Commun. Math. Phys. 149, 433–462 (1992)

Communications in
Mathematical
Physics

Perturbation Theory for Kinks

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Received July 2, 1990; in revised form August 26, 1991

Abstract. In this paper we prove the validity of formal asymptotic results on perturbation theory for kink solutions of the sine-Gordon equation, originally obtained by McLaughlin and Scott. We prove that for appropriate perturbations, of size ε in an appropriate norm, slowly varying in time in the rest frame of the kink, the shape of the kink is unaltered in the L^∞ norm to $O(\varepsilon)$ for a time of $O\left(\frac{1}{\varepsilon}\right)$. The kink parameters, which represent its velocity and centre, evolve slowly in time in the way predicted by the asymptotics. The method of proof uses an orthogonal decomposition of the solution into an oscillatory part and a one-dimensional "zero-mode" term. The slow evolution of the kink parameters is chosen so as to suppress secular evolution of the zero-mode.

Section 1. Introduction and Statement of Results

In this paper we prove the validity of formal asymptotic results due to McLaughlin and Scott (1978) and Karpman and Solov'ev (1981) for appropriate nonlinear perturbations of the sine-Gordon equation:

$$\theta_{TT} - \theta_{XX} + \sin\theta + \varepsilon g = 0\,. \tag{1.1}$$

More precisely, we prove the existence, for long but finite times, of solutions to this equation which approximate travelling waves of the unperturbed equation, with parameters evolving slowly in time under the action of the perturbation εg. The travelling waves of interest are uniformly moving kinks. Kinks are members of a two parameter family of solutions of the unperturbed equation:

$$\theta_{TT} - \theta_{XX} + \sin\theta = 0 \tag{1.2}$$

★ Partially supported as a graduate student at Princeton University of NSF grant 215 6211

given by $\theta = \theta_K\left(\frac{X - uT - C}{\sqrt{1 - u^2}}\right)$ for $u \in (-1, 1)$, $C \in \mathbf{R}$. Here θ_K is the C^∞ function

$$2n\pi + 4\arctan e^Z \tag{1.3}$$

which solves $\theta_K'' = \sin\theta$ with boundary conditions

$$\theta_K \to 2n\pi, \quad Z \to -\infty; \quad \theta_K \to 2(n+1)\pi, \quad Z \to +\infty \tag{1.4}$$

for $n \in \mathbf{Z}$. The energy and momentum in the field at time T are given by integration of the corresponding densities:

$$E = \int \tfrac{1}{2}\theta_T^2 + \tfrac{1}{2}\theta_X^2 + (1 - \cos\theta) \quad \text{(energy)}\,, \tag{1.5}$$

$$P = \int \theta_T \theta_X \quad \text{(momentum)}\,. \tag{1.6}$$

Evaluating these for the kink solutions we find for $m = 8$, $E = \frac{m}{\sqrt{1 - u^2}}$, $P = \frac{mu}{\sqrt{1 - u^2}}$ so that the kink obeys the energy momentum relationship for a relativistic particle $E^2 = P^2 + m^2$. We therefore refer to m as the mass of the kink, which is to be thought of as a particle-like object in a background radiation field.

We remark that the solutions of interest are not square integrable themselves due to the boundary conditions at spatial infinity. The natural space for solutions is $\theta \in H^1_{\mathrm{loc}}(\mathbf{R})$, $\theta_T \in L^2(\mathbf{R})$. However it is part of a general philosophy expounded in Parenti et al. (1977) that more can be said, namely that the boundary conditions are preserved in an L^2 sense. To be precise we have the following local existence theorem:

Theorem. *Consider initial data $\theta(0, X) \in H^1_{\mathrm{loc}}(\mathbf{R})\, \theta_T(0, X) \in L^2(\mathbf{R})$ for* (1.1) *with the property that*

$$(\theta(0, X) - \theta_K(X)) \in L^2(\mathbf{R})$$

assume further that g is a smooth function of T, X, θ. Then there exists a unique local solution $\theta(T, X)$ such that

$$(\theta(T, X) - \theta_K(X)) \in L^2(\mathbf{R})$$

and such that $T \to (\theta_T, \theta_X) \in L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$ is a strongly continuous map.

Proof. See Parenti et al. (1977) or Martin (1976).

A method which is useful for the understanding of nonlinear waves is to decompose the solution into solitary (spatially localised) and radiative components, and then study the interaction between them. For the case of the sine-Gordon equation, the method of inverse scattering yields, at a formal asymptotic level, very detailed information on this decomposition. This is useful for understanding the asymptotic behaviour of solutions at large times (see e.g. Novikov et al. (1984) or Eckhaus (1980)), and perturbation problems (Maclaughlin and Scott (1978)). One of the reasons for interest in kinks is that (together with breathers) they dominate the large time behaviour of solutions to the initial value problem, due to the dispersive character of the radiation component. To be more precise the radiation component decays like $\sim T^{-1/2}$, i.e. as the Klein Gordon equation, whereas the kink (solitary component) does not decay at all (see also Stuart (1991)). These analyses suggest questions which can be asked about solitary waves in general (non-integrable) wave equations. The case of the non-linear Schrödinger equation has been discussed in Weinstein (1985) and Soffer and Weinstein (1990). In particular, in the latter paper dispersive properties of the linear equation are used to prove stability and scattering results for solitary waves and radiation on an infinite time interval. The case of the kink is in some ways simpler, although being one dimensional the effects of dispersion are weaker, and not sufficient to give results globally in time. There are two interesting stability questions for kinks which can be given mathematically rigorous answers without using inverse scattering information: stability of kinks with respect to perturbation of the initial data (see Henry et al. (1982)), and stability with respect to perturbation of the equation, which is the problem addressed in this paper. Before stating the theorem and giving the proof we give a discussion of the general features of these two problems:

Stability with Respect to Initial Data. Here one considers the initial value problem for the sine Gordon equation with near kink initial data:

$$\theta_{TT} - \theta_{XX} + \sin\theta = 0 \quad \theta(0, X) = \theta_0(X) \quad \theta_T(0, X) = \theta_{0,T}(X) . \tag{1.7}$$

The crucial point (Benjamin (1972)) is that the appropriate kind of stability is *form stability*, i.e. stability modulo translation. To express this mathematically, consider the distance function for $\psi \in H^1_{\mathrm{loc}}(\mathbf{R})$:

$$d(\psi) \equiv \min_{c \in \mathbf{R}} \| \psi - \tau_c \theta_K \|_{H^1} ,$$

where τ_c is the translation operator: $\tau_c \psi(x) = \psi(x + c)$. Form stability is rigorously expressed in the following theorem which states that this distance function is non-increasing:

Theorem (*Henry et al.* 1982). *There exists a number r such that if $\theta_0, \theta_{0,T}$ satisfy $\theta_0 \in H^1_{\mathrm{loc}}(\mathbf{R})$, $\theta_{0,X} \in L^2(\mathbf{R})$, $\theta_{0,T} \in L^2(\mathbf{R})$ having finite energy (as defined in (1.5)) and, most importantly, $d(\theta_0) < r$, then there exists a unique global weak solution θ such that $(\theta, \theta_T) \in H^1_{\mathrm{loc}} \oplus L^2$, $T \to (\theta_X, \theta_T) \in L^2 \oplus L^2$ is strongly continuous and $d(\theta(T, \cdot)) < r \;\forall T$.*

Remark. One can picture the motion as being particle motion in an infinite dimensional potential valley, motion along the valley corresponding to translation of the kink and motion up the hills corresponding to the oscillatory radiation. This is expressed mathematically by the following properties of the linearised equation. The sine-Gordon equation linearised about a kink is

$$\tilde{\theta}_{TT} + L\tilde{\theta} = 0 ,$$

where $L = -\partial_X^2 + \cos\theta_K(X)$ has spectrum consisting of:

– zero as the unique eigenvalue, with eigenfunction θ_K'. This so-called zero mode arises due to translation invariance, and its presence is the infinitesimal version of the fact that the appropriate distance function for the stability statement involves minimising over the centre of the kink. Notice that, from monotonicity of the kink, $\theta_K' > 0$ so it is a simple lowest eigenvalue.

– continuous spectrum $[1, \infty)$. This gives motion in the subspace orthogonal to the zero mode. This motion is oscillatory in time, not exponentially decaying (as for corresponding parabolic problems), making the nonlinear treatment somewhat subtle. It is important that the continuous spectrum does not reach zero – it starts at 1 because of the fact that $\cos\theta_K \to 1$ as $|X| \to \infty$. This part of the spectrum corresponds to radiation.

This has the consequence that for

$$\psi \in H^1, \quad \int \psi\theta_K' = 0 \Rightarrow \int \psi L\psi \geqq \int \psi^2 \tag{1.8}$$

which is important for this paper.

Remark. The process of minimising over the centre of the kink can be regarded at an infinitesimal level, as forcing the motion into the oscillatory subspace by appropriate choice of the modulation of the parameters.

Remark. The proof in Henry et al. (1982) does not depend on complete integrability, and the theorem is stated for a more general class of nonlinearity. The result of this paper goes immediately over to this class of nonlinearities.

Stability with Respect to Perturbation of Equation. It is the aim of this paper to prove the modulational stability in an appropriate norm of kink solutions under the action of an appropriate class of perturbations εg over an appropriate time interval. Thus we consider an initial value problem for (1.1) with initial data close to a kink, and ask for how long, and in what norm, does the solution look like a kink, possibly with velocity and centre changing in time. Modulational stability means the kink parameters are expected to evolve in time – see (1.20), (1.21). The precise meaning to be attached to the word appropriate is part of the problem – see the statement of the main theorem and the comments at the end of this section, as well as the following discussion. The proof does not depend on complete integrability and could easily be written out for the class of nonlinearities of the equation considered in Henry et al. (1982). Before launching into the proof we discuss why such a statement is expected to be true. As for stability with respect to initial data, the basic point is the spectrum of the linearised operator L. In the argument of Henry et al. one can heuristically think of stability as being due to the possibility of choosing the centre of the kink (as a function of time) in such a way as to push the linearised motion onto the oscillatory subspace, i.e. orthogonal to the zero mode. We would like to do a similar thing, but there are two immediate difficulties:

Question (*a*). The kink will presumably accelerate under the influence of the perturbation, so as well as choosing the variation of the centre it will be necessary to choose the variation of the velocity, presumably in such a way as to force the linearised motion onto the oscillatory subspace. What are the equations for the modulation of u, C?

Question (*b*). Even if the motion is confined to the oscillatory subspace, does the perturbation produce a large response over long times? The answer to this leads to the important heuristic requirement that the perturbation be *slowly varying in the Lorentz rest frame of the kink* (see the end of this section for the most general types of perturbation allowed).

To answer these questions we first describe the basic strategy of the proof. We make an ansatz for the solution

$$\theta(T,X)=\theta_K(Z)+\varepsilon\tilde{\theta}(T,X),\quad Z=\frac{X-\int^T u-C(T)}{\sqrt{1-u^2}}, \tag{1.9}$$

$$C(T)=C_0(\varepsilon T)+\varepsilon\tilde{C},$$

$$u(T)=u_0(\varepsilon T)+\varepsilon\tilde{u}(T)\ \left(\Rightarrow p=\frac{u}{\sqrt{1-u^2}}=p_0(\varepsilon T)+\varepsilon\tilde{p}(T)\right). \tag{1.10}$$

The aim is to bound $\tilde{\theta}$, $\tilde{u}$, $\tilde{C}$ for times of $O\left(\frac{1}{\varepsilon}\right)$ by appropriate choice of the time variation of u, C. So we calculate the equation for $\tilde{\theta}$ by substitution into (1.1) as:

$$\tilde{\theta}_{TT}-\tilde{\theta}_{XX}+\cos\theta_K(Z)\tilde{\theta}=f^{(1)}+\varepsilon f^{(2)}\equiv f(T,Z,\tilde{\theta},\dot{C},\ddot{C},p,\dot{p},\ddot{p}), \tag{1.11}$$

where the function f is written out explicitly in appendix zero. As noted above the crucial point is the behaviour of the linearised equation – in particular the fact that, orthogonal to the zero mode, the motion is oscillatory. This suggests the following *zero-mode/oscillatory mode orthogonal decomposition*:

$$\tilde{\theta}(T,X)=\alpha(T)\theta_K'(Z)+\theta^*(T,X)\quad \int\theta_K'(Z)\theta^*(T,X)dX=0. \tag{1.12}$$

The strategy for the proof is now given by the following answers to the questions above:

Answer to question (a). It turns out that an appropriate choice of $\frac{du}{dT}$, $\frac{dC}{dT}$ allows the supression of the dominant growth in α, i.e. in the zero mode. This can be seen by projecting Eq. (1.11) in the two direction corresponding to the action of the infinitesimal symmetries on the kink: θ_K', $Z\theta_K'$ corresponding to translation and Lorentz invariance. This leads to identities in appendix two which are used to estimate α in the proof of the central Lemma 2.3. This calculation is a rigorous version of a formal asymptotic argument for the modulation equations due to McLaughlin and Scott which is outlined below, after the statement of the main theorem.

