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Brownian Motion with Restoring Drift: The Petit and Micro-Canonical Ensembles

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Abstract: Let $f(Q)$ be odd and positive near $+\infty$. Then the non-linear wave equation $\partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 + f(Q) = 0$, considered on the circle $0 \leq x < L$, can be written in Hamiltonian form $Q^\bullet = \partial H/\partial P$, $P^\bullet = -\partial H/\partial Q$ with

$$P = Q^\bullet \quad \text{and} \quad H = \tfrac{1}{2}\int_0^L (Q')^2 + \int_0^L F(Q) + \tfrac{1}{2}\int_0^L P^2\,;$$

the corresponding flow preserves the (suitably interpreted) "petit ensemble" $e^{-H} d^\infty Q d^\infty P$; and, for $L \uparrow \infty, Q$ settles down to the stationary diffusion with infinitesimal operator $\frac{1}{2}\partial^2/\partial Q^2 + m(Q)\partial/\partial Q, m$ being the logarithmic derivative of the ground state of $-d^2/dQ^2 \mid F(Q)$. This diffusion is the "Brownian motion with restoring drift"; see McKean-Vaninsky [1993(1)]. For reasons suggested by the paper of Lebowitz-Rose-Speer [1988] on NLS, it is interesting to study the "micro-canonical ensemble" obtained by restricting to the sphere $\int_0^L Q^2 = N$ and making $L \uparrow \infty$ with fixed $D = N/L$. Now, for $F(Q)/Q^2 \to \infty$, the same type of diffusion appears, but with drift arising from the modified potential $F(Q) + cQ^2, c$ being chosen so that the mean of Q^2 is the assigned number D. The proof employs Döblin's method of "loops" [1937] and steepest descent. The same is true for $F(Q) = m^2 Q^2$, only now the proof is elementary. The outcome is also the same if $F(Q)/Q^2 \to 0$, provided D is smaller than the petit canonical mean of Q^2; for D larger than this mean, the matter is more subtle and the outcome is unknown.

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1. Introduction

Let $f(Q)$ be odd and positive near $+\infty$ and let $F(Q)=\int_0^Q f$. The recipe

$$dM_L = \exp\left\{-\int_{-L/2}^{L/2} F[Q(x)]\,dx\right\} \times \frac{\exp\left\{-\frac{1}{2}\int_{-L/2}^{L/2}[Q'(x)]^2\,dx\right\}}{(2\pi 0+)^{\infty/2}}\,d^\infty Q$$

defines a measure on the space of continuous paths $x \to Q(x)$ of period L. The second factor is "circular" Brownian measure, obtained by conditioning the "standard" Brownian motion so that $Q(L/2)=Q(-L/2)$ and distributing this common value h over the line according to the infinite measure $(2\pi L)^{-1/2}\times dh$, i.e., if E_{00} is the "tied" Brownian mean and if $I(Q)$ is any reasonable function of the path, then

$$\int I(Q)\frac{\exp\left\{-\frac{1}{2}\int(Q')^2\right\}}{(2\pi 0+)^{\infty/2}}\,d^\infty Q = \int_{-\infty}^{\infty} E_{00}[I(Q+h)]\,\frac{dh}{\sqrt{2\pi L}}\,.$$

The first factor is a mere density; it controls the partition function Z_L = the total mass, the latter being finite or infinite together with $\int_0^\infty e^{-LF(h)}dh$. This type of measure figures in the (petit) canonical ensemble for the classical wave equation $\partial^2Q/\partial t^2-\partial^2Q/\partial x^2+f(Q)=0$: in fact, with the Hamiltonian $H=\frac{1}{2}\int P^2+\frac{1}{2}\int(Q')^2+\int F(Q)$, you have $Q^\bullet=P=\partial H/\partial P$ and $P^\bullet=Q''-f(Q)=-\partial H/\partial Q$, and the canonical measure

$$e^{-H}d\,\mathrm{vol} = e^{-\int F(Q)}\frac{\exp\left\{-\frac{1}{2}\int P^2\right\}}{(2\pi 0+)^{\infty/2}}\,d^\infty P \times \frac{\exp\left\{-\frac{1}{2}\int (Q')^2\right\}}{(2\pi 0+)^{\infty/2}}\,d^\infty Q$$

is invariant under the flow, suitably interpreted; see McKean-Vaninsky [1993(1)].

$Z_L<\infty$ for $L\uparrow\infty$ is now assumed, and M_L is re-expressed in terms of a Brownian motion with (restoring) drift. $Z_L<\infty$ implies $F(\pm\infty)=\infty$, so $\mathfrak{G}_0=-(1/2)d^2/dQ^2+F(Q)$ has pure point spectrum $\lambda_0(\mathfrak{G}_0)<\lambda_1(\mathfrak{G}_0)<$ etc. $\uparrow\infty$ and (positive) ground state ψ_0 with $\int\psi_0^2(Q)\,dQ=1$, say. Let m be the logarithmic derivative of ψ_0, compute $d\lg\psi_0[Q(x)]$ in the Brownian manner:

$$\begin{aligned} d\lg\psi_0[Q(x)] &= (\lg\psi_0)'(Q)\,dQ+\tfrac{1}{2}(\lg\psi_0)''(Q)\,(dQ)^2 \\ &= m(Q)\,dQ-\tfrac{1}{2}m^2(Q)\,dx+[F(Q)-\lambda_0(G_0)]\,dx\,, \end{aligned}$$ [1]

and integrate from $-L/2$ to $+L/2$, noting that $\int d\lg\psi_0=0$. It follows that

$$\begin{aligned} dM_L = \exp&\left\{\int m(Q)\,dQ-\tfrac{1}{2}\int m^2(Q)\,dx\right\} \\ &\times\frac{\exp\left\{-\frac{1}{2}\int(Q')^2\right\}}{(2\pi 0+)^{\infty/2}}\,d^\infty Q\,, \end{aligned}$$

