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Geometry of Geometrically Finite One-Dimensional Maps

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Abstract. We study the geometry of certain one-dimensional maps as dynamical systems. We prove the property of bounded and bounded nearby geometry of certain $C^{1+\alpha}$ one-dimensional maps with finitely many critical points. This property enables us to give the quasisymmetric classification of geometrically finite one-dimensional maps.

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1. Introduction

Two smooth maps f and g from a one-dimensional C^2-Riemannian manifold M into itself are topologically conjugate if there is a homeomorphism h from M onto itself such that $f \circ h = h \circ g$. A nontrivial problem [10] asked by Sullivan was about whether the conjugating map h is necessarily quasisymmetric [1]. (We note that when f and g are both holomorphic and expanding maps on a domain in the Riemann sphere, then h is quasiconformal [1] because of bounded geometry property [9, 10].) In [4 and 5], we studied this kind of problem for an interval map with one critical point. In this paper, we generalize the results of [4 and 5] to geometrically finite maps which are certain one-dimensional maps with finitely many critical points (see Definition 3 in Sect. 2). We prove that the induced sequence of nested partitions of M by a geometrically finite map (see Sect. 2 for the definition) has bounded and bounded nearby geometry (see Definition 2 in Sect. 2).

Theorem A. *Suppose f from M into itself is geometrically finite and $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then η is of bounded and bounded nearby geometry.*

A homeomorphism h from M onto itself is *quasisymmetric* [1] if there is a constant $K > 0$ so that for any two points x and y in M,

$$K^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq K,$$

where $z = (x + y)/2$ is the midpoint of x and y. Two maps f and g are quasisymmetrically conjugate if they are topologically conjugate and the conjugacy is quasisymmetric. From Milnor and Thurston's paper [8], any topological class of geometrically finite maps is determined by its kneading invariant. Using bounded and bounded nearby geometry, we can further prove that the quasisymmetric classes of geometrically finite maps are determined by their kneading invariants.

Theorem B. *Suppose f and g from M into itself are geometrically finite and topologically conjugate. They are then quasisymmetrically conjugate.*

Before proving these theorems we will prove two important lemmas in Sect. 3 to estimate the nonlinearity of the iterates of a geometrically finite map. The reader may refer to [6] for a more general version of Lemma 2 in Sect. 3.

2. Geometrically Finite One-Dimensional Maps

Suppose M is the interval $[-1, 1]$ or the unit circle S^1 and f from M into itself is a C^1 map. A point $c \in M$ is said to be critical if $f'(c) = 0$ and it is said to have power law type at c if there is a number $\gamma > 1$ such that

$$\lim_{x \mapsto c+} \frac{f'(x)}{|x - c|^{\gamma - 1}} \quad \text{and} \quad \lim_{x \mapsto c-} \frac{f'(x)}{|x - c|^{\gamma - 1}}$$

have nonzero limits A and B. Here γ and $\tau = A/B$ are called the exponent and asymmetry of f at c [5]. Let $C = \{c_1, c_2, \dots, c_l\}$ be the set of critical points of f. Henceforth, we will assume that all the critical points of f are of power law type and $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_l\}$ is the set of corresponding exponents. Furthermore, we assume that f maps the boundary of M (if it is not empty) into itself and the one-sided derivative of f at every boundary point of M is nonzero.

Definition 1. We say that f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if
$(*)$ the derivative f' of f is α-Hölder continuous, and
$(**)$ for every critical point c_i of f, there is a small neighborhood U_i of c_i in M so that $r(x) = f'(x)/|x - c_i|^{\gamma_i - 1}$ is α-Hölder on $\{x < c_i\} \cap U_i$ and on $\{x > c_i\} \cap U_i$.

Suppose the set of critical orbits $CO = \bigcup_{i=0}^{\infty} f^{\circ i}(C)$ is finite. Then $\eta_1 = \{L_1, \dots, L_d\}$, the closures of the intervals in the complement of CO in M, is a Markov partition $\left(\text{namely } f(L_i) = \bigcup_{i_k} L_{i_k} \text{ for every } L_i \in \eta_1\right)$. Let $\eta_n = \{I | f^{\circ n}(I) = L_i$ for some L_i and $f^{\circ n}|I$ is a homeomorphism$\}$. We call η_n the n^{th}-partition and $\eta = \{\eta_n\}_{n=1}^{\infty}$ *the induced sequence of nested partitions* of M by f.

