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Renormalization Group and the Ginzburg-Landau Equation

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Abstract. We use Renormalization Group methods to prove detailed long time asymptotics for the solutions of the Ginzburg-Landau equations with initial data approaching, as $x \to \pm\infty$, different spiraling stationary solutions. A universal pattern is formed, depending only on this asymptotics at spatial infinity.

1. Introduction

Parabolic PDE's often exhibit universal scaling behavior in long times: the solution behaves as $u(x,t) \sim t^{-\frac{\alpha}{2}} f^*(t^{-\frac{\beta}{2}} x)$ as $t \to \infty$, where the exponents α and β and the function f^* are *universal*, i.e. independent on the initial data and equation, in given classes. This fact has an explanation in terms of the Renormalization Group (RG) [10, 11, 4], very much like the similar phenomenon in statistical mechanics.

In [4] a mathematical theory of this RG was developed and here we would like to apply these ideas to a concrete situation, namely the Ginzburg-Landau equation

$$\dot{u} = \partial^2 u + u - |u|^2 u \,, \tag{1}$$

where $u: \mathbf{R} \times \mathbf{R} \to \mathbf{C}$, is complex, $\partial = \frac{\partial}{\partial x}$ and the dot denotes the time derivative. Equation (1) has a two parameter family of stationary solutions

$$u_{q\theta}(x) = \sqrt{1-q^2} e^{i(qx+\theta)} \,, \tag{2}$$

and a natural question is to inquire about the time development of initial data $u(x)$ which approach two solutions at $\pm\infty$:

$$\lim_{x \to \pm\infty} |u(x) - u_{q_\pm \theta_\pm}(x)| = 0 \,. \tag{3}$$

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This problem has been extensively studied for Eq. (1), with u *real* and $u(-\infty) = 1$, $u(\infty) = 0$ (i.e. $q_- = 0$, $q_+ = 0$) [1,3]. Then the solution takes the form of a propagating front. This occurs because $u(x) = 1$ is a stable stationary solution, while $u(x) = 0$ is an unstable one.

For complex u, the solutions (2) are stable for $q^2 < \frac{1}{3}$ (the Eckhaus stable domain). An interesting problem is to analyze the behavior of a front connecting a stable and an unstable solution for complex u, since (2) would then describe a non-trivial pattern emerging in the laboratory frame. A "phase diagram" giving the values of $q_\pm$ for which fronts exist can be found in [7] as well as an interesting review of related problems. The linear stability analysis of such fronts has been carried out in [2,9]. Going beyond the linear analysis seems to require new mathematical techniques, since the study of fronts for real u [1,3] depend crucially on the applicability of the maximum principle.

We believe that the RG is the right method to study such problems, and illustrate this in the present paper by considering a related question, suggested in [6], namely we take $q_\pm$ in (3) small, belonging to the Eckhaus stable domain.

In [6], it was shown that these boundary conditions do not produce phase-slips, i.e. the amplitude of u does not vanish. However, it was unknown whether $u(x,t)$ converges for t large. We show that actually, it does not converge but oscillates. We prove a detailed long time asymptotics for the solution of (1) in this situation. As a consequence, we shall show that, for any interval I,

$$\sup_{x\in I} |u(x,t) - e^{i\sqrt{t}\phi^*} u_{q^*\theta^*}(x)| \leq \frac{C_I}{\sqrt{t}}, \tag{4}$$

where the constants q^*, ϕ^*, and θ^* depend only on the boundary conditions (3). For the detailed asymptotics, see theorem in Sect. 3.

2. The Renormalization Group Idea

Following [6], we write

$$u = (1-s)e^{i\phi}\,. \tag{1}$$

Equation (1.1) becomes in these variables

$$\begin{aligned} \dot{s} &= \partial^2 s - 2s + 3s^2 - s^3 + (\partial\phi)^2 - s(\partial\phi)^2 \equiv \partial^2 s - 2s + F(s, \partial\phi)\,, \\ \dot{\phi} &= \partial^2\phi - \frac{2(\partial\phi)(\partial s)}{1-s} \equiv \partial^2\phi + G(s, \partial s, \partial\phi) \end{aligned} \tag{2}$$

with the initial data (it will be convenient to take the initial time as $t = 1$)

$$\lim_{x\to\pm\infty} s(x,1) = s_\pm\,, \qquad \lim_{x\to\pm\infty} |\phi(x,1) - \phi_\pm x - \theta_\pm| = 0\,, \tag{3}$$

where $2s_\pm = F(s_\pm, \phi_\pm)$. We will specify below the precise space of initial data (3). We will prove the following asymptotics as $t \to \infty$:

$$\phi(x,t) = \sqrt{t}\phi^*\left(\frac{x}{\sqrt{t}}\right) + \theta^*\left(\frac{x}{\sqrt{t}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \tag{4}$$

$$s(x,t) = s^*\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}}r^*\left(\frac{x}{\sqrt{t}}\right) + \mathcal{O}\left(\frac{1}{t}\right) \tag{5}$$

again in a certain Banach space. The functions ϕ^*, θ^*, s^*, r^* are *universal*, depending on the initial data only through the boundary conditions (3). They are smooth and therefore u will have the asymptotics (1.4), with

$$\phi^* = \phi^*(0), \qquad q^* = \phi^{*'}(0), \qquad \theta^* = \theta^*(0).$$

Before going to the proofs, we will explain in a heuristic fashion the RG idea behind the proof.

There are two mechanisms giving rise to the asymptotics (4), (5): the diffusive approach of ϕ to the scaling form (4) and the "slaving" of s to follow whatever ϕ is doing, due to the linear $-2s$ term in (2). We will now explain the first mechanism in the context of an equation for ϕ only. We will see in the next section that this is precisely what the slaving mechanism produces: s in (2) will be effectively "slaved" to a function of $\partial\phi$ only.

Thus, we consider the equation

$$\dot{\phi} = (1 + a(\partial\phi, \partial^2\phi))\partial^2\phi, \qquad \phi(x,1) = f(x) \tag{6}$$

with the boundary conditions (3) for the initial data f. We assume that a is analytic around the origin.

The RG analysis of (6) proceeds as follows. We fix some (Banach) space of initial data $\mathscr{S}$. Next, we pick a number $L > 1$ and set

$$\phi_L(x,t) = L^\alpha \phi(Lx, L^2 t), \tag{7}$$

where α will be chosen later and ϕ solves (6) with the initial data $f \in \mathscr{S}$. The RG map $R: \mathscr{S} \to \mathscr{S}$ (this has to be proven!) is then

$$(Rf)(x) = \phi_L(x,1). \tag{8}$$

Note that ϕ_L satisfies

$$\dot{\phi}_L = (1 + a_L(\partial\phi_L, \partial^2\phi_L))\partial^2\phi_L, \tag{9}$$

with $a_L(x,y) = a(L^{-1-\alpha}x, L^{-2-\alpha}y)$.

We may now iterate R to study the asymptotics of (6). R depends, besides α, on L and a. Let us denote this by $R_{L,a}$. We have then the "semigroup property"

$$R_{L^n,a} = R_{L,a_{L^{n-1}}} \circ \ldots \circ R_{L,a_L} \circ R_{L,a}. \tag{10}$$

Each R on the RHS involves a solution of a fixed time problem and the long time problem on the LHS is reduced to an iteration of these. Letting $t = L^{2n}$, we have

$$\phi(x,t) = t^{-\frac{\alpha}{2}}(R_{L^n,a}f)(xt^{-1/2}). \tag{11}$$

Now one tries to show that there exists an α such that

$$a_{L^n} \to a^*, \qquad R_{L^n,a}f \to f^*, \tag{12}$$

where

$$R_{L,a^*} f^* = f^* \tag{13}$$

is the fixed point of the RG, corresponding to the scale-invariant equation $\dot{\phi} = (1 + a^*)\partial^2\phi$. Then, rescaling x, the asymptotics of the original problem is given by

$$\phi(xt^{1/2}, t) \sim t^{-\frac{\alpha}{2}} f^*(x) \tag{14}$$

What are α and f^*? To understand this, consider first the trivial case $a = 0$. This is just the diffusion equation with initial conditions increasing linearly at infinity. This problem is of course exactly soluble. We have

$$\begin{aligned} t^{-1/2}\phi(\sqrt{t}x, t+1) &= \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{1}{4}(y-x)^2} f(\sqrt{t}y)dy\,, \\ &\to_{t\to\infty} \phi_- x + (\phi_+ - \phi_-)(x + 2\partial)e(x) \equiv \phi_0^*(x)\,, \end{aligned} \tag{15}$$

where $e(x) = \int\limits_{-\infty}^{x} e^{-\frac{1}{4}y^2} \frac{dy}{\sqrt{4\pi}}$. In RG terminology, we have the "Gaussian" fixed point ϕ_0^* corresponding to this $\phi_\pm$ boundary condition problem. It is easy to check that ϕ_0^* is a fixed point for the map (7) with $\alpha = -1$.

How about $a \neq 0$? We take again $\alpha = -1$ whence the fixed point equation is

$$\dot{\phi} = (1 + a^*(\partial\phi))\partial^2\phi\,, \tag{16}$$

where $a^*(\cdot) = a(\cdot, 0)$ and the fixed point is the scale invariant solution

$$\phi(x, t) = \sqrt{t}\phi^*\left(\frac{x}{\sqrt{t}}\right). \tag{17}$$

We get $\left(\text{replacing } \frac{x}{\sqrt{t}} \text{ by } z\right)$

$$(1 + a^*)\frac{d^2}{dz^2}\phi^* + \frac{1}{2}z\frac{d}{dz}\phi^* - \frac{1}{2}\phi^* = 0 \tag{18}$$

with $a^* = a^*\left(\frac{d}{dz}\phi^*\right)$ and, for small $\phi_\pm$, we look for a solution

$$\phi^* = \phi_0^* + \varrho\,, \tag{19}$$

where $\varrho(\pm\infty) = 0$ and ϕ_0^* is the Gaussian solution (15), which solves (18) with $a^* = 0$. This is easy to solve (see Proposition 1 below or the proposition in Sect. 4 of [4]).

Consider then θ^*. We set

$$\phi(x, t) = \phi^*(x, t) + \theta(x, t)\,, \tag{20}$$

where, with an abuse of notation, $\phi^*(x,t) = \sqrt{t}\phi^*\left(\frac{x}{\sqrt{t}}\right)$ and ϕ^* is given by (19); ϕ solves (6). Then,

$$\dot{\theta} = \partial^2\theta + (a\partial^2\phi - a^*\partial^2\phi^*) \tag{21}$$

with $\theta(\pm\infty) = \theta_\pm$. Now we set

$$\theta_L(x,t) = \theta(Lx, L^2t) \tag{22}$$

which, using $\partial^l\phi^*(Lx, L^2t) = L^{1-l}\partial^l\phi^*(x,t)$, satisfies the equation

$$\begin{aligned}\dot{\theta}_L = \partial^2\theta_L + L(a(\partial\phi^* + L^{-1}\partial\theta_L, L^{-1}\partial^2\phi^* + L^{-2}\partial^2\theta_L)\\ \times(\partial^2\phi^* + L^{-1}\partial^2\theta_L) - a^*(\partial\phi^*)\partial^2\phi^*)\,.\end{aligned} \tag{23}$$

Thus, reasoning as above, we expect

$$\theta_{L^n} \to \theta^*\,, \tag{24}$$

where $\theta^*(x,t) = \theta^*\left(\frac{x}{\sqrt{t}}\right)$ satisfies the $L\to\infty$ form of (23):

$$\dot{\theta}^* = \partial^2\theta^* + a^*\partial^2\theta^* + (a_x(\partial\phi^*,0)\partial\theta^* + a_y(\partial\phi^*,0)\partial^2\phi^*)\partial^2\phi^*\,. \tag{25}$$

This is a linear equation, easy to solve, whose solution is a small perturbation of the "Gaussian" solution [which solves (25) with $a = 0$],

$$\theta_0^*(z) = \theta_- + (\theta_+ - \theta_-)e(z)\,. \tag{26}$$

Actually, $\theta_0^*(z) = \phi_0^{*'}(z)$, where ϕ_0^* is given by (15), and $\theta_0^*(z)$ is a fixed point of the map

$$(R_L\theta)(z) = (e^{(L^2-1)\partial^2}\theta)(Lz)\,, \tag{27}$$

i.e. of (7), (8) with $\alpha = 0$ and $a = 0$.