Answer to question (b). To see how to estimate the oscillatory part, we notice the fact that once the zero mode is excluded we can think of (1.11) as an infinite dimensional version of the perturbed 1-D oscillator:

$$* \qquad \ddot{X}+X=f(\varepsilon T)+\varepsilon g(X)$$

which remains bounded for times of $O\left(\frac{1}{\varepsilon}\right)$. A proof of this which nicely generalises to our case is as follows—if we calculate the rate of change of energy $E=\dot{X}^2+X^2$ and then integrate again, integrating by parts the f term, we find the inequality

$$** \qquad |E(T)|\leqq E(0)+E^{1/2}(0)|f(0)|+|f(\varepsilon T)||E^{1/2}(T)| + \varepsilon\int_0^T|\dot{f}(\varepsilon T')X-g(X)\dot{X}|\,dT'$$

from which long time boundedness of X can be deduced from Gronwall's inequality, for suitable f, g. To find an "energy" for our case, which allows a generalisation of this proof, recall that if we consider the linearised equation about a static kink:

$$\tilde{\theta}_{TT} - \tilde{\theta}_{XX} + \cos\theta_K(X)\tilde{\theta} = 0$$

then the energy given by

$$\int_{-\infty}^{+\infty} \frac{\tilde{\theta}_T^2 + \tilde{\theta}_X^2 + \cos\theta_K\tilde{\theta}^2}{2}\,dX$$

is preserved. Next consider linearisation about a uniformly moving kink at speed u – the relativistic version of this result is the conservation of the following *linearised energy*:

$$E(T) \equiv \int_{-\infty}^{+\infty} \frac{\tilde{\theta}_T^2 + \tilde{\theta}_X^2 + \cos\theta_K(Z)\tilde{\theta}^2}{2} + u\tilde{\theta}_T\tilde{\theta}_X dZ\,. \tag{1.13}$$

It turns out (see Lemma 2.1) that for times of order $\frac{1}{\varepsilon}$ E is bounded if α is bounded, just as in $**$, as long as the perturbation is slowly varying in time in the rest frame of the kink – this means it should be a function of Z, εT as described at the end of this section in detail. This gives the answer to question (b). We emphasize that the nonlinearity couples the zero-modes and oscillatory modes, so estimations of E, α must be done in tandem. The significance of the quantity E stems from the following proposition:

Proposition 1.1. *There exists a number c such that*:

$$\|\tilde{\theta}(T)\| \leqq c\|\gamma(T)\|(\|E^{1/2}(T)\| + \|\alpha(T)\|)\,, \tag{1.14}$$

where

$$\|\tilde{\theta}(T)\| = \max_{T'\in[0,T]} \{|\tilde{\theta}_T(T')|_2 + |\tilde{\theta}(T')|_{H^1}\}\,, \tag{1.15}$$

where $|\tilde{\theta}(T)|_2^2 = \int|\tilde{\theta}(T,X)|^2\,dZ$ *and* $|\tilde{\theta}(T)|_{H^1}^2 \equiv |\tilde{\theta}(T)|_2^2 + |\tilde{\theta}_X(T)|_2^2$ *and for functions of time f*:

$$\|f(T)\| \equiv \max_{T'\in[0,T]} |f(T')|\,. \tag{1.16}$$

Proof. See Appendix 1.

Thus we see that to estimate $\tilde{\theta}$ in H^1 (and hence pointwise by Sobolev's lemma) we only need to estimate E and α. The estimation of E similar to ** is carried out in Lemma 2.1, while in Lemma 2.3 we find equations for u_T, C_T which ensure α grows slowly. These are the crucial steps which lead to the following:

Main Theorem. *Let the perturbation* $g = g(\theta)$ *be a smooth function such that* $g_0(Z) \equiv g(\theta_K(Z)) \in L^2(dZ)$. *Then for sufficiently small* ε *there exists* $T_* = O\left(\frac{1}{\varepsilon}\right)$ *such* *that for* $T \leqq T_*$ *there is a unique solution to the initial value problem:*

$$\theta_{TT} - \theta_{XX} + \sin\theta + \varepsilon g = 0\,, \tag{1.17}$$

$$\theta(0, X) = \theta_K(Z(0)) + \varepsilon\tilde{\theta}(0, X),$$

$$\theta_T(0, X) = \frac{-u(0)}{\sqrt{1 - u(0)^2}}\,\theta_K'(Z(0)) + \varepsilon\tilde{\theta}_T(0, X)\,, \tag{1.18}$$

where $(\tilde{\theta}(0, X), \tilde{\theta}_T(0, X)) \in H^1 \oplus L^2$, *of the form:*

$$\theta(T, X) = \theta_K(Z) + \varepsilon\tilde{\theta}(T, X) \qquad Z = \frac{X - \int^T u - C(T)}{\sqrt{1 - u^2}}\,, \tag{1.19a}$$

where $\tilde{\theta} \in C([0, T_*], H^1)$, $\tilde{\theta}_T \in C([0, T_*], L^2)$ *and*

$$C(T) = C_0(\varepsilon T) + \varepsilon\tilde{C}$$

$$u(T) = u_0(\varepsilon T) + \varepsilon\tilde{u}(T)\ \left(\Rightarrow p = \frac{u}{\sqrt{1 - u^2}} = p_0(\varepsilon T) + \varepsilon\tilde{p}(T)\right) \tag{1.19b}$$

with $\tilde{p}, \tilde{u}, \tilde{C}, \frac{d\tilde{u}}{dT}, \frac{d\tilde{C}}{dT}, |\tilde{\theta}|_{H^1(\mathbf{R})}$ *bounded independent of* ε, *and* u_0, C_0 *the solutions of the zeroth order modulation equations:*

$$m\frac{dC_0}{dT} = \varepsilon u_0(1 - u_0^2)\int_{-\infty}^{+\infty} g(\theta_K(Z))Z\theta_K'(Z)dZ \quad C_0(0) = C(0)\,, \tag{1.20}$$

$$m(u_0(1 - u_0^2)^{-1/2})_T \equiv mp_{0,T} = \varepsilon\int_{-\infty}^{+\infty} g(\theta_K(Z))\theta_K'(Z)dZ \quad u_0(0) = u(0)\,, \tag{1.21}$$

where $p_0 = \frac{u_0}{\sqrt{1 - u_0^2}}$ *is the momentum per unit mass and* $m = 8$ *is the mass of the kink defined in* (1.5), (1.6). ■

Proof. This is carried out in Sect. 2. Derivation of certain identities and local existence theory is in the appendices.

Remark. It will appear in the proof that one bounds not only $\|\tilde{\theta}\|$ but also the momentum $p = \frac{u}{\sqrt{1 - u^2}}$ on the time interval $[0, T_*]$. From this it follows that the velocity is bounded away from one, i.e. the kink velocity cannot approach the speed of light. The perturbation theory is not expected to be valid at the speed of light. The perturbation theory is however valid near $u = 0$ which is not a preferred point from the relativistic viewpoint. This is in spite of the fact that the asymptotic derivation of the modulation equations due to McLaughlin and Scott (1978), which is given below, degenerates at $u = 0$.

Remark. If the further condition is added that the initial data satisfies

$$\tilde{\theta}_{XX}(0, X), \tilde{\theta}_{TX}(0, X) \in L^2 \oplus L^2\,,$$

then strong differentiability in time of the solution can be deduced. Indeed one can produce a smooth classical soolution for smooth initial data – see (Stuart (1990)) for full details. The basic point is that by Sobolev's lemma the solution is bounded pointwise, so the growth in higher order Sobolev norms can be controlled.

Remark. The proof works for much more general types of perturbation, as described at the end of this section. We also emphasize again that the proof works for more general nonlinearities than sin, e.g. the class used in (Henry et al. 1982).

Formal Calculation of the Modulation Equations. We now present a formal calculation of the modulation equations from MacLaughlin and Scott (1978). The idea is to use the basis of solutions to the linearised equation obtained from the inverse scattering method (by differentiation with respect to the scattering data.) This basis divides into "discrete" and "continuous" modes, the discrete modes being given by differentiation with respect to the parameters u, C. The philosophy is that it is important not to excite the discrete modes if the response is to be small, but the continuous modes do not matter as they are oscillatory in time. Thus the modulation equations are chosen to make sure that the error term f does not excite the discrete modes. The discrete modes are given by differentiation with respect to the kink parameters u, C which form the discrete part of the scattering data. This gives two orthogonality relations for the error term f which give, to highest order in ε, the modulation equations (1.20), (1.21). The calculation is most simply done by writing Eq. (1.1) in first order from:

$$\frac{d}{dT}\begin{pmatrix} \theta \\ \theta_T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{XX} - \sin(\cdot) & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon g \end{pmatrix}.$$

We now substitute in an ansatz corresponding to (1.9), which leads to the following linearised equation:

$$*** \qquad \frac{d}{dT}\begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}_T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\partial_{XX} + \cos\theta_K(Z) & 0 \end{pmatrix} + \begin{pmatrix} \theta_{K,C}\dot{C} + \theta_{K,u}\dot{u} \\ g + \theta_{K,TC}\dot{C} + \theta_{K,TU}\dot{u} \end{pmatrix}.$$

Here we are considering Z, defined in (1.9), as a function of u, C, T with $\int_0^T u$ thought of as a function of T only. We have two elements of the null space of this linearised equation provided by differentiation with respect to u, C. We will apply the Fredholm alternative; however the linear operator is not self-adjoint so we take the null space of the adjoint operator instead. This leads to the following two vectors:

$$n_1 \equiv \begin{pmatrix} \theta_{K,Tu} \\ -\theta_{K,u} \end{pmatrix} \qquad n_2 \equiv \begin{pmatrix} \theta_{K,TC} \\ -\theta_{K,C} \end{pmatrix} \sim \frac{1}{u}\begin{pmatrix} \theta_{K,TT} \\ -\theta_{K,T} \end{pmatrix} + O(\varepsilon).$$

We now require that the inhomogeneous term in *** be orthogonal to these elements. This gives the modulation equations (1.20), (1.21). We remark that the method is like a time-dependent Fredholm alternative; however the interesting point is that the inverse scattering approach gives an enormous number of solutions to the linearised equations, but only the two corresponding to n_1, n_2 are used for the orthogonality conditions. This is because the other solutions are oscillatory, and therefore supposed to be less excitable. This is only the case if the perturbation is slowly varying in time. The result of the paper can be thought of as a rigorous justification for this time-dependent Fredholm alternative. Lemma 2.3 provides a rigorous version of the above calculation, while the fact that the radiation is not excited by slowly varying perturbations is expressed rigorously by Lemma 2.1.

Summary of the Rest of the Paper. The heart of the paper is Lemma 2.3, which gives an a priori estimate for the growth of the zero mode, α (defined by the decomposition (1.12)), subject to the modulation equations (1.20), (1.21). Together with the a priori estimate of the linearised energy E in Lemma 2.1, this gives, via Proposition 1.1, an a priori estimate for $\tilde{\theta}$ for times of order $1/\varepsilon$. These estimates depend on identities which are proved in appendix two. There is a difficulty however in that this a priori estimate is for the coupled ODE-PDE system formed by the equation for $\tilde{\theta}$, (1.11), and the modulation equations. These are coupled because the full modulation equations for u, C, as opposed to u_0, C_0, contain $O(\varepsilon)$ terms depending on $\tilde{\theta}$. Thus to complete the argument it is necessary to prove the local existence of solutions to this system in an appropriate norm for a continuation based on an a priori estimate. It is also necessary to check that these solutions satisfy the a priori estimate. This is done in appendix three.