Definition 2. The induced sequence $\eta = \{\eta_n\}_{n=1}^{\infty}$ is said to be of bounded geometry if there is a constant $K > 0$ so that the ratio $|J|/|I| \geq K$ for every pair $J \subset I$ with

$J \in \eta_{n+1}$ and $I \in \eta_n$. And it is said to be of bounded nearby geometry if there is a constant $K > 0$ so that the ratio $|J_1|/|J_2| \geq K$ for every pair J_1 and J_2 in η_n with a common endpoint.

Remark 1. We use B_f to denote the largest possible value of K in this definition.

Let λ_n be the maximum of lengths of the intervals in η_n. Then η_n is said to tend to zero exponentially if there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for all positive integers n.

Definition 3. We say that f is geometrically finite if
(1) f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$,
(2) $CO = \bigcup_{i=0}^{\infty} f^{\circ i}(C)$ is finite,
(3) no critical point is a periodic point of f,
(4) η_n tends to zero exponentially.

Remark 2. (2) and (3) are equivalent to the statement that f has only finitely many critical points and every critical point is preperiodic.

3. Estimates of Nonlinearity

We need the naive distortion lemma for one-dimensional maps.

The Naive Distortion Lemma. *Suppose g from V into M is a $C^{1+\alpha}$ map for some $0 < \alpha \leq 1$ and $a_0 = \inf_{x \in V} |g'(x)| > 0$. Let $b_0 = \sup_{x \neq y \in V} \dfrac{|g'(x) - g'(y)|}{|x-y|^\alpha} < \infty$. Then for any two sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ in V,*

$$\log\left(\prod_{i=1}^{n}\left|\frac{g'(x_i)}{g'(y_i)}\right|\right) \leq \frac{b_0}{a_0}\sum_{i=1}^{n}|x_i - y_i|^\alpha .$$

Proof. The proof of this lemma is easy for

$$\begin{aligned}\log\left(\prod_{i=1}^{n}\left|\frac{g'(x_i)}{g'(y_i)}\right|\right) &\leq \sum_{i=1}^{n}\left|\log|g'(x_i)| - \log|g'(y_i)|\right| \\ &\leq \sum_{i=1}^{n}\frac{1}{a_0}|g'(x_i) - g'(y_i)| \leq \sum_{i=1}^{n}\frac{b_0}{a_0}|x_i - y_i|^\alpha .\end{aligned}$$

Suppose f from M into itself is geometrically finite., Next two lemmas provide estimates for the nonlinearity of the iterates of f.

Let $U = \bigcup_{i=1}^{l} U_i$ be the union of U_i in (**) of Definition 1 and $V = \overline{M \setminus U}$ be the closures of the complement of U in M. By (4) of Definition 3, we can take U_i so that every U_i consists of two intervals in η_{n_0} (for a fixed integer n_0) and, without loss of generality, we may assume that $\bar{U} \cap \left(\bigcup_{i=1}^{\infty} f^{\circ i}(C)\right) = \emptyset$.

Lemma 1. *There is a constant $K > 0$ such that if $f^{\circ i}(y)$ and $f^{\circ i}(x)$ are in the same connected component of V for every $i = 0, 1, \ldots, n-1$, then*

$$\log\left(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\right) \leq K .$$

Proof. It follows directly from the naive distortion lemma.

For the fixed integer n_0 and x and y in an interval $\tilde{I}$ in η_{n_0}, we use $I_{xy} \subset \tilde{I}$ to denote the interval bounded by x and y.

Lemma 2. *There is a constant $K > 0$ so that if f from I_{xy} to $f^{\circ n}(I_{xy})$ is injective and $f^{\circ n}(I_{xy})$ is a subinterval of some U_i, then*

$$\log\left(\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|}\right) \le K .$$

Remark 3. A more general version of this lemma appears in [6].