Finally, one sets

$$\phi(x,t) = \phi^*(x,t) + \theta^*(x,t) + \psi(x,t) = \sqrt{t}\phi^*\left(\frac{x}{\sqrt{t}}\right) + \theta^*\left(\frac{x}{\sqrt{t}}\right) + \psi(x,t)\,, \tag{28}$$

and one studies the asymptotics of ψ in the same manner. The last two steps (θ and ψ) were carried out in [4] in great detail. We will now combine this RG approach with the slaving principle to arrive at (4), (5).

3. The Full Model

We may summarize the previous discussion in a more conventional language: we look for solutions of (2.2) of the form

$$\begin{aligned}\phi(x,t) &= \phi^*(x,t)+\theta^*(x,t)+\psi(x,t)\,,\\ s(x,t) &= s^*(x,t)+r^*(x,t)+v(x,t)\,,\end{aligned}\tag{1}$$

with

$$\begin{aligned}&\phi^*(x,t)=\sqrt{t}\phi^*\left(\frac{x}{\sqrt{t}}\right), \qquad \theta^*(x,t)=\theta^*\left(\frac{x}{\sqrt{t}}\right),\\ &s^*(x,t)=s^*\left(\frac{x}{\sqrt{t}}\right), \qquad r^*(x,t)=\frac{1}{\sqrt{t}}r^*\left(\frac{x}{\sqrt{t}}\right)\end{aligned}\tag{2}$$

$\Big($we stick to this abuse of notation in order not to proliferate symbols, but we shall use $\frac{d}{dz}$ or prime for the derivative with respect to the single argument of ϕ^*,θ^*, etc...., as opposed to ∂ for the partial derivative with respect to $x\Big)$. The boundary conditions are

$$\begin{aligned}&\lim_{x\to\pm\infty}|\phi^*(x)-\phi_\pm x|=0\,, \qquad \lim_{x\to\pm\infty}|\theta^*(x)-\theta_\pm|=0\,,\\ &\lim_{x\to\pm\infty}|s^*(x)-s_\pm|=0\,, \qquad \lim_{x\to\pm\infty}r^*(x)=0\,,\end{aligned}\tag{3}$$

ϕ^* and θ^* are solved by a fixed point method near the Gaussian solutions

$$\begin{aligned}&\phi_0^*(z)=\phi_- z+(\phi_+-\phi_-)\left(z+2\frac{d}{dz}\right)e(z)\,,\\ &\theta_0^*(z)=\theta_-+(\theta_+-\theta_-)e(z)\,,\end{aligned}\tag{4}$$

and s^* and r^* are "slaved" to them.

Thus, let us set

$$s^*=s^*(\partial\phi^*)\,,\tag{5}$$

where s^* solves the algebraic equation $-2s^*+F(s^*,\partial\phi^*)=0$, which is equivalent to $1-s^*=\sqrt{1-(\partial\phi^*)^2}$, so that s^* is of order $\frac{1}{2}(\partial\phi^*)^2$ for small $(\partial\phi^*)$. Note that, by (2), $\partial\phi^*(x,t)=\phi^{*'}\left(\frac{x}{\sqrt{t}}\right)$, so $s^*\left(\frac{x}{\sqrt{t}}\right)=s^*(\phi^{*'})$. Then ϕ^* is the solution of $\dot{\phi}^*=\partial^2\phi^*+G(s^*,\partial s^*,\partial\phi^*)$, i.e., using (2),

$$\begin{aligned}B\phi^* &\equiv \left(-\frac{d^2}{dz^2}-\frac{1}{2}z\frac{d}{dz}+\frac{1}{2}\right)\phi^* = -2\frac{\left(\frac{d}{dz}s^*\right)\left(\frac{d}{dz}\phi^*\right)}{1-s^*}\\ &= -2\frac{\left(\frac{d}{dz}\phi^*\right)^2\left(\frac{d^2}{dz^2}\phi^*\right)}{1-\left(\frac{d}{dz}\phi^*\right)^2}\,.\end{aligned}\tag{6}$$

We set

$$\phi^* = \phi_0^* + \varrho \tag{7}$$

and solve the fixed point equation (6) in the space of C^N functions equipped with the norm

$$\|\varrho\|_N = \max_{0\le m\le N} \sup_z \left| \frac{d^m}{dz^m} \varrho(z) \right| e^{\frac{z^2}{8}} . \tag{8}$$

Then we have

Proposition 1. *For any N, there exists an $\varepsilon > 0$ such that, for $|\phi_\pm| \le \varepsilon$ in (3), (6) has a unique solution with*

$$\|\varrho\|_N \le C\varepsilon^3 , \tag{9}$$

and then

$$\|s^*\|_N \le C\varepsilon^2 . \tag{10}$$

Next, we turn to θ^* and r^*. Set

$$\phi = \phi^* + \theta , \qquad s = s^* + r , \tag{11}$$

whereby

$$\dot\theta = \partial^2\theta + G(s, \partial s, \partial\phi) - G(s^*, \partial s^*, \partial\phi^*) \tag{12}$$

and

$$\dot r = \partial^2 r - 2r + F(s, \partial\phi) - F(s^*, \partial\phi^*) + H_1 , \tag{13}$$

where

$$H_1 = \partial^2 s^* - \partial_t s^* = t^{-1}\left(s^{*''}\left(\frac{x}{\sqrt t}\right) + \left(\frac{x}{2\sqrt t}\right)\left(s^{*'}\left(\frac{x}{\sqrt t}\right)\right)\right). \tag{14}$$

Note that $H_1(x,t) = t^{-1} H_1\left(\frac{x}{\sqrt t}\right)$. r^* is now taken [recall (2)] to be the part of (13) proportional to $t^{-1/2}$: we solve the linear equation

$$2r^* = DF_{s^*,\partial\phi^*}(r^*, \partial\theta^*) \tag{15}$$

for $r^*(\partial\theta^*(x,t)) = \frac{1}{\sqrt t} r^*\left(\theta^{*'}\left(\frac{x}{\sqrt t}\right)\right)$, where DF is the derivative of F, i.e.

$$DF_{s^*,\partial\phi^*}(r^*, \partial\theta^*) = 6s^*r^* - 3s^{*2}r^* + 2\partial\phi^*\partial\theta^* - 2s^*\partial\phi^*\partial\theta^* - (\partial\phi^*)^2 r^* .$$