Notation. Throughout this paper we use the following notation:

– The *fast* time variable is T, while $t = \varepsilon T$ is the *slow* variable; these are not treated here as independent variables, as in the multiscale method, but as alternatives to simplify notation. A function of t with derivative bounded independent of ε will be referred to as *slowly varying*. The same applies for spatial functions with $x = \varepsilon X$.

– For functions of time, f, we use a dot to denote $\frac{1}{\varepsilon}\frac{df}{dT} = \frac{df}{dt} \equiv \dot{f}$.

– We shall frequently use the change of variables

$$Z = \frac{X - \int^{T} u(\varepsilon T')\,dT' - C(T)}{\sqrt{1 - u(T)^2}} = \frac{X - \Xi(T)}{\sqrt{1 - u(T)^2}} \quad S = T \quad s = \varepsilon S = t = \varepsilon T \tag{1.22}$$

in which S is used to distinguish what is being kept constant on differentiation, i.e. $\frac{\partial}{\partial S} = \left.\frac{\partial}{\partial T}\right|_Z$.

– As usual we call $\gamma = (1 - u^2)^{-1/2}$. u represents the speed of the kink, and has magnitude less than one; it is related to the momentum (per unit mass) by $p = \gamma u$ and $u = \frac{p}{\sqrt{1 + p^2}}$ in terms of which $\gamma = \sqrt{1 + p^2}$. The same notation with subindex 0 will be used for the $O(1)$ terms in the expansions (1.19).

– The mass and moment of inertia of the kink are defined by:

$$m = \int_{-\infty}^{+\infty} \theta_K'^2(Z)\,dZ (= 8), \quad I = \int_{-\infty}^{+\infty} Z^2 \theta_K'^2(Z)\,dZ\,. \tag{1.23}$$

– We will use the norms for $\tilde{\theta}$:

$$\|\tilde{\theta}(T)\| = \max_{T' \in [0,T]} \{|\tilde{\theta}_T(T')|_2 + |\tilde{\theta}(T')|_{H^1}\} , \tag{1.24}$$

where $|\tilde{\theta}(T)|_2^2 = \int |\tilde{\theta}(T,X)|^2 \, dZ$ and $|\tilde{\theta}(T)|_{H^1}^2 \equiv |\tilde{\theta}(T)|_2^2 + |\tilde{\theta}_X(T)|_2^2$, and for functions of time f:

$$\|f(T)\| \equiv \max_{T' \in [0,T]} |f(T')| , \tag{1.25}$$

so in particular $\| |g|_2(T)\| \equiv \max_{[0,T]} |g|_2$. Notice that $dZ = \gamma dX$ is used in the definition of the L^2 norm. For uniformly bounded momentum this is equivalent to $L^2(dX)$.

– The perturbation term evaluated at the kink is denoted:

$$g_0 \equiv g(\theta_K(Z)) . \tag{1.26}$$

To denote maxima over time we shall use the $\|$ notation as above.

– Finally we will work within the following finite energy subspace:

$$\begin{aligned} S^K(T) \equiv \{&(\tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X, p, C) \in C([0,T], L^2(\mathbf{R}))^3 \\ &\oplus C^2([0,T])^2 : \|\tilde{\theta}(T)\|, \|p(T)\| \leqq K\} . \end{aligned} \tag{1.27}$$

It will be clear from the method of proof that the main theorem is valid under the following conditions which are weaker than those in the statement above.

More General Conditions on the Perturbation. We can take far more general perturbations than those given in the statement of the main theorem. For example we may take g to be of one of the following forms:

first type $\quad g = g(\varepsilon T, \varepsilon X, \theta) = g(t, x, \theta)$.

second type $\quad g = g(\varepsilon T, Z, \theta) = g(t, Z, \theta)$,

where g is a differentiable function

$$g : \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} . \tag{1.28}$$

We define in the first case

$$g_0(t, x, Z) \equiv g(t, x, \theta_K(Z))$$

and in the second case

$$g_0(t, Z) \equiv g(t, Z, \theta_K(Z)) .$$

We assume further that there exists a time interval $[0, T_+] = \left[0, \frac{t_+}{\varepsilon}\right]$, where t_+ is independent of ε, on which g, g_0:

$$\| |g_0|_2(T_+)\|, \left\| \left|\frac{\partial g_0}{\partial s}\right|_2 (T_+)\right\|, \left\| \left|\frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon}\right|_2 (T_+)\right\| \leqq A , \tag{1.29}$$

where A is a number independent of ε, and for $(\tilde{\theta}, p) \in S^K(T_+)$,

$$\left\| \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 (T_+) \right\| \leqq C(K) \left\| \left| \left(\frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right)_S \right|_2 (T_+) \right\| \leqq C(K) . \tag{1.30}$$

These are easily seen to be satisfied for g as described in the statement of the main theorem. Here p is the momentum defined in the previous notation sub-section. We will carry through the proof for the second type of perturbation, as this is the most extreme.

Remark (*slowly varying perturbation*). The interpretation of these conditions is that the perturbation is allowed to be a nonlinear function of θ, and also to have appropriate slow variation in time, and fast variation in space. This must be interpreted with care. *Any fast variation in space must be in the local Lorentz rest frame*, otherwise relative to the kink the time variation will appear to be fast. In the notation we have been using, this means g takes either of the forms above or a combination thereof. The second form, with g depending on Z, may seem strange from the standpoint of perturbation theory – how can the perturbation know where the kink is, and what its speed is? The answer is that asymptotic theories due to Neu (1987), also discussed in Stuart (1991) give rise to perturbations of this form. ■

Section 2

In this section we will prove the main theorem stated in the introduction. We recall that we have made a "zero-mode/oscillatory-mode" decomposition of the error $\tilde{\theta}$ in (1.12). We also recall from Proposition 1.1 that $\tilde{\theta}$ is estimated in H^1 by E, α. The estimation of E is contained in Lemma 2.1, and that of α in Lemma 2.3, where the modulation equations first appear. These lemmas rely on identities derived in appendix two. Lemma 2.3 leads to a coupled ODE-PDE system for u, C, $\tilde{\theta}$. Local existence for this system is proved in appendix three together with the fact that the growth of the momentum $p = \dfrac{u}{\sqrt{1-u^2}} = \gamma u$ is determined by A to $O(\varepsilon)$ – see Theorem 2.5. As remarked in the introduction, boundedness of the momentum is needed for validity of the asymptotics. Combination of this local existence with the basic estimates finally proves the theorem. We remark that although the results in Lemmas 2.1, 2.3 are stated as a priori estimates for smooth solutions, they are valid far more generally – in particular for the weak solutions to the ODE-PDE system constructed in appendix three: see Theorem 2.5.

Notation. We define

$$W(T) \equiv \frac{\|\alpha(T)\|}{1 + \|E^{1/2}(T)\|}, \quad Y \equiv \frac{\|E(T)\|}{1 + \|E^{1/2}(T)\|} . \tag{2.1}$$

Notice that $\|E^{1/2}\| \leqq 1 + Y$.

Lemma 2.1 (Spectral Estimate). *Let $\tilde{\theta}$ be a smooth solution to Eq.* (1.11) *compactly supported in space, with u, C smooth functions of time having $|u| < 1$. Fix $\varepsilon_1 > 0$ then*

$\exists c_1, c_2 > 0$, *independent of* ε, *such that for* $\varepsilon < \varepsilon_1$:

$$E(T) \leqq E(0) + c_1 \|\tilde{\theta}(T)\| \{ \| |g_0|_2(T) \| + \|\gamma\|^3 (1 + \|\dot{u}\| + \|\dot{C}\|) \}$$
$$+ \varepsilon c_2 \int_0^T \|\tilde{\theta}\| \, \|\gamma\|^4 \left\{ |g_{0,t}|_2 + \left\| \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 (T_+) \right\| + \|\tilde{\theta}\|^3 \right.$$
$$\left. + \|\dot{u}\|^2 + \|\dot{C}\|^2 + \|\ddot{C}\| + \|\ddot{u}\| \right\}.$$

Proof. This follows from the spectral identity, Lemma A2.1, by estimating all but one of the integrals in a direct way using the formulae for f in (A0.1–3):

$$|\int M(\tilde{\theta}_T + u\tilde{\theta}_X)\, dZ| \leqq \text{const.} \|\tilde{\theta}\| \, \|\gamma\|^4 (1 + \|\dot{C}\|^2 + \|\dot{u}\|^2 + \|\ddot{C}\| + \|\ddot{u}\|),$$

$$|\int N(\tilde{\theta}_T + u\tilde{\theta}_X)\, dZ| \leqq \text{const.} \|\tilde{\theta}\| \left(\|\tilde{\theta}\|^2 + \varepsilon \|\tilde{\theta}\|^3 + \left\| \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 (T_+) \right\| \right),$$

$$|\int \tilde{\theta}^2 (\cos\theta_K(Z)(X_S - u))_X \, dZ| \leqq \varepsilon\text{const.} \|\tilde{\theta}\|^2 \|\gamma\|^2 (1 + \|\dot{u}\| + \|\dot{C}\|),$$

$$|\int (\tilde{\theta}_T^2 + \tilde{\theta}_X^2)(u + X_S)_X \, dZ| \leqq \varepsilon\text{const.} \|\tilde{\theta}\|^2 \|\gamma\|^2 \|\dot{u}\|,$$

$$|\int \tilde{\theta}_T \tilde{\theta}_X (\varepsilon\dot{u} - (uX_S)_X)\, dZ| \leqq \varepsilon\text{const.} \|\tilde{\theta}\|^2 \|\gamma\|^2 \|\dot{u}\|,$$

where X_S is defined in (A2.1). The only term which is treated differently is the term $\int\int f^1(\tilde{\theta}_T + u\tilde{\theta}_X)$. Since we want to obtain longtime behaviour using a Gronwall type estimate a finite term in the integrand is unacceptable. However we can get rid of this by integrating by parts as we did for the 1-D oscillator in the introduction, and taking advantage of the fact that f^1 is slowly varying in time, when Z is held constant. Some care is needed because $\tilde{\theta}$ is expressed as a function of T, X not Z, so the relevant integration by parts formula is

$$\int_0^T \int f^1(\tilde{\theta}_T + u\tilde{\theta}_X)\, dZ\, dT = \int_{-\infty}^{+\infty} f^1 \tilde{\theta}\, dZ \Big|_0^T - \varepsilon \int_0^T \int \tilde{\theta} \frac{\partial f^1}{\partial s} + f^1(\dot{C} - \gamma u\dot{u})\tilde{\theta}_X dZ\, dT,$$

which leads to the estimate:

$$\left| \int_0^T \int f^{(1)}(\tilde{\theta}_T + u\tilde{\theta}_X)\, dZ\, dT' \right| \leqq \text{const.} \|\tilde{\theta}(T)\| \{ \| |g_0|_2(T) \| + \|\gamma\|^3 (1 + \|\dot{u}\| + \|\dot{C}\|) \}$$
$$+ \varepsilon\text{const.} \int_0^T \|\tilde{\theta}\| \, \|\gamma\|^4 \{ 1 + |g_{0,s}|_2 + \|\dot{u}\|^2$$
$$+ \|\dot{C}\|^2 + \|\ddot{C}\| + \|\ddot{u}\| \}\, dT',$$

which completes the proof. ∎

Corollary 2.2. *Under the conditions of the previous lemma* $\exists c_3 > 0$ *and a polynomial* P_1 *in* 4 *variables, independent of* ε, *such that for* $\varepsilon < \varepsilon_1$,

$$Y(T) \leqq Y(0) + c_3 (1 + W(T)) \{ |g_0|_2 + \|\gamma\|^3 (\|\dot{u}\| + \|\dot{C}\|) \}$$
$$+ \varepsilon \int_0^T (1 + W) \|\gamma\|^4 P_1(\|\dot{u}\|, \|\dot{C}\|, \|\ddot{u}\|, \|\ddot{C}\|)(1 + \|\tilde{\theta}\|^3)\, dT'.$$

The coefficients of the polynomial depend only on

$$\| |g_{0,s}|(T)\|, \| |g_0|(T)\|, \left\| \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 (T) \right\|$$

while c_3 is a fixed number. ■

Proof. Follows by inserting (2.1) into the previous lemma, taking a maximum over time, and then dividing by $1 + \|E^{1/2}\|$. ■

