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$N=2$ Topological Gauge Theory, the Euler Characteristic of Moduli Spaces, and the Casson Invariant

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Abstract. We discuss gauge theory with a topological $N=2$ symmetry. This theory captures the de Rham complex and Riemannian geometry of some underlying moduli space $\mathcal{M}$ and the partition function equals the Euler number $\chi(\mathcal{M})$ of $\mathcal{M}$. We explicitly deal with moduli spaces of instantons and of flat connections in two and three dimensions. To motivate our constructions we explain the relation between the Mathai-Quillen formalism and supersymmetric quantum mechanics and introduce a new kind of supersymmetric quantum mechanics based on the Gauss-Codazzi equations. We interpret the gauge theory actions from the Atiyah-Jeffrey point of view and relate them to supersymmetric quantum mechanics on spaces of connections. As a consequence of these considerations we propose the Euler number $\chi(\mathcal{M})$ of the moduli space of flat connections as a generalization to arbitrary three-manifolds of the Casson invariant. We also comment on the possibility of constructing a topological version of the Penner matrix model.

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1. Introduction

The purpose of this paper is to investigate in some detail the properties of gauge theories with an $N=2$ topological supersymmetry (models of this type have appeared previously in [58, 12, 62]). As we will show, these theories describe the de Rham complex and Riemannian geometry of some underlying moduli space $\mathscr{M}$, in contrast with the standard $N=1$ gauge theories [57] which model the deformation complex of $\mathscr{M}$ and capture the geometry of the Atiyah-Singer [6] universal bundle (see e.g. [9, 38, 12, 11]). The most important property of this class of theories is that formally the partition function of the corresponding $N=2$ action $S_{\mathscr{M}}$ equals the Euler number of $\mathscr{M}$,

$$Z(S_{\mathscr{M}})=\chi(\mathscr{M}),$$

i.e. the Euler characteristic of the de Rham complex of $\mathscr{M}$. That $N=2$ theories may have this property was first suggested by Witten [60], who has recently shown [61] that the twisted Kazama-Suzuki models [39] calculate the Euler number of vector bundles over the moduli space of Riemann surfaces.

All this is, of course, quite reminiscent of the properties of supersymmetric quantum mechanics [56]. That $N=2$ topological gauge theory is indeed closely related to supersymmetric quantum mechanics on spaces of connections is seen most clearly within the framework of the Mathai-Quillen formalism [44] (as applied to topological field theories by Atiyah and Jeffrey [5], see also [16]). As pointed out by Atiyah and Jeffrey, this formalism (whose relation with supersymmetric quantum mechanics we will explain in Sect. 2) can be used to define some kind of *regularized Euler number* $\chi_s(E)$ of a vector bundle E, depending on a section s of E, in cases where the classical (co)homological or differential geometric definitions are not terribly useful, e.g. when E is infinite-dimensional. Moreover, the integral expression for $\chi_s(E)$ can be interpreted as the partition function of a topological field theory.

In order to illustrate this we will briefly review the classical Mathai-Quillen formalism (Sect. 2.1) and then apply it formally (in the spirit of Atiyah and Jeffrey) to the infinite-dimensional loop space LM of a manifold M and its tangent bundle. We will show that for a special class of sections s the regularized Euler number $\chi_s(LM)$ is precisely the partition function of supersymmetric quantum mechanics and hence, in particular, equal to the rigorously defined Euler number $\chi(M)$ of M (Sect. 2.2). Conversely, it is possible to derive the general form of the Mathai-Quillen representative of the Euler character of a finite-dimensional vector bundle from supersymmetric quantum mechanics and these two observations allow us to clarify considerably the meaning of the regularized Euler number of an infinite-dimensional vector bundle.

There is yet another way of obtaining the Euler number of some manifold from supersymmetric quantum mechanics whose classical counterpart is based on a combination of the Gauss-Bonnet theorem with the Gauss-Codazzi equations.

These describe the curvature of some embedded submanifold in terms of the curvature of the ambient manifold and the extrinsic curvature (second fundamental form) of the submanifold. The idea is thus to embed the manifold M into some space Y (e.g. Euclidean space R^k for k sufficiently large) whose curvature is known and to combine supersymmetric quantum mechanics on the latter with a supersymmetric delta function imposing the restriction to M. As this construction appears not to have been discussed in the literature before, and as it is the prototype of the procedure we will adopt when considering gauge theories, we explain it in the case of $S^2 \subset R^3$ in Sect. 2.3 (see [17] for the general case).

Having recreated supersymmetric quantum mechanics, which can be regarded as the simplest example of a topological field theory [13, 10, 11], in this way it is tempting to apply these ideas to spaces of connections to construct topological gauge theories. Precisely, this has been done by Atiyah and Jeffrey [5] who showed that the action of Donaldson theory [57] can be interpreted as the Mathai-Quillen realization of the Euler number of an infinite-dimensional vector bundle of self-dual two forms over the orbit space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of connections.

Our main interest here will be in theories where the bundle in question is (related to) the tangent bundle of $\mathcal{A}/\mathcal{G}$. In this context Atiyah and Jeffrey have shown that the three-dimensional topological gauge theories of [58, 14, 8, 12] can be interpreted as Lagrangian descriptions of the Euler number $\chi_s(\mathcal{A}/\mathcal{G})$ for the section $s(A) = *F_A$ of $T(\mathcal{A}/\mathcal{G})$. In the case that the underlying three-manifold M is a homology three-sphere (so that the non-trivial flat connections, the zeros of s, are irreducible) this is in agreement with Witten's identification of the partition function of these theories with the Casson invariant [1] and Taubes' observation [52] that the Casson invariant can be interpreted as the Euler number of $\mathcal{A}/\mathcal{G}$ defined by (a suitable perturbation of) the vectorfield $*F_A$.

This theory is already an $N=2$ model in disguise [58, 12] and we will show that it has the feature in common with supersymmetric quantum mechanics that its partition function can be identified with the Euler number $\chi(\mathcal{M})$ of some finite-dimensional space $\mathcal{M}$, in this case the moduli space of flat connections. In conjunction with the considerations of [58, 52] this suggests that also the Casson invariant could in general be defined as $\chi(\mathcal{M})$. We will come back to this proposal in Sect. 4.3.

It will become clear in the course of this paper that there are a number of features peculiar to the case of flat connections in three dimensions. However, the construction of $N=2$ actions with the property that the partition function Z is equal to the Euler number of some moduli space $\mathcal{M}$ is (formally, i.e. ignoring analytical questions) completely general and not limited to this example. To illustrate this we will also construct these $N=2$ gauge theories in the somewhat simpler, although perhaps geometrically less transparent, context of flat connections in two dimensions and instantons. The "simpler" here refers to the fact that the deformation complex is "short" in these examples.

As our proof that $Z=\chi(\mathcal{M})$ will be based on the Gauss-Bonnet theorem and the Gauss-Codazzi equations (i.e. we show explicitly that Z reduces to an integral over $\mathcal{M}$ of the exponential of the Riemann curvature $\mathcal{R}_{\mathcal{M}}$ of $\mathcal{M}$, determined from the embedding of $\mathcal{M}$ into $\mathcal{A}/\mathcal{G}$) we will review some of the more elementary aspects of Riemannian geometry of $\mathcal{A}/\mathcal{G}$ and $\mathcal{M}$ in Sect. 3.1.

We then show (Sect. 3.2) how to construct Lagrangian descriptions of these geometries in terms of $N=2$ superfields (see [37] for the superfield formulation of topological field theories). The actions will essentially consist of two parts. One is universal, i.e. common to all $N=2$ topological gauge theories, and describes the Riemannian geometry of $\mathscr{A}/\mathscr{G}$. It is the counterpart of the supersymmetric quantum mechanics action for Y or R^k mentioned above. The other part depends on the choice of moduli space $\mathscr{M}$. It serves to restrict the theory to $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ in an $N=2$ invariant way and corresponds to the delta function imposing the restriction to $M \subset Y$.

As in the $N=1$ theories there is a considerable degree of freedom and arbitrariness in the specific choice of Lagrangian. And as the construction of $N=1$ Witten type topological field theories, Lagrangian realizations of cohomological field theories defined by intersection theory on some moduli space, is well understood [41, 9, 14, 45, 12] (and the significance of having a particular Lagrangian realization at one's disposal should not be overemphasized) we will be rather brief about these matters here. A fairly detailed analysis of these actions, in particular with regard to questions of gauge fixing, can be found in the lectures [53].

In Sect. 4 we complete the circle of ideas involving $N=2$ topological gauge theories, the Mathai-Quillen formalism and supersymmetric quantum mechanics. We interpret the topological actions of Sect. 3.2 from the Atiyah-Jeffrey point of view (Sect. 4.1) and show that they also can be regarded as Mathai-Quillen realizations of Euler numbers of certain infinite-dimensional vector bundles, regularized to give (as in the case of supersymmetric quantum mechanics) the Euler number of some finite-dimensional space, in this case of the moduli space $\mathscr{M}$ in question.

In order to understand the emergence and role of de Rham cohomology in these theories we also explain their relation with supersymmetric quantum mechanics on $\mathscr{A}/\mathscr{G}$. In particular, we will see that the quantum mechanics theory associated with the space $\mathscr{A}/\mathscr{G}$ of gauge orbits on a three-manifold M (and a particular section of the tangent bundle of the loop space of $\mathscr{A}/\mathscr{G}$) is nothing other than Donaldson theory on $M \times S^1$, as could have been anticipated from [3]. In the topologically trivial sector this theory in turn is equivalent to a three-dimensional topological gauge theory as (in accordance with general properties of supersymmetric quantum mechanics) only the time-independent modes contribute to the partition function. As we will explain in more detail in [17] this theory is, as expected, precisely the previously constructed $N=2$ gauge theory of flat connections. This gives an alternative demonstration of the relation between Floer (instanton) homology and the Casson invariant.

We then turn our attention to the Casson invariant itself (Sect. 4.3). We review the intersection theory definition of the Casson invariant [1] and its relation with Taubes' gauge-theoretic definition [52]. We comment on the generalizations suggested in the mathematics literature and then make some remarks on the structure of the moduli space $\mathscr{M}$ of flat connections in three dimensions. In the light of this, we then look at the status of our suggestion that $\chi(\mathscr{M})$ be regarded as an appropriate generalization of the Casson invariant to arbitrary three-manifolds.

Finally, let us point out that the property $Z=\chi(\mathscr{M})$ also immediately brings to mind the Penner matrix model [49, 23] and suggests the possibility of constructing a topological version of this theory. This is work in progress and we will comment on this possibility, as well as on other possible applications and generalizations, only briefly in Sect. 5.

2. The Mathai-Quillen Formalism and Supersymmetric Quantum Mechanics

We will now briefly explain the Mathai-Quillen formalism in the finite-dimensional case (see [44] for details and [5, 11, 16] for discussions in the context of topological field theories). We then explain the relation between the Mathai-Quillen formalism and supersymmetric quantum mechanics (Sect. 2.2) and describe the Gauss-Codazzi form of supersymmetric quantum mechanics (Sect. 2.3).

2.1. *The Mathai-Quillen Formalism*

We start with some classical material (see e.g. [18]). Recall that an oriented $2m$-dimensional real vector bundle E over a manifold X has an Euler class $e(E) \in H^{2m}(X, \mathbf{Z})$. If $\dim X = 2m$, this class can be evaluated on (the fundamental class $[X]$ of) X to give the Euler characteristic (or Euler number)

$$\chi(E) = e(E)[X].$$

In particular, if $E = TX$, the tangent bundle of X, $\chi(TX) \equiv \chi(X)$ is the Euler number of X. There are two concrete ways of thinking about $\chi(E)$. On the one hand, the Gauss-Bonnet-Chern theorem provides one with an explicit differential form representative $e_\nabla(E)$ of $e(E)$ constructed from the Pfaffian of the curvature $\Omega = \Omega^\nabla$ of a connection ∇ on E, such that

$$\chi(E) = \int_X e_\nabla(E). \tag{2.1}$$

On the other hand, $\chi(E)$ can be computed as the number of zeros of a generic section s of E (counted with signs),

$$\chi(E) = \sum_{x : s(x) = 0} \pm 1. \tag{2.2}$$

If $E = TX$ (and hence s a vector field on X) this is the content of the classical Hopf theorem. A more general formula,

$$\chi(E) = \int_X e_{s,\nabla}(E), \tag{2.3}$$

obtained by Mathai and Quillen, interpolates between the two quite different descriptions (2.1) and (2.2). Here $e_{s,\nabla}$ is a closed $2m$-form on X, depending on both a section s and a connection ∇, with the following properties: if s is the zero section of E, then $e_{s,\nabla} = e_\nabla$ and (2.3) reduces to (2.1); if one replaces s by ts, with $t \in \mathbf{R}$, and evaluates (2.3) in the limit $t \to \infty$ using the stationary phase approximation, (2.2) is reproduced. Moreover, $e_{s,\nabla} \equiv e_s$ (we will suppress the dependence on the connection ∇ in the following) is the pullback to X via s of a closed form U on the total space E of the vector bundle, $e_s = s^* U$. U is a representative of the Thom class [18] of E but, unlike the classical Thom class which has compact support in the fibre directions, U is Gaussian shaped along the fibres [cf. (2.4) below].