Thus, $r^*(\partial\theta^*)$ is of order $\varepsilon\partial\theta^*$, and we let θ^* solve

$$\dot\theta^* = \partial^2\theta^* + DG_{s^*,\partial s^*,\partial\phi^*}(r^*, \partial r^*, \partial\theta^*) , \tag{16}$$

where DG is the derivative of G, i.e.

$$DG_{s^*,\partial s^*,\partial\phi^*}(r^*, \partial r^*, \partial\theta^*) = -\frac{2\partial s^*\partial\phi^*}{(1-s^*)^2} r^* - \frac{2\partial s^*}{1-s^*}\partial\theta^* - \frac{2\partial\phi^*}{1-s^*}\partial r^* ,$$

and we get

$$\left(- \frac{d^2}{dz^2} - \frac{1}{2} z \frac{d}{dz} \right) \theta^* = - \frac{2 s^{*'} \phi^{*'} r^*}{(1-s^*)^2} - \frac{2 s^{*'} \theta^{*'}}{1-s^*} - \frac{2 \phi^{*'} r^{*'}}{1-s^*} . \tag{17}$$

This is solved by setting

$$\theta^* = \theta_0^* + \tau , \tag{18}$$

and we have

Proposition 2. *For any N, there exists an $\varepsilon > 0$ such that, for $|\theta_\pm| \leq \varepsilon$ in* (3), *and s^*, ϕ^* as in Proposition 1,* (16) *has a unique solution with*

$$\|\tau\|_N \leq C \varepsilon^3 , \tag{19}$$

and then

$$\|r^*\|_N \leq C \varepsilon^2 . \tag{20}$$

We can now write the final equations that we will study. In accordance with (1), we set

$$\theta = \theta^* + \psi , \qquad r = r^* + v , \tag{21}$$

whence, from (12), (13), (15), and (16), we get the equations

$$\begin{aligned} \dot{\psi} &= \partial^2 \psi + [G(s, \partial s, \partial \phi) - G(s^* + r^*, \partial(s^* + r^*), \partial(\phi^* + \theta^*))] \\ &\quad + [G(s^* + r^*, \partial(s^* + r^*), \partial(\phi^* + \theta^*)) \\ &\quad - G(s^*, \partial s^*, \partial \phi^*) - DG_{s^*, \partial s^*, \partial \phi^*}(r^*, \partial r^*, \partial \theta^*)] \\ &= \partial^2 \psi + A(v, \partial v, \partial \psi) + D , \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{v} &= \partial^2 v - 2v + [F(s, \partial \phi) - F(s^* + r^*, \partial(\phi^* + \theta^*))] \\ &\quad + [F(s^* + r^*, \partial(\phi^* + \theta^*)) - F(s^*, \partial \phi^*) - DF_{s^*, \partial \phi^*}(r^*, \partial \theta^*)] + H \\ &= \partial^2 v - 2v + B(v, \partial \psi) + E , \end{aligned} \tag{23}$$

where $H = H_1 + H_2$ and

$$\begin{aligned} E &= F(s^* + r^*, \partial(\phi^* + \theta^*)) - F(s^*, \partial \phi^*) - DF_{s^*, \partial \phi^*}(r^*, \partial \theta^*) + H , \\ H_2 &= \partial^2 r^* - \partial_t r^* = t^{-3/2} \left(r^{*''} \left(\frac{x}{\sqrt{t}} \right) + \left(\frac{x}{2\sqrt{t}} \right) \left(r^{*'} \left(\frac{x}{\sqrt{t}} \right) \right) \right), \end{aligned} \tag{24}$$

and we have separated the inhomogeneous terms in (22), (23) for future convenience.

We will now specify the initial data for (2.2) that we will consider. We take a perturbation of a "Gaussian" data satisfying the boundary conditions,

$$\phi(x) \equiv \phi(x, 1) = \phi_0^*(x) + \theta_0^*(x) + \psi(x) , \tag{25}$$

$$s(x) \equiv s(x, 1) = s^*(\partial \phi_0^*(x)) + r^*(\partial \theta_0^*(x)) + v(x) , \tag{26}$$

and describe now a Banach space of ψ and v.

Due to the fact that Eq. (22), (23) involve second derivatives in the nonlinear terms (v is coupled to $\partial \psi$ and ψ to ∂v), it is useful to use the Fourier transform to control smoothness. Due to the fact that they involve functions like ϕ^* that do not decay at infinity, we need to be careful about the spatial behavior at infinity. We need norms that combine these two aspects, i.e. the *phase space* behavior of ψ and v.

Thus, let χ be a non-negative C^∞ function on $\mathbf{R}$ with compact support on the interval $(-1, 1)$, such that its translates by $\mathbf{Z}$, $\chi_n = \chi(\cdot - n)$, form a partition of unity on $\mathbf{R}$. For $f \in C^2$, we then introduce the norms, for $j = 1, 2$,

$$\|f\|^{(j)} = \sup_{n \in \mathbf{Z}, k \in \mathbf{R}, i \le j} (1 + n^4)(1 + k^2) |\widehat{\chi_n \partial^i f}(k)| . \tag{27}$$

Roughly, $\|f\|^{(j)} < \infty$ means that f falls off at least as x^{-4} at infinity and $\hat{f}(k)$ as k^{-2}. Note that r^*, the derivatives of θ^* and s^* and $\partial^2 \phi^*$ have a norm bounded by ε (actually, by ε^2 for s^* and r^*). To check this, use the explicit form (2.15), (2.26) of ϕ_0^*, θ_0^* and the bound $\|f\|^{(j)} \le C \|f\|_N$ for $N \ge j + 2$. The n^4 could be changed to anything increasing faster than n^γ, for $\gamma > 1$, but not faster than $e^{\frac{n^2}{8}}$ [coming from (8)]. These norms for different choices of χ may be shown to be equivalent. The reason we need different norms for v and ψ has to do with the slaving and will be explained below.

Our main theorem may now be stated.