[1] $(dQ)^2=dx$, in accord with Itô's lemma: in fact, under the conditioning $Q(L)=Q(0)$, $Q(x)=B(x)-(x/L)B(L)$ with a standard Brownian motion B

up to a numerical factor which is dropped, and here you recognize the law of the (circular) diffusion with infinitesimal operator $\mathfrak{G} = (1/2)\partial^2/\partial Q^2 + m(Q)\partial/\partial Q$,[2] conditioned as before so that $Q(L/2) = Q(-L/2)$, this common value being distributed over the line according to the finite measure $p(L, h, h)\,dh$.[3] The total mass is now

$$Z_L = \int_{-\infty}^{\infty} p(L, h, h)\,dh = \sum_{n=0}^{\infty} e^{\lambda_n(\mathfrak{G})L}$$

with $\lambda_n(\mathfrak{G}) = -\lambda_n(\mathfrak{G}_0) + \lambda_0(\mathfrak{G}_0)$, as appears from the similarity $-\psi_0\mathfrak{G}\psi_0^{-1} = G_0 - \lambda_0(\mathfrak{G}_0)$, and from $\lambda_0(\mathfrak{G}) = 0$ it is plain that $Z_L = 1 + o(1)$ for $L \uparrow \infty$. The drift $m(Q)$ is odd and negative for large $Q > 0$, i.e., it pulls Q back to rest ($Q = 0$), and it comes as no surprise that, for $L \uparrow \infty$, M_L tends to the laws M_∞ of the stationary diffusion with infinitesimal operator G and invariant distribution $\psi_0^2(Q)\,dQ$;[4] see McKean-Vaninsky [1993(1)].

The same ideas apply to the nonclassical wave equation NLS (= cubic Schrödinger) for $L < \infty$. Now $\sqrt{-1}\partial Q/\partial t = -\partial^2 Q/\partial x^2 \pm |Q|^2 Q$ with Hamiltonian $H = \int |Q'|^2 \pm \frac{1}{2}\int |Q|^4$,[5] and the canonical measure

$$e^{-H} d\,\mathrm{vol} = e^{\mp(1/2)\int |Q|^4} \frac{\exp\{-\int |Q'|^2\}}{(2\pi 0+)^\infty} d^\infty(\text{real } Q)\, d^\infty(\text{imag } Q)$$

is invariant under the flow,[6] with this difference: that, with the lower (focussing) sign, $Z_L = \infty$. This divergence prompted Lebowitz-Rose-Speer [1988] to introduce the micro-canonical ensemble, obtained by conditioning the canonical measure upon a fixed value N of the constant of motion $\int |Q|^2$, and to speculate that the temperature-dependent micro-canonical ensemble with $e^{-H/T}$ in place of e^{-H} might exhibit a phase transition in the thermodynamic limit as $L \uparrow \infty$ with fixed $D = N/L$, favoring solitons/radiation at low/high temperature. Their program is in its infancy and it seemed profitable to make a trial run with the canonical measure M_L introduced at the start, artifically conditioned upon $\int Q^2 = N$. Denote this micro-canonical ensemble by $M_{N/L=D}$ and take the thermodynamic limit, as above. The result is as predicted by Gibb's principle of equivalence of ensembles (maximal entropy production): *as $L \uparrow \infty$ with fixed $D = N/L$, the micro-canonical ensemble $M_{N/L=D}$ approaches the petit ensemble M_∞^* based upon $F^*(Q) = F(Q) + cQ^2$, the number c being adjusted so that $M_\infty^*(Q)^2 = D$.* The rest of the paper is mostly occupied by the proof of this fact under the (more or less) necessary condition that $Q^{-2}F(Q)$ tends to $+\infty$ with Q, but see Sect. 2 which deals with the special case $F(Q) = m^2Q^2/2$.

What happens for $Q^{-2}F(Q) = o(1)$ is, in part, more subtle. The recipe stands with a suitable choice of $c \geq 0$ if $D \leq M_\infty(Q^2)$, but if $D > M_\infty(Q^2)$ it cannot be so. Then c could only be negative and $F^*(Q) = cQ^2[1 + o(1)]$ acts as a repulsive force far out; as such, it is incapable of producing a stationary diffusion, so while the micro-canonical ensemble makes sense and its thermodynamic limit surely exists, something else must come out. The same problem arises in an aggravated form, for KdV with

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[2] This is the Cameron-Martin formula [1945] as adapted to diffusions by Girsanov [1960]; see, for example, McKean [1969]

[3] p is the transiton density of the diffusion, i.e., it is the (smooth) elementary solution of $\partial p/\partial x = \mathfrak{G}p$

[4] $\mathfrak{G}^\dagger\psi_0^2 = 0$

[5] $\sqrt{-1}Q^\bullet = \partial H/\partial Q^*$ with $*$ = conjugation

[6] McKean-Vaninsky [1993(2)]

$F(Q) = -Q^3$, and for focussing NLS with $F(Q) = -Q^4$ and a 2-dimensional Bessel process in place of the Brownian motion. Both cases fall naturally into the micro-canonical format, $\int Q^2$ being a constant of motion, and the micro-canonical ensemble makes sense since $E[\exp \int Q^{6-} \mid \int Q^2 = N] < \infty$,[7] but the outcome of the thermodynamic limit is unknown.

The present condition $Q^{-2}F(Q) \uparrow \infty$ permits a simple attack (steepest descent) capitalizing upon the diffusion format expounded above and an allied method (loops) of Döblin [1937]. The next section (2) presents the explicitly solvable example $F(Q) = m^2Q^2/2$; it is a continuous analog of the spherical model of Berlin-Kac [1952], for which see also the finer results of Molchanov-Sudarev [1975]. Section 3 is preparatory to the main computation (Sect. 4). Section 5 isolates the local limit theorem which lies at the bottom of it all and places it in the context of Martin boundaries. Gibbs' principle occupies the final section (6); compare, esp. Kolmogorov [1949] and also Dobrushin-Tirozzi [1977] for its novel insistence upon the fact that, for Gibbsian-Markovian lattice fields, the local central limit theorem and its superficially weaker integral form are really one and the same.