At this point it will be necessary to introduce some more notation: we let ξ^a denote fibre coordinates of E, χ_a corresponding Grassmann odd variables, and Ω^{ab} the curvature two-form of E, and we regard E as a vector bundle associated to the principal G bundle P with standard fibre V, $E = P \times_G V$. Then the Mathai-Quillen form U can be written as a fermionic integral over the χ's,

$$U=\pi^{-m}e^{-\xi^2}\int d\chi e^{\chi_a\Omega^{ab}\chi_b/4+\mathrm{i}d\xi^a\chi_a}. \tag{2.4}$$

U is a representative of the Thom class in the G-equivariant cohomology $H_G^{2m}(V)$ of V and can be regarded as a G-equivariant form on $P\times V$ whose horizontal part descends to the Thom form on E. In our (somewhat careless) notation s^*U is obtained from U simply by replacing ξ by $s(x)$. We can now see explicitly that if we take s to be the zero section of the tangent bundle TX, (2.4) coincides with the standard Gauss-Bonnet integrand as the fermionic integral over χ serves to pick out the highest form part $\sim(\Omega^{ab})^m$ of $\exp\chi_a\Omega^{ab}\chi_b$ which can then be integrated over M to yield $\chi(E)$. We will explain in Sect. 2.3 how to obtain the general Mathai-Quillen formula (2.4) from supersymmetric quantum mechanics.

2.2. Supersymmetric Quantum Mechanics from the Mathai-Quillen Formalism and vice versa

In finite dimensions (2.4) may perhaps be regarded as an unnecessary complication since one has the simple classical formula (2.1) at one's disposal. But, as Atiyah and Jeffrey have pointed out, (2.4) acquires particular significance in the case of infinite-dimensional bundles where expressions like (2.1) are quite hopeless but where it may be possible to give a meaning to (2.3) for a suitable choice of section s. Equation (2.3) can then be regarded as defining a regularized Euler number $\chi_s(E)$, which is, however, no longer necessarily independent of s. If s is a section canonically associated with E, $\chi_s(E)$ may nevertheless carry interesting (topological) information. Indeed, in all the examples to be discussed in this paper we will find that $\chi_s(E)$ is actually the (rigorously defined) Euler number of some finite-dimensional vector bundle and, as such, certainly has topological significance [see e.g. Eqs. (2.11, 4.2, 4.6)]. We will see later that this is a general feature (and, in a way, the quintessence) of the Mathai-Quillen formalism whenever the zero set of the section s is finite-dimensional.

Clearly, the Mathai-Quillen formalism is closely related to supersymmetric quantum mechanics. To make this analogy more precise let us consider, as our first infinite-dimensional example, the case where X is the loop space $X=LM=\{x(t):S^1\to M\}$ of a finite-dimensional Riemannian manifold M and E is its tangent bundle $T(LM)$. The fibre $T_x(LM)$ at a loop $x(t)$ can be identified with the space $\Gamma(x^*(TM))$ of sections of the pullback of the tangent bundle of M to S^1, i.e. with the space of vector fields on M restricted to the image of the loop $x(t)$. Hence a natural section of the tangent bundle $T(LM)$ is $s_0(x)(t)=\dot{x}(t)$ which we will use to tentatively define the regularized Euler characteristic $\chi_s(LM)$ of LM. With this choice of section the exponent in (2.4),

$$\xi^2-\chi_a\Omega^{ab}\chi_b/4-\mathrm{i}d\xi^a\chi_a \tag{2.5}$$

(summation over the fibre indices now includes an integration over t) becomes

$$S(x)=\int_0^\beta dt[\dot{x}(t)^2-\tfrac{1}{2}\bar{\psi}_i(t)R^{ij}_{kl}(x(t))\psi^k(t)\psi^l(t)\bar{\psi}_j(t)+2i\bar{\psi}_k(t)\nabla_t\psi^k(t)]\,. \tag{2.6}$$

Here we have replaced Lorentz by tangent space indices using the vielbein e^a_k corresponding to the fibre metric implicit in (2.5), $\bar{\psi}_k=e^a_k\chi_a/2$ [this also converts the prefactor in (2.4) to $(2\pi)^{-m}$], and we have replaced $dx^k(t)$ by the anticommuting variable $\psi^k(t)$ [so that (2.3) will now also include an explicit integral over ψ^k]. But (2.6) is nothing other than the standard action of $N=2$ supersymmetric quantum mechanics (see e.g. [56, 2]).[1] The action usually considered is actually slightly more general, depending on a potential function W on M, and can be obtained from (2.5) by choosing, instead of the above section $s_0(x)$, $s_W(x)(t)=\dot{x}(t)+W'(x(t))$. This option will turn out to be essential in our considerations in Sect. 4.2. More generally still, one can replace W' by an arbitrary section V of TM (vector field) and this will allow us to rederive (2.4) (valid, after all, for an arbitrary section s of $E=TM$).

In either case the regularized Euler number of LM, defined via the Mathai-Quillen formalism, is precisely the partition function of $N=2$ supersymmetric quantum mechanics on M which, as is well known, is the Euler number $\chi(M)$ of M. The standard way of seeing this is to start with the definition of $\chi(M)$ as the Euler characteristic of the de Rham complex of M,

$$\chi(M)=\sum_{k=0}^{2m}(-)^k b_k(M) \tag{2.7}$$

[here $b_k(M)=\dim H^k(M,\mathbf{R})$ is the k^{th} Betti number of M], and to rewrite this as the Witten index

$$\chi(M)=\operatorname{tr}(-)^F e^{-\beta H} \tag{2.8}$$

of the Laplace operator $H=\Delta$ on differential forms (or of its generalization, defined by the twisted exterior derivative $d_W=e^{-W}de^W$ [56]). One then uses the Feynman-Kac formula to represent this as a supersymmetric path integral with the action (2.6) (or its generalization) and periodic boundary conditions on the anticommuting variables ψ^k [due to the insertion of $(-)^F$ in (2.8)]. Using the β-independence of (2.8) it can be shown that only the zero modes of the action contribute to the partition function (the contributions from the non-zero modes cancelling exactly between the bosonic and fermionic fields), and the evaluation of the remaining finite-dimensional integral then gives the right-hand side of either (2.1) or (2.2), i.e. a path integral proof of either the Gauss-Bonnet or the Poincaré-Hopf theorem. It is interesting to note that these two rather different classical formulae for $\chi(M)$ simply correspond to a different choice of section (with fixed connection ∇) in the Mathai-Quillen expression $e_{s,\nabla}(LM)$ for the Euler number $\chi_s(LM)$ of the loop space of M.

So far we have derived the action of supersymmetric quantum mechanics by formally applying the Mathai-Quillen formalism to LM, and we have indicated how to rederive the classical formulae (2.1, 2.2) for the Euler number $\chi(M)$ from the resulting action. What is still lacking to complete the picture is a derivation of the general (finite-dimensional) Mathai-Quillen formula (2.4) from supersymmetric quantum mechanics and this can be done along the following lines. Consider the quantum mechanics action corresponding to the section $s(x)(t)=\dot{x}(t)+\alpha V(x(t))$, where V is a vector field on M and $\alpha\in\mathbf{R}$ is some real parameter. Introduce a

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[1] There is a slight clash in notation here. In the literature on supersymmetric quantum mechanics this model is usually referred to as $N=1$. In the context of topological field theories, however, it is more convenient and conventional to count Majorana charges. In the same way the standard $N=1$ topological gauge theories are the field theoretic cousins of the $N=\frac{1}{2}$ (Dirac operator) model of [2]

multiplier field B to write the bosonic part of the action as $\int_0^\beta (\dot{x}+\alpha V(x))B - B^2/2$. The rest of the action is as in (2.6) with the addition of the term $\alpha\bar{\psi}_i V_k V^i \psi^k \equiv \alpha\bar{\psi}\cdot dV$ (in our notation we will not distinguish between the vector field V and its metric dual one-form). Now scale B and $\bar{\psi}$ by $\beta^{-1/2}$. Setting $\alpha=\beta^{-1/2}$, the contributions from all the non-zero-modes can again be shown to cancel identically in the limit $\beta\to 0$ and one is left with an action of the form

$$B^2+(VB+\bar{\psi}\cdot dV)+(\text{curvature terms}) \tag{2.9}$$

which – upon integrating over B – reproduces precisely the Mathai-Quillen formula (2.3, 2.4).

In light of the above let us now make a few more comments concerning the Mathai-Quillen formalism and the significance of the regularized Euler number $\chi_s(E)$ of an infinite-dimensional vector bundle E. First of all we want to draw attention to the fact that what is a section in the Mathai-Quillen formalism is in other contexts called a Nicolai map. All Witten type topological field theories have a complete Nicolai map [13, 11] and it is known that in these theories the partition function can be reduced to a sum (integral) over the zeros of the Nicolai map. In the present case this is either $(s_0(x)(t)=\dot{x}(t))$ the space M of constant paths $\dot{x}=0$ or $(s_W(x)(t)=\dot{x}(t)+W'(x(t)))$ the set of critical points of W. Indeed, by squaring and integrating one sees that $\dot{x}+W'(x)=0$ implies $\dot{x}=W'(x)=0$ (this we will refer to as the "squaring argument" in the following). In this case one obtains the Euler number of M in the form

$$\chi(M)=\sum_k \chi(M_W^{(k)}), \tag{2.10}$$

where the $M_W^{(k)}$ are the connected components of the critical point set of W and relative orientations are to be taken into account. In (2.10) the critical point set of W can also be replaced by the zero set of any (not necessarily gradient) vector field V on M. The corresponding quantum mechanics action also has a Nicolai map. The above squaring argument fails, however, as the cross-term $\dot{x}\cdot V$ does not necessarily integrate to zero. Thus the partition function reduces not to a delta function onto the critical points but only to a Gaussian [as in (2.4, 2.9)], albeit arbitrarily sharply peaked around the critical points of V. The analogue of (2.10) is then reproduced in the limit $\alpha\to\infty$.

Whichever section (action) we use, what we have found is that the Mathai-Quillen formalism reproduces supersymmetric quantum mechanics when applied formally to the loop space LM of a finite-dimensional manifold M, and that the regularized Euler number of LM (in the sense of Atiyah and Jeffrey) is

$$\chi_s(LM)=\chi(M). \tag{2.11}$$

Conversely, we have seen that we can derive the Mathai-Quillen generalization $e_{s,V}$ (2.3, 2.4) of the Gauss-Bonnet integrand from supersymmetric quantum mechanics.

Equation (2.11) shows that (2.10) is in a way also the essence of the definition of the regularized Euler number $\chi_s(LM)$. One *defines* $\chi(LM)$ to be equal to the Euler number of the zero set of some vector field on LM. In the finite-dimensional case this is, according to (2.10), not a definition but an equality. Here, if one chooses the section of $T(LM)$ to be any of those discussed above one recovers the result (2.11).

This is a general feature of the Mathai-Quillen formalism in the context of infinite-dimensional bundles: any "reasonable" definition of the regularized Euler number (any choice of section with a finite-dimensional zero-set) will equate it to the Euler number of some finite-dimensional vector bundle over the zero-locus of the section. The latter is, of course, well-defined and unique, while it is the identification of the former with the latter which is not unique. In fact, there is no good reason for different choices of sections s making $\chi_s(E)$ well-defined to give the same result in general. We will encounter examples of this in later parts of this paper.

2.3. The Gauss-Codazzi Form of Supersymmetric Quantum Mechanics

There is yet one more form for the Euler character that can be obtained from these supersymmetric quantum mechanics models. We could wish to determine the Euler character of a manifold M by embedding it into an ambient space Y whose curvature tensor is known (e.g. into Euclidean space R^k with k large enough) and then use the Gauss-Codazzi equations (cf. below) to determine the curvature tensor of M, the Gauss-Bonnet theorem then giving an explicit result for the Euler character of M.