Theorem. *For any $\delta > 0$ there is an $\varepsilon > 0$ such that for $|\phi_\pm|$, $|\theta_\pm|$, $\|\psi\|^{(2)}$, $\|v\|^{(1)} \le \varepsilon$, Eq.* (2.2) *have a unique solution satisfying*

$$\begin{aligned} \lim_{t \to \infty} t^{\frac{1}{2}(1-\delta)} \|\phi(\sqrt{t}\cdot, t) - \sqrt{t}\phi^*(\cdot) - \theta^*(\cdot)\|^{(2)} &= 0 , \\ \lim_{t \to \infty} t^{1-\delta} \left\| s(\sqrt{t}\cdot, t) - s^*(\phi^{*'}(\cdot)) - \frac{1}{\sqrt{t}} r^*(\theta^{*'}(\cdot)) \right\|^{(1)} &= 0 , \end{aligned} \tag{28}$$

where ϕ^* and θ^* are given in Propositions 1 and 2 and s^* in (5) *and* (15).

Remark 1. The convergence in the norm (27) implies convergence in L^1 and in L^∞, see Eq. (47) and (48) below, applied to $i = 0$. Thus, using $1 - s^* = \sqrt{1 - (\partial\phi^*)^2}$, we have, for any interval I,

$$\sup_{x \in I} |u(x, t) - \sqrt{1 - {q^*}^2} e^{i(xq^* + \sqrt{t}\phi^*(0) + \theta^*(0))}| \le C_I t^{-\frac{1}{2} + \delta} . \tag{29}$$

The "anomalous" term $\phi^*(0)$ equals $\pi^{-\frac{1}{2}}(\phi_+ - \phi_-) + \mathcal{O}(\varepsilon^3)$ and $q^* = (\partial\phi^*)(0) = \frac{1}{2}(\phi_+ + \phi_-) + \mathcal{O}(\varepsilon^2)$.

Remark 2. The theorem does not depend much on the specific form of F and G in (2.2). For a more general application of the slaving principle, see [5]. One obtains easily more refined asymptotics, as in [4], namely the error in the first equation in (28) is $t^{-\frac{1}{2}}\psi^*(\cdot) + \mathcal{O}(t^{-1+\delta})$.

Proof. Using the propositions (where we always assume that we have N as large as needed), we can replace in (25), (26), ϕ_0^*, θ_0^* by the true fixed points ϕ^*, θ^*, and still denote by ψ, v the remainder. We consider Eq. (22, 23) and solve them by the RG method. Thus, let

$$\begin{aligned} \psi_n(x,t) &= \psi(L^n x, L^{2n}t), \qquad \Psi_n(x) = \psi_n(x,1), \\ v_n(x,t) &= L^n v(L^n x, L^{2n}t), \qquad V_n(x) = v_n(x,1). \end{aligned} \tag{30}$$

We have

$$\Psi_{n+1}(x) = \psi_n(Lx, L^2); \qquad V_{n+1}(x) = Lv_n(Lx, L^2), \tag{31}$$

and ψ_n, v_n are the solutions of the equations

$$\dot{\psi}_n = \partial^2 \psi_n + A_n(v_n, \partial v_n, \partial \psi_n) + D_n, \tag{32}$$

$$\dot{v}_n = \partial^2 v_n - L^{2n} v_n + B_n(v_n, \partial \psi_n) + E_n, \tag{33}$$

obtained from (22, 23) upon the scaling (30). Concretely, let $s_n \equiv s^* + L^{-n}r^* + L^{-n}v_n$ and $\partial\phi_n \equiv \partial\phi^* + L^{-n}\partial(\theta^* + \psi_n)$. Then, using $s^*(L^n x, L^{2n}t) = s^*(x,t)$ and the corresponding scaling properties of $r^*, \partial\phi^*, \partial\theta^*$, we get

$$\begin{aligned} A_n &= L^{2n}[G(s_n, L^{-n}\partial s_n, \partial\phi_n) - (\ldots)|_{\psi_n = v_n = 0}] \\ &= \sum_{|\mathbf{p}|>0} L^{nd_\mathbf{p}} a_\mathbf{p} v_n^{p_1} \partial v_n^{p_2} \partial \psi_n^{p_3}, \end{aligned} \tag{34}$$

where $p_2, p_3 = 0, 1$, $|\mathbf{p}| = p_1 + p_2 + p_3$ and

$$\begin{aligned} a_\mathbf{p} &= 2(-)^{1+p_1} \frac{(\partial s^* + L^{-n}\partial r^*)^{1-p_2}(\partial\phi^* + L^{-n}\partial\theta^*)^{1-p_3}}{(1 - s^* - L^{-n}r^*)^{1+p_1}}, \\ d_\mathbf{p} &= 1 - p_1 - p_2 - p_3. \end{aligned} \tag{35}$$

In the same way

$$D_n = \sum_{|\mathbf{p}|>1} L^{nd_\mathbf{p}} \tilde{a}_\mathbf{p} r^{*p_1} \partial r^{*p_2} \partial\theta^{*p_3}, \tag{36}$$

$\tilde{a} = a|_{r^* = \theta^* = 0}$. Note that in (34) each term has $d_\mathbf{p} \le 0$, while in (36), we have $d_\mathbf{p} \le -1$. B_n has a similar expansion

$$\begin{aligned} B_n &= L^{3n}[F(s_n, \partial\phi_n) - (\ldots)|_{v_n = \psi_n = 0}] \\ &= \sum_{|\mathbf{p}|>0} L^{n(3-p_1-p_2)} b_\mathbf{p} v_n^{p_1} \partial\psi_n^{p_2} \end{aligned} \tag{37}$$

[where in fact $|\mathbf{p}| \le 3$ see (2.2)]. Finally,

$$E_n = \sum_{|\mathbf{p}|>1} L^{n(3-p_1-p_2)} \tilde{b}_\mathbf{p} r^{*p_1} \partial\theta^{*p_2} + L^n H_1 + H_2. \tag{38}$$

Note that here we have $3 - p_1 - p_2 \le 2$ in (37) and $3 - p_1 - p_2 \le 1$ in (38). We will show inductively in n that

$$\|\Psi_n\|^{(2)}, \|V_n\|^{(1)} \le \left(\frac{C}{L}\right)^n \varepsilon \equiv \varepsilon_n. \tag{39}$$

Here and below, C denotes a generic constant, which may change from place to place. Note that (39) implies the theorem, since, for any δ, we can choose L large enough so that (28) holds, remembering that $t = L^{2n}$; ε will be chosen in the course of the proof, in a L-dependent way.