2. An Example

The micro-canonical ensemble for the Ornstein-Uhlenbeck process with mass m provides the simplest illustration; it corresponds to $F(Q) = m^2Q^2/2$. The petit partition function is found to be

$$Z = \int \frac{\exp\{-\frac{1}{2}\int[(Q')^2 + m^2Q^2]\}}{(2\pi 0+)^{\infty/2}} d^\infty Q = \frac{1}{2\,\mathrm{sh}(mL/2)}\,.$$

Fix a compact test function ϕ and, to evaluate the micro-canonical mean[8]

$$M_{N/L=D}[e^{\int \phi Q}] = \frac{M_L[e^{\int \phi Q}, \int Q^2 = N]}{Z_{N/L=D} = M_L[\int Q^2 = N]}\,,$$

take the transform

$$\int_0^\infty e^{-c^2N/2} M_L\left[e^{\int \phi Q}, \int Q^2 = N\right] dN$$

$$= Z_+^{-1} \int \frac{e^{-(1/2)\int[(Q')^2 + m_+^2 Q^2]}}{(2\pi 0+)^{\infty/2}} e^{\int \phi Q} d^\infty Q \times Z_+ \quad \text{with} \quad m_+^2 = m^2 + c^2$$

$$= e^{(1/2)\int \phi K \phi} \times \frac{1}{2\,\mathrm{sh}(m_+ L/2)}$$

in which K is the (periodic) Green's function for $-d^2/dx^2 + m_+^2$, and invert it with respect to the variable $s = c^2/2$ to obtain

$$M_{N/L=D}[e^{\int \phi Q}] = Z_{N/L=D}^{-1} \frac{1}{2\pi\sqrt{-1}} \int_{-i\infty}^{+i\infty} \frac{e^{sN}\, e^{(1/2)\int \phi K \phi}}{2\,\mathrm{sh}(m_+ L/2)} ds\,.$$

[7] See Lebowitz-Rose-Speer [1988]

[8] $M(I)$ and the like are expectations $M(I) = \int I\, dM$; $M[\int Q^2 = N] = (\partial/\partial N) M[\int Q^2 \le N]$ and the like stand for densities

Now comes the descent:

$$\frac{e^{sN}}{2\,\mathrm{sh}(m_+L/2)} = \frac{e^{L(sD-(1/2)\sqrt{m^2+2s})}}{1-e^{-L\sqrt{m^2+2s}}}$$

is analytic in the s-plane cut from $-\infty$ to $-m^2/2$, and the integral is controlled by the contribution from the critical point $s = 1/8D^2 - m^2/2$ of function $sD - \frac{1}{2}\sqrt{m^2+2s}$ figuring in the top exponent,[9] with the result that

$$\lim_{L\uparrow\infty} M_{N/L=D}[e^{\int \phi Q}] = e^{\frac{1}{2}\int \phi K \phi},$$

where K is now the *whole-line* Green's funtion for $-d^2/dx^2 + m_*^2$ with $m_* = 1/2D$, i.e., the thermodynamic limit simply changes the mass from m to $m_* = 1/2D$. This agrees with the statement of Sect. 1: for mass $m, \psi_0^2(Q) = (m/\pi)^{1/2} e^{-mQ^2}$ and $\int \psi_0^2 Q^2 = 1/2m$ so that mass $1/2D$ produces mean square D. The genereal computation is, naturally, more difficult; it is carried out in the next two sections under the stated condition $Q^{-2}F(Q) \uparrow \infty$.

3. Preliminaries

The restriction of the petit measure to the micro-canonical ensemble and the implementation of the thermodynamic limit require a few preparations, both general and technical.

Generalities. The first task is to explain why the conditioning by the value of $\int Q^2 = N$ makes sense. $x \to Q(x)$ is the diffusion with infinitesimal operator $\mathfrak{G} = (1/2)\partial^2/\partial Q^2 + m(Q)\partial/\partial Q$. The motion $x \to \left[Q(x), I(x) = \int_0^x Q^2\right]$ is also a diffusion, but now its infinitesimal operator $\mathfrak{G}_+ = \mathfrak{G} + Q^2\partial/\partial I$ is degenerate in that $\partial^2/\partial I^2$ is missing from the top of it. This is not troublesome: the double commutator $[\partial/\partial Q, [\partial/\partial Q, Q^2\partial/\partial I]] = 2\partial/\partial I$ reproduces the missing vector field and now a deep theorem of Hörmander [1967] guarantees that the joint density[10] $p(x, A, B, I) = P\left[Q(x) = B, \int_0^x Q^2 = I \,\middle|\, Q(0) = A\right]$ is smooth in all its variables and also positive, provided only that $x > 0$.[11] This fact dispels any anciety as to the propriety of the micro-canonical ensemble: for example, if $0 < x_1 < x_2 < L$, then the micro-canonical measure of the event $(a_1 \le Q(x_1) < b_1) \cap (a_2 \le Q(x_2) < b_1)$ is nothing but

$$Z^{-1} \int_{-\infty}^{+\infty} dQ_0 \int_{a_1}^{b_1} dQ_1 \int_{a_2}^{b_2} dQ_2 \int_{I_1+I_2\le N} d^2 I$$
$$\times\, p(x_1, Q_0, Q_1, I_1)\, p(x_2 - x_1, Q_1, Q_2, I_2)\, p(L - x_2, Q_2, Q_0, N - I_1 - I_2)$$

9 $(sD - \frac{1}{2}\sqrt{\text{etc.}})'' = 1/16D^3 > 0$ at the critical point; the line of integration is bent so as to pass through the latter

10 $P\left[Q(x) = Q, \int_0^x Q^2 = I\right]$ means $(\partial/\partial Q)(\partial/\partial I)P\left[Q(x) \le Q, \int_0^x Q^2 \le I\right]$. This kind of unconventional but handy notation is used throughout