The Gauss-Codazzi equations express the curvature of M in terms of the curvature of the ambient space Y and the second fundamental form (extrinsic curvature) of the embedding $i: M \hookrightarrow Y$ of M into Y. The second fundamental form of (M, i) is a section K of $\mathrm{Sym}^2(T^*M)\otimes N_M$ (N_M is the normal bundle to TM in $TY|_M = i^*TY$) defined by

$$K_M(X, V) = (\nabla_{i_*X} i_* V)^{\perp}, \tag{2.12}$$

where $X, V \in TM$, ∇ is the Levi-Cività connection on Y, and $(\cdot)^{\perp}: i^*TY \to N_M$ the projection. The Gauss-Codazzi equations[2] now state that in terms of K_M and the curvature $\mathscr{R}_Y$ of Y the curvature $\mathscr{R}_M$ of M is given by

$$\begin{aligned}\langle \mathscr{R}_M(X, V)Z, W\rangle &= \langle \mathscr{R}_Y(X, V)Z, W\rangle \\ &\quad + (\langle K_M(V, Z), K_M(X, W)\rangle - (X \leftrightarrow V)).\end{aligned} \tag{2.13}$$

The supersymmetric quantum mechanics action S_M yielding $\chi(M)$ in terms of the integral of (2.13) will itself have a form resembling that of the Gauss-Codazzi equations. It will consist of the standard action $S = S_Y$ (2.6), describing the curvature of Y and corresponding to the first line of (2.13), and of a term S_M^0 performing the restriction to $M \subset Y$ in a supersymmetric way and giving rise to the extrinsic curvature terms. This idea is easily carried out in general [17] and will also underly our construction of gauge theory actions in Sect. 3.2 (with $M \to \mathscr{M}$ and $Y \to \mathscr{A}/\mathscr{G}$). Here we will, for concreteness, consider first the example of S^2 embedded in R^3.

In the case of $Y = R^3$ (2.6) becomes

$$S_Y(x) = \int dt [2i\dot{x}(t)B(t) + B(t)^2 + i\bar{\psi}_k(t)\partial_t \psi^k(t)], \tag{2.14}$$

where we have introduced a multiplier field $B(t)$. The path integral associated with this action is ill defined, being the product of infinity [due to the presence of $x(t)$ zero modes] and zero (due to the ψ and $\bar{\psi}$ zero modes). Now we wish to cut out the

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[2] Actually, the Gauss part of the Gauss-Codazzi equations; the Codazzi equations express the normal part of the curvature [$W \in N_M$ in (2.13)] in terms of K_M and its derivative

loop space of S^2, LS^2. To do this we work with $N=2$ superfields,

$$X^i(t,\theta,\bar{\theta})=x^i(t)+\theta\psi^i(t)+\bar{\theta}\bar{\psi}^i(t)+\theta\bar{\theta}B^i(t) \tag{2.15}$$

and

$$b(t,\theta,\bar{\theta})=\lambda(t)+\theta\sigma(t)+\bar{\theta}\bar{\sigma}(t)+\theta\bar{\theta}b(t), \tag{2.16}$$

and add to the action (2.14) the following ($M=S^2$)

$$S_M^0=\int dt d\theta d\bar{\theta} b(t,\theta,\bar{\theta})(X(t,\theta,\bar{\theta})^2-1). \tag{2.17}$$

In terms of components this addition is essentially a delta function constraint on the paths so that they lie in $LS^2 \hookrightarrow LR^3$, plus similar constraints on the tangents. By standard arguments we need only restrict our attention to the zero mode sector of the theory, and in this limit the partition function becomes

$$\int d\Phi \exp[ib(x^2-1)+B^2+2i\sigma x\cdot\bar{\psi}-2i\bar{\sigma}x\cdot\psi+2i\lambda(x\cdot B-\psi\cdot\bar{\psi})], \tag{2.18}$$

where Φ designates all the constant modes. The integral over b restricts us to S^2 while the integrals over σ and $\bar{\sigma}$ restrict ψ and $\bar{\psi}$ to be tangents to S^2. Finally, the B and λ integrals yield the exponent

$$(\psi\cdot\bar{\psi})(\psi\cdot\bar{\psi}), \tag{2.19}$$

which is the curvature term appearing in (2.6) with the Gauss-Codazzi constant curvature that S^2 inherits from R^3. In this case the connection with the Mathai-Quillen construction arises at the level of taking the section $s_0=\dot{x}$ of $T(LR^3)|_{LS^2}$ after the pullback from $T(LR^3)$ via $i: S^2 \hookrightarrow R^3$. This will be explained in more detail in the gauge theory context in Sect. 4.2.

More generally, assume that $M\subset Y$ is (locally) given by

$$M=\{x\in Y: F^a(x)=0,\ a=1,\dots,\dim(Y)-\dim(M)\}$$

(the relation between the formulae arising from this implicit description and that in terms of an explicit embedding is explained e.g. in [15]). Introducing coordinate and Lagrange multiplier superfields X^μ and Λ_a as above, one finds [17] that the action

$$S_{M\subset Y}=S_Y+\alpha\int dt\int d\theta d\bar{\theta}\Lambda_a(t,\theta,\bar{\theta})F^a(X(t,\theta,\bar{\theta})), \tag{2.20}$$

can be reduced to

$$S_{M\subset Y}=(\tfrac{1}{4}R^{\mu\nu}_{\varrho\sigma}+\tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}\nabla_\alpha\partial_\varrho F^a(F^{-1})_{ab}\nabla_\beta\partial_\sigma F^b)\bar{\psi}_\mu\psi^\varrho\bar{\psi}_\nu\psi^\sigma \tag{2.21}$$

by manipulations analogous to those performed in the case of S^2. Here ψ^μ and $\bar{\psi}_\nu$ are now tangent to M and F^{ab} is the matrix $F^{ab}=\partial_\mu F^a\partial_\nu F^b g^{\mu\nu}$. The description of $M\subset Y$ in terms of the F^a is valid at points where $\det(F^{ab})\neq 0$ so that F^{ab} is indeed invertible there. The right-hand side of (2.21) is precisely the Gauss-Codazzi expression for the curvature of M.

The upshot of this is that we have indeed recovered the Euler characteristic as the Gauss-Bonnet integral *via* the explicit Gauss-Codazzi form for the curvature.

Finally, we mention that one can also construct quantum mechanics actions $S_{Z\to M}$ for Riemannian submersions $Z\to M$ instead of embeddings, deriving the O'Neill equations [48] in this case instead of the Gauss-Codazzi equations – for an illustration of this in the gauge theory context see [17].

There is one subtlety in the above prescription which we have ignored so far. Namely that the introduction of the constraint (2.17) has actually lowered the symmetry of the theory from $N=2$ to $N=1$. The easiest way to see this is to note that (2.14) is invariant under the simultaneous interchange $\psi \to \bar{\psi}$ and $\bar{\psi} \to \psi$ while (2.17) has the symmetry $\psi \to \bar{\psi}$ and $\bar{\psi} \to -\psi$ (one also needs to swap σ and $\bar{\sigma}$). Nevertheless, the Hamiltonian (which has no time derivatives) keeps the $N=2$ symmetry of the superfields manifest. This property will arise again in our analysis of the Euler character of moduli spaces of flat connections over 3-manifolds.

3. $N=2$ Topological Gauge Theories and the Euler Characteristic of Moduli Spaces of Connections

In this section we will construct a topological gauge theory with the property that its partition function is the Euler characteristic of some underlying moduli space $\mathcal{M}$. We begin with a brief review of the Riemannian geometry of the spaces of connections involved (see e.g. [7, 33, 47, 51]). We then construct the actions roughly according to the recipe explained in Sect. 2.3 without worrying too much about the geometrical origin of the action and its relation with supersymmetric quantum mechanics. The connection with the various ideas of Sect. 2 will be explained in Sect. 4.

3.1. Riemannian Geometry of Spaces of Connections…

Let (M, g) be a compact, oriented, Riemannian manifold, $P \to M$ a principal G bundle over M, G a compact semisimple Lie group and $\mathbf{g}$ its Lie algebra. We denote by $\mathcal{A}$ the space of (irreducible) connections on P, by $\mathcal{G}$ the infinite-dimensional gauge group of vertical automorphisms of P (modulo the center of G), by $\Omega^k(M, \mathbf{g})$ the space of k-forms on M with values in the adjoint bundle $\operatorname{ad} P := P \times_{\mathrm{ad}} \mathbf{g}$ and by d_A the covariant exterior derivative. The spaces $\Omega^k(M, \mathbf{g})$ have natural scalar products defined by the metric g on M (and the corresponding Hodge operator $*$) and an invariant scalar product tr on $\mathbf{g}$, namely

$$\langle X, Y \rangle = \int_M \operatorname{tr}(X * Y), \qquad X, Y \in \Omega^k(M, \mathbf{g}). \tag{3.1}$$

The tangent space $T_A\mathcal{A}$ to $\mathcal{A}$ at a connection A can be identified with $\Omega^1(M, \mathbf{g})$. At each point $A \in \mathcal{A}$, $T_A\mathcal{A}$ can be split into a vertical part $V_A = \operatorname{Im}(d_A)$ (tangent to the orbit of $\mathcal{G}$ through A) and a horizontal part $H_A = \operatorname{Ker}(d_A^*)$ [the orthogonal complement of V_A with respect to the scalar product (3.1)]. Explicitly this decomposition of $X \in \Omega^1(M, \mathbf{g})$ into its vertical and horizontal parts is

$$X = d_A G_A^0 d_A^* X + (X - d_A G_A^0 d_A^* X), \tag{3.2}$$

where $G_A^0 = (d_A^* d_A)^{-1}$ is the Green's function of the scalar Laplacian (which exists if A is irreducible).

Working with appropriate Sobolev spaces of connections (we will not indicate this explicitly in the following) it can be shown that the space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes $[A]$ of connections is a smooth Hausdorff-Hilbert manifold. It is often convenient to identify the tangent space $T_{[A]}\mathcal{A}/\mathcal{G}$ with H_A for some representative A of the gauge equivalence class $[A]$. $\mathcal{G}$ acts on $\mathcal{A}$ isometrically and preserving the above decomposition so that (3.1) induces a metric on $\mathcal{A}/\mathcal{G}$ making the principal projection $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{G}$ a Riemannian submersion.

The Riemannian curvature of $\mathscr{A}/\mathscr{G}$ can now be computed straightforwardly in a variety of ways and is [51, 7, 47]

$$\langle \mathscr{R}_{\mathscr{A}/\mathscr{G}}(X,Y)Z,W\rangle = \langle *[\bar{X},*\bar{W}], G_A^0 * [\bar{Y},*\bar{Z}]\rangle - \langle *[\bar{Y},*\bar{W}], G_A^0 * [\bar{X},*\bar{Z}]\rangle + 2\langle *[\bar{W},*\bar{Z}], G_A^0 * [\bar{X},*\bar{Y}]\rangle \quad (3.3)$$

(with $W,X,Y,Z \in T_{[A]}\mathscr{A}/\mathscr{G}$ and $\bar{W},\bar{X},\bar{Y},\bar{Z}$ local horizontal extensions of their lifts to H_A).

We now turn our attention to certain finite-dimensional (moduli) subspaces $\mathscr{M}$ of $\mathscr{A}/\mathscr{G}$. Obvious examples that come to mind are moduli spaces of flat connections ($F_A \equiv dA + \frac{1}{2}[A,A]=0$), Yang-Mills connections ($d_A^* F_A = 0$) and (in dimension 4) instantons ($P_+F_A \equiv \frac{1}{2}(1+*)F_A=0$). Following Groisser and Parker [33], who discussed the case of instantons, we will describe the Riemannian curvature $\mathscr{R}_{\mathscr{M}}$ of $\mathscr{M}$ in terms of $\mathscr{R}_{\mathscr{A}/\mathscr{G}}$ (3.3) and the second fundamental form (extrinsic curvature) of the embedding $i:\mathscr{M}\hookrightarrow \mathscr{A}/\mathscr{G}$, using the Gauss-Codazzi equations (cf. Sect. 2.3). At this point it is rather awkward to continue in this generality and we will therefore deal explicitly now with the moduli spaces $\mathscr{M}_2$ and $\mathscr{M}_I$ of flat connections in two dimensions and instantons. Afterwards we will treat the moduli space $\mathscr{M}_3$.