Thus, let us assume (39) for n and prove it for $n+1$. We need to solve (32, 33) with initial data $\Psi_n(x)$, $V_n(x)$ up to time L^2. This is done by writing (32, 33) as integral equations:

$$\begin{aligned}\psi_n(t) &= (e^{(t-1)\partial^2}\Psi_n + \int_0^{t-1} ds e^{s\partial^2} D_n(t-s)) + \int_0^{t-1} ds e^{s\partial^2} A_n(t-s)\\ &\equiv \psi_n^0 + \mathscr{A}_n(\psi_n, v_n)\,,\\ v_n(t) &= (e^{(t-1)(\partial^2-2L^{2n})}V_n + \int_0^{t-1} ds e^{s(\partial^2-2L^{2n})} E_n(t-s))\\ &\quad + \int_0^{t-1} ds e^{s(\partial^2-2L^{2n})} B_n(t-s)\\ &\equiv v_n^0 + \mathscr{B}_n(\psi_n, v_n)\,,\end{aligned} \tag{40}$$

with obvious notations, and where ψ_n^0, v_n^0 regroup the first two terms; as in [8], we use the contraction mapping with the norms

$$\|w\|_L^{(j)} = \sup_{t\in[1,L^2]} \|w(\cdot,t)\|^{(j)}\,. \tag{41}$$

We shall show that $T(\psi, v) = (\psi_n^0, v_n^0) + (\mathscr{A}_n(\psi, v), \mathscr{B}_n(\psi, v))$ maps the ball $B = \{(\psi, v)\, \|\psi - \psi_n^0\|_L^{(2)} + \|v - v_n^0\|_L^{(1)} \le \varepsilon_n\}$ into itself and is a contraction there.

We need to estimate the norms of $\mathscr{A}$ and $\mathscr{B}$. Consider $\mathscr{A}$ first, and a generic term in (34):

$$\alpha(x,\mathbf{p}) = \int_0^{t-1} ds \int dy e^{s\partial^2}(x-y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y)\,, \tag{42}$$

where $F_{\mathbf{p}}(y) = (v^{p_1}\partial v^{p_2}\partial\psi^{p_3})(y, t-s)$ and $a_{\mathbf{p}}(y) = a_{\mathbf{p}}(y, t-s)$. We localize the y variable:

$$\alpha(x,\mathbf{p}) = \sum_{m\in\mathbf{Z}} \int_0^{t-1} ds \int dy e^{s\partial^2}(x-y)\chi_m(y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y) \equiv \sum_{m\in\mathbf{Z}} \alpha_m(x,\mathbf{p})\,. \tag{43}$$

We want to bound, for $i \le 2$,

$$\beta_{ml}(\mathbf{p}) = \sup_k |(1+k^2)\chi_l \widehat{\partial^i \alpha_m}(k,\mathbf{p})|\,. \tag{44}$$

We distinguish between $|l-m| \ge 2$ and $|l-m| < 2$.

(A) Let first $|m-l| \geq 2$. Then χ_m and χ_l have disjoint supports and $e^{s\partial^2}(x-y)$ is smooth uniformly in s. We write

$$\beta_{ml}(\mathbf{p}) = \sup_k \left| \int dx e^{-ikx} \int_0^{t-1} ds \int dy (1-\partial_x^2)\,(\chi_l(x)\partial_x^i e^{s\partial^2}(x-y)) \right.$$
$$\left. \times\, \chi_m(y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y) \right|, \tag{45}$$

and estimate the various factors on the RHS.

First, we use, for $j \leq 4$, $s \leq L^2 - 1$,

$$|\partial^j e^{s\partial^2}(x-y)| \leq C_L e^{-|m-l|}, \tag{46}$$

where C_L denotes a constant depending on L; (46) holds on the support of χ_m and χ_l. To bound $F_{\mathbf{p}}(y)$, note first that our norms (27) imply the following L^∞ and L^1 bounds. First, for any function w,

$$\begin{aligned} |\partial^i w(x)| &= \sum_l |\chi_l(x)\partial^i w(x)| \leq \sum_l \int dk |\widehat{\chi_l \partial^i w}(k)| \\ &\leq \sum_l (1+l^4)^{-1} \int dk (1+k^2)^{-1} \|w\|^{(j)} \leq C\|w\|^{(j)} \end{aligned} \tag{47}$$

for $i \leq j$, and secondly, by Schwarz

$$\int |\chi_m \partial^i w| \leq (2\int |\chi_m \partial^i w|^2)^{1/2} = (2\int |\widehat{\chi_m \partial^i w}|^2)^{1/2} \leq \frac{C\|w\|^{(j)}}{1+m^4}. \tag{48}$$

We will see below that

$$\|\psi_n^0\|_L^{(n)}, \|v_n^0\|_L^{(1)} \leq C\varepsilon_n \tag{49}$$

so that, for (ψ, v) in B,

$$\|\psi\|_L^{(2)} + \|v\|_L^{(1)} \leq (2C+1)\varepsilon_n\,. \tag{50}$$

Finally, from (35), (2.15), (2.26) and the propositions, we have

$$\|a_{\mathbf{p}}\|_\infty \leq C^{1+p_1}\varepsilon^{3-2p_2-p_3}\,. \tag{51}$$

Now we bound (45): the x integral is controlled by $\chi_l(x)$ or its derivatives, the s integral is less than L^2 and we use (47) with $w = v$ or ψ for all factors in $F_{\mathbf{p}}(y)$ except one, for which we use (48):

$$\sum_{|\mathbf{p}|>0} L^{-nd\mathbf{p}} \beta_{ml}(\mathbf{p}) \leq C_L \varepsilon (\|v\|_L^{(1)} + \|\chi\|_L^{(2)}) e^{-|m-l|}(1+m^4)^{-1}\,. \tag{52}$$

We get ε from (51) if p_2 or $p_3 = 0$, or, otherwise, from (50), because we have then a nonlinear term in $\|v\|_L^{(1)}, \|\psi\|_L^{(2)}$.
(B) Let now $|m-l| < 2$. The difficulty is that we do not have (46) for s close to zero. We will use Fourier transforms. Let $\phi_m \in C_0^\infty(\mathbf{R})$ be such that $\phi_m \chi_m = \chi_m$ and denote $\phi_m a_{\mathbf{p}}$ by f_m. Then

$$|\widehat{\chi_l \partial^i \alpha_m}(k,\mathbf{p})| \leq \int dsdpdq |\hat{\chi}_l(k-p)|\,|p|^i e^{-sp^2} |\hat{f}_m(p-q)\widehat{\chi_m F_{\mathbf{p}}}(q)| \,. \tag{53}$$

Let us consider the various factors on the RHS. Since χ is C^∞ with compact support, we have

$$|\hat{\chi}_l(k-p)| = |e^{-i(k-p)l}\hat{\chi}(k-p)| \leq C_r(1+|k-p|^r)^{-1} \tag{54}$$

for any r.