Proof. The ratio $|(f^{\circ n})'(x)|/|(f^{\circ n})'(y)|$ equals the product $\prod_{i=0}^{n-1} |f(x_i)|/|f'(y_i)|$, where $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$. We divide this product into two products,

$$\prod_{x_i, y_i \in V} \frac{|f'(x_i)|}{|f'(y_i)|} \quad \text{and} \quad \prod_{x_i, y_i \in U} \frac{|f'(x_i)|}{|f'(y_i)|} .$$

From the naive distortion lemma, there is a constant $K_1 > 0$ so that

$$\log\left(\prod_{x_i, y_i \in V} \frac{|f'(x_i)|}{|f'(y_i)|}\right) \le K_1 .$$

To estimate the second one, we write $\prod_{x_i, y_i \in U} |f'(x_i)|/|f'(y_i)| = I \cdot II \cdot III$, where

$$I = \prod_{x_i, y_i \in U} \left(\frac{|x_i - c_{k_i}|^{\gamma_{k_i}}}{|f(x_i) - f(c_{k_i})|} \frac{|f(y_i) - f(c_{k_i})|}{|y_i - c_{k_i}|^{\gamma_{k_i}}}\right)^{m_{k_i}},$$

$$II = \prod_{x_i, y_i \in U} \left(\frac{|y_i - c_{k_i}|^{\gamma_{k_i}-1}}{|f'(y_i)|} \frac{|f'(x_i)|}{|x_i - c_{k_i}|^{\gamma_{k_i}-1}}\right),$$

and

$$III = \prod_{x_i, y_i \in U} \left(\frac{|f(x_i) - f(c_{k_i})|^{m_{k_i}}}{|f(y_i) - f(c_{k_i})|^{m_{k_i}}}\right),$$

where $m_k = (\gamma_{k_i} - 1)/\gamma_{k_i}$ if x_i and y_i are in U_{k_i}. If we take

$$g_i'(x) = \frac{|x - c_{k_i}|^{\gamma_{k_i}}}{|f(x) - f(c_{k_i})|} \quad \text{or} \quad \frac{|x - c_{k_i}|^{\gamma_{k_i}-1}}{f'(x)},$$

then from $(**)$ of Definition 1 and the naive distortion lemma, there is a constant $K_2 > 0$ such that

$$\log(I \cdot II) \le K_2 .$$

Now let us concentrate on the estimate of

$$\log\left(\prod_{x_i, y_i \in U} \left(\frac{|f(x_i) - f(c_{k_i})|}{|f(y_i) - f(c_{k_i})|}\right)^{m_{k_i}}\right).$$

Write

$$\frac{f(x_i)-f(c_{k_i})}{f(y_i)-f(c_{k_i})}=1+\frac{f(x_i)-f(y_i)}{f(y_i)-f(c_{k_i})}$$

then

$$\log\left(\prod_{x_i,y_i\in U}\left(\frac{|f(x_i)-f(c_{k_i})|}{|f(x_i)-f(c_{k_i})|}\right)^{m_{k_i}}\right)\le\sum_{s=1}^{r-1}\frac{1}{m_{k_{i_s}}}\log\left(1+\frac{|f(x_{i_s})-f(y_{i_s})|}{|f(x_{i_s})-f(c_{k_{i_s}})|}\right)$$
$$\le K_3\sum_{s=1}^{r-1}\frac{|f(x_{i_s})-f(y_{i_s})|}{|f(x_{i_s})-f(c_{k_{i_s}})|},$$

where $i_1<i_2<\ldots<i_{r-1}<n$ and $K_3>0$ is a constant. Let $i_r=n$. Using Lemma 1, there is a constant $K_4>0$ such that for $0\le s<r$,

$$\frac{|f(x_{i_s})-f(y_{i_s})|}{|f(x_{i_s})-f(c_{k_{i_s}})|}\le K_4\frac{|x_{i_{s+1}}-y_{i_{s+1}}|}{|y_{i_{s+1}}-f^{\circ(i_{s+1}-i_s)}(c_{k_{i_s}})|}.$$

Suppose $D>0$ is the distance between U and the post-critical orbits $\bigcup_{i=1}^{\infty}f^{\circ i}(C)$. For $0\le s<r-1$, since $y_{i_{s+1}}$ is in U,

$$\frac{|x_{i_{s+1}}-y_{i_{s+1}}|}{|y_{i_{s+1}}-f^{\circ(i_{s+1}-i_s)}(c_{k_{i_s}})|}\le K_4\frac{|x_{i_{s+1}}-y_{i_{s+1}}|}{D}.$$

For $s=r-1$, by the hypothesis, y_{i_r} is in U too. So

$$\frac{|x_{i_r}-y_{i_r}|}{|y_{i_r}-f^{\circ(i_r-i_s)}(c_{k_{i_{r-1}}})|}\le K_4\frac{|x_{i_r}-y_{i_r}|}{D}.$$

Hence there is a constant $K_5>0$ such that

$$\log\left(\prod_{x_i,y_i\in U}\frac{|f(x_i)-f(c_{k_i})|^{m_{k_i}}}{|f(x_i)-f(c_{k_i})|^{m_{k_i}}}\right)\le K_5.$$

Combining all the estimates together we get a constant $K>0$ satisfying the lemma.