The formal (Zariski) tangent space $T_{[A]}\mathscr{M}_2$ ($T_{[A]}\mathscr{M}_I$) can be identified with the subspace of $T_{[A]}\mathscr{A}/\mathscr{G} \sim H_A = \{X \in \Omega^1(M,\mathbf{g}) \colon d_A * X = 0\}$ satisfying the linearized equations $d_A X = 0$ (respectively $P_+ d_A X = 0$). Put differently, one has $T_{[A]}\mathscr{M} \sim \mathbf{H}_A^1$, where $\mathbf{H}_A^k$ is the k^{th} cohomology group of the flat connection or instanton deformation complex,

$$0 \longrightarrow \Omega^0(M,\mathbf{g}) \xrightarrow{d_A} \Omega^1(M,\mathbf{g}) \xrightarrow{d_A} \Omega^2(M,\mathbf{g}) \longrightarrow 0\,, \quad (3.4)$$

$$0 \longrightarrow \Omega^0(M,\mathbf{g}) \xrightarrow{d_A} \Omega^1(M,\mathbf{g}) \xrightarrow{P_+ d_A} \Omega^2(M,\mathbf{g})_+ \longrightarrow 0 \quad (3.5)$$

[note that $d_A^2=0$ ($P_+d_A^2=0$) for A flat (an instanton)]. Although we shall not be too concerned with the analytical properties of these moduli spaces (see, however, Sect. 4.3 for remarks on the structure of the not so well understood moduli spaces $\mathscr{M}_3$) we mention that the zeroth cohomology groups of (3.4) and (3.5) as well as the second cohomology group of (3.4) (by Poincaré duality) are zero for A irreducible, and that the second cohomology group $\mathbf{H}_A^2 = \operatorname{Ker} P_+ d_A (P_+ d_A)^*$ of (3.5) can be shown to be zero at irreducible connections for particular [4] and generic [29] metrics. This allows to establish the local smoothness of $\mathscr{M}$ in the neighbourhood of irreducible connections via the implicit function theorem. In particular, at smooth points of $\mathscr{M}$ the dimension of $\mathscr{M}$, the dimension of $\mathbf{H}_A^1$ and the index of (3.4, 3.5) all agree. For more information see [25, 31, 11].

Vectors in $T_{[A]}\mathscr{M}$ can be represented by elements $\bar{X}_A \in \Omega^1(M,\mathbf{g})$ of the form

$$\bar{X}_A = X_A - d_A^* G_A^2 d_A X_A\,, \quad (3.6)$$

$$\bar{X}_A = X_A - (P_+ d_A)^* G_A^2 P_+ d_A X_A\,, \quad (3.7)$$

where $X_A \in H_A$ and G_A^2 is the Green's function of the second Laplacian $d_A d_A^*$ ($(P_+d_A(P_+d_A)^*)$) of the deformation complex (3.4, 3.5). Indeed, one easily verifies that e.g. $\bar{X}_A$ as given by (3.6) satisfies $d_A\bar{X}_A = d_A^*\bar{X}_A = 0$. Using (3.6, 3.7), the extrinsic curvature $K_{\mathscr{M}}$ can be computed to be

$$K_{\mathscr{M}}(X,Y) = -d_A^* G_A^2 [\bar{X},\bar{Y}]\,, \quad (3.8)$$

$$K_{\mathscr{M}}(X,Y) = -(P_+d_A)^* G_A^2 P_+ [\bar{X},\bar{Y}]\,. \quad (3.9)$$

Together with the Gauss-Codazzi equation (2.13), Eq. (3.3) and

$$\langle K_{\mathscr{M}}(Y,Z), K_{\mathscr{M}}(X,W)\rangle = \langle [\bar{Y},\bar{Z}], G_A^2[\bar{X},\bar{W}]\rangle\,, \tag{3.10}$$

$$\langle K_{\mathscr{M}}(Y,Z), K_{\mathscr{M}}(X,W)\rangle = \langle P_+[\bar{Y},\bar{Z}], G_A^2 P_+[\bar{X},\bar{W}]\rangle\,, \tag{3.11}$$

this allows us to express the Riemann curvature tensor of $\mathscr{M}_2$ and $\mathscr{M}_I$ entirely in terms of the Green's functions G_A^0 and G_A^2. And it is in precisely this form that we will derive $\mathscr{R}_{\mathscr{M}}$ from a suitable Lagrangian in the next section.

So far we have dealt with two examples of moduli spaces of connections whose deformation complexes (3.4, 3.5) are "short" and where the obstructing cohomology groups $\mathbf{H}_A^k$, $k \neq 1$ are (generically) zero. This will, of course, not always be the case and one may wonder how much of the above nevertheless remains valid under more general circumstances. For concreteness let us consider the moduli space $\mathscr{M}_3=\mathscr{M}_3(M,G)$ of flat G-connections on a three-manifold M. Its deformation complex is [like (3.4)] the twisted de Rham complex of M,

$$0 \longrightarrow \Omega^0(M,\mathbf{g}) \xrightarrow{d_A} \Omega^1(M,\mathbf{g}) \xrightarrow{d_A} \Omega^2(M,\mathbf{g}) \xrightarrow{d_A} \Omega^3(M,\mathbf{g}) \longrightarrow 0\,. \tag{3.12}$$

In this case, however, significantly less information can be extracted from it than in the examples discussed previously. In particular, although the formal tangent space $T_{[A]}\mathscr{M}$ can still be identified with $\mathbf{H}_A^1$, the index of (3.12) (being zero by Poincaré duality) provides no information on the dimension of $\mathscr{M}$. The cohomology groups $\mathbf{H}_A^0$ and $\mathbf{H}_A^3$ of (3.12) are zero at irreducible connections, but there will certainly be no vanishing theorem for $\mathbf{H}_A^2$ in general as $\mathbf{H}_A^2 \sim \mathbf{H}_A^1$ and $\mathbf{H}_A^1 \neq 0$ is a necessary condition for having a non-zero-dimensional moduli space. General results on the structure of the smooth and singular parts of $\mathscr{M}_3$ appear not to be known. In Sect. 4.3 we shall mention some partial information that can be extracted from the existing literature on representation varieties of finitely generated groups. In the following we simply assume that we are working with the smooth part of $\mathscr{M}_3$ only.

Turning now to Riemannian geometry let us first look at the analogue of (3.6) in the present case. Although the Laplacian on two-forms, $\Delta_A^2 = d_A^* d_A + d_A d_A^*$, whose Green's function enters into (3.6), has a non-trivial kernel, (3.6) makes sense in three dimensions as it stands since Δ_2 is certainly invertible on $\operatorname{Im} d_A$ (hence in particular on $d_A X_A$). Thus Eqs. (3.8) and (3.10) also remain valid provided that one thinks of G_A^2 as including a projection onto the orthogonal complement [with respect to (3.1)] of $\mathbf{H}_A^2$ in $\Omega^2(M,\mathbf{g}) = \operatorname{Im} d_A \oplus \operatorname{Im} d_A^* \oplus \mathbf{H}_A^2$. In conclusion we see that despite the additional complications present in the case of flat connections in three (and higher) dimensions, (3.8) and (3.10) remain valid and the expression for the Riemann tensor $\mathscr{R}_{\mathscr{M}}$ is formally identical to that given above for the moduli space $\mathscr{M}_2$.

3.2. ... and its Lagrangian Realization

We will now explain how to construct the topological action $S_{\mathscr{M}}$ capturing the geometry of some moduli space $\mathscr{M}$ described in the previous section. In analogy with the Gauss-Codazzi equation (2.13) and the considerations of Sect. 2.3 $S_{\mathscr{M}}$ will essentially consist of two parts. One of them, $S_{\mathscr{A}/\mathscr{G}}$, is universal, i.e. common to all gauge theories with a topological $N=2$ symmetry, and describes the Riemannian geometry of $\mathscr{A}/\mathscr{G}$ (much in the same way as $N=1$ topological gauge theories all have a part in common which describes the geometry of the Atiyah-Singer [6] universal bundle [9, 38, 12, 11]). The other, $S^0_{\mathscr{M}}$ [corresponding to the extrinsic curvature contribution in (2.13)] will depend on the particular moduli space chosen.

We introduce an $N=2$ superconnection [58, 12]

$$\hat{A}=A_\mu(x,\theta,\bar{\theta})dx^\mu+A_\theta(x,\theta,\bar{\theta})d\theta+A_{\bar{\theta}}(x,\theta,\bar{\theta})d\bar{\theta} \tag{3.13}$$

with components

$$\begin{aligned} A_\mu(x,\theta,\bar{\theta})&=A_\mu(x)+\theta\psi_\mu(x)+\bar{\theta}\bar{\psi}_\mu(x)+\theta\bar{\theta}\Sigma_\mu(x)\,,\\ A_\theta(x,\theta,\bar{\theta})&=\xi(x)+\theta\phi(x)+\bar{\theta}\varrho(x)+\theta\bar{\theta}\eta(x)\,,\\ A_{\bar{\theta}}(x,\theta,\bar{\theta})&=\bar{\xi}(x)+\theta\bar{\varrho}(x)+\bar{\theta}\bar{\phi}(x)+\theta\bar{\theta}\bar{\eta}(x)\,. \end{aligned} \tag{3.14}$$

In this formulation the fields carry a natural trigrading (a,b,c), where the first entry is the conventional form degree while the second and third entries correspond to the θ and $\bar{\theta}$ weights, respectively. For our purposes, however, it will be sufficient and more convenient to assign a bigrading to the fields in such a way that $A=A_\mu(x)dx^\mu$, θ and $\bar{\theta}$ are $(1,0)$-, $(0,-1)$-, and $(0,1)$-forms, respectively. This determines e.g. $\psi=\psi_\mu dx^\mu$ and ϕ to be $(1,1)$- and $(0,2)$-forms, which is in agreement with the ghost number assignments of the standard $N=1$ theories.

From $\hat{A}$ we can construct the supercurvature form $\hat{F}$ as

$$\hat{F}=\hat{d}\hat{A}+\tfrac{1}{2}[\hat{A},\hat{A}] \tag{3.15}$$

$(\hat{d}=dx^\mu\partial_\mu+d\theta\partial_\theta+d\bar{\theta}\partial_{\bar{\theta}})$, which transforms homogeneously $(\delta\hat{F}=[\hat{F},\hat{\Lambda}])$ under the supergauge transformation

$$\delta\hat{A}=\hat{d}\hat{\Lambda}+[\hat{A},\hat{\Lambda}]\,. \tag{3.16}$$

We will use this supergauge transformation to set $\xi=\bar{\xi}=\varrho-\bar{\varrho}=0$. This reduces (3.16) to the ordinary gauge symmetry $(\hat{d}\hat{\Lambda}=d\hat{\Lambda})$ which we will keep manifest together with the $N=2$ symmetry. As a consequence of the above gauge choice the $N=2$ generators s and $\bar{s}$ are now the superspace derivatives ∂_θ and $\partial_{\bar{\theta}}$ supplemented by field dependent gauge transformations. For instance, instead of $sA=\psi$, $s\psi=0$ one now has $sA=\psi$, $s\psi=-d_A\phi$, which is the familiar equivariant supersymmetry of Donaldson theory [57, 38] and any other $N=1$ topological gauge theory.

We pause here to explain the relationship between the equivariant and non-equivariant versions of topological field theories (for any N). By standard gauge covariance arguments the calculation of any gauge invariant observable does not depend on the gauge chosen. If one therefore calculates expectation values of observables that are both shift supersymmetric and gauge invariant, in either version of the theory the results will necessarily be the same. In particular, for the theory at hand the partition function may be viewed as the expectation value of 1, which is certainly gauge and shift invariant. The alternative of keeping the complete set of fields [plus the $N=2$ multiplets of Faddeev-Popov ghosts required to gauge fix the symmetry (3.16)] has been worked out in [53] and the results, of course, agree with those presented here.