For $\hat{f}_m$, note that due to (2.15), (2.26) and the propositions,

$$\int |(1+(-\partial^2)^r)\phi_m a_{\mathbf{p}}(x)| dx \leq C^{1+p_1}\varepsilon^{3-2p_2-p_3} \tag{55}$$

for all $r \leq \dfrac{N-1}{2}$, whence, taking N large enough in Proposition 1, 2,

$$|\hat{f}_m(p-q)| \leq CC^{1+p_1}\varepsilon^{3-2p_2-p_3}(1+(p-q)^{2r})^{-1} \tag{56}$$

for any r. Also, $\int ds|p|^i e^{-sp^2} \leq CL^2$ if $i \leq 2$, so, we provided we can show

$$|\widehat{\chi_m F_{\mathbf{p}}}(q)| \leq C^{|\mathbf{p}|}(\|v\|_L^{(1)})^{p_1+p_2}(\|\psi\|_L^{(2)})^{p_3}(1+m^4)^{-1}(1+q^2)^{-1}\,, \tag{57}$$

we can perform the convolutions in (53) to get

$$\sum_{|\mathbf{p}|>0} L^{+nd\mathbf{p}}\beta_{ml}(\mathbf{p}) \leq C_L\varepsilon(\|v\|_L^{(1)} + \|\psi\|_L^{(2)})\,(1+m^4)^{-1} \tag{58}$$

Equations (52) and (58) yield, combined with (50),

$$\|\mathscr{A}_n(\psi, v)\|^{(2)} \leq C_L\varepsilon\varepsilon_n\,. \tag{59}$$

To prove (57), use

$$|\widehat{\chi_m \partial^i w}(k)| \leq (1+m^4)^{-1}(1+k^2)^{-1}\|w\|^{(j)}$$

and

$$|\widehat{\partial^i w}(k)| = \left|\sum_n \widehat{\chi_n \partial^i w}(k)\right| \leq \frac{C\|w\|^{(j)}}{1+k^2}$$

for $w = v$ or ψ, and perform the convolutions using

$$\int dp(1+(k-p)^2)^{-1}(1+p^2)^{-1} \leq C(1+k^2)^{-1}\,. \tag{60}$$

For $\mathscr{B}_n$ the analysis is similar, but we have to take avantage of the "mass" term, $-2L^{2n}$, in (40), in order to control the terms with $3 - p_1 - p_2 = 2$ in (37) or $3 - p_1 - p_2 = 1$ in (38). It is also here that the difference in the norms (27) enters.

The only change in (A) above is in (46):

$$|\partial^j e^{s(\partial^2 - L^{2n})}(x - y)| \le C_L e^{-sL^{2n}} e^{-|m-l|} \tag{61}$$

(for $|l - m| \ge 2$).

For (B), we integrate by parts $\partial_x = -\partial_y$ in (45) (recall $i \le 1$ here), and we get $\partial_y(\chi_m(y)\alpha_{\mathbf{p}}(y)F_{\mathbf{p}}(y))$. After going to the Fourier transform, as in (53), we see that we can use (54), (56) when $\chi_m(y)$ or $\alpha_{\mathbf{p}}(y)$ is replaced by its derivative. We also have (57) for $F_{\mathbf{p}}(y)$ replaced by its derivative, because F does not involve derivatives of v and contains only the first derivative of ψ. It is here that we use a different norm on v and ψ. Hence proceedings as before, we end up estmating

$$\sup_k (1 + k^2) \int_0^{t-1} ds e^{-s(k^2 + L^{2n})}(1 + k^2)^{-1} \le CL^{-2n} \,. \tag{62}$$

We also get L^{-2n} from the integral over s of (61). This cancels the L^{2n} in the leading term of (37). Therefore,

$$\|\mathscr{B}_n(\psi, v)\|^{(1)} \le C_L \varepsilon\varepsilon_n \,. \tag{63}$$

The bounds (49) for the inhomogeneous terms are proved in the same way, remembering (for D_n, E_n) that $d_{\mathbf{p}} \le -1$ in (36) and $3 - p_1 - p_2 \le 1$ in (38). For H_1, H_2, it is easy to see that their norm is bounded by $C\varepsilon^2$, because derivatives of s^*, r^* are bounded using (10), (20); Ψ_n, V_n are discussed below.

Thus, we have shown that T maps B into itself (take ε small enough, depending on L). The contraction is proved in the same way:

$$\begin{aligned} \|\mathscr{A}(\psi_1, v_1) - \mathscr{A}(\psi_2, v_2)\|^{(2)} &\le C_L \varepsilon(\|\psi_1 - \psi_2\|^{(2)} + \|v_1 - v_2\|^{(1)}) \,, \\ \|\mathscr{B}(\psi_1, v_1) - \mathscr{B}(\psi_2, v_2)\|^{(1)} &\le C_L \varepsilon(\|\psi_1 - \psi_2\|^{(2)} + \|v_1 - v_2\|^{(1)}) \,. \end{aligned} \tag{64}$$

To complete the induction for (39), we need to study the inhomogeneous terms in (40).