11 Krylov [1987] can be consulted for such matters

with normalizer

$$Z = \int_{-\infty}^{\infty} p(L, Q, Q, N)\, dQ\,.$$

Now the statement to be proved is that, *for any nice function $H(Q)$ depending upon a limited sample of the path, such as $Q(x)$: $0 \le x \le 1$, the micro-canonical mean*

$$M_{N/L=D}(H) = Z^{-1} \int_{-\infty}^{\infty} E_Q\left[H, Q(L) = Q, \int_0^L Q^2 = N\right] dQ$$

tends, as $L \uparrow \infty$ with fixed $N/L = D$, to the mean

$$M_\infty^*(H) = \int_{-\infty}^{\infty} [\psi_0^*(Q)]^2\, dQ\, E_Q^*(H)$$

for the stationary diffusion with infinitesimal operator $\mathfrak{G}^*$ arising from $F^*(Q) = F(Q) + cQ^2$, the number c being adjusted so that $M_\infty^*(Q^2) = \int [\psi_0^*(Q)]^2 Q^2$ takes the prescribed value D.[12] Naturally, it is necessary to check that this adjustment can really be made. The fact is plain if $F(Q) = m^2Q^2/2$ since $M_\infty(Q^2) = 1/2m$, as noted before. Now consider the general case $Q^{-2}F(Q) \uparrow \infty$. For $c \uparrow \infty$, the ground state ψ_0 for $F^*(Q) = F(Q) + cQ^2$ satisfies

$$c \int Q^2\psi_0^2 \le \tfrac{1}{2} \int (\psi_0')^2 + \int F^*(Q)\psi_0^2 \le \tfrac{1}{2} \int (\psi')^2 + \int F(Q)\psi^2 + c \int Q^2\psi^2$$

for any nice function ψ with $\int \psi^2 = 1$, from which it appears that $D = \int Q^2\psi_0^2$ can be made as small as you like: just concentrate ψ in the vicinity of $Q = 0$. Contrariwise for $c \downarrow -\infty$ the same appraisal shows that $\int Q^2\psi_0^2$ can be made as big as you like by spreading ψ^2 out. Now pick c so that $\int Q^2\psi_0^2 = D$ and note that the micro-canonical ensemble is not changed by the adjustment $F(Q) \to F(Q) + cQ^2$: because of the conditioning $\int Q^2 = N$, it washes out above and below. This means that the proof needs to be made *only for* $c = 0$, i.e., when $N/L = D$ is already the mean $M_\infty(Q^2)$ for the original petit ensemble.

Idea of the Proof. This is easy to describe if $M_\infty(Q^2) = D$. Let H depend upon $Q(x)$: $0 \le x \le 1$, say. Then

$$\begin{aligned} M_{N/L=D}(H) &= Z^{-1} \int_{-\infty}^{\infty} E_Q\left[H, Q(L) = Q, \int_0^L Q^2 = N\right] dQ \\ &= Z^{-1} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \int_0^N dI\, E_Q\left[H, Q(1) = Q', \int_0^1 Q^2 = I\right] \\ &\quad \times P_{Q'}\left[Q(L-1) = Q, \int_0^{L-1} Q^2 = N - I\right]. \end{aligned}$$

[12] $\int \psi_0^2 Q^2 < \infty$ as soon as $Q^{-2}F(Q) \uparrow \infty$ in view of $\int (\psi_0')^2 + \int F(Q)\psi_0^2 = \lambda_0(\mathfrak{G}_0)$

It is to be shown that, for $L \uparrow \infty$ with fixed $N/L = D$,

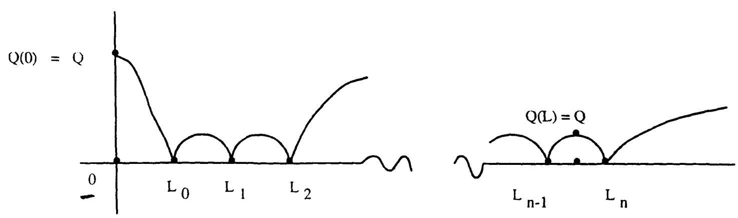
$$P_{Q'}\left[Q(L) = Q, \int_0^L Q^2 = N\right]$$

is proportional to $L^{-1/2}\psi_0^2(Q) \times [1 + o(1)]$, with the result that

$$M_{N/L=D}(H) \cong \frac{\int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \int_0^{\infty} dI\, E_Q\left[H, Q(1) = Q', \int_0^1 Q^2 = I\right] \psi_0^2(Q)}{\int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \int_0^{\infty} dI\, P_Q\left[Q(1) = Q', \int_0^1 Q^2 = I\right] \psi_0^2(Q)}$$
$$= M_\infty(H).$$

Let L_0 be the passage time from $Q(0) = Q'$ to $Q = 0$ and let $L_1 > L_0$, $L_2 > L_1$ etc. be the successive "loop times" for passing from $Q = 0$ to $Q = 1$ and back; see Fig. 1.

Fig. 1



The individual loops $Q(x){:}\, L_{n-1} \leq x \leq L_n$ are independent and identically distributed;

$$D = M_\infty(Q^2) = \lim_{L\uparrow\infty} L^{-1} \int_0^L Q^2 = \lim_{n\uparrow\infty} L_n^{-1} \int_0^{L_n} Q^2 = \frac{E_0\left(\int_0^{L_1} Q^2\right)}{E_0(L_1)},$$

by the law of large numbers; and the diffusion settles (exponentially fast) into its stationary regime, so it is natural to hope that

$$P_{Q'}\left[Q(L) = Q, \int_0^L Q^2 = N\right]$$
$$= P_{Q'}\left[\int_0^L Q^2 = N \,\middle|\, Q(L) = Q\right] \times P_{Q'}[Q(L) = Q]$$
$$\cong P_{Q'}\left[\int_0^L Q^2 = N\right] \times \psi_0^2(Q),$$

in which the first factor simplifies itself, in response to the law of large numbers and the (local) central limit theorem, as follows:

$$P_{Q'}\left[\int_0^L Q^2 = N\right] = P_{Q'}\left[\sum_{k=1}^{n}\int_{L_{k-1}}^{L_k}(Q^2 - D) = 0\right] \quad \text{with} \quad n = [L/E_0(L_1)]$$

$$= P_{Q'}\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_{L_{k-1}}^{L_k}(Q^2 - D) = 0\right]\frac{1}{\sqrt{n}}$$

$$\cong \frac{1}{\sqrt{2\pi\sigma^2 n}} \quad \text{with} \quad \sigma^2 = E_0\left(\int_0^{L_1} Q^2 - DL_1\right)^2,$$

i.e.,

$$P_{Q'}\left[Q(L) = Q, \int_0^L Q^2 = N\right] \cong \frac{1}{\sqrt{2\pi\sigma^2 L/E_0(L_1)}} \times \psi_0^2(Q).$$