From the supercurvature $\hat{F}$ with components

$$
\begin{aligned}
F_0 &\equiv \tfrac{1}{2} F_{\mu\nu}(x,\theta,\bar{\theta}) dx^\mu dx^\nu \\
&= F_A - \theta d_A \psi - \bar{\theta} d_A \bar{\psi} + \theta\bar{\theta}(d_A \Sigma + [\psi,\bar{\psi}]) , \qquad (3.17)\\
F_\theta &\equiv F_{\mu\theta}(x,\theta,\bar{\theta}) dx^\mu \\
&= (\psi - \theta d_A \phi + \bar{\theta}(\Sigma - d_A \varrho) + \theta\bar{\theta}(d_A \bar{\eta} + [\phi,\bar{\psi}] - [\varrho,\psi])), \qquad (3.18)
\end{aligned}
$$

...

we can construct topological actions from the various contributions to the super Yang-Mills action $\int_{(M,\theta,\bar{\theta})} \hat{F} * \hat{F}$ (where $*$ is a suitably defined super Hodge operator). As in conventional $N=1$ topological theories most of these terms are inessential and have no influence on the dynamics or calculation of correlation functions. The one term that will be important for us is

$$S_{\mathscr{A}/\mathscr{G}} = \int_M d\theta d\bar{\theta} F_\theta * F_{\bar{\theta}} \qquad (3.19)$$

which provides propagators for all the components of A_θ and $A_{\bar{\theta}}$. The "dynamics" for $A_\mu(x,\theta,\bar{\theta})$, which, of course, depends on the choice of $\mathscr{M}$, will be specified in terms of F_0 or $A(x,\theta,\bar{\theta})$ [cf. (3.24) and (3.30) below]. $S_{\mathscr{A}/\mathscr{G}}$ is manifestly gauge invariant and $N=2$ supersymmetric and given explicitly in terms of components by

$$
\begin{aligned}
S_{\mathscr{A}/\mathscr{G}} = \int_M & d_A \phi * d_A \bar{\phi} - d_A \varrho * d_A \varrho + \eta d_A * \psi + \bar{\eta} d_A * \bar{\psi} \\
& + \phi[\bar{\psi}, *\bar{\psi}] + \bar{\phi}[\psi, *\psi] - 2\varrho[\psi, *\bar{\psi}] + \Sigma * \Sigma . \qquad (3.20)
\end{aligned}
$$

Let us now analyze this action. The equations of motion of η and $\bar{\eta}$ tell us that $d_A * \psi = d_A * \bar{\psi} = 0$ so that ψ and $\bar{\psi}$ can be interpreted as horizontal tangent vectors to $\mathscr{A}$, i.e. elements of H_A (cf. Sect. 3.1). The Gaussian integral over ϱ generates a term $[\psi, *\bar{\psi}] G_A^0 * [\psi, *\bar{\psi}]$, and similarly the integrals over ϕ and $\bar{\phi}$ contribute the term $[\bar{\psi}, *\bar{\psi}] G_A^0 * [\psi, *\psi]$. Away from reducible connections there will be no scalar zero modes to worry about so that effectively the action $S_{\mathscr{A}/\mathscr{G}}$ now takes the form

$$S_{\mathscr{A}/\mathscr{G}} = \int_M ([\psi, *\bar{\psi}] G_A^0 * [\psi, *\bar{\psi}] + [\bar{\psi}, *\bar{\psi}] G_A^0 * [\psi, *\psi] + \Sigma * \Sigma) , \qquad (3.21)$$

where ψ and $\bar{\psi}$ are gauge fixed pointwise at A, i.e. satisfy $d_A * \psi = d_A * \bar{\psi} = 0$. The important observation is now that the first two terms of (3.21) are precisely the combinations of Green's functions appearing in the expression (3.3) for the Riemann tensor $\mathscr{R}_{\mathscr{A}/\mathscr{G}}$ so that (by slight abuse of notation) we can rewrite (3.21) more succinctly as

$$S_{\mathscr{A}/\mathscr{G}} = \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M \Sigma * \Sigma . \qquad (3.22)$$

It is in this sense that the universal contribution $S_{\mathscr{A}/\mathscr{G}}$ to the action of any $N=2$ topological gauge theory captures the Riemannian geometry of $\mathscr{A}/\mathscr{G}$.

We will now explain how to construct the $\mathscr{M}$-dependent part $S_{\mathscr{M}}^0$ of the action. The role of $S_{\mathscr{M}}^0$ is to restrict the gauge fields to $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$. Performing this restriction in an $N=2$ invariant way automatically provides the extrinsic curvature contribution to the Gauss-Codazzi equation (2.13).

We will first discuss the example $\mathscr{M}_2$ of flat connections in two dimensions. The obvious way to impose the condition $F_A=0$ in the path integral is to introduce a term $\sim \int_M BF_A$ into the action where B is a scalar field. In order to do this in a manifestly $N=2$ invariant way we introduce a scalar superfield

$$\hat{B}=u+\theta\chi+\bar{\theta}\bar{\chi}+\theta\bar{\theta}B \tag{3.23}$$

and consider the action

$$S^0_{\mathscr{M}}=\int_M d\theta d\bar{\theta}\hat{B}F_0 \tag{3.24}$$

[F_0 is the two-form part of the supercurvature as given in (3.17)]. Written out in components (3.24) is

$$S^0_{\mathscr{M}}=\int_M BF_A-\chi d_A\bar{\psi}+\bar{\chi}d_A\psi+u(d_A\Sigma+[\psi,\bar{\psi}])\,. \tag{3.25}$$

We see that the B-integral (or equation of motion) forces A to be flat while the χ- and $\bar{\chi}$-integrals tell us that ψ and $\bar{\psi}$ satisfy the linearized flatness conditions $d_A\psi=d_A\bar{\psi}=0$. Together with the previously established conditions $d_A*\psi=d_A*\bar{\psi}=0$ this means that the solutions ψ_A and $\bar{\psi}_A$ to these equations represent elements of $T_{[A]}\mathscr{M}_2$.

Adding $S^0_{\mathscr{M}}$ to $S_{\mathscr{A}/\mathscr{G}}$ with an arbitrary coefficient α,

$$S_{\mathscr{M}}=S_{\mathscr{A}/\mathscr{G}}+\alpha S^0_{\mathscr{M}} \tag{3.26}$$

(this action should, of course, still be supplemented by gauge fixing terms which we will, however, not write explicitly) and integrating over Σ and u one finds that the only contribution from $S^0_{\mathscr{M}}$ to the action $S_{\mathscr{M}}$ (in addition to the above constraints on the fields A, ψ, and $\bar{\psi}$) is the *α-independent* term $-[\psi,\bar{\psi}]*G^2_A[\psi,\bar{\psi}]$ (it can be checked that α also drops out of the measure, as it should by supersymmetry). This is precisely the term (3.10) quadratic in the extrinsic curvature which appears in the Gauss-Codazzi equation (2.13). Thus what we have achieved so far in this section can [with the same *caveat* as that preceding (3.22)] be summarized by the equation

$$S_{\mathscr{M}}(A,\psi,\bar{\psi})=\mathscr{R}_{\mathscr{M}} \tag{3.27}$$

with A representing a point $[A]$ of $\mathscr{M}_2$ and $\psi=\psi_A$ and $\bar{\psi}=\bar{\psi}_A$ representing tangents to that point.

The evaluation of the partition function

$$Z(S_{\mathscr{M}})=\int_{[A]\in\mathscr{M}} D[A]\int D\psi_A\int D\bar{\psi}_A e^{iS_{\mathscr{M}}(A,\psi_A,\bar{\psi}_A)} \tag{3.28}$$

is now straightforward and proceeds exactly as in the case of supersymmetric quantum mechanics [2, 30]. There are an equal number $d(\mathscr{M})=\dim\mathscr{M}_2$ of ψ and $\bar{\psi}$ Grassmann-odd zero modes which can be soaked up, provided that $d(\mathscr{M})$ is even, by expanding the exponential to $(d(\mathscr{M})/2)^{\text{th}}$ order. The remaining integral is then of the Gauss-Bonnet form $\int_{\mathscr{M}}\mathscr{R}^{d(\mathscr{M})/2}_{\mathscr{M}}$. If, on the other hand, $d(\mathscr{M})$ is odd (this does not occur for two-dimensional surfaces) the partition function is zero. Moreover, if $\mathscr{M}$ is not connected (as will frequently be the case) then care has to be taken with the relative signs of the contributions from the connected components of $\mathscr{M}$.

In any case one finds (possibly up to a numerical factor depending only on the dimension but not on the nature of $\mathcal{M}$)[3] that the partition function of the $N=2$ topological gauge theory defined by the action (3.26) = (3.20) + (3.25) is the Euler characteristic of the moduli space $\mathcal{M}_2$,

$$Z(S_{\mathcal{M}})=\chi(\mathcal{M}_2)\,! \tag{3.29}$$

In the case of instantons all that needs to be changed in the above derivation is to replace the scalar superfield $\hat{B}$ by a selfdual two-form superfield $\hat{B}_+$ [giving rise to (3.11) instead of (3.10)]. And for flat connections in $n>2$ dimensions one could use the action (3.24) with $\hat{B}$ an $(n-2)$-form. In that case, however, additional gauge fixing terms are required because of the "Bianchi" symmetry $\delta B_{n-2}=d_A A_{n-3}$ of the action $\int_M B_{n-2}F_A+\dots$. This is a manifestation of the extended length of the deformation complex (the twisted de Rham complex in n dimensions) and will give rise to a plethora of new zero modes (corresponding to the cohomology groups $\mathbf{H}^k_A$, $k>1$ of the deformation complex). Nevertheless, ignoring these complications one will then formally find $Z(S_{\mathcal{M}})=\chi(\mathcal{M}_n)$ as well. But it should be borne in mind that this result is on a much less secure footing than (3.29) where the zero modes are under control and their significance (reducibility) is well understood.

In three dimensions, however, another procedure is available, technically because of the fact that the required multiplier B is a one-form so that it can be incorporated into the $N=2$ multiplet of A. This is a feature shared by topological gauge theories based on the moduli space of Yang-Mills connections *in any dimension* [12] because (like F_A in $n=3$ dimensions) $d_A * F_A$ is an $(n-1)$-form. This also means that the "obvious" $N=1$ topological theories associated with these moduli spaces will automatically have an underlying $N=2$ symmetry.

Thus in three dimensions, instead of introducing $\hat{B}$ and using the analogue of (3.24), one can use a super Chern-Simons action [58] and we will choose

$$\begin{aligned} S^0_{\mathcal{M}} &= \tfrac{1}{2}\int_M d\theta d\bar{\theta} A(x,\theta,\bar{\theta})dA(x,\theta,\bar{\theta})+\tfrac{2}{3}A(x,\theta,\bar{\theta})^3 \\ &= \int_M BF_A+\bar{\psi}d_A\psi \end{aligned} \tag{3.30}$$

[we have now called the $\theta\bar{\theta}$-component of $A(x,\theta,\bar{\theta})$ B instead of Σ as it plays the role the multiplier field B_{n-2} plays in the formulation (3.24)]. For the theories based on the Yang-Mills moduli space the appropriate action would, of course, have been the super Yang-Mills action $\int_M d\theta d\bar{\theta} F_0 * F_0$ [cf. (3.17)].

The classical equations of motion that one obtains from the action (3.30) are (none too surprisingly) $F_0=0$. In particular, the gauge fields are flat while the classical ψ and $\bar{\psi}$ configurations represent, as above, tangents to the moduli space $\mathcal{M}_3$. Note that in addition to the A, ψ, and $\bar{\psi}$ zero modes we will have an equal number of B zero modes as the solution to the equation of motion $d_A B+[\psi,\bar{\psi}]=0$ will only be unique up to the addition of one forms $X\in\Omega^1(M,\mathbf{g})$ satisfying the linearized flatness equation $d_A X=0$.

The above considerations have to be modified slightly once we add (3.30) to the action $S_{\mathcal{A}/\mathcal{G}}$ (3.22). The only essential modification is that the $\Sigma^2\equiv B^2$ term in the

[3] Following the analysis of the normalization of the zero mode integrals in [2, 30] this factor can be shown to be 1

action $S_{\mathscr{A}/\mathscr{G}}$ now implies that the path integral is only Gaussian (and not delta-function) peaked around the moduli space $\mathscr{M}_3$ (although this Gaussian can be arbitrarily close to a delta-function since the coefficient α of $S^0_{\mathscr{M}}$ is arbitrary). This will be reflected in the fact that the quantum fluctuations of A, which did not make any appearance in the delta-function examples discussed previously, will play an important role in the analysis below. The presence of the B^2 term also ensures that the B zero modes are damped in the path integral, only contributing to its overall normalization – a rather welcome feature at this point as all the relevant geometry is already encoded into the zero modes of A, ψ, $\bar{\psi}$ and undamped Grassmann even B zero modes would have just been a nuisance.