The main task is to show

$$\|R_0 \Psi\|^{(2)} = \|(e^{(L^2 - 1)\partial^2} \Psi_n)(L\cdot)\|^{(2)} \le \frac{C}{L} \|\Psi_n\|^{(2)} \,. \tag{65}$$

It is now easier to work in the x representation. We have

$$R_0 g = \int G(x, y) g(y) dy \tag{66}$$

with

$$G(x, y) = L^{-1}(4\pi(1 - L^{-2}))^{-1/2} e^{-\frac{1}{4}(1 - L^{-2})^{-1}(x - L^{-1}y)^2} \,. \tag{67}$$

We need to bound

$$\begin{aligned}\|R_0 g\|^{(2)} &= \sup_{k,l,i\le 2} (1+l^4)\left|\int dx e^{ikx}\int dy(1-\partial_x^2)(\chi_l(x)\partial_x^i G(x,y))g(y)\right|\\ &\le CL^{-1}\|g\|^{(2)}.\end{aligned} \tag{68}$$

We have

$$\begin{aligned}|\dots| &\le \sum_m \int dxdy|(1-\partial_x^2)(\chi_l(x)\partial_x^i G(x,y))\chi_m(y)g(y)|\\ &\le CL^{-1}\sum_m e^{-|l-L^{-1}m|}(1+m^4)^{-1}\|g\|^{(2)} \le CL^{-1}\frac{\|g\|^{(2)}}{1+l^4}\end{aligned} \tag{69}$$

for L large enough. We used (48) (with $\partial^i w$ replaced by g) and (67). To get the last inequality, consider separately $|m|\le L$ and $|m|\ge L$. The first sum is controlled by $(1+m^4)^{-1}$ and the second is a Riemann sum which can be evaluated as an integral. Hence, (68) follows.

The corresponding term for v in (40) is estimated similarly (it is of course superexponentially small), and the other contributions to ψ_n^0 and v_n^0 are $\mathcal{O}(\varepsilon\varepsilon_n)$, as are those of $\mathcal{A}$ and $\mathcal{B}$, see (59), (63). Equation (39) is proved. □

Proof of the Propositions. Consider first (9). Using (6), (7), and $B\phi_0^*=0$, we have the equation

$$\begin{aligned}B\varrho &= G\left(s^*\left(\frac{d}{dz}(\phi_0^*+\varrho)\right), \frac{d}{dz}s^*\left(\frac{d}{dz}(\phi_0^*+\varrho)\right), \frac{d}{dz}(\phi_0^*+\varrho)\right)\\ &\equiv g\left(\frac{d}{dz}\varrho, \frac{d^2}{dz^2}\varrho\right).\end{aligned} \tag{70}$$

Put

$$h = B\varrho \tag{71}$$

and solve

$$h = g\left(\frac{d}{dz}B^{-1}h, \frac{d^2}{dz^2}B^{-1}h\right) \equiv h_0 + N(h) \tag{72}$$

in the Banach space defined by (8). h_0 is the value of g at $h=0$, i.e.

$$h_0 = G\left(s^*\left(\frac{d}{dz}\phi_0^*\right), \frac{d}{dz}s^*\left(\frac{d}{dz}\phi_0^*\right), \frac{d}{dz}\phi_0^*\right).$$

In $N(h)$ we encounter terms like $\frac{d}{dz}B^{-1}h$ and $\frac{d^2}{dz^2}B^{-1}h$. We write the latter as $\frac{d^2}{dz^2}B^{-1}h = (-B - \frac{1}{2}z\frac{d}{dz}+\frac{1}{2}\Big)B^{-1}h = -h + \left(\frac{1}{2}-\frac{1}{2}z\frac{d}{dz}\right)B^{-1}h$. Thus,

$$N(h) = R\left(h, B^{-1}h, \frac{d}{dz}B^{-1}h, z\frac{d}{dz}B^{-1}h\right) \tag{73}$$

with $R: \mathbf{C}^4 \to \mathbf{C}$ analytic near zero. Thus all we now need is

Lemma. *The operators* B^{-1}, $\frac{d}{dz}B^{-1}$, *and* $z\frac{d}{dz}B^{-1}$ *are bounded in the norm* (8).

This lemma was proved in [4] for $A = B - \frac{1}{2}$, but the proof holds for B as well. From the explicit form of G, of ϕ_0^* in (2.15), and of $s^*\left(\frac{d}{dz}\phi_0^*\right)$ in (5), we have $\|h_0\|_N \le C\varepsilon$ while $\|N(h)\|_N \le C(\varepsilon^2\|h\|_N + \varepsilon\|h\|_N^2 + \|h\|_N^3)$. So, we can use the contraction mapping principle in a ball whose radius is of order ε^3 around h_0. This proves (9). The bound (10) holds because s^* is of order $\left(\frac{d}{dz}\phi^*\right)^2$ and $\frac{d}{dz}\phi_0^*$ is of order ε.

Proposition 2 is proven in the same way. It is enough to note that θ_0^* solves (17) with the RHS equal to zero and, setting $h = A\tau$, with A as in [4], we get an equation like (72), with $N(h)$ linear in h and where h_0 = the RHS of (17) with θ^* replaced by θ_0^*; we have the bounds $\|h_0\|_N \le C\varepsilon^3$ for $N = N' - 1$, where N' is the N of Proposition 1, and $\|N(h)\|_N \le C\varepsilon^2\|h\|_N$. □

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References

1. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusion arising in population genetics. Adv. Math. **30**, 33–76 (1978)
2. Ben-Jacob, E., Brand, H., Dee, G., Kramers, L., Langer, J.S.: Pattern propagation in nonlinear dissipative systems. Physica **14**D, 348–364 (1985)
3. Bramson. M.: Convergence of solutions of the Kolmogorov equation to traveling waves. Mem. of the Am. Math. Soc. **44**, nr. 285, 1–190 (1983)
4. Bricmont, J., Kupiainen, A., Lin, G.: Renormalization group and asymptotics of solutions of nonlinear parabolic equations. To appear in Commun. Pure Appl. Math.
5. Bricmont, J., Kupiainen, A.: In preparation
6. Collet, P., Eckmann, J.-P.: Solutions without phase-slip for the Ginsburg-Landau equation. Commun. Math. Phys. (1992)
7. Collet, P., Eckmann, J.-P.: Space-time behaviour in problems of hydrodynamic type: a case study. Preprint (1992)
8. Collet, P., Eckmann, J.-P., Epstein, H.: Diffusive repair for the Ginsburg-Landau equation. Helv. Phys. Acta **65**, 56–92 (1992)
9. Dee, G.: Dynamical properties of propagating front solutions of the amplitude equation. Physica **15**D, 295–304 (1985)
10. Goldenfeld, N., Martin, O., Oono, Y., Liu, F.: Anomalous dimensions and the renormalization group in a nonlinear diffusion process. Phys. Rev. Lett. **64**, 1361–1364 (1990)
11. Goldenfeld, N., Martin, O., Oono, Y.: Asymptotics of partial differential equations and the renormalization group. To appear in the Proc. of the NATO ARW on Asymptotics beyond all orders. Ed. by S. Tanveer, Plenum Press

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