We now see what is required. We want an estimate for α which can be combined with this result to give an a priori estimate for the H^1 norm of $\tilde{\theta}$ using Lemma 2.1. We obtain this by choosing the time evolution of u, C in a particular way, as given in the following lemma:

Lemma 2.3. *Consider a smooth solution of the perturbed sine-Gordon equation* (1.11), *compactly supported in space, for which the time evolution of u, C obey the equations*:

$$m\dot{C} = u(1-u^2) \int_{-\infty}^{+\infty} g(t, Z, \theta_K(Z))\theta'_K(Z)\,dZ\,, \tag{2.2}$$

$$\begin{aligned} m(\gamma u)_t = {} & \int_{-\infty}^{+\infty} g(t, Z, \theta_K(Z))\theta'_K(Z)\,dZ - \varepsilon m\gamma^3 u\dot{u}\dot{C} \\ & - \varepsilon \int_{-\infty}^{+\infty} (G_1^T\tilde{\theta} + G_2^T\tilde{\theta}_T)\,dZ - \varepsilon \int_{-\infty}^{+\infty} N\theta'_K(Z)\,dZ\,, \end{aligned} \tag{2.3}$$

where $G_{1,2}^T$ are given in (A2.13–14), *and N in* (A0.2). *Fix $\varepsilon_1 > 0$ then α, the coefficient of the translation mode, defined in* (1.12), *satisfies the following inequality $\forall \varepsilon < \varepsilon_1$:*

$$\begin{aligned} |\alpha(T)| \leqq {} & |\alpha(0)| + c_2 E(T)^{1/2} \\ & + \varepsilon(1 + \gamma|\dot{u}|) \int_0^T |\gamma|^4 P_2(|\dot{u}|, |\ddot{u}|, |\dot{C}|)\Bigg(1 + \|\tilde{\theta}(T')\|^3 \\ & + \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 \Bigg)\,dT'\,, \end{aligned} \tag{2.4}$$

where c_2 is a fixed number and P_2 is a fixed polynomial whose coefficients do not depend on ε.

Corollary 2.4. *Under the hypotheses of Lemma* 2.3 *there exist a fixed number c_4 and a polynomial P_3, whose coefficients depend only on* $\left\| \left| \frac{g(\theta_K + \varepsilon\tilde{\theta}) - g_0}{\varepsilon} \right|_2 (T) \right\|$ *such that for $\varepsilon < \varepsilon_1$:*

$$\begin{aligned} \|\alpha(T)\| \leqq {} & |\alpha(0)| + c_4(1 + Y(T)) + \varepsilon(1 + \|\gamma\|\,\|\dot{u}\|) \int_0^T \|\gamma\|^4 P_3(\|\dot{C}\|, \|\dot{u}\|, \|\ddot{u}\|) \\ & \times (1 + \|\tilde{\theta}\|^3)\,dT'\,, \end{aligned}$$

$$\begin{aligned} W(T) \leqq {} & W(0) + c_4 + \varepsilon(1 + \|\gamma\|\,\|\dot{u}\|) \int_0^T \|\gamma\|^4 P_3(\|\dot{C}\|, \|\dot{u}\|, \|\ddot{u}\|) \\ & \times (1 + \|\tilde{\theta}\|^2(W + Y))\,dT'\,. \end{aligned}$$

Remark. The two important things about these estimates are, firstly, that an ε appears before the integrand, and secondly, the $O(1)$ term is $\sim E^{1/2}$ which does not depend on α as far as the method of estimation goes. Notice that (2.2), (2.3) reduce to (1.20)–(1.21) as $\varepsilon \to 0$. ■

Proof. The proof is divided into seven parts, of which the first three are devoted to proving the following *formula for* α which does not depend on (2.2), (2.3):

$$\begin{aligned}\frac{m\alpha}{\gamma} &= -\gamma u \int Z\theta'_K(\tilde{\theta}_T - u\tilde{\theta}_X)\,dX + \int_0^T \int (\gamma u Z\theta'_K f + \gamma u H)\,dX\,dT' \\ &\quad + \varepsilon\gamma^2 u\dot{u} \int_0^T \int_{-\infty}^{+\infty} \tilde{\theta}\theta'_K dX - \varepsilon \int_0^T \int_{-\infty}^{+\infty} (\gamma\dot{C} - \gamma^2 u\dot{u}Z)\tilde{\theta}\theta''_K dK \\ &\quad - \varepsilon \int_0^T mu\dot{u}\gamma^2\alpha + \varepsilon \int_0^T \int_{-\infty}^{+\infty} (G_1^S\tilde{\theta} + G_2^S\tilde{\theta}_T)\,dX\,dT'\,, \end{aligned} \tag{2.5}$$

where H is given by (2.7) below. To prove this we use the two symmetry mode identities (A2.11), (A2.12), with a special choice of ψ, ϕ. In Steps 4–7 we choose the time evolution of u, C in such a way that this formula leads to the desired estimate.

Step One. Substitute for $\int \tilde{\theta}\theta''_K(Z)$ in the scaling mode identity from the translation mode identity, leading to:

$$\begin{aligned}\psi \int_{-\infty}^{+\infty} Z\theta'_K(\tilde{\theta}_T - u\tilde{\theta}_X)\,dX \Big]_T &= \int_0^T \int \left(\psi Z\theta'_K f + \psi\theta''_K\tilde{\theta} - \frac{\psi}{\gamma u}\theta'_K\tilde{\theta}_T + \psi H\right) dT'\,dX \\ &\quad + \varepsilon \int_0^T \int_{-\infty}^{+\infty} (G_1^S\tilde{\theta} + G_2^S\tilde{\theta}_T)\,dX\,dT'\,, \end{aligned} \tag{2.6}$$

where

$$H = \frac{1}{\phi\gamma u} \int_0^T \int_{-\infty}^{+\infty} \phi\theta'_K f + \varepsilon G_1^T\tilde{\theta} + \varepsilon G_2^T\tilde{\theta}_T dX\,dT'\,. \tag{2.7}$$

Step Two. Notice that from (1.12) we have $\alpha = \frac{\gamma}{m}\int_{-\infty}^{+\infty} \tilde{\theta}\theta'_K(Z)\,dX$ which implies

$$\int (\theta'_K(Z)\tilde{\theta}_T - \gamma u\theta''_K(Z)\tilde{\theta})\,dX = \frac{m}{\gamma}\frac{d\alpha}{dT} + \varepsilon \int_{-\infty}^{+\infty} \{(\gamma\dot{C} - \gamma^2 u\dot{u}Z)\theta''_K\tilde{\theta} - \gamma^2 u\dot{u}\theta'_K\tilde{\theta}\}\,dK\,. \tag{2.8}$$

Step Three. Comparing the results of the two previous steps, we see that if we choose $\phi = 1$, $\psi = \gamma u$, then $\dfrac{d\alpha}{dT}$ appears in the integrand on the right-hand side of (2.6). Integrating this by parts leads to the formula for α.

Step Four (*choice of* $\dot{u}$). The plan should be clear by now; to obtain a long time Gronall type inequality, we need to choose the time evolution of u, C in such a way that the $O(1)$ terms in the integrand in the above expression vanish. Now write $f = f^{(1)} + \varepsilon M + \varepsilon N$, as in appendix zero, and integrate explicitly the terms involving $f^{(1)}$, M:

$$\int \theta'_K(f^{(1)} + \varepsilon M)\,dZ = m(\gamma u)_t - \int \theta'_K g_0\,dZ + m(\gamma\dot{C})_t\,. \tag{2.9}$$

The only subtle point is that, as already discussed, in the formulae for $\dot{u}$, $\dot{C}$ we do not want any second derivatives to occur. To take account of this it is necessary to integrate by parts the $\ddot{C}$ term in H which arises from the M term, giving:

$$\gamma u H = m\dot{C} + \int_0^T \left\{ -\int_{-\infty}^{+\infty} \theta_K' g dX + \frac{m(\gamma u)_t}{\gamma} + \varepsilon m\gamma^2 u\dot{u}\dot{C} + \varepsilon \int_{-\infty}^{+\infty} (G_1^T \tilde{\theta} + G_2^T \tilde{\theta}_T + N\theta_K') dX \right\} dT' . \tag{2.10}$$

Now to make α bounded, we must first choose $\dot{u}$ so that H is bounded. This leads to the equation:

$$m(\gamma u)_t = \int_{-\infty}^{+\infty} g_0(t, Z)\theta_K'(Z)\, dZ - \varepsilon m\gamma^2 u\dot{u}\dot{C} - \varepsilon \int_{-\infty}^{+\infty} (G_1^T \tilde{\theta} + G_2^T \tilde{\theta}_T)\, dZ - \varepsilon \int_{-\infty}^{+\infty} N\theta_K'(Z)\, dZ , \tag{2.11}$$

which Eq. (2.3) of the lemma statement. Notice that this choice implies $\gamma u H = m\dot{C}$.

Step Five (*choice of* $\dot{C}$). Next we calculate $\int (f^{(1)} + \varepsilon M) Z\theta_K'$ using (A0.1–A0.3), leading to the formula:

$$\int_{-\infty}^{+\infty} Z\theta_K'(Z) f dX = mu\gamma\dot{C} - \int_{-\infty}^{+\infty} gZ\theta_K' dX + \frac{\varepsilon m\dot{C}^2\gamma}{2} + 3\varepsilon I\gamma^3 u^2\dot{u}^2 - \frac{\varepsilon I}{\gamma}((u\dot{u}\gamma^2)_t) + \varepsilon \int_{-\infty}^{+\infty} NZ\theta_K' dX ,$$

where $I = \int Z^2 \theta_K'^2(Z)\, dZ$ as defined in the introduction. The idea now is to choose $\dot{C}$ so that the formula for $\alpha(T)$ contains only finite terms evaluated at T, or terms of the form $\varepsilon \int_0^T q(T')$, so that the Gronwall inequality can be used. Thus we put $\gamma u H = m\dot{C}$ and the previous equation into the formula for α (2.5) leading to the choice:

$$mu\gamma\dot{C} + \frac{m\dot{C}}{\gamma u} - \int_{-\infty}^{+\infty} g_0 Z\theta_K' dX = 0$$

which is the equation for $\dot{C}$ given in the lemma conditions (Eq. 2.2).

Step Six. Now returning to α we are left with the following formula:

$$\begin{aligned} \alpha = &-\frac{\gamma^2 u}{m} \int_{-\infty}^{+\infty} Z\theta_K'(\tilde{\theta}_T - u\tilde{\theta}_X)\, dX \\ &+ \frac{\varepsilon\gamma^3 u\dot{u}}{m} \int_0^T \int_{-\infty}^{+\infty} \tilde{\theta}\theta_K' dX - \frac{\varepsilon\gamma}{m} \int_0^T \int_{-\infty}^{+\infty} (\gamma\dot{C} - \gamma^2 u\dot{u}Z)\tilde{\theta}\theta_K'' dX \\ &+ \varepsilon\gamma \int_0^T \left\{ \frac{\gamma^2 um\dot{C}^2 + 4\gamma^4 u^3\dot{u}^2 I}{2m} - \frac{Iu}{m}((u\dot{u}\gamma^2)_t) \right\} dT' \\ &- \varepsilon\gamma \int_0^T u\dot{u}\gamma^2 \alpha dT' + \frac{\varepsilon\gamma}{m} \int_0^T \int_{-\infty}^{+\infty} (G_1^S \tilde{\theta} + G_2^S \tilde{\theta}_T)\, dX\, dT' \\ &+ \frac{\varepsilon\gamma}{m} \int_0^T \int_{-\infty}^{+\infty} \gamma u Z\theta_K' N\, dX\, dT' . \end{aligned} \tag{2.12}$$

There is now one last important point. In the estimate stated in the lemma the finite first term is $\sim E^{1/2}$, i.e. does not contain α, as is necessary to be a good estimate. Thus we must decompose the first line in the above formula, and move over to the right-hand side the part involving α. This is now done in the last step:

Step Seven. We now write:

$$\int Z\theta_K'(\tilde{\theta}_T - u\tilde{\theta}_X)\,dX = \int Z\theta_K'(\tilde{\theta}_T + u\tilde{\theta}_X)\,dX + 2u\int (Z\theta_K')_X\tilde{\theta}\,dX\,. \tag{2.13}$$

Now using the orthogonal decomposition $\tilde{\theta} = \alpha\theta_K' + \theta^*$ we find

$$-\frac{\gamma^2 u}{m}\int_{-\infty}^{+\infty} Z\theta_K'(\tilde{\theta}_T - u\tilde{\theta}_X)\,dX = -\gamma^2u^2\alpha - \frac{\gamma^2 u}{m}\int Z\theta_K'(\tilde{\theta}_T + u\tilde{\theta}_X)\,dX$$
$$-\frac{2\gamma^2u^2}{m}\int (Z\theta_K')_X\theta^*\,dX\,. \tag{2.14}$$

Putting the α term on the right-hand side gives $(1+\gamma^2u^2)\alpha = \gamma^2\alpha$, so there is a cancellation of γ^2. Finally the lemma is proved by use of Lemma A1.1a to bound $|\tilde{\theta}_T + u\tilde{\theta}_X|_2, |\theta^*|_2$. ■

Remark. The formula for $\dot{C}$ depends on u while that for $\dot{u}$ contains $u, \dot{u}, \dot{C}$ though only to $O(\varepsilon)$.

Remark. We have now obtained a coupled system of equations, (2.15)–(2.17) below, for which we need a local exsistence theorem. We will then be able to prove the main theorem by combining with the estimates of Lemmas 2.1, 2.3 and their corollaries. We quote the following theorem which is proved in appendix three:

Theorem 2.5 (Local Existence). *Consider the* ODE-PDE *system*:

$$\tilde{\theta}_{TT} - \tilde{\theta}_{XX} + \cos\theta_K(Z)\tilde{\theta} = f^{(1)} + \varepsilon f^{(2)} = f(T, Z, \tilde{\theta}, \dot{C}, \ddot{C}, p, \dot{p}, \ddot{p})\,, \tag{2.15}$$

$$m\dot{p} = m(\gamma u)_t = \int_{-\infty}^{+\infty} g_0(t,Z)\theta_K'(Z)\,dZ - \varepsilon m\gamma^3 u\dot{u}\dot{C}$$
$$-\varepsilon\int_{-\infty}^{+\infty}(G_1^T\tilde{\theta} + G_2^T\tilde{\theta}_T)\,dZ - \varepsilon\int_{-\infty}^{+\infty} N\theta_K'(Z)\,dZ \tag{2.16}$$

$$m\dot{C} = u(1-u^2)\int_{-\infty}^{+\infty} g_0(t,Z)Z\theta_K'(Z)\,dZ \tag{2.17}$$

with g as described in section one. Then for any initial data satisfying:

$$|p(0)|, |\tilde{\theta}(0)|_{H^1} + |\tilde{\theta}_T(0)|_2 \leqq \frac{K}{2}, \tag{2.18}$$

there exists $\varepsilon_2(K, A)$ such that for $\varepsilon < \varepsilon_2$ there is a weak solution $(\tilde{\theta}, p, C) \in S^K(T_{\text{loc}})$ consisting of $p, C \in C^2([0, T_{\text{loc}}])$, $\tilde{\theta} \in C([0, T_{\text{loc}}], H^1)$, $\tilde{\theta}_T \in C([0, T_{\text{loc}}], L^2)$ with p, C also satisfying (A3.15–16) *such that for $T < T_{\text{loc}}$ there exist numbers $c_5(A)$, $c_6(K, A)$ such that*:

$$\|\dot{C}(T)\| \leqq c_5(A)\,, \tag{2.19}$$

$$\|\ddot{C}(T)\|, \|\dot{p}(T)\| \leqq c_5(A) + \varepsilon c_6(K, A)\,, \tag{2.20}$$

$$\|\gamma(T)\|, \|p(T)\| \leqq (|p(0)| + (c_5(A) + \varepsilon c_6(K, A))(1 + t))\,, \tag{2.21}$$

$$\|\ddot{p}(T)\| \leqq c_6(K, A)\,. \tag{2.22}$$

Furthermore the solutions satisfy the identities in appendix two and hence Lemmas 2.1–2.3, *and their corollaries.* ■

Proof. This follows from Theorem A3.1 in the third appendix and obvious estimates of (A3.12–16). ■

Remark. The important thing is that $T_{\text{loc}} = T_{\text{loc}}(K, A)$ only. A is determined by g through (1.29), so basically the energy determines the continuation. An important point is what exactly do we mean by energy? We mean both the energy of the kink, determined by p or $\gamma = \sqrt{1 + p^2}$, and the linearised "field" energy, related to $\|\tilde{\theta}\|$. As noted in the introduction, the perturbation theory is not expected to be valid as $|u| \to 1$, which corresponds to $\gamma, |p| \to \infty$. Thus it is very natural that the ability to continue local solutions depends on bounds obtained for the momentum as well as $\|\tilde{\theta}\|$. These are provided by (2.21). To see where this comes from the reader must refer to the reformulation of (2.15)–(2.17) in appendix three (Eqs. (A3.12)–(A3.16)).

We now use this to prove the basic theorem by obtaining an a priori bound for the solution, which shows that for initial data such that

$$|p(0)| + |\tilde{\theta}(0)|_{H^1}| + |\tilde{\theta}_T(0)|_2 \leqq \kappa \,, \tag{2.23}$$

then for K large enough any solution obeys:

$$\|p(T_*)\|, \|\tilde{\theta}(T_*)\| \leqq \frac{K}{2} \tag{2.24}$$

for a time $T_* = O\left(\frac{1}{\varepsilon}\right)$. It is then possible to continue the local existence up to this time. The important thing is that K is independent of ε, depending only on A, κ. Recall that A is defined by (1.29–1.30). We wish to combine Corollaries 2.2–2.4 to estimate $\|\tilde{\theta}\|$ using Eq. (2.3) and Proposition 1.1, together with appendix one. Thus we introduce the quantity:

$$G(T) \equiv W + Y + \|\alpha\| + \|\gamma\| \tag{2.25}$$

and try to obtain a long-time Gronwall inequality for this.

Now using $\|\gamma\| \geqq 1$ we find from (2.2) that $\|E^{1/2}\| \leqq G$, from which it follows by Corollary A1.2 and (2.21) that

$$\|p\| + \|\tilde{\theta}(T)\| \leqq \{1 + c(|p(0)|) + (c_5(A) + \varepsilon c_6(K, A))(1 + t)G\} \,. \tag{2.26}$$

Theorem 2.6 (Estimate for 2.15–2.17). *Consider a solution to the system* (2.15)–(2.17) *such that* $(p, \tilde{\theta}) \in S^K(T_+)$. *Fix* $\varepsilon_1(K, A)$ *as in Lemmas* 2.1, 2.3, *then for* $T < T_+, \varepsilon < \varepsilon_1$ *there exist numbers* $P(A), Q(K, A), R(K, A)$, *independent of* ε, *such that*

$$G(T) - G(0) \leqq P + \varepsilon Q + \varepsilon \int_0^T RG(T')\,dT' \,.$$

Remark. The important thing is that $P = P(A)$ only. The Gronwall inequality then gives the required property of the solution.

Proof. This is obtained by combining the estimates in Corollaries 2.2–2.4 with the definitions (2.1), (2.2), (2.25) – all the integral terms are dealt with in the obvious way. The important thing is the finite term which depends only on A, not K. Thus we emphasise these terms, and write the integral terms as $\varepsilon \int$:

$$Y \leqq Y(0) + c_3\{A + 2(|p(0)| + (c_5(A) + \varepsilon c_6(K, A))(1 + t))^4 \times (1 + |W(0)| + c_4)\} + \varepsilon \int ,$$

$$W \leqq W(0) + c_4 + \varepsilon \int ,$$

$$\|\alpha\| \leqq |\alpha(0)| + c_4(1 + c_3(A + Y)) + \varepsilon \int ,$$

$$\|\gamma\| < 1 + \|p\| < 1 + \kappa + cc_5(A)t + \varepsilon \int ,$$

where $\varepsilon < \varepsilon_1, t_+ = \varepsilon T_+$. Addition of these gives the required property. ■

Corollary 2.7. *Fix A, κ, then $\exists \varepsilon_3(K, A)$, $K_{cr}(\kappa, A)$ independent of ε, such that for*

$$K > K_{cr}, \quad \varepsilon < \varepsilon_*(K, A) = \min(\varepsilon_2, \varepsilon_3)$$

there exists $T_* = O\left(\frac{1}{\varepsilon}\right)$ such that a solution to (2.15)–(2.17) *with initial data satisfying* (2.23) *obeys the estimate*

$$\|p(T_*)\|, \|\tilde{\theta}(T_*)\| < K/2 .$$

Proof. Recall ε_2 appeared in Theorem 2.5. We apply the Gronwall inequality to the a priori estimate:

$$G(T) \leqq (P + \varepsilon Q + G(0))e^{\varepsilon RT} ,$$

so that as long as the logarithm is positive we can, by (2.26), choose:

$$T_* = \min\left(T_+, \frac{1}{\varepsilon R}\ln\left(\frac{K}{2(1 + c(|p(0)| + (c_5(A) + \varepsilon c_6(K, A))(1 + t)))(P + \varepsilon Q + G(0))}\right)\right),$$

$$\varepsilon_3 = \min\left(\frac{c_5(1 + t_+) + \kappa}{c_6(1 + t_+)}, \frac{P + G(0)}{Q}\right),$$

$$K_{cr} = 8(1 + 2c(\kappa + c_5(1 + t_+)))(P + G(0)) . \tag{2.27}$$

Recall that T_+ is the time for which the assumptions hold, presumed to be of order $\frac{1}{\varepsilon}$ so that t_+ is a fixed number independent of ε. ■

We now combine this with the local existence theorem to find that the perturbation theory is valid up to a time $T_* = O\left(\frac{1}{\varepsilon}\right)$:

Theorem 2.8. *Consider the perturbed sine-Gordon equation with the perturbation satisfying the conditions in the statement of the main theorem, the more general ones given at the end of section one, and with initial data of the form* (1.18) *subject to* (2.23). *Then given any $A, \kappa > 0$ from* (1.29), *$\exists \varepsilon_*(K, A, \kappa)$, $K_{cr}(K, A, \kappa)$ such that*

$\forall \varepsilon < \varepsilon_* \exists T_* = O\left(\frac{1}{\varepsilon}\right)$ *such that if* u_0, C_0 *are determined by* (1.20, 21) *there is a solution to the initial value problem* (1.17–18) *on* $[0, T_*]$ *of the form in* (1.19a, b) *with* $(\tilde{\theta}, p) \in S^K(T_*)$ *and* $\|\tilde{p}(T_*)\|, \|\tilde{C}(T_*)\| \leqq$ const.(K, A). ■

Proof. To prove this we now combine Theorem A.3 with the previous corollary. This gives a solution to (2.15–2.17) with $\tilde{\theta}, p \in S^{K/2}(T_*)$. Recall that $p = \gamma u$. This then implies that over any time interval of order $\frac{1}{\varepsilon}$ the solutions to Eqs. (2.16–2.17) are within ε of the solutions of (1.20), (1.21), i.e.

$$\|\tilde{p}(T_*)\|, |\tilde{C}(T_*)\| \leqq \text{const.}(K, A)$$

independent of ε. ■

Proof of Main Theorem. The existence part of the main theorem stated in the introduction follows directly from this lemma. Uniqueness of the solution is standard. It is proved in Martin (1976) or can be proved by energy estimates (with a smoothing of the data as in appendix three). ■

Appendix Zero

Here we display the explicit form of the error terms in (1.11):

$$f^{(1)} \equiv -g_0(t, Z) - 2\gamma u(\gamma \dot{C} - u\dot{u}\gamma^2 Z)\theta_K''(Z) + ((\gamma u)_t + \gamma^3 u^2 \dot{u})\theta_K'(Z) \quad \text{(A0.1)}$$

and $f^{(2)} = M + N$, where

$$N \equiv \sin\theta_K(Z)\frac{1-\cos\varepsilon\tilde{\theta}}{\varepsilon^2} + \cos\theta_K(Z)\frac{\varepsilon\tilde{\theta} - \sin\varepsilon\tilde{\theta}}{\varepsilon^2} - \left\{\frac{g(t, Z, \theta) - g_0(t, Z)}{\varepsilon}\right\} \quad \text{(A0.2)}$$

with

$$M \equiv -(\gamma\dot{C} - u\dot{u}\gamma^2 Z)^2\theta_K''(Z) - (Z(u\dot{u}\gamma^2)_t - (\gamma\dot{C})_t - \gamma^3 u\dot{u}\dot{C} + (u\dot{u}\gamma^2)^2 Z)\theta_K' \,. \quad \text{(A0.3)}$$