With these remarks in mind let us now complete the analysis of this action. We expand A, ψ, and $\bar{\psi}$ about their classical configurations

$$A = A_c + A_q, \qquad \psi = \psi_c + \psi_q, \qquad \bar{\psi} = \bar{\psi}_c + \bar{\psi}_q. \tag{3.31}$$

The only terms of interest in the action are

$$\mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M (B * B + \alpha B d_{A_c} A_q + \alpha[\bar{\psi}_c, \psi_c] A_q). \tag{3.32}$$

Integrating over the B and A_q fields allows us to write the action in its final form

$$S_{\mathscr{M}} = \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M [\bar{\psi}, \psi] * G^2_{A_c}[\bar{\psi}, \psi] = \mathscr{R}_{\mathscr{M}}, \tag{3.33}$$

leading, as above, to the result

$$Z(S_{\mathscr{M}}) = \chi(\mathscr{M}_3). \tag{3.34}$$

4. Geometry of $N=2$ Topological Gauge Theories

In the previous section we discussed the geometry of moduli spaces of connections and constructed topological actions which describe this geometry. At this point, however, it may not yet be clear why these actions do the job. By linking the construction of Sect. 3.2 with the ideas of Sect. 2, we will now try to provide a more geometric explanation of the origin of these actions. In particular, we will interpret these actions from the Atiyah-Jeffrey point of view as being Mathai-Quillen representatives of regularized Euler numbers of certain infinite-dimensional bundles. These we can then, according to the calculations of Sect. 3.2, identify with some finite-dimensional Euler number [as in the case of supersymmetric quantum mechanics, cf. (2.11)]. Although this interpretation "explains" the actions to a certain extent, it does not immediately shed any light on the question in which way these theories can be regarded as arising from some infinite-dimensional supersymmetric quantum mechanics theories, and this we will try to remedy in Sect. 4.2 and, from a more computational point of view, in [17]. These considerations will naturally lead us to the generalization of the Casson invariant mentioned in the introduction (Sect. 4.3).

4.1. The Atiyah-Jeffrey Interpretation

Here the idea is to show that the action $S_{\mathscr{M}} = S_{\mathscr{A}/\mathscr{G}} + S^0_{\mathscr{M}}$ (3.20, 3.25, 3.30) has the form of the Mathai-Quillen exponent (2.5) for a suitable bundle E and section s.

This can be done in either of two ways: by exploiting the geometry of the principal fibration $P \to X$ ($\mathscr{A} \to \mathscr{A}/\mathscr{G}$) and its associated vector bundles to manipulate (2.5) into the form (3.20)+(3.25, 3.30) with the complete field content, or (more simply, but also less elegantly) by reducing $S_{\mathscr{M}}$ to the form (2.5). The former has been explained in great detail by Atiyah and Jeffrey in the case of Donaldson theory and the three-dimensional theory of flat connections discussed above. For simplicity we will focus on the latter here, which essentially amounts to performing the manipulations of Sect. 3.2 [or, equivalently, those leading to (2.9) in the case of quantum mechanics].

We begin with the three-dimensional theory. This is the richest of the models discussed in Sect. 3 and also geometrically the most transparent (reflected in the fact that no auxiliary fields were required in the construction of the action). Recalling (3.22) and (3.30) we see that we can already write $S_{\mathscr{M}}$ in reduced form as

$$\begin{aligned} S_{\mathscr{M}} &= \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M BF_A + \bar{\psi} d_A \psi + B * B \\ &= -\tfrac{1}{4} \int_M F_A * F_A + \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M \bar{\psi} d_A \psi \,. \end{aligned} \tag{4.1}$$

This is precisely of the form (2.5), i.e. of the form

$$-\xi^2 + \chi_a \Omega^{ab} \chi_b / 4 + \mathrm{i} d \xi^a \chi_a \,,$$

for $s(A) = *F_A \in \Omega^1(M, \mathbf{g})$ provided that we rescale $\bar{\psi}$ appropriately [to identify the third term of (4.1) note that $\delta F_A = d_A \delta A$ and that $\bar{\psi}$ plays the role of χ]. s is a section of the tangent bundle $T\mathscr{A}$ of $\mathscr{A}$ which passes down to a section of $T(\mathscr{A}/\mathscr{G})$ as $d_A^* * F_A = 0$ by the Bianchi identity $d_A F_A = 0$. In fact, $s(A) = *F_A$ is the gradient vectorfield of the Chern-Simons functional and as such enters into the definition of Floer cohomology [27] as well as into Taubes' interpretation of the Casson invariant [52]. Recalling the discussion of Sect. 2, we see that the partition function $Z(S_{\mathscr{M}})$ can be regarded as the regularized Euler number $\chi_s(\mathscr{A}/\mathscr{G})$ of $\mathscr{A}/\mathscr{G}$. On the other hand, from the previous section we already know that $Z(S_{\mathscr{M}})$ localizes onto the zeros of s, i.e. onto flat connections, and yields the Euler number of $\mathscr{M}$ via the zero section of $T\mathscr{M}$ and the Gauss-Bonnet theorem. Thus here we have yet another example in which the (ambiguous) regularized Euler number of an infinite-dimensional vector bundle equals the Euler number of a finite-dimensional vector bundle [cf. (2.11)],

$$\chi_s(\mathscr{A}/\mathscr{G}) = \chi(\mathscr{M}) \,. \tag{4.2}$$

Moreover, we know from [52] that for M a homology three-sphere (see Sect. 4.2 for the definition) and $G = SU(2)$

$$\chi_s(\mathscr{A}/\mathscr{G}) = \lambda(M) \,, \tag{4.3}$$

where $\lambda(M)$ is the *Casson invariant* [1] of M (in accordance with more recent work on the Casson invariant [21, 55] we have dropped a factor of 1/2 in the definition of $\lambda(M)$). This identification as well as the implications of (4.2, 4.3) will be discussed further in Sects. 4.2 and 4.3.

In two dimensions the relevant part of the (reduced) action is

$$\begin{aligned} S_{\mathscr{M}} &= \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M BF_A + u(d_A \Sigma + [\psi, \bar{\psi}]) + \Sigma * \Sigma + \bar{\psi} d_A \chi \\ &= \mathscr{R}_{\mathscr{A}/\mathscr{G}} + \int_M BF_A + u[\psi, \bar{\psi}] - \tfrac{1}{4} d_A u * d_A u + \bar{\psi} d_A \chi \,. \end{aligned} \tag{4.4}$$

As we have had to introduce a scalar superfield $\hat{B}$ in addition to the superconnection $\hat{A}$, we expect the base space of the bundle in question to be something like $\mathscr{A}\times\Omega^0(M,\mathbf{g})$ instead of $\mathscr{A}$. The tangent space to $\mathscr{A}\times\Omega^0(M,\mathbf{g})$ at a point (A,u) is $T_{(A,u)}(\mathscr{A}\times\Omega^0(M,\mathbf{g}))=\Omega^1(M,\mathbf{g})\oplus\Omega^0(M,\mathbf{g})$, and (4.4) suggests the section $s(A,u)=(*d_Au,*F_A)$. However, this is strictly correct only if we add a B^2-term to the action (4.4) (which can be done in an $N=2$ invariant way).

In the present case (the "delta function gauge," cf. [12, 11]) the correct geometrical picture is obtained by integrating out B and working directly with the bundle $T\mathscr{A}|_{\mathscr{F}}\times T\Omega^0(M,\mathbf{g})$, where $\mathscr{F}=\{A\in\mathscr{A}\colon F_A=0\}$ is the space of flat connections. The above section now becomes $s(A,u)=(*d_Au,0)$. It gives rise to the $s^2\sim\int_M d_Au*d_Au$ term of (4.4) as well as to the remaining two terms

$$\int_M u[\psi,\bar{\psi}]+\bar{\psi}d_A\chi=-\int_M\bar{\psi}\delta(d_Au) \tag{4.5}$$

which correspond to the third term $\sim\chi ds$ of (2.5), with $\delta\triangleq\partial_\theta$ denoting the exterior derivative $\delta A=\psi$, $\delta u=\chi$ on $\mathscr{A}\times\Omega^0(M,\mathbf{g})$.

Away from reducible connections the zeros of this section are precisely the flat connections: $s(A,u)=(0,0)\Leftrightarrow F_A=0,\ u=0$. It passes down to a section of $T(\mathscr{A}/\mathscr{G})|_{\mathscr{M}}$ as $d_A^*(*d_Au)=0$ for $A\in\mathscr{F}$. Note that s does *not* define a section of $T\mathscr{M}$, but rather of the normal bundle $N_{\mathscr{M}}$ of $T\mathscr{M}$ in $T(\mathscr{A}/\mathscr{G})|_{\mathscr{M}}$. Thus the action (4.4) of our two-dimensional $N=2$ topological gauge theory can be regarded as the Mathai-Quillen representative of the regularized Euler number $\chi_s(N_{\mathscr{M}})$ of the normal bundle $N_{\mathscr{M}}$. Again, by the calculation of Sect. 3.2, we know that this choice of section regularizes this Euler number to be

$$\chi_s(N_{\mathscr{M}})=\chi(\mathscr{M}). \tag{4.6}$$

In the case of instantons in four dimensions everything runs as above provided that $\Omega^0(M,\mathbf{g})$ is replaced by $\Omega^2(M,\mathbf{g})_+$.

4.2. $N=2$ Topological Gauge Theories, Floer Cohomology, and Supersymmetric Quantum Mechanics on $\mathscr{A}/\mathscr{G}$

The considerations of the preceding section show that $N=2$ topological gauge theories are based on the tangent bundle geometry of $\mathscr{A}/\mathscr{G}$, in agreement with the calculations of Sect. 3.2 which exhibited a relation between these theories and the Riemannian geometry of $\mathscr{A}/\mathscr{G}$. This already makes these theories much closer to supersymmetric quantum mechanics than, say, Donaldson theory where the bundle in question [5] is not related to the tangent bundle of $\mathscr{A}/\mathscr{G}$.

However, the analogy is not yet perfect. In order to gain a better understanding of the emergence of de Rham cohomology in our $N=2$ models we will now construct supersymmetric quantum mechanics on $\mathscr{A}/\mathscr{G}$ along the lines of Sect. 2 (i.e. *via* a section of the tangent bundle of the loop space of $\mathscr{A}/\mathscr{G}$). Alternatively, we can use the Gauss-Codazzi form of supersymmetric quantum mechanics (Sect. 2.3) to get to the desired moduli space not via the zeros of a section but by means of a supersymmetric delta function in the path integral. For the space of gauge orbits $\mathscr{A}^3/\mathscr{G}^3$ on a three-manifold M we find that the resulting quantum mechanics model for a particular choice of section is precisely $N=1$ Donaldson theory on $M\times S^1$ which reduces to the three-dimensional $N=2$ theory in the limit that the circle (time) shrinks to zero (in particular, their partition functions are equal). This gives a relation between the $N=2$ symmetry, de Rham theory on $\mathcal{M}_3$ and Floer cohomology. As all this is really just a reinterpretation of the transition from the Hamiltonian [3] to the Lagrangian [57] description of Donaldson theory, we will not construct the quantum mechanics action in detail (see [17]), concentrating instead on the features relevant for us here. Analogous considerations can be carried out, *mutatis mutandis*, in other dimensions.

We begin with the space $\mathcal{A}^3$ of connections on a (trivial) principal G-bundle over a three-manifold M and would like to interpret its loop space as the space of connections on some bundle over $M\times S^1$. The first thing we should decide is whether to work with $L\mathcal{A}^3$ (eventually modded out by $L\mathcal{G}^3$) or with $L(\mathcal{A}^3/\mathcal{G}^3)$. The difference between the two is that in the latter case the connections are required to be periodic in time only up to a gauge transformation, and it is that space we have to work with if we are interested in non-trivial bundles on $M\times S^1$. To see this, note that $L(\mathcal{A}^3\times\mathcal{G}^3)$ is not connected if there are large gauge transformations on M,

$$\pi_0(L(\mathcal{A}^3/\mathcal{G}^3))=\pi_0(\mathcal{G}^3)\,,$$

and these can be used as clutching functions to construct non-trivial bundles over the *mapping cylinder* $M\times S^1$ of M. $L(\mathcal{A}^3/\mathcal{G}^3)$ represents the disjoint union of gauge equivalence classes of connections on all isomorphism classes of bundles of $M\times S^1$ and contains $L\mathcal{A}^3/L\mathcal{G}^3$ as its trivial component. Since we are not going to worry about gauge fixing in the following it is most convenient to work equivariantly on $L\mathcal{A}^3$. However, the difference between the two spaces will occasionally be crucial and we will draw attention to it when that occurs.