It remains to hope, in addition, that the shift $L \to L-1$ and $N \to N-I$ will not disturb this appraisal, and it is the content of Sect. 4 that this is perfectly correct, *but be warned*: if the adjustment $M_\infty(Q^2) = D$ is *not made* beforehand, the present method predicts that $c = -\Delta D/2\sigma^2$, and this is not correct as it takes $c \to \infty$ to make $D \downarrow 0$ – in short, the conditioning $Q(L) = Q$ must now distort the Gaussian law, and that is not so easy to track.

Technicalities. These are needed in Sect. 4; at a first reading, just note the facts and pass on. $Q(0) = 0$, T is the passage time to 1, Q is a free variable, and $\mathfrak{t}(x, Q)$ is the local time $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\,\text{meas}(0 \le x' \le x : Q \le Q(x') < Q + \varepsilon)$.[13]

Item 1. $E_0(e^{rT}) < \infty$ *for small values of* $r > 0$.

Proof. $\mathfrak{G}$ has pure point spectrum and it is the same if you kill the particle at $Q = 1$, only now the ground state is displaced from $\lambda_0(\mathfrak{G}) = 0$ to $c < 0$. The rest follows from

$$P_0(T = x) = -(\partial/\partial x)\int_{-\infty}^{1} p(x, 0, Q)\,dQ = -\int_{-\infty}^{1} \mathfrak{G}^{\dagger} p(x, 0, Q)\,dQ$$

$$= -\left[\frac{1}{2}\frac{\partial}{\partial Q} - m(Q)\right] p(x, 0, Q)\Big|_{-\infty}^{1}$$

$$= -\tfrac{1}{2}\,p_3(x, 0, 1) \le \text{constant} \times e^{cx},$$

p being the transition density for the killed diffusion.[14]

[13] See Itô-McKean [1965] for such matters
[14] See Itô-McKean [1965] for details

Item 2. $E_0[\mathfrak{t}(T,Q)] = \int\limits_{Q^+}^{1} \psi_0^{-2} \times 2\psi_0^2(Q)$ *for* $Q \le 1$, Q^+ *being the larger of* Q *and* 0; *also* $E_0[\mathfrak{t}^2(T,Q)] = \int\limits_{Q^+}^{1} \psi_0^{-2} \int\limits^{1} \psi_0^{-2} \times \psi_0^4(Q)$.

Proof. $E_0(\mathfrak{t}) = \int\limits_0^{\infty} p(x,0,Q)\,dx$ is nothing but the Green's function $G(0,Q)$ for the operator $\mathfrak{G}$ [15] restricted to $Q \le 1$, with killing at $Q = 1$, and from $\mathfrak{G} = (1/2)\psi_0^{-2}(\partial/\partial Q)\psi_0^2(\partial/\partial Q)$, it is easy to see that

$$G(Q',Q) = \int\limits_{\bullet}^{1} \psi_0^{-2} \times 2\psi_0^2(Q)\,,$$

"spot" being the larger of Q and Q'. The second evaluation is similar.

Item 3. $E_0\left[\int\limits_0^T x\,d\mathfrak{t}(x,Q)\right] \le \int\limits_0^1 \psi_0^{-2} \int\limits^1 \psi_0^{-2} \int\limits_{-\infty}^1 2\psi_0^2 \times 2\psi_0^2(Q)$.

Proof. $E_0(\text{etc.}) = \int\limits_0^{\infty} xp(x,0,Q) = \int\limits_{-\infty}^{1} G(0,Q')G(Q',Q)\,dQ'$; now use the explicit form of G from Item 2.

Item 4.[16] $E_0\left[\int\limits_0^T I(x)\,d\mathfrak{t}(x,Q)\right] \le \int\limits_0^1 \psi_0^{-2} \int\limits^1 \psi_0^{-2} \int\limits_{-\infty}^1 Q^2 2\psi_0^2 \times 2\psi_0^2(Q)$.

Proof. See Item 3.

Item 5. $E_0[e^{-\alpha T - \beta I(T)}]$ *is of modulus* < 1 *for* α *and* β *in the closed (right-hand) half-plane, the origin expected; moreover, it vanishes as* α *and/or* β *tend to* ∞ *in that region.*

Proof. The expectation can be of modulus 1 only if both α and β are real and $P_0[\alpha T + \beta I \in 2\pi Z] = 1$.[17] But a typical path starting at 0 and hitting 1 for the first time will immediately overshoot, so that if Q is such a path with $\alpha T + \beta I = 2\pi n$, then it is the same for any nearby path. But this is manifestly wrong if it would happen with positive probability. The rest follows from the deeper fact that T and $I = I(T)$ have a joint density: indeed, $h(Q) = E_Q[e^{-\alpha T - \beta I}]$ satisfies $\mathfrak{G}h = (\alpha + \beta Q^2)h$ for $Q < 1$, which is to say that the (possibly fictitious) density $p = P_Q[T = x, I = N]$ is a weak solution of $\partial p/\partial x = \mathfrak{G}p - Q^2 \partial p/\partial N$, and now Hörmander [1967] guarantees that p is an honest function.