Remark 4. From the proof of Lemma 2, one can see that the distortion of f along an orbit is controlled by $|x_n-y_n|/\operatorname{dist}\left(y_n,\bigcup_{i=1}^{\infty}f^{\circ i}(C)\right)$ even if the orbit may visit the neighborhood U of the set C of critical points of f many times where dist means the distance. The reader may compare this with the Koebe distortion property in one complex variable [2].

4. Bounded and Bounded Nearby Geometry

We prove that the sequence $\{\eta_n\}_{n=1}^{\infty}$ of nested partitions of M by a geometrically finite map f has bounded and bounded nearby geometry in this section.

Theorem A. *Suppose f from M into itself is geometrically finite and $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then η is of bounded and bounded nearby geometry.*

Proof. Let n_0 be the fixed integer in Sect. 3 (before Lemma 1) and $K_1 > 0$ be the minimum of ratios $|J|/|I|$ for $J \subset I$ with $J \in \eta_{j+1}$ and $I \in \eta_j$ for $1 \leq j \leq n_0$.

For a pair $J \subset I$ with $J \in \eta_{k+1}$ and $I \in \eta_k$ and $n = k - n_0 > 0$, let $J_i = f^{\circ i}(J)$ and $I_i = f^{\circ i}(I)$ for $i = 0, \dots, n$. Then $J_n \in \eta_{n_0+1}$ and $I_n \in \eta_{n_0}$. We consider the intervals $\{I_0, \dots, I_n\}$ in two cases: (i) no one of them is in U and (ii) at least one of them is in U.

In the case (i), applying Lemma 1, there is a constant $K_2 > 0$, such that

$$\frac{|(f^{\circ n})'(y)|}{|(f^{\circ n})'(x)|} \geq K_2$$

for x and y in I. This implies that

$$\frac{|J|}{|I|} \geq K_3 = K_2 K_1 .$$

In the case (ii), let $l \leq n$ be the greatest integer so that $I_l \subset U$. We note that $I_i \subset V$ for $i = l+1, \dots, n$. Applying Lemma 1 again as in the case (i), we have that

$$\frac{|J_{l+1}|}{|I_{l+1}|} \geq K_3 .$$

Suppose I_l is contained in U_i. Because $f|U_i$ is comparable with the map $q_i(x) = |x - c_i|^{\gamma_i} + f(c_i)$, there is a constant $K_4 > 0$ (only depends on K_3) so that

$$\frac{|J_l|}{|I_l|} \geq K_4 .$$

Now applying Lemma 2, we have a constant $K_5 > 0$ so that

$$\frac{|(f^{\circ(n-l)})'(y)|}{|(f^{\circ(n-l)})'(x)|} \geq K_5$$

for x and y in I. This implies that

$$\frac{|J|}{|I|} \geq K_6 = K_5 K_4 .$$

Hence η is of bounded geometry.

To prove η is of bounded nearby geometry, let $n_1 > n_0$ be an integer such that if a pair J_1 and J_2 in η_{n_1} with a common endpoint then either both of them are in U or no endpoints of J_1 and J_2 are critical points of f. Suppose $K_7 > 0$ is the minimum of ratios $|J_1|/|J_2|$ for J_1 and J_2 with a common endpoint for $1 \leq j \leq n_1$.

Now for $k \geq n_1$ and J_1 and J_2 with a common endpoint, let $J_{1,i} = f^{\circ i}(J_1)$ and $J_{1,i} = f^{\circ i}(J_2)$ for $i = 0, \dots, n = k - n_1$. We consider $\{J_{1,i}\}_{i=0}^{n}$ and $\{J_{2,i}\}_{i=0}^{n}$ in two cases: (a) for some $0 < l \leq n$, $J_{1,l} = J_{2,l}$ and (b) $J_{1,i}$ and $J_{2,i}$ are all different for $i = 0, \ldots, n$.