Appendix One

In this appendix we prove Proposition 1.1 as a simple corollary of the following lemmas:

Lemma A1.1a.

$$E = \int_{-\infty}^{+\infty}\left\{\frac{1}{2}(\tilde{\theta}_T + u\tilde{\theta}_X)^2 + \frac{1}{2}\tilde{\theta}L\tilde{\theta}\right\}dZ$$

$$\geqq \frac{1}{2}|\tilde{\theta}_T + u\tilde{\theta}_X|_2^2 + \frac{1}{2}|\theta^*|_2^2 \,.$$

Lemma A1.1b.

$$E \geqq \int_{-\infty}^{+\infty}\frac{(1-|u|)(\tilde{\theta}_T^2 + \tilde{\theta}_X^2) + \cos\theta_K(Z)\tilde{\theta}^2}{2}dZ \,,$$

which are immediate consequences of (1.8). *Thus we see that control over E and α gives control over the H^1 norm of $\tilde{\theta}$. In fact*:

Corollary A1.2. *There exists a number c such that*

$$\|\tilde{\theta}(T)\| \leqq c\|\gamma(T)\|(\|E^{1/2}(T)\| + \|\alpha(T)\|)\,.$$

Proof.

$$|\tilde{\theta}|_2^2 = |\theta^*|_2^2 + \frac{m\alpha^2}{\gamma} \leqq 2E + \frac{m\alpha^2}{\gamma}\,,$$

so since $|\cos\theta_K| \leqq 1$ we find from Lemma 2.1b,

$$|\tilde{\theta}_T|_2^2 + |\tilde{\theta}_X|_2^2 \leqq \frac{2}{1-|u|}(E + |\tilde{\theta}|_2^2)\,,$$

so all together we find

$$|\tilde{\theta}_T|_2 + |\tilde{\theta}|_{H^1} \leqq \frac{c}{1-|u|}(E^{1/2} + \alpha)$$

for some constant c. The result follows from $\gamma = (1-u^2)^{-1/2}$. ■

Appendix Two. Three Identities

In this appendix we prove three identities by which we can estimate the oscillatory and secular time evolution of the appropriate "modes." We recall that we have made the basic "zero-mode/oscillatory-mode" decomposition in (1.12), and we know from the proposition that $\tilde{\theta}$ is estimated in terms of E, α. Thus we derive identities for E, α here. The first identity relates to the evolution of E, while the second two give the projection of Eq. (1.11) in the two directions $\theta_K', Z\theta_K'$ spanned by the infinitesimal variation of kink parameters. We remark that the identities in this section, while derived as a priori estimates for smooth solutions are valid far more generally. In particular they are valid for the class of weak solutions constructed in appendix three.

Now recall that for a uniformly moving kink (i.e. u, C constant) the energy E is constant if $\tilde{\theta}$ is a solution of $\tilde{\theta}_{TT} - \tilde{\theta}_{XX} + \cos\theta_K(Z)\tilde{\theta} = 0$. The first identity consists of the generalisation of this to the case where the kink parameters u, C vary slowly in time due to the appearance of an inhomogeneous term f on the right-hand side. This gives the following

Lemma A2.1 (Spectral Identity). *Let $\tilde{\theta}$ be a smooth solution of Eq.* (1.11) *and let E be as in* (1.13). *Then for slowly varying u, C we have the following identity*:

$$E(T) - E(0) = \int_0^T \int \left\{ (\tilde{\theta}_T\tilde{\theta}_X)\left(\varepsilon\dot{u} - \left(u\frac{\partial X}{\partial S}\right)_X\right) - \left(\frac{\tilde{\theta}_T^2 + \tilde{\theta}_X^2}{2}\right)\left(u + \frac{\partial X}{\partial S}\right)_X \right.$$
$$\left. - \frac{\tilde{\theta}^2}{2}\left(\cos\theta_K(Z)\left(\frac{\partial X}{\partial S} - u\right)\right)_X + f(\tilde{\theta}_T + u\tilde{\theta}_X)\right\} dZ\,dT\,,$$

where f is as given in appendix zero and

$$\frac{\partial X}{\partial S} = u + \varepsilon(\dot{C} - \gamma u \dot{u} Z) \tag{A2.1}$$

form (2.2). ■

Proof. Follows from a calculation of the time derivative of E, followed by integration by parts in space. Notice that the integral is defined with respect to Z so since $\tilde{\theta} = \tilde{\theta}(T, X)$, we use the formula

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial S} + \frac{\partial X}{\partial S}\frac{\partial}{\partial X}$$

to differentiate under the integral sign. (See (1.13).) ■

We next give two identities which relate to the time evolution in the direction of the symmetry modes θ_K', $Z\theta_K'$. To do this it will be useful to write Eq. (1.11) in first order form. Let

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \equiv \begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}_T \end{pmatrix}, \tag{A2.2}$$

then

$$\frac{\partial U}{\partial T} = \begin{pmatrix} 0 & 1 \\ \partial_X^2 - \cos\theta_K(Z) & 0 \end{pmatrix} U + \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{A2.3}$$

where f is as defined in appendix zero.

Lemma A2.2. *Let U be a smooth solution of the first order Eq.* (A2.3), *compactly supported in space, and let $V \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ be a pair of smooth functions, then*:

$$\int_{-\infty}^{+\infty} (U_1 V_1 + U_2 V_2)\, dX \Big]_0^T - \int_0^T\!\!\int \left\{ \frac{\partial V_1}{\partial T} + \frac{\partial^2 V_2}{\partial X^2} - \cos\theta_K V_2 \right\} U_1\, dX\, dT$$
$$- \int_0^T\!\!\int \left\{ \frac{\partial V_2}{\partial T} + V_1 \right\} U_2\, dX\, dT = \int_0^T\!\!\int V_2 f\, dX\, dT\,. \tag{A2.4}$$

Proof. This is obtained by doing the calculation for the adjoint of the first order differential operator in (A2.3), and leaving in nonzero boundary terms. ■

We now apply this for two special V's. The first V is obtained by differentiation with respect to C and corresponds to translation invariance

$$V^T = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = \phi(t) \begin{pmatrix} \gamma u \theta_K''(Z) \\ \theta_K'(Z) \end{pmatrix}, \tag{A2.5}$$

where ϕ is a slowly varying function of time to be chosen later. For the other we could differentiate with respect to the velocity parameter, but this gives rise to undesirable time dependence, so instead we just take the part of this which is independent of the translation mode. As noted above this is given by the action of the scaling vector field, so we take the second V to be:

$$V^S = \begin{pmatrix} V_1^S \\ V_2^S \end{pmatrix} = \psi \begin{pmatrix} \gamma u\theta_K'(Z) + \gamma uZ\theta_K''(Z) \\ Z\theta_K'(Z) \end{pmatrix}, \tag{A2.6}$$

where ψ is a slowly varying function of time to be chosen later.

Remark. In both of these the first component is, to order ε, the negative of the time derivative of the second component. Thus the factor multiplying U_2 in the integrand in (A2.4) is $O(\varepsilon)$, and the factor multiplying U_1 reduces to

$$-\frac{\partial^2 V_2}{\partial T^2} + \frac{\partial^2 V_2}{\partial X^2} - \cos\theta_K(Z)\, V_2\,,$$

i.e. the error to which V_2 is a solution to the linearised equation. For the translation case this is $O(\varepsilon)$, while in the scaling case there is a finite error due to the noncommutativity of the scaling vector field and the d'Alembertian. This is important for the equations of motion as we shall see later. The precise results are:

The translation mode.

$$V_1^T + \frac{\partial V_2^T}{\partial T} = \varepsilon\phi(\gamma^2 u\dot{u}Z - \gamma\dot{C})\theta_K''(Z) + \varepsilon\dot{\phi}\theta_K'(Z)\,, \tag{A2.7}$$

$$\frac{\partial V_1^T}{\partial T} + \frac{\partial^2 V_2^T}{\partial X^2} - \cos\theta_K(Z)\, V_2 = \varepsilon\phi(\gamma^2 u\dot{u}Z - \gamma\dot{C})\gamma u\theta_K'''(Z) + \varepsilon(\phi(\gamma u)_t + \dot{\phi}\gamma u)\theta_K''(Z)\,. \tag{A2.8}$$

The scaling mode

$$V_1^S + \frac{\partial V_2^S}{\partial T} = \varepsilon\psi(\gamma^2 u\dot{u}Z - \gamma\dot{C})(\theta_K'(Z) + Z\theta_K''(Z)) + \varepsilon\dot{\psi}Z\theta_K'(Z)\,, \tag{A2.9}$$

$$\frac{\partial V_1^S}{\partial T} + \frac{\partial^2 V_2^S}{\partial X^2} - \cos\theta_K V_2^S = 2\psi\theta_K'' + \varepsilon\psi(\gamma^2 u\dot{u}Z - \gamma\dot{C})(2\gamma u\theta_K'' + \gamma uZ\theta_K''')$$
$$+ \varepsilon(\psi(\gamma u)_t + \dot{\psi}\gamma u)(Z\theta_K'' + \theta_K')\,. \tag{A2.10}$$

Using these formulae we now obtain the following

Lemma A2.3 (Symmetry Mode Identities). *Let $\tilde{\theta}$ be a smooth solution to Eq.* (1.11), *compactly supported in space. Then we have the following two identities valid for ϕ, ψ arbitrary slowly varying functions of time:*

Translation mode identity.

$$\phi \int_{-\infty}^{+\infty} (\gamma u\theta_K''(Z)\tilde{\theta} + \theta_K'(Z)\tilde{\theta}_T)\, dX \Bigg]_0^T = \int_0^T \phi \int \theta_K'(Z) f\, dX\, dT$$
$$+ \varepsilon \int_0^T \int (G_1^T\tilde{\theta} + G_2^T\tilde{\theta}_T)\, dX\, dT\,. \tag{A2.11}$$

Scaling mode identity.

$$\psi \int_{-\infty}^{+\infty} Z\theta_K'(Z)(\tilde{\theta}_T - u\tilde{\theta}_X)\,dX \Bigg]_0^T = \int_0^T \psi \int (Z\theta_K'(Z)f + 2\theta_K''(Z)\tilde{\theta})\,dX\,dT$$

$$+ \varepsilon \int_0^T \int (G_1^S\tilde{\theta} + G_2^S\tilde{\theta}_T)\,dX\,dT\,, \qquad \text{(A2.12)}$$

where

$$G_1^T = \phi(\gamma^2 u\dot{u}Z - \gamma\dot{C})\gamma u\theta_K''(Z) + (\phi(\gamma u)_t + \dot{\phi}\gamma u)\theta_K''(Z)\,, \qquad \text{(A2.13)}$$

$$G_2^T = \phi(\gamma^2 u\dot{u}Z - \gamma\dot{C})\theta_K''(Z) + \dot{\phi}\theta_K'(Z)\,, \qquad \text{(A2.14)}$$

$$G_1^S = \psi(\gamma^2 u\dot{u}Z - \gamma\dot{C})(2\gamma u\theta_K''(Z) + \gamma uZ\theta_K'''(Z))$$
$$+ (\psi(\gamma u)_t + \dot{\psi}\gamma u)(Z\theta_K''(Z) + \theta_K'(Z))\,, \qquad \text{(A2.15)}$$

$$G_2^S = \psi(\gamma^2 u\dot{u}Z - \gamma\dot{C})(\theta_K'(Z) + Z\theta_K''(Z)) + \dot{\psi}Z\theta_K'(Z)\,. \qquad \text{(A2.16)}$$

Appendix Three. Local Existence

In this appendix the main result is Theorem 2.5 (here rewritten as Theorem A3.1)—local existence for the system (A3.1)–(A3.3) in the appropriate spaces—H^1 for $\tilde{\theta}$. This is a consequence of the boundedness and Lipshitz properties of the F_i expressed in Lemmas A3.2–A3.4. The method of proof is to produce a sequence of iterates which are uniformly bounded and form a Cauchy sequence.

Discussion. We have now obtained the following set of equations:

$$\tilde{\theta}_{TT} - \tilde{\theta}_{XX} + \cos\theta_K(Z)\tilde{\theta} = f^{(1)} + \varepsilon f^{(2)} = f(T, Z, \tilde{\theta}, \dot{C}, \ddot{C}, p, \dot{p}, \ddot{p})\,, \qquad \text{(A3.1)}$$

$$m\dot{p} = m(\gamma u)_t = \int_{-\infty}^{+\infty} g_0(t, Z)\theta_K'(Z)\,dZ - \varepsilon m\gamma^3 u\dot{u}\dot{C}$$

$$- \varepsilon \int_{-\infty}^{+\infty} (G_1^T\tilde{\theta} + G_2^T\tilde{\theta}_T)\,dZ - \varepsilon \int_{-\infty}^{+\infty} N\theta_K'(Z)\,dZ\,, \qquad \text{(A3.2)}$$

$$m\dot{C} = u(1 - u^2) \int_{-\infty}^{+\infty} g_0(t, Z)Z\theta_K')(Z)\,dZ\,. \qquad \text{(A3.3)}$$

There are two things that we want to do with these equations:

(1) Show that there exist local solutions to this system with $\| p(T_*) \|$ and $\| \tilde{\theta}(T_*) \|$ bounded and with a continuation theorem depending only on these norms.
(2) Show that these solutions satisfy the identities in appendix two necessary for the proof of the estimates in Lemmas 2.1–2.4. This is done by considering the limits of approximate identities for the smoothed iterates.

It will then be possible to prove the main theorem by continuation of the local solutions by showing that the estimates of Lemmas 2.1, 2.3 and their corollaries imply that the norms $\| p(T) \|$ and $\| \tilde{\theta}(T) \|$ remain bounded for times of order $\frac{1}{\varepsilon}$ – see section two.

The proof of the local existence theorem is not difficult once the equations are rewritten in the right way. The important thing is that the modulation equations can be solved for $\dot{p}, \dot{C}$ as nice functions of $\tilde{\theta}, \tilde{\theta}_T, p, C$. Differentiation of these expressions gives formulae for $\ddot{p}, \ddot{C}$ which can be substituted into f, the right-hand side of (A3.1), leaving the system in an appropriate form for existence theory. We now write this out explicitly:

Calculation of First Derivatives. The formula for $\dot{C}$ is precisely that given by the equation of motion; it is convenient to introduce the variable $p = \dfrac{u}{\sqrt{1-u^2}} = \gamma u$ in place of u, in terms of which $\gamma = \sqrt{1+p^2}$ and

$$m\dot{C} = p\gamma^{-3} \int g_0(t, Z) Z\theta_K'(Z)\, dZ\,. \tag{A3.4}$$

The formula for $\dot{p}$ can be found from (A3.2) by using the formulae for G_1^T, G_2^T in (A2.13–14) to bring all the terms involving $\dot{p}$ onto the left, leading to:

$$\dot{p} = \frac{U}{D}\,, \tag{A3.5}$$

where

$$U = \int g_0 \theta_K' dZ - \varepsilon \int N\theta_K' dZ + \varepsilon \dot{C} p\gamma \int \theta_K'' \tilde{\theta} dZ + \varepsilon\gamma \dot{C} \int \theta_K'' \tilde{\theta}_T dZ\,, \tag{A3.6}$$

and

$$\begin{aligned} D = m &+ \varepsilon \int \theta_K'' \tilde{\theta} dZ + \varepsilon p^2 \gamma^{-4} \int g_0 Z\theta_K' dZ \\ &+ \varepsilon\gamma^{-2} p \int Z\theta_K'' \tilde{\theta}_T dZ + \varepsilon\gamma^{-2} p^2 \int Z\theta_K'' \tilde{\theta} dZ\,. \end{aligned} \tag{A3.7}$$

Calculation of Second Derivatives. The second derivative of C can be expressed in terms of $\dot{u}$ via the formula

$$m\ddot{C} = \frac{p}{\gamma^3}\left(\int g_0 Z\theta_K' dZ\right)_t + \left(\frac{p}{\gamma^3}\right)_t \int g_0 Z\theta_K' dZ\,, \tag{A3.8}$$

and hence in terms of $u, C, \tilde{\theta}$ via the previous paragraph.

The situation with $\ddot{p}$ is more complicated because $\tilde{\theta}_{TT}$ appears on the right-hand side on differentiation, which is unacceptable because we want to estimate everything in terms of $\|\tilde{\theta}\|$. Thus we substitute for $\tilde{\theta}_{TT}$ from Eq. (A3.1), and remove the terms involving $\ddot{p}$ to the right-hand side. Thus we define the part of f not involving $\ddot{p}$ as f^*:

$$f^* \equiv f + \frac{p\ddot{p}}{\gamma^2} Z\theta_K'(Z)\,. \tag{A3.9}$$

Now we differentiate the above formula for $\dot{p}$, and inspection shows that since $Z\theta_K'$ is orthogonal to $Z\theta_K''$, $\ddot{p}$ occurs only from U_T. This leads to:

$$\ddot{p} = \frac{U^*}{D + \dfrac{\varepsilon m p \dot{C}}{2\gamma}} \equiv \frac{U^*}{D^*}\,, \tag{A3.10}$$

where

$$\begin{aligned}U^* \equiv\ & \dot{p} \int \theta_K''(\tilde{\theta}_T + X_S\tilde{\theta}_X)\,dZ + \varepsilon(\gamma^{-4}p^2)_t \int g_0 Z\theta_K' dZ \\ & + \varepsilon\dot{p}p^2\gamma^{-4} \int g_{0,t} Z\theta_K' dZ - \gamma^{-2}p\dot{p} \int (X_S Z\theta_K'')_X \tilde{\theta}_T dZ \\ & - \varepsilon\dot{p}(\gamma^{-2}p)_t \int Z\theta_K''\tilde{\theta}_T dZ - \gamma^{-2}\dot{p}p \int (Z\theta_K'')_X \tilde{\theta}_X dZ \\ & - \gamma^{-2}\dot{p}p \int Z\theta_K''(\cos\theta_K\tilde{\theta} - f^*)\,dZ - \int g_{0,t}\theta_K' dZ + \int N_S\theta_K' dZ \\ & - \varepsilon(\dot{C}p\gamma)_t \int \theta_K''\tilde{\theta}\,dZ - \dot{C}p\gamma \int \theta_K''(\tilde{\theta}_T + X_S\tilde{\theta}_X)\,dZ \\ & - \varepsilon(\gamma\dot{C})_t \int \theta_K''\tilde{\theta}_T dZ + \gamma\dot{C} \int ((\theta_K'')_X\tilde{\theta}_X + \cos\theta_K\tilde{\theta}\theta_K'')\,dZ \\ & + \gamma\dot{C} \int X_S\theta_K''\tilde{\theta}_T dZ - \gamma\dot{C} \int \theta_K'' f^* dZ\,, \end{aligned} \tag{A3.11}$$

where X_S is defined in (A2.1), rewritten as a function of $p, C, \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X$ using (A3.4)–(A3.5). The point of these formulae is not the details, but the fact that they allow us to write the system (A3.1)–(A3.3) in more natural form with only lower derivatives on the right-hand side. This leads us to consider the system:

$$\frac{dp}{dT} = \varepsilon\dot{p} = F_1(T, C(T), p(T), \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X(T))\,, \tag{A3.12}$$

$$\tilde{\theta}_{TT} - \tilde{\theta}_{XX} + \tilde{\theta} = F_2(T, C(T), p(T), \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X(T))\,, \tag{A3.13}$$

$$\frac{dC}{dT} = \varepsilon\dot{C} = F_3(T, C(T), p(T), \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X(T))\,, \tag{A3.14}$$

where $F_2 = f + (1 - \cos\theta)\tilde{\theta}$, rewritten as a function of the stated arguments using (A3.4)–(A3.11). We shall also write:

$$\frac{d^2p}{dT^2} = \varepsilon^2\ddot{p} = F_4(T, C(T), p(T), \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X(T))\,, \tag{A3.15}$$

$$\frac{d^2C}{dT^2} = \varepsilon^2\ddot{C} = F_5(T, C(T), p(T), \tilde{\theta}, \tilde{\theta}_T, \tilde{\theta}_X(T))\,, \tag{A3.16}$$

in terms of which

$$F_2 = (1 - \cos\theta_K(Z))\tilde{\theta} + f(T, Z, \tilde{\theta}, C, p, \varepsilon^{-1}F_3, \varepsilon^{-2}F_5, \varepsilon^{-1}F_1, \varepsilon^{-2}F_4)\,. \tag{A3.17}$$

We will now prove the following local existence theorem which is equivalent to Theorem 2.5 used in the proof of the main theorem.

Theorem A3.1. *Let g be as described in the introduction, then for any initial data satisfying*:

$$|p(0)|, |\tilde{\theta}(0)|_{H^1} + |\tilde{\theta}_T(0)|_2 \leqq \frac{K}{2}\,, \tag{A3.18}$$

there exists a weak solution to (A3.12)–(A3.14) *consisting of*

$$p, C \in C^2([0, T_{\mathrm{loc}}]),\ \tilde{\theta} \in C([0, T_{\mathrm{loc}}], H^1),\quad \tilde{\theta}_T \in C([0, T_{\mathrm{loc}}], L^2)$$

with p, C also satisfying (A3.15–16) *such that for* $T < T_{\text{loc}}$,

$$\|p(T)\|, \|\tilde{\theta}(T)\| \leqq K\,. \tag{A3.19a}$$

Furthermore the solutions satisfy the identities in appendix two and hence Lemmas 2.1–2.4. ■

Remark. Notice the slightly surprising fact that p, C are C^2 even for such weak solutions. This is because even though $\tilde{\theta}$ is not strongly differentiable in time, the formulae for derivatives of p, C involve inner products with very rapidly decreasing functions like θ'_K so only differentiability with respect to some distributional topology is needed.

Proof. We use the iteration scheme:

$$\tilde{\theta}^{(i+1)}_{TT} - \tilde{\theta}^{(i+1)}_{XX} + \tilde{\theta}^{(i+1)} = F^i_3\,, \tag{A3.20}$$

$$\dot{p}^{(i+1)} = F_1(t, p^{(i)}, C^{(i)}, \tilde{\theta}^{(i)}, \tilde{\theta}^{(i)}_T, \tilde{\theta}^{(i)}_X) \equiv F^i_1\,, \tag{A3.21}$$

$$m\dot{C}^{(i+1)} = F_3 = u^{(i)}(1 - u^{(i)2}) \int_{-\infty}^{+\infty} gZ\theta'_K(Z)\,dZ \equiv F^i_3\,, \tag{A3.22}$$

with initial data:

$$p^{(i+1)}(0) = p(0) \quad C^{(i+1)}(0) = C(0) \quad \tilde{\theta}^{(i)}(0, X) = \tilde{\theta}^{(i)}(X) \quad \tilde{\theta}^{(i)}_T 0, X) = \tilde{\theta}_{0,T}(X)\,, \tag{A3.23}$$

where $\tilde{\theta}^{(i)}_0, \tilde{\theta}^{(i)}_{0,T}$ are C^∞_0 functions which satisfy:

$$|\tilde{\theta}^{(i)}_0 - \tilde{\theta}(0,\cdot)|_{H^1} \leqq \text{const.}\,2^{-i} \quad |\tilde{\theta}^{(i)}_{0,T} - \tilde{\theta}_T(0,\cdot)|_{L^2} \leqq \text{const.}\,2^{-i} \tag{A3.24}$$

and F^i_3 are C^∞_0 approximations to F_3 evaluated at the previous iterates:

$$|F^i_3 - F_3(t, p^{(i)}, C^{(i)}, \tilde{\theta}(i), \tilde{\theta}^{(i)}_T, \tilde{\theta}^{(i)}_X)|_2 \leqq \text{const.} \times 2^{-i}\,. \tag{A3.25}$$

This is possible as the previous iterates are C^∞_0 and the other terms in F_3 are at least L^2 using the asssumption in (1.29)–(1.30). We do this in order to prove the integral identities of appendix two by approximation of those for the iterates, for which integration by parts is justified.