---

[15] $\mathfrak{G}G = -1$

[16] $I(x) = \int\limits_0^x Q^2$

[17] Z is now the integers

4. The Thermodynamic Limit

Fix a nice function H depending upon $Q(x) : 0 \le x \le 1$, say, and form its micro-canonical mean

$$
\begin{aligned}
M_{N/L=D}(H) &= Z^{-1} \int E_Q\left[H, Q(L) = Q, \int_0^L Q^2 = N\right] dQ \\
&= Z^{-1} \int dQ \int dQ' \int_0^N dN' \\
&\quad \times E_Q\left[H, Q(L') = Q', \int_0^{L'} Q^2 = N'\right] \\
&\quad \times P_{Q'}\left[Q(L-L') = Q, \int_0^{L-L'} Q^2 = N - N'\right]
\end{aligned}
$$

with $1 \le L' \le 2 < L$, say. Now the micro-canonical mean of $Q^2(x)$ is $D = N/L$, independently of $0 \le x < L$,[18] so only a small error is incurred if the integration is restricted to a big box in $QQ'N'$-space; in particular, it is permitted to replace the QQ'-density in the top of the micro-canonical mean by the double convolution

$$
\begin{aligned}
I * II = &\int_0^L dL' \int_0^N dN' E_Q\left[H, Q(L') = Q', \int_0^{L'} Q^2 = N'\right] A(L') B(N') \\
&\times P_{Q'}\left[Q(L-L') = Q, \int_0^{L-L'} Q^2 = N - N'\right]
\end{aligned}
$$

in which A and B are smooth compact functions, with A vanishing outside $[1, 2]$, $\int_1^2 A = 1$, and $B = 1$ on a long interval containing $N' = 0$.

The introduction of A and B and the integration over $1 \le L' \le 2$ looks artificial but is essential to the method. It makes I smooth and compact, with transform $\widehat{I} = \int_0^\infty \int_0^\infty e^{-\alpha L' - \beta N'} I \, dL' \, dN'$ which is both analytic in C^2 and class $C_\downarrow^\infty(\sqrt{-1}R^2)$. This decay, or something like it, must be present to justify the descent: without it, the inverse transform simply masks the realities; moreover, *it cannot be obtained from* $\widehat{II}$, as is easily confirmed for the case of pure Brownian motion: in fact, with positive

[18] $M_{N/L=D}[Q^2(x)] = L^{-1} \int_0^L M[Q^2(x')] \, dx' = L^{-1} M\left[\int_0^L Q^2\right] = N/L = D$

imaginary $\alpha = \sqrt{-1}p$ and $\beta = \sqrt{-1}q/2$,

$$\int_0^\infty e^{-\alpha x} dx \int_0^\infty e^{-\beta I} dI\, P_0 \left[Q(x) = 0, \int_0^x Q^2 = I \right]$$

$$= \sqrt{\frac{m}{\pi}} \int_0^\infty \frac{e^{-\alpha x} dx}{\sqrt{\operatorname{sh} mx}} \quad \text{with} \quad m = \sqrt{2\beta} = q e^{\sqrt{-1}\pi/4}$$

$$= \frac{1}{\sqrt{\pi m}} \int_0^\infty e^{-(p/q)\times e^{\sqrt{-1}\pi/4}} \frac{dx}{\sqrt{\operatorname{sh} x}}$$

behaves like $q^{-1/2}$ or like $p^{-1/2}$ according as p/q is of moderate size or $p/q \uparrow \infty$.
Now, in the language of the local time,

$$\mathfrak{t}(x, Q) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \operatorname{meas}(0 \le x' \le x : Q \le Q(x') < Q + \varepsilon),$$

$$\widehat{II} = E_{Q'} \left[\int_0^\infty e^{-\alpha L} e^{-\beta \int_0^L Q^2} d\mathfrak{t}(L, Q) \right].$$

This may be put into a convenient form by cutting up the half-line $0 \le L < \infty$ according to the passage time $L_0 = \min[x : Q(x) = 0]$ and the succeeding "loop times" $L_1 < L_2 <$ etc. depicted in Fig. 1. Write $I(x) = \int_0^x Q^2$ and $I_n = I(L_n)$ for $n = 0, 1, 2, 3$, etc. Then

$$\widehat{II} = E_{Q'} \left[\int_0^{L_0} e^{-\alpha x} e^{-\beta I(x)} d\mathfrak{t}(x, Q) \right]$$

$$+ E_{Q'}[e^{-\alpha L_0 - \beta I_0}] \times \sum_{n=0}^\infty E_0[e^{-\alpha L_n - \beta I_n}] \times E_0 \left[\int_0^{L_1} e^{-\alpha x - \beta I} d\mathfrak{t} \right]$$

$$= E_{Q'} \left[\int_0^{L_0} e^{-\alpha x - \beta I} d\mathfrak{t} \right] + \frac{E_{Q'} \left[\int_{L_0}^{L_1} e^{-\alpha x - \beta I} d\mathfrak{t} \right]}{1 - E_0[e^{-\alpha L_1 - \beta I_1}]}$$

$$= \widehat{III} + \widehat{IV},$$

in which the independence of loops was used to reduce the sum in line 2 to a geometrical series, and the outlying factors, representing the initial passage from Q' to 0 and the final passage from 0 to Q, were combined to produce $E\left[\int_{L_0}^{L_1} \text{etc. } d\mathfrak{t} \right]$.
Now the technical Items 2, 3, 4 show that $\widehat{III}$ is of class $C^1(\sqrt{-1}R^2)$, so the same is true of the product $\widehat{I}\,\widehat{III}$, with the added feature that this product and its gradient

are summable. It follows that

$$I * III = -\frac{1}{4\pi^2} \int\limits_{\sqrt{-1}R^2} e^{\alpha L+\beta N}\widehat{III} = O(L^{-1}),$$

this estimate being independent of QQ' in the samll, as you will readily check. The same remarks apply, in part, to $\hat{I}\widehat{IV}$, only now the inverse transform must be taken as in Fig. 2a to avoid the root of $e(\alpha,\beta) = E_0[e^{-\alpha L_1-\beta I_1}] = 1$ at $\alpha = \beta = 0$; see Item 5. Now comes the descent.

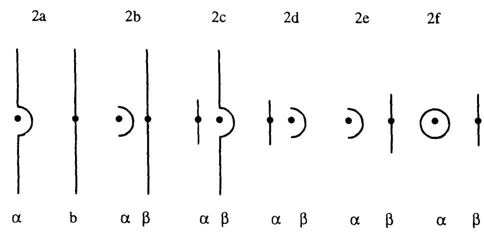


Fig. 2

Step 1 removes the vertical α-segments in Fig. 2a with an error of magnitude L^{-1}, as above, leaving the integral depicted in Fig. 2b. Step 2 distorts Fig. 2b into Fig. 2c. Step 3 removes the vertical β-segments at like cost, leaving the integral depicted in Fig. 2d. Step 4 distorts Fig. 2d into Fig. 2e. Now $e(\alpha,\beta) = 1$ has, for small real β, a simple root $\alpha = \alpha(\beta) = -\beta D + \sigma^2\beta^2/2 + \text{etc.}$ with positive $\sigma^2 = E_0(I_1 - L_1 D)^2/E_0(L_1)$, as you will check,[19] and you may take the β-segment in Fig. 2e so short that $\alpha(\beta)$ lies inside the α-circle seen there. Step 5 estimates the integral over the omitted left-hand arc of that with the aid of technical Items 1 and 2: if the radius r is small, then the top of $\widehat{IV}$ is of magnitude $\leq E_{Q'}(e^{rL_0})E_0[e^{rL_1}\mathfrak{t}(L_1,Q)]$, and as this is finite, so the integral is controlled by $\int_{\pi/2}^{3\pi/2} e^{r\cos\theta L}d\theta = O(L^{-1})$. Step 6 is now permitted, which is to integrate as in Fig. 2f, evaluating the α-integral as $2\pi\sqrt{-1}\times$ the residue at the pole $\alpha(\beta) = -\beta D + \sigma^2\beta^2/2 + \text{etc.}$: up to errors of magnitude L^{-1},

$$\begin{aligned} I * IV &= -\frac{1}{4\pi^2}\int\limits_{\updownarrow} e^{\beta N}d\beta \int\limits_{\odot} e^{\alpha L}d\alpha \hat{I}\cdot\widehat{IV} \\ &= -\frac{1}{4\pi^2}\int\limits_{\updownarrow} e^{\beta N}d\beta 2\pi\sqrt{-1}\, e^{-LD\beta+L\sigma^2\beta^2/2+\text{etc.}}\hat{I}(0)\,\frac{E_0[\mathfrak{t}(L_1,Q)]}{E_0(L_1)} + o(1), \end{aligned}$$

in which the $o(1)$ is controlled by the size of the β-segment. But $N = DL$, so the β-integral is nothing but

$$\frac{1}{2\pi\sqrt{-1}} \int\limits_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} e^{L\sigma^2\beta^2/2}d\beta = \frac{1}{\sqrt{2\pi\sigma^2 L}},$$

---

19 $-E_0(L_1) = \partial e/\partial\alpha$ at $\alpha = \beta = 0$ does not vanish. $\alpha'(0) = -E_0(I_1)/E_0(L_1) = -D$

up to an exponentially small error; also, $E_0[\mathfrak{t}(L_1, Q)]/E_0(L_1) = \psi_0^2(Q)$, as you may check either by the technical Item 2 or by the law of large numbers;[20] and all this estimation is independent of QQ' in the small, so that the density $I * II$ in the top of the micro-canonical mean can be replaced by

$$\int_0^\infty dL' \int_0^\infty dN' E_Q\left[H, Q(L') = Q', \int_0^{L'} Q^2 = N'\right] A(L')B(N') \frac{\psi_0^2(Q)}{\sqrt{2\pi\sigma^2 L}} .$$

You can even remove the B and integrate over all $QQ' \in R^2$, at the cost of a small error relative to Z, and so obtain

$$M_{N/L=D}(H) \cong \frac{\int_{-\infty}^{\infty} \psi_0^2(Q) E_Q(H) \times (2\pi\sigma^2 L)^{-1/2}}{Z = \int_{-\infty}^{\infty} \psi_0^2(Q) E_Q(1) \times (2\pi\sigma^2 L)^{-1/2}} = M_\infty(H) ,$$

the same principles having been applied to the bottom as to the top. The proof is finished.

5. Martin Boundaries and the Local Limit Theorem

Amateurs of the Martin boundary[21] will recognize the ratio

$$Z^{-1} P_{Q'}\left[Q(L - L') = Q, \int_0^{L'} Q^2 = N - N'\right] :$$

assuming it has a limit $h(L', Q', N')$ for suitable, $L \uparrow \infty$, $N \uparrow \infty$, and Q, you expect this function to solve

$$0 = \left[\frac{\partial}{\partial L'} + \frac{1}{2}\frac{\partial^2}{\partial Q'^2} + m(Q')\frac{\partial}{\partial Q'} + (Q')^2 \frac{\partial}{\partial N'}\right] h = \left(\frac{\partial}{\partial L'} + \mathfrak{G}_+\right) h .$$

Now for fixed $1 \le L' < \infty$, $L \uparrow \infty$, and general $N/L = D$, the micro-canonical mean

$$M_{N/L=D}(H) = \int E_Q\left[H, Q(L') = Q', \int_0^{L'} Q^2 = N'\right]$$

$$\times Z^{-1} P_{Q'}\left[Q(L - L') = Q, \int_0^{L-L'} Q^2 = N - N'\right]$$

[20] $E_0[\mathfrak{t}(L_1, Q)]/E_0(L_1) = \lim_{L\uparrow\infty} L^{-1} \int_0^L p(x, 0, Q)\, dx = \psi_0^2(Q)$

[21] See Williams [1979] for such matters

tends to the petit mean $M^*_\infty(H)$ corresponding to $F^*(Q) = F(Q) + cQ^2$, c being adjusted so that $M^*_\infty(Q^2) = D$. The latter can be expressed as

$$M^*_\infty(H) = \int E_Q\left[H, Q(L') = Q', \int_0^{L'} Q^2 = N'\right] \times \psi_0^*(Q')\psi_0^*(Q)\,\frac{\psi_0(Q)}{\psi_0(Q')}\,e^{-LN'}\,e^{(\lambda_0^*-\lambda_0)L'},$$

since the density dP_Q^*/dP_Q restricted to the field of $Q(x) : 0 \le x \le L'$ is

$$\frac{\exp\left[\int_0^{L'} m^* dQ - \frac{1}{2}\int_0^{L'}(m^*)^2\right]}{\exp\left[\int_0^{L'} m\,dQ - \frac{1}{2}\int_0^{L'} m^2\right]} = \frac{\exp\left[\int_0^{L'} d\lg\psi_0^*(Q) - \int_0^{L'} F^*(Q) + \lambda_0^* L'\right]}{\exp\left[\int_0^{L'} d\lg\psi_0(Q) - \int_0^{L'} F(Q) + \lambda_0 L'\right]}$$

$$= \frac{\psi_0^*(Q')}{\psi_0^*(Q)}\,\frac{\psi_0(Q)}{\psi_0(Q')}\,e^{-c\int_0^{L'} Q^2}\,e^{(\lambda_0^*-\lambda_0)L'},$$

in which $Q = Q(0)$ and $Q' = Q(L')$, so for fixed $Q, L \uparrow \infty$, and $N/L = D$, you expect that

$$\lim_{L\uparrow\infty} Z^{-1}P_{Q'}\left[Q(L-L') = Q', \int_0^{L-L'} Q^2 = N - N\right]$$

$$= \psi_0^*(Q')\psi_0^*(Q)\,\frac{\psi_0(Q)}{\psi_0(Q')}\,e^{-cN'}\,e^{(\lambda_0^*-\lambda_0)L'} \equiv h\,.$$

This is the "local limit/renewal theorem"; compare Feller [1971]. The function h is, in fact, a solution of $(\partial/\partial L' + \mathfrak{G}_+)h = 0$, and the formula is perfectly correct: indeed, the results of Krylov [1987] show that the functions $Z^{-1}P_{Q'}$[etc.] form a locally compact family and, to identify the limit unambiguously, you have only to choose H to be a general test function in the variables Q, Q', and N', L' being fixed. These remarks suggest that it may be interesting to study the full Martin boundary of the space-time diffusion $x \to \left[x, Q(x), I(x) = \int_0^x Q^2\right]$ with infinitesimal operator $\partial/\partial x + \mathfrak{G}_+ = \partial/\partial x + \frac{1}{2}\partial^2/\partial Q^2 + m\partial/\partial Q + Q^2\partial/\partial I$. This is left to the future except to note that for the Ornstein-Uhlenbeck process with mass m, the minimal space-time functions are of the form

$$h = \exp\left[\alpha Q\,e^{m'x} - \frac{\alpha^2}{2}\,\frac{e^{2m'x}-1}{2m'} - \beta x + \beta\gamma^2 - \gamma I/2\right]$$

with $\gamma \ge -m^2$, $2\beta = m \pm \sqrt{\gamma + m^2}$, and either $\alpha = 0$ or else $m' = \mp(1/2)\sqrt{\gamma + m^2}$; in particular, the boundary is a topological plane. Note that the function h is more general than the "micro-canonical" functions suggested by the thermodynamic limit; the latter arise precisely for $\alpha = 0$. It is natural to conjecture that the general boundary is similar, but nothing more is known about it.

6. Gibbs' Principle

Let P_0 be the law of the original stationary diffusion of the petit canonical ensemble for $L = \infty$, and let P be the law of any other stationary process (Markovian or not). Let Δ be the density of P with respect to P_0, both restricted to the field of $Q(x):0 \leq x \leq L$. The rate of entropy production of P relative to P_0 is

$$h = \lim_{L\uparrow\infty} -\frac{1}{L} \int \Delta \lg \Delta \, dP_0 = \lim_{L\uparrow\infty} -\frac{1}{L} \int \lg \Delta \, dP \,;$$

$h \leq 0$ in any case, with the understanding that $h = -\infty$ if Δ does not make sense. Gibbs principle[22] asserts that, in the class of all laws P with fixed mean square $\int Q^2 \, dP = D$, the micro-canonical ensemble makes h biggest. This is easy to prove.[23] Let Δ_* and h_* be the corresponding objects with the micro-canonical ensemble P_* in place of P_0. Then

$$h - h_* = \lim_{L\uparrow\infty} \frac{1}{L} \int \lg \frac{\Delta_*}{\Delta} \, dP$$

is independent of P: in fact, using for reference the free Brownian motion with starting point distributed by the infinite measure dh, the density of P_0 is[24]

$$\begin{aligned} \psi_0^2(Q) \, e^{\int m_0 \, dQ - (1/2)\int m_0^2} &= \psi_0^2(Q) \, e^{\int d \lg \psi_0} \, e^{-\int (F - \lambda_0)} \\ &= \psi_0(Q)\psi_0(Q') \, e^{-\int (F-\lambda_0)} \quad \text{with} \quad Q' = Q(L) \,, \end{aligned}$$

so

$$\frac{\Delta_*}{\Delta} = \frac{dP_0}{dP_*} = \frac{\psi_0(Q)\psi_0(Q')}{\psi_*(Q)\psi_*(Q')} \, e^{-c\int Q^2 + (\lambda_0 - \lambda_*)L}$$

and

$$\begin{aligned} \int \lg \frac{\Delta_*}{\Delta} \, dP &= 2 \int \lg \frac{\psi_0(Q)}{\psi_*(Q)} \, dP - cDL + (\lambda_0 - \lambda_*)L \\ &= O(1) + L(\lambda_0 - \lambda_* - cD) \,. \end{aligned}$$

But then it is the same to maximize h_* as to maximize h, and it is easy to see that $h_* < 0$ unless $\Delta_* = 1$, i.e., unless $P = P_*$.

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[22] Gibbs [1902]

[23] Varadhan showed us this nice way

[24] The integral $\int$ is extended from $x = 0$ to $x = L$

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