The obvious section to start off with is (as in Sect. 2) $s_0(A)(t)=\dot{A}(t)$. The corresponding Mathai-Quillen action (2.5) [i.e. the action (2.6) with $x(t)$ replaced by $A(t)$] has, however, still got a divergent partition function. In fact, it would regularize the Euler number of $\mathcal{A}^4/\mathcal{G}^4=L\mathcal{A}^3/L\mathcal{G}^3$ to be $\chi_s(\mathcal{A}^4/\mathcal{G}^4)=\chi(\mathcal{A}^3/\mathcal{G}^3)$, which is not yet well defined. An alternative way of seeing this is to note that the dimensional reduction of this action gives precisely what we called $S_{\mathcal{A}^3/\mathcal{G}^3}$ in Sect. 3.2 – thus our four-dimensional action still requires addition of a term corresponding to $S_{\mathcal{M}}$.

The most natural way to try to do this is to change the section s_0 to s_W for some gauge invariant potential function W on $\mathcal{A}^3$. In the setting of Sect. 2 this did not change the result: the partition function of supersymmetric quantum mechanics is well defined and independent of the choice of W, a statement equivalent to the classical formula (2.10). In the present case the left-hand side of (2.10) is not yet well defined, but we can make sense of it (regularize it further) by *defining* it to be equal to the right-hand side of (2.10) for some choice of W (cf. the discussion at the end of Sect. 2.2).

A natural candidate for W is the Chern-Simons functional

$$W(A)=\mathrm{CS}(A)\equiv\int_M \mathrm{Ad}A+\tfrac{1}{3}A[A,A]\,. \tag{4.7}$$

$\mathrm{CS}(A)$ is not quite gauge invariant (it changes by a constant proportional to the winding number under large gauge transformations) but its derivative is, and this is sufficient for our purposes. This choice of potential defines the section $s_{\mathrm{CS}}(A)(t)=\dot{A}(t)+*F_{A(t)}$ of $T(L\mathcal{A}^3)$.

By general results on supersymmetric quantum mechanics [11] (or explicit calculation) one finds that the partition function of the action corresponding to s_{CS} localizes onto the zeros of s_{CS} (in the present case possibly modified by terms required for four-dimensional gauge invariance which is not guaranteed by the Mathai-Quillen formalism – this will not affect any of our conclusions). If $A(t)$ is periodic in t (this means that we are in the topologically trivial sector) the same "squaring argument" as in Sect. 2 shows that these are precisely the time-independent flat connections on M. In the topological non-trivial sectors the "squaring argument" fails and there are non-trivial solutions to the equation (the gradient equation of the Chern-Simons functional)

$$\frac{d}{dt}A(t) = -*F_{A(t)}\,. \tag{4.8}$$

Equation (4.8) (usually read as an equation on $M\times\mathbf{R}$) is nothing other than the instanton equation in the $A_0=0$ gauge and plays a fundamental role in defining the relative Morse indices of Floer's instanton (co)homology [27, 3] and also provides the link between the three-dimensional Floer cohomology groups and the four-dimensional Donaldson invariants [24] associated with the moduli spaces of instantons (see also [3, 11] for the definition of these invariants and their relation with Floer theory).

This already suggests that we have just reinvented the wheel and that the four-dimensional topological gauge theory we have constructed here is nothing other than Donaldson theory. That this is indeed the case becomes immediately obvious by noticing that the Hamiltonian of the corresponding quantum mechanics action is the Laplacian of δ_{CS}, the exterior derivative δ on $\mathscr{A}^3$ twisted by the Chern-Simons functional, and hence precisely the Hamiltonian of Donaldson theory [3, 57]. Alternatively, this can, of course, be seen directly at the level of the action [17]. There are some things that we ought nevertheless to check to be sure that all the pieces fit. Firstly, we know from the four-dimensional standpoint that the partition function of the Donaldson theory vanishes if the index of the deformation complex (the formal dimension of the instanton moduli space) is not zero. An essential ingredient in the construction is then that this index be zero. On the other hand, to get sensible results for the three-dimensional theory the only sector in the four-dimensional theory which contributes must be the one with trivial second Chern class. We will now see how these two requirements take care of each other. The index is equal to [4]

$$p_1 - \tfrac{1}{2}\dim G(\chi(M_4)+\sigma(M_4))\,. \tag{4.9}$$

Here p_1 is the first Pontryagin number of the adjoint bundle $\mathrm{ad}\,P$ [equal to $8k$, k the instanton number, for $G=SU(2)$] and $\sigma(M_4)$ is the signature of M_4. This is the signature of the intersection form on $H^2(M_4,\mathbf{Z})$ or the number of self-dual minus the number of anti-self-dual harmonic two-forms on M_4. The Euler characteristic of a four-manifold of the form $M_4=M_3\times S^1$ is zero as $\chi(M_3\times S^1)=\chi(M_3)\chi(S^1)=0$. By the same multiplicative property of the signature [36, Theorem 8.2.1] $\sigma(M_3\times S^1)$ also vanishes. Explicitly, this can be seen as follows: the Künneth formula tells us that $H^2(M_3\times S^1;R)$ is isomorphic to $H^1(M_3;R)\oplus H^1(M_3;R)$, and if $\{\omega_i\}$ form a basis for $H^1(M_3;R)$ then (symbolically) a basis for $H^2(M_3\times S^1;R)$ is $\{\omega_i\oplus\omega_i, \omega_i\oplus-\omega_i\}$. The first entry forms a basis for the space of self-dual harmonic two forms H^2_+ while the second is a basis for that of anti-self-dual harmonic two forms H^2_-: they necessarily have the same dimension.[4] Hence we find that the index is non-zero for all $p_1 \neq 0$. Notice that while the dimension formula (4.9) tells us that the instanton moduli space is formally zero for $p_1=0$ it tells us nothing about the dimension of the space of flat connections. In fact, the index of the flat connection deformation complex is the sum of the instanton and anti-instanton indices for $p_1=0$ and is given by $-\dim G\chi(M_4)$ (the index of the twisted de Rham complex) which is zero for $M_4 = M_3 \times S^1$.

This also settles the question raised above whether we should work with $L(\mathscr{A}^3/\mathscr{G}^3)$ or $L\mathscr{A}^3/L\mathscr{G}^3$: the theories are identical in the sector with $p_1=0$, which is (according to the above) the only one that will contribute to the partition function, so for our purposes both alternatives are equivalent. In the topologically trivial sector the partition function reduces to an integral over the moduli space $\mathscr{M}_3(M,G)$ for an arbitrary three-manifold M. There the twisted exterior derivative $\delta_{CS}=\delta+\int_M F_A\delta A$ reduces to the ordinary exterior derivative, the Hamiltonian to the Laplacian on $\mathscr{M}_3$, and the partition function is (independently of the radius of the circle) the Euler characteristic $\chi(\mathscr{M})$ of the de Rham complex of $\mathscr{M}$.

That this agrees with the partition function of the three-dimensional $N=2$ theory is no coincidence. In fact, we can expand all fields in Fourier modes along the circle. Integrating out the non-constant modes the resulting three-dimensional action is precisely the action constructed in Sect. 3.2. In the light of our previous considerations and those of [58, 14] (the three-dimensional theory is the dimensional reduction of Donaldson theory) this is not very surprising and the calculational details can be found in [17]. This finally also establishes the sought-for direct relation between this $N=2$ topological gauge theory and supersymmetric quantum mechanics on spaces of connections.

Another part of the puzzle that fits in place is that ignoring time derivatives the $N=1$ symmetry of Donaldson theory extends to an $N=2$ symmetry which is enjoyed by the Hamiltonian (i.e. the Lagrangian of the three-dimensional theory) just as we found when embedding S^2 into R^3 in Sect. 2.3. Indeed, as $H \sim \int T_{00}$ and $T_{00}=\{Q, V_{00}\}$ for some V_{00} (by the fundamental property $T_{\alpha\beta}=\{Q, V_{\alpha\beta}\}$ of topological field theories [57]) we could imagine that $H=\{Q,\bar{Q}\}$, where $\bar{Q}$ is nilpotent and leaves the Hamiltonian invariant. That this is indeed the case was established by Witten in [57].

Returning to the three-dimensional discussion, let us momentarily assume that M is a homology three-sphere, i.e. an orientable closed three-manifold with $H_1(M,\mathbf{Z})=0$, and that the gauge group is $G=SU(2)$. In that case, non-trivial flat connections are irreducible [in fact, a reducible connection, defined by a reducible element of $\mathrm{Hom}(\pi_1(M),G)$ would factor through to an element of $\mathrm{Hom}(\pi_1(M),U(1)) \approx H_1(M,U(1))=0$]. In this setting the Floer cohomology groups, the cohomology groups of (a perturbation of) the twisted exterior derivative δ_{CS}, are well defined and coincide (as in ordinary supersymmetric quantum mechanics) with the ground states of the above Hamiltonian. In particular, therefore, the partition function of this theory is (ignoring problems with the trivial connection) the Euler characteristic $\chi_F(M)$ of the Floer complex which is known [3, 52] to be

$$\chi_F(M)=\lambda(M). \tag{4.10}$$

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[4] Alternatively, we note that $\mathrm{Tr}\int RR$ vanishes with the product metric which implies the vanishing of $\sigma(M_3 \times S^1)$ by the Hirzebruch signature theorem

Note that the calculation of the Euler characteristic of the Floer complex requires only the topologically trivial sector (flat connections) although the definition of the individual instanton homology groups depends crucially on all the topologically non-trivial sectors (instantons). This is entirely analogous to ordinary supersymmetric quantum mechanics on a manifold M: the Euler number $\chi(M)$ can be calculated in terms of the fixed points of some vector field alone whereas instanton paths connecting these fixed points enter into the computation of the homology groups of M [56].

The consequences of the intriguing equations $\chi_s(\mathscr{A}^3/\mathscr{G}^3)=\chi(\mathscr{M}_3)$ (4.2), $\chi_s(\mathscr{A}^3/\mathscr{G}^3)=\lambda(M)$ (4.3), and $\chi_F(M)=\lambda(M)$ (4.10) will be explored in the following section, after we have recalled the definition and some properties of the Casson invariant.

4.3. The Casson Invariant and its Generalization

In this section we will deal exclusively with the three-dimensional theory defined by the action (3.20)+(3.30). We have seen above that formally the partition function of this theory yields the Euler characteristic of the moduli space $\mathscr{M}_3$ of flat connections via the Gauss-Codazzi equations and the Gauss-Bonnet formula. On the other hand, we will see below that (again formally) the partition function is the Casson invariant if M is an integral homology three-sphere. Of course, these two observations taken together immediately suggest a generalization of the Casson invariant to arbitrary three-manifolds. But in order to substantiate this suggestion, there are some problems that need to be overcome at a purely mathematical level before one can try to assert whether $\chi(\mathscr{M})$ is a meaningful and useful generalization of the Casson invariant. In particular, one needs to

a) define what one means by $\chi(\mathscr{M})$ when $\mathscr{M}$ is not a smooth manifold but perhaps (at best) an orbifold stratification (in the sense of Kirwan [40]), and

b) compare candidate definitions of $\chi(\mathscr{M})$ with already existing extensions of the Casson invariant to certain more general classes of three-manifolds (rational homology spheres [21, 55], homology lens spaces [19]).

We have no definite solutions to offer to these problems but we will provide some background information and preliminary suggestions below which we believe will play a role in the resolution of these issues.

In addition to these mathematical issues (which are completely independent of the field theoretic considerations by which we were led to them) there are problems with the field theoretic realization of these topological (differential) invariants. In particular, in order to be able to assert that the partition function really calculates the Casson invariant (in the simplest case of homology spheres) or the Euler number, one needs to

c) come to terms with the contributions from the trivial connection and other reducible connections.

At first, however, our considerations will be formal. Let us assume for the time being that there are no ψ and $\bar{\psi}$ zero modes (i.e. no non-trivial solutions ψ_A to the equations $d_A\psi=d_A*\psi=0$, etc.). Then the partition function will simply reduce to a sum of contributions from the points of $\mathscr{M}$ [cf. (3.28)], which – by supersymmetry – are plus or minus one,

$$Z(S_{\mathscr{M}})=\sum_{\mathscr{M}}\pm 1 \tag{4.11}$$

[the contribution of the trivial connection is ill-defined at this point and is assumed to be excluded from the sum (4.11) until further notice]. A look at the action $S^0_{\mathcal{M}} = \int_M BF_A + \bar{\psi} d_A \psi$ reveals that the *relative* signs are determined by the (mod 2) spectral flow of the operator d_A, the same spectral flow that defines the relative Morse indices of Floer homology [27, 3]. Therefore, $Z(S_{\mathcal{M}})$ equals the Euler characteristic $\chi_F(M)$ of the Floer complex. Since d_A is the Hessian of the Chern-Simons functional whose first derivative defines the gradient vector field $*F_A$ on $\mathscr{A}/\mathscr{G}$, we also see rather directly that $Z(S_{\mathcal{M}})$ can be regarded as defining the regularized Euler number $\chi_s(\mathscr{A}/\mathscr{G})$, as we, of course, already know more generally from Sect. 4.1.