In (a), let l be the smallest such integer, then the common endpoint of $J_{1,l-1}$ and $J_{2,l-1}$ is a critical point of f. It is easy to see that there is a constant $K_8 > 0$ such that

$$\frac{|J_{1,l-1}|}{|J_{2,l-1}|} \geq K_8 .$$

Since $J_{1,l-1} \cup J_{2,l-1}$ is a subinterval of U, by applying Lemma 2, we have a constant $K_9 > 0$ such that

$$\frac{|(f^{\circ(l-1)})'(y)|}{|(f^{\circ(l-1)})'(x)|} \geq K_9$$

for x and y in $J_1 \cup J_2$. This implies that

$$\frac{|J_1|}{|J_2|} \geq K_{10} = K_9 K_8 .$$

In (b), by using almost the same arguments as those in the proof of bounded geometry, we have a constant $K_{11} > 0$ such that

$$\frac{|J_1|}{|J_2|} \geq K_{11} .$$

Hence η is of bounded nearby geometry. This completes the proof of Theorem A.

5. Quasisymmetric Conjugacy

Using bounded and bounded nearby geometry, we prove that any topological class of geometrically finite maps is actually a quasisymmetric class.

Theorem B. *Suppose f and g from M into itself are geometrically finite and topologically conjugate. They are then quasisymmetrically conjugate.*

Remark 5. Some other interesting results about quasisymmetric classification have been proved in [3, 7, and 11].

Proof. Suppose h is the conjugacy between f and g, namely $h \circ f = g \circ h$, and B_f and B_g are the constants in Remark 1. Let $\eta_f = \{\eta_{n,g}\}_{n=1}^{+\infty}$ and $\eta_g = \{\eta_{n,g}\}_{n=1}^{+\infty}$ be the induced sequence of nested partitions of M by f and g respectively.

For any $x < y$ in M, let $z = (x+y)/2$ be the midpoint of x and y and $N > 0$ be the smallest integer such that there is an interval I in $\eta_{N,f}$ contained in $[x, y]$. Let $\tilde{I}$ be the interval in $\eta_{N-1,f}$ containing I. Then the union of $\tilde{I}$ and one of its adjacent intervals in $\eta_{N-1,f}$ contains $[x, y]$ (see Figs. 1–3). Because of bounded and bounded nearby geometry of η_g (and refer to Fig. 1–3), there is a constant $K_1 = K_1(B_f) > 0$ such that

$$\frac{|h(I)|}{|h([x,z])|} \geq K_1 \quad \text{and} \quad \frac{|h(I)|}{|h([z,y])|} \geq K_1 .$$

Fig. 1

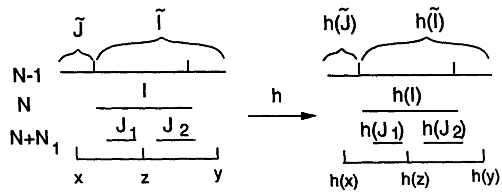


Fig. 2

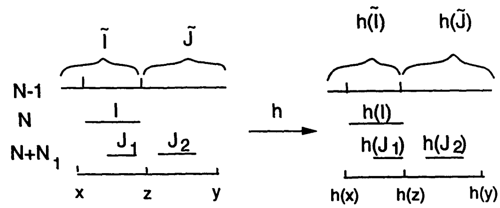
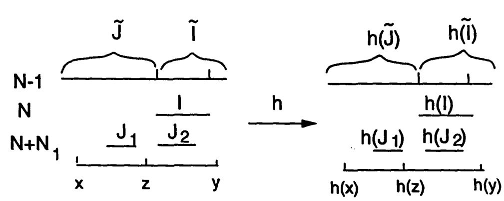


Fig. 3



Because $\eta_{n,f}$ tends to zero exponentially and η_f is of bounded and bounded nearby geometry, we can find a constant integer $N_1 = N_1(B_f) > 0$ such that there are intervals J_1 and J_2 in η_{N+N_1} contained in $[x, z]$ and $[z, y]$, respectively. This implies that $h(J_1)$ and $h(J_2)$ are contained in $h([x, z])$ and $h([z, y])$ respectively. Because of bounded and bounded nearby geometry of η_g again, there is a constant $K = K(N_1, B_g) > 0$ (see Fig. 2–3) such that

$$K^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq K ,$$

which shows that h is quasisymmetric.

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