal{L}_{r,n}(p_1, \ldots, p_{n-1}), \quad t \in \mathbb{C} .$$

Then, for $|p_1|, \ldots, |p_{n-1}| \leqq \eta$, $\mathcal{B}_n$ is analytic in t at least within the disk $|t| < (\text{const} \cdot (\max\{2, \eta\})^{12})^{-1}$, where the "const" is the same as the one in (3.19).

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