It is a result of Taubes [52] that this topological invariant agrees (possibly up to a sign) with the Casson invariant [1] $\lambda(M)$,

$$Z(S_{\mathcal{M}}) = \lambda(M) \tag{4.12}$$

(again, provided that the trivial connection is excluded from the sum). Actually Taubes also fixes the absolute sign. This requires considerations involving perturbations of the trivial connection, and we will not enter into these here.

Casson's original definition of $\lambda(M)$ was somewhat different, involving Heegard splittings of M along a Riemann surface Σ_g, and intersection theory in $\mathcal{M}(\Sigma_g, SU(2))$. We will now show how his definition can be recovered from the path integral point of view (this is taken from [11]). Imagine splitting M along a Riemann surface Σ_g, i.e. $M = M_1 \#_{\Sigma_g} M_2$, where M_1 and M_2 are handlebodies (solid Riemann surfaces). Then – according to the general principles of quantum field theory – the path integral over connections on the manifold M_1 with boundary $\partial M_1 = \Sigma_g$ will define a wave function Ψ_1 having support on those flat connections on Σ_g which extend to flat connections on M_1, i.e. on the Lagrangian submanifold $\mathcal{M}(M_1, SU(2))$ of the symplectic manifold $\mathcal{M}(\Sigma_g, SU(2))$. Likewise the path integral over connections on M_2 will produce a wave function Ψ_2 having support on $\mathcal{M}(M_2, SU(2)) \subset \mathcal{M}(\Sigma_g, SU(2))$. The partition function $Z(S_{\mathcal{M}})$ can then be computed as the scalar product

$$Z(S_{\mathcal{M}}) = \int_{\mathcal{M}(\Sigma_g, SU(2))} \Psi_1^* \Psi_2 , \tag{4.13}$$

and evidently only receives contributions from flat connections on Σ_g which extend to both M_1 and M_2 or – in other words – from flat connections on M. If these flat connections are isolated, (4.13) is a sum over these points, their contributions being determined as in [52]. The key point in Taubes' work is to show that the relative intersection numbers of $\mathcal{M}(M_1, SU(2))$ and $\mathcal{M}(M_2, SU(2))$ in $\mathcal{M}(\Sigma_g, SU(2))$ can be determined from the spectral flow of d_A. Denoting the total intersection number in $\mathcal{M}$ by $\#_{\mathcal{M}}$, (4.12) then implies

$$\lambda(M) = \#_{\mathcal{M}(\Sigma_g, SU(2))}(\mathcal{M}(M_1, SU(2)), \mathcal{M}(M_2, SU(2))) , \tag{4.14}$$

which is precisely Casson's original definition (up to the factor of 1/2 mentioned above).

If the moduli space $\mathcal{M}(M, SU(2))$ is non-zero-dimensional, the Casson invariant is defined as follows. As the dimension of $\mathcal{M}(M_k, SU(2))$ is $3g-3$ and that of $\mathcal{M}(\Sigma_g, SU(2))$ is $6g-6$, the moduli spaces $\mathcal{M}(M_k, SU(2))$ can be perturbed (isotoped) into general position to intersect in isolated points, and the Casson invariant is now defined as in (4.14). The significance of M being a homology three sphere is that the isotopies can be chosen to avoid the reducible connections. From the gauge theory point of view this can be mimicked by a perturbation of the vector field $*F_A$. On the other hand, the path integral is already well defined in this case and calculates the Euler number $\chi(\mathcal{M}(M, SU(2)))$. This strongly suggests that, for homology spheres, the Casson invariant is equal to the Euler number of the moduli space (and thus has a definition which makes no reference to isotopies).[5]

In recent years some effort has gone into generalizing the Casson invariant to other groups G or to more general classes of three-manifolds (see e.g. [21, 55, 19]). One is then inevitably confronted with the presence of non-trivial reducible flat connections. In Casson's approach this is problematic because $\mathcal{M}(M_1, G) \cap \mathcal{M}(M_2, G)$ now meets the singularities of $\mathcal{M}(\Sigma_g, G)$ and the definition of the intersection numbers requires more care. Alternatively, in Taubes' approach (which has not yet been worked out in a more general setting) a more delicate perturbation theory would be required to deal with the zero modes of d_A at these points. Important progress was made by Walker [55] who extended the definition of the Casson invariant to rational homology three-spheres ($H_*(M, \mathbf{Q}) = H_*(S^3, \mathbf{Q})$), and by Cappell, Lee, and Miller [21] who generalized it to arbitrary semisimple Lie groups G. In all these generalizations, $\lambda(M)$ is no longer necessarily an integer but a rational number.

From a different angle we have seen at various points in this paper that it is natural to propose the Euler characteristic $\chi(\mathcal{M}(M, G))$ as a generalization of the Casson invariant. In the case of isolated irreducible flat connections these two definitions coincide [as the spectral flow of d_A indeed measures the relative tangent space (point) orientations]. Additionally, naive evaluation of (4.13) in the case that the moduli space is non-zero-dimensional also yields $\chi(\mathcal{M}(M, G))$, independently of any isotopies. Indeed, the same topological theory that formally gives us the Casson invariant if the underlying three-manifold M is a homology sphere formally computes the Euler number of $\mathcal{M}(M)$ via the Gauss-Codazzi equations and the Gauss-Bonnet theorem in general.

This suggestion by itself does, of course, not solve any of the technical problems inherent in the definition of $\chi(\mathcal{M})$ for the types of spaces arising as moduli spaces of flat connections on three-manifolds. Nevertheless, we hope that this proposal is concrete enough to be useful and perhaps guide future investigations, as the problem is now more specifically that of finding a "good" definition of $\chi(\mathcal{M})$. The difficulties one encounters when trying to find such a definition are all related in one way or another to the fact that the singularity-structure of the spaces $\mathcal{M}(M, G)$ is not well understood.

For instance, in three dimensions singularities are not only due to reducible representations of $\pi_1(M)$ but also to the relations satisfied by the generators of $\pi_1(M)$. For information about the fundamental groups of three-manifolds see [35, 28]. More generally, these moduli spaces have been studied in the context of representation varieties of finitely generated groups (see the monograph [43] and the contributions in [32]). However, again little is known about the nature of the singularities away from irreducible representations.

Thinking now more concretely about defining the Euler number of singular spaces we want to mention the encouraging result that there is a Gauss-Bonnet theorem for V-manifolds (orbifolds) [50] which calculates the virtual Euler

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5 In the case of Seifert fibered homology spheres it has indeed been shown by Fintushel and Stern that the Casson invariant equals the Euler number – see the "Note Added" at the end of the paper

characteristic of an orbifold as defined e.g. in [50, 20, 54, 46, 34] in various contexts. The virtual Euler number is different from the topological Euler number of an orbifold and is no longer necessarily an integer if the orbifold is not a smooth manifold, but a rational number. This checks with the properties of the Casson invariant away from integral homology spheres. And if the moduli space $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$ is an orbifold then the metric induced on $\mathcal{M}$ by the metric on $\mathcal{A}/\mathcal{G}$ will be an orbifold metric (in the sense of [50]) and therefore the evaluation of the partition function (i.e. of the Gauss-Bonnet integrand) will give rise to the virtual (orbifold) Euler number of $\mathcal{M}$.

Additional circumstantial evidence in favour of our suggestion could be provided by showing that – at least formally – the Euler number $\chi(\mathcal{M}(M, G))$ has in general properties similar to those satisfied by the Casson invariant, e.g. under the operations of connected sums or reversing orientations. One of the most important properties of the Casson invariant (apart from being a differential invariant) is its nice behaviour under Dehn surgery on knots and its relation with the Alexander polynomial. This is a property that one may not wish to give up, but unfortunately also one that seems to be rather difficult to prove for $\chi(\mathcal{M})$.

Whatever the outcome of these investigations will be, we hope that thinking in terms of traditional differential-geometric concepts will contribute to the understanding of the Casson invariant and its generalizations.

5. Concluding Remarks: The Penner Model and Other Open Questions and Generalizations

In this paper we have drawn together a number of threads to construct a topological gauge theory with the property that its partition function is the Euler number $\chi(\mathcal{M})$ of some given finite-dimensional moduli space $\mathcal{M}$ of connections. Among these threads were supersymmetric quantum mechanics, its relation with the Mathai-Quillen formalism, the Gauss-Codazzi equations for $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$, a superfield construction of $N=2$ topological gauge theories, and the Atiyah-Jeffrey interpretation of topological field theories.

Along the way we have also obtained some results which are potentially interesting outside the context of topological field theories as well. In particular, we have introduced a new kind of supersymmetric quantum mechanics based on (or: deriving) the Gauss-Codazzi equations of classical Riemannian geometry. We believe that we have also clarified the concept of the regularized (Mathai-Quillen) Euler number $\chi_s(E)$ of an infinite-dimensional vector bundle E, introduced by Taubes and Atiyah-Jeffrey, by showing that under very general conditions $\chi_s(E)$ can be identified with the rigorously and unambiguously defined Euler number of some finite-dimensional vector bundle. Combined with the fact that the Casson invariant of a homology three-sphere can be interpreted as $\chi_s(\mathcal{A}/\mathcal{G})$ and with the observation that the partition function of one and the same topological gauge theory (formally) yields either the Casson invariant or the Euler number $\chi(\mathcal{M})$ of the moduli space of flat connections, this led us to suggest $\chi(\mathcal{M})$ as a generalization of the Casson invariant to other classes of three-manifolds.

In the previous section we have mentioned some of the technical problems one encounters when attempting to a) make this suggestion more precise from a purely mathematical point of view, and b) put the field theoretic considerations on a slightly more rigorous footing.

In addition to these technical questions there are a number of other open problems and avenues for future research. In particular, we want to draw attention to the possibility of constructing a topological counterpart of the Penner matrix model [49]. This is a hermitian matrix model whose partition function calculates the virtual (orbifold) Euler characteristic of the moduli space of Riemann surfaces of genus g [34]. In analogy with Distler's observation [22] that topological gravity [42] is a bosonized form of Liouville theory coupled to $c=-2$ matter one may speculate on the existence of a topological field theory describing Liouville theory coupled to the $c=1$ model conjectured by Distler and Vafa [23] to describe the continuum limit of the Penner model. Such a topological theory would be characterized by the property that its partition function is the Euler number of moduli space (which can be described in terms of connections), precisely the property shared by the topological models discussed in this paper.

Another open problem in this context is whether this or other topological $N=2$ models in two dimensions can be described as "twisted" $N=4$ models. It is known that $N=2$ models can be twisted to $N=1$ topological theories [59, 26] and that twisted $N=3$ theories describe supersymmetric $N=1$ topological theories while twisted $N=4$ theories in two dimensions appear to describe non-supersymmetric topological $N=2$ theories [63]. We hope to report on progress along these lines in the future.

More immediate generalizations of the models discussed in this paper are e.g. supersymmetric extensions or a reformulation of the three-dimensional model of flat connections on R^3 and the hyperbolic three-plane to describe moduli spaces of monopoles (see [14, 8]). It is also possible to construct topological $N=2$ sigma models and these have the expected property of describing the Riemannian geometry of spaces of sigma model instantons. The details are in either case not too difficult and are left to the reader.

Finally, we want to mention that it is possible to add a Chern-Simons term to the three-dimensional $N=2$ action of flat connections. The resulting action has still got a topological $N=2$ symmetry (albeit slightly different from the one considered in this paper). The partition function is now not the number of flat connections counted with signs but rather a sum over flat connections weighted by signs and phases (the exponential of the Chern-Simons invariant of the flat connection). This raises the question, with which we conclude this paper, if this is an interesting refinement of the Casson invariant of homology spheres.

Note added (July 1992). In [64] (see also [65, 66]), Fintushel and Stern investigate the instanton homology of Seifert fibred homology three-spheres. One of their results is that, in this case, the Casson invariant indeed equals the Euler characteristic of the moduli space of flat connections [64, Theorem 4.3]. We regard this as further circumstantial evidence in favour of our suggestion of Sect. 4.3.

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