To prove the local existence we find a time interval on which all the iterates are uniformly bounded. We then show that they converge on this interval. We also calculate the approximate identities corresponding to those in appendix two for each iterate so their limit can be taken. The convergence of the iterates depends as usual on the Lipshitz properties of the right-hand sides. These follow directly from the following fact:

Lemma A3.2. *Consider smooth functions* $\tilde{\theta}, p, C$ *such that* $(\tilde{\theta}, p) \in S^K(T_+)$: *then* $\exists \varepsilon_2(K, A)$ *such that* $\forall \varepsilon < \varepsilon_2\ \exists c(A), c(K, A)$ *independent of* ε, *such that for* $T \leqq T_+$:

$$\|\dot{p}(T)\| \leqq c(A) + \varepsilon c(K, A)\,, \tag{A3.26}$$

$$\|\gamma(T)\|, \|\dot{C}(T)\|, \|\ddot{C}(T)\|, \|\ddot{p}\| \leqq c(K, A)\,. \tag{A3.27}$$

Proof. Since $(\tilde{\theta}, p) \in S^K(T_+)$ (1.21) we know from the hypotheses on g in (1.29)–(1.30), that all the terms in the numerators in (A3.12)–(A3.15) are bounded by some number depending on K, A. Given this the lemma depends on the fact that $m = 8$ is a fixed number while the denominators D, D^* which occur in (A3.12–15) are schematically of the form

$$m + B(K, A)\varepsilon, \quad m + B^*(K, A)\varepsilon\,, \tag{A3.28}$$

so choosing

$$\varepsilon_2(K, A) < \min\left(\frac{8}{|B(K, A)|}, \frac{8}{|B^*(K, A)|}\right) \tag{A3.29}$$

we can bound all the expressions for $\varepsilon < \varepsilon_2$ using the formula

$$\frac{a\varepsilon + b}{c\varepsilon + d} = \frac{b}{d} + \int^{\varepsilon} \frac{ad - bc}{(cz + d)^2}\, dz\,.$$

Further inspection of (A3.12) gives (A3.26). Notice from (A3.11) that further derivatives rise to $O\left(\frac{1}{\varepsilon}\right)$ terms so that in the phraseology introduced at the end of section one, p and $\dot{p}$ are slowly varying but $\ddot{p}$ is not since $p_{ttt} = O\left(\frac{1}{\varepsilon}\right)$. ■

Lemma A3.3 (Boundedness of F_i). *With the conditions of Lemma* A3.2 *the functions* $\{F_i\}_1^3$ (A3.12)–(A3.14) *are bounded in the following norms*:

$$\| F_{1,3}(T_+)\| \leqq K_*, \quad \| |F_2|_2(T_+)\| \leqq K_*\,, \tag{A3.30}$$

where for $\varepsilon < \varepsilon_2(K, A)$, K_* is a number depending on K, A, t_+, where $t_+ = \varepsilon T_+$. ■

Lemma A3.4 (Lipshitz Properties of F_i). *Under the conditions of Lemma* A3.2 *with two triples of smooth functions $\tilde{\theta}_i, C_i, p_i$ for which $(\tilde{\theta}_i, p_i) \in S^K(T_+)$ we have the following local Lipshitz bounds for the F_i:*

$$|F_{1,3}(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_{1,3}(T, C_2, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X})| \leqq \mathrm{Lip}|C_1 - C_2|\,,$$

$$|F_{1,3}(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_{1,3}(T, C_1, p_2, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X})| \leqq \mathrm{Lip}|p_1 - p_2|\,,$$

$$|F_{1,3}(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_{1,3}(T, C_1, p_1, \tilde{\theta}_2, \tilde{\theta}_{2,T}, \tilde{\theta}_{2,X})|$$

$$\leqq \mathrm{Lip}(|\tilde{\theta}_1 - \tilde{\theta}_2|_{H^1} + |\tilde{\theta}_{1,T} - \tilde{\theta}_{2,T}|_2)\,,$$

$$|F_2(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_2(T, C_2, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X})|_2 \leqq \mathrm{Lip}|C_1 - C_2|\,,$$

$$|F_2(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_2(T, C, p_2, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X})|_2 \leqq \mathrm{Lip}|p_1 - p_2|\,,$$

$$|F_2(T, C_1, p_1, \tilde{\theta}_1, \tilde{\theta}_{1,T}, \tilde{\theta}_{1,X}) - F_2(T, C, p, \tilde{\theta}_2, \tilde{\theta}_{2,T}, \tilde{\theta}_{2,X})|_2$$

$$\leqq \mathrm{Lip}(|\tilde{\theta}_1 - \tilde{\theta}_2|_{H^1} + |\tilde{\theta}_{1,T} - \tilde{\theta}_{2,T}|_2)$$

for some number Lip(K, A). ■

We take the zeroth iterates to be constant in time at the initial values and then generate further iterates from A3.20–22. The iterates are smooth and compactly supported in space on account of the smoothings introduced in A3.24–25. The basic estimates for these iterates are:

$$\|\tilde{\theta}^{(i)}(T)\| \leqq \|\tilde{\theta}^{(i)}(0)\| + \int_0^T |F_2(T')|\,dT' ,$$

$$|p^{(s)}(T)| \leqq |p^{(s)}(0)| + \int_0^T |F_1^{(s)}(T')|\,dT' , \tag{A3.31}$$

$$|C^{(i)}(T)| \leqq |C^{(i)}(0)| + \int_0^T |F_3^{(i)}(T')|\,dT' . \tag{A3.32}$$

Uniform Boundedness in H^1. The first step is to find a time interval on which all the iterates of $\tilde{\theta}$ are bounded in H^1 and also the momentum by K. We see from these estimates that if $(\tilde{\theta}^{(i-1)}, p^{(i-1)}) \in \chi^K(T)$ then from Lemma A3.3 and (A3.31–32) we have:

$$\|\tilde{\theta}^i(T)\| \leqq K/2 + \int_0^T K_* ,$$

$$\|p^i(T)\| \leqq K/2 + \int_0^T K_* ,$$

and since the zeroth iterate is bounded by $K/2$ we find by induction that for $T < T_{\mathrm{loc}} = \min\left(T_+, T_K = \dfrac{K}{2K_*}\right)$ all the iterates are bounded by K.

Convergence of the Iterates. We now show the convergence of the iterates obtained in the H^1 norm. Convergence follows from the Lipshitz properties of F_i expressed in Lemma A3.4. Indeed if we take the difference between successive equations for the iterates and apply (A3.31–32) we find

$$\|\tilde{\theta}^{(i+1)} - \tilde{\theta}^{(i)}\| \leqq \mathrm{const.}(1+T)2^{-i} + 3\mathrm{Lip}\int_0^T (|p^{(i)} - p^{(i)}| + |C^{(i)} - C^{(i-1)}| + \|\tilde{\theta}^{(i)} - \tilde{\theta}^{(i-1)}\|)\,dT'$$

$$|p^{(i+1)} - p^{(i)}| \leqq 3\mathrm{Lip}\int_0^T (|p^{(i)} - p^{(i)}| + |C^{(i)} - C^{(i-1)}| + \|\tilde{\theta}^{(i)} - \tilde{\theta}^{(i-1)}\|)\,dT'$$

from which convergence in H^1 follows using the fact that if $Y(0)$ is constant and

$$Y(i) \leqq \int_0^T Y(i-1) + \mathrm{const.}2^{-i} ,$$

then

$$Y(i) \leqq Y(0)\frac{T^i}{i!} + \mathrm{const.}2^{-i}e^{2T} \to 0 \quad \text{as } i \to \infty . \tag{A3.33}$$

Notice that convergence is uniform on $[0, T_{\mathrm{loc}}]$.

Convergence to a Weak Solution. The fact that the solution solves the equation weakly follows from the fact that $\tilde{\theta}$ converges pointwise by Sobolev's lemma and g is continuous so Lebesgue's theorem allows passage to the limit.

Regularity. The strong continuity in time, i.e.:

$$\tilde{\theta} \in C([0, T], H^1(\mathbf{R})) , \tag{A3.34}$$

$$\tilde{\theta}_T \in C([0, T], L^2(\mathbf{R})) \tag{A3.35}$$

follows from the uniform convergence of the iterates in the interval $[0, T_{\mathrm{loc}}]$. In order to deduce strong differentiability in time, however, it is necessary to assume an extra spatial derivative for the initial data. This can indeed be done, as far as the initial data allows, producing smooth solutions if necessary, see ([Stuart 1990]).

More surprising is the fact that the limit functions p, C are twice differentiable in time even though $\tilde{\theta}$ is not (strongly) differentiable. The results for C follow from those for p on account of (A3.4), so we concentrate on p. The differentiability of p is easy since the limit of Eq. (A2.31) can be taken using uniform convergence of $p^{(i)}$, $C^{(i)}$, $\tilde{\theta}^{(i)}$ giving uniform convergence of $\dot{p}^{(i)}$ to $\dot{p}$. The interesting thing is that $\ddot{p}^{(i)} \to \ddot{p}$ in $C([0, T_{\mathrm{loc}}])$. This is because the difficulty arises with the presence of terms involving $\tilde{\theta}^{(i)}_{TT}$ in the formula obtained for $\ddot{p}^{(i)}$ by differentiation of (A3.21). However such terms always occur in the form

$$\int_{\mathbf{R}} Q\tilde{\theta}^{(i)}_{TT}\, dX$$

with Q a smooth rapidly decaying function (in space). Thus using Eq. (A3.20), and integrating by parts, allows passage to the limit uniformly in time. This leads to the conclusion that the formula (A3.10) for $\ddot{p}$ is valid for the weak solution constructed and p, C are twice differentiable.

Validity of Identities in Appendix Two. Finally we give a specimen calculation to show that the identities of Lemmas A2.1–A2.3 are valid. We carry this out for Lemma A2.1. For the $(i+1)^{\text{th}}$ iterate is in C_0^∞ so all integration by parts is justified leading to:

$$\begin{aligned}
E^{(i+1)}(T) = \int_0^T \int \Bigg\{ & (\tilde{\theta}^{(i+1)}_T \tilde{\theta}^{(i+1)}_X)(\varepsilon \dot{u}^{(i+1)} - \left(u^{(i+1)} \frac{\partial X}{\partial S}\right)^{(i+1)}_X\Bigg) \\
& - \left(\frac{(\tilde{\theta}^{(i+1)}_T)^2 + (\tilde{\theta}^{(i+1)}_X)^2}{2}\right)\left(u^{(i+1)} + \frac{\partial X}{\partial S}\right)^{(i+1)}_X \\
& - \left(\frac{\tilde{\theta}^{(i+1)2}}{2}\left(\cos\theta_K(Z^{(i+1)})\left(\frac{\partial X}{\partial S} - u\right)^{(i+1)}\right)_X\right. \\
& + f(\tilde{\theta}^{(i+1)}_T + u^{(i+1)}\tilde{\theta}^{(i+1)}_X)\Bigg\} dZ^{(i+1)}\, dT \\
& + \int\int (\tilde{\theta}^{(i+1)}_T + u^{(i+1)}\tilde{\theta}^{(i+1)}_X)(\tilde{\theta}^{(i)} - \tilde{\theta}^{(i+1)}) \\
& + \cos\theta_K(Z^{(i+1)})\tilde{\theta}^{(i+1)} - \cos\theta_K(Z^{(i)})\tilde{\theta}^{(i)})\, dZ^{(i+1)}\, dT
\end{aligned}$$

from which it is clear that the convergence of $\tilde{\theta}^{(i)}$ in the $\|\ \|$ norm is sufficient to ensure the validity of the identity. A similar argument applies for Lemma A2.3, using the fact that convergence of the coefficient of the zero mode coefficient $\alpha^{(i)} = \int \theta'_K(Z^{(i)})\tilde{\theta}^{(i)}$ follows from the strong L^2 convergence of $\tilde{\theta}^{(i)}$.

Acknowledgements. I would like to thank Professors Majda, McLaughlin and Weinstein for encouragement and advice during the course of this work. In addition thanks to the referee for comments which have improved the exposition and clarified a couple of points.

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Communicated by A. Jaffe