Commun. Math. Phys. 144, 53–86 (1992)

Communications in
Mathematical
Physics

On the Symplectic Geometry of the Super Teichmüller Space*

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Received July 18, 1990; in revised form April 8, 1991

Abstract. The geometry of the Teichmüller space of the super Riemann surfaces is examined. The Weil–Petersson Kähler form is calculated in terms of the super coordinate functions which provide local coordinates for the super Teichmüller space. It is shown that the Kähler form on the super Teichmüller space is closed.

1. Introduction

The purpose of this paper is to show that the Weil–Petersson Kähler form on the super Teichmüller space for the compact super Riemann surfaces of genus $h \geqq 2$ is closed. As for the ordinary (non-super) Riemann surfaces, it is known that the Teichmüller space is a complex Kähler manifold, and hence the Kähler form is closed. A Kähler metric, defined in terms of the Petersson product for the automorphic forms, was introduced by Weil [1]. The Weil–Petersson metric may naturally project to the moduli space because it is invariant under the covering transformations.

Wolpert [2] gave a description of the Fenchel–Nielsen deformation [3] in terms of quasiconformal mappings. The Fenchel–Nielsen vector fields t_*, which are generators of the deformation, were found to be related to the geodesic length functions l_*, introduced by Fricke and Klein [4] to provide local coordinates for the Teichmüller space. He showed a duality formula,

$$\omega(t_*,) = -dl_*, \tag{1}$$

* Work partially supported by Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture No. 02302020
** Work partially supported by Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture No. 02740145

where ω is the Weil–Petersson Kähler form, and also the cosine formula,

$$\omega(t_\alpha, t_\beta) = t_\alpha l_\beta = \sum_{p \in \alpha \# \beta} \cos\theta_p, \tag{2}$$

where the sum is over the cosines of the angles at the intersections of the geodesics α and β on the Riemann surface. He also evaluated the Lie derivatives $t_\alpha t_\beta l_\gamma$ in terms of the hyperbolic geometry of the geodesics and showed the quadratic reciprocity relation,

$$t_\alpha t_\beta l_\gamma + t_\beta t_\gamma l_\alpha + t_\gamma t_\alpha l_\beta = 0. \tag{3}$$

This identity leads to the conclusion that ω is a closed 2-form,

$$d\omega = 0. \tag{4}$$

Here we will present the analogous results for the super Teichmüller space. We shall begin with a review on the super Beltrami equation for the super Riemann surfaces along the line of our previous analysis [5] in the next section and we refer to the super Teichmüller space in Sect. 3. In Sect. 4 we evaluate the Lie derivatives of super cross ratios and Grassmann odd super Möbius invariants [6] along geodesics. We show a superanalog of the linear reciprocity relation of the twist derivative in Sect. 5. The quadratic reciprocity relation is also presented and its proof is given in Sect. 6. In Sect. 7 we introduce the Weil–Petersson Kähler form ω on the super Teichmüller space and we show that it is actually a closed 2-form. The final section is devoted to discussions.

2. Preliminaries

In this section we will review the Beltrami equation for the super Riemann surfaces along the line of our previous analysis [5].

A super Riemann surface (SRS) having compact body with $h \geqq 2$ holes is represented by a homogeneous space $SH/S\Gamma$ [7–9] with a superanalog of the Poincaré geometry. The super complex upper half-plane SH is the universal covering space of the SRS with one even and one odd complex coordinates z and θ, respectively,

$$SH = \{\vec{z} = (z, \theta) \mid \operatorname{Im} z > 0\}. \tag{5}$$

Note that $\operatorname{Im} z > 0$ means that $\operatorname{Im} z_0 > 0$ with z_0 being the body part of z. We shall use such a convention for inequalities throughout this paper for simplicity. $S\Gamma$ is a super Fuchsian group, a discrete subgroup of superconformal automorphism $SPL(2, \mathbb{R})$ of SH. The supergroup $SPL(2, \mathbb{R})$ consists of such transformations as,

$$\begin{aligned} z \to \tilde{z} &= \frac{az+b}{cz+d} + \theta \frac{\alpha z + \beta}{(cz+d)^2}, \\ \theta \to \tilde{\theta} &= \frac{\alpha z + \beta}{cz+d} + \theta \frac{1 + \frac{1}{2}\beta\alpha}{cz+d}, \end{aligned} \tag{6}$$

where a, b, c and d are Grassmann even and α and β are Grassmann odd parameters with[1]

$$ad - bc = 1, \quad a, b, c, d \in \mathbb{R},$$
$$\bar{\alpha} = i\alpha, \quad \bar{\beta} = i\beta. \tag{7}$$

Our convention of the "super" real axis $\mathbb{R}_s$ is[2]

$$\mathbb{R}_s = \{\vec{z} \,|\, \mathrm{Im}\, z = 0,\ \bar{\theta} = i\theta\}, \tag{8}$$

and hence all the parameters in (7) are real in this sense. Note that the above transformation (6) is, of course, superanalytic and also a superconformal transformation,

$$D\tilde{z} - \tilde{\theta} D\tilde{\theta} = 0, \tag{9}$$

$$D \equiv \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}. \tag{10}$$

A super Fuchsian group $S\Gamma$ is generated by $2h$ elements $\{A_i, B_i\}$ $(i = 1, 2, \ldots, h)$ satisfying a condition,

$$\prod_{i=1}^{h} (A_i B_i A_i^{-1} B_i^{-1}) = 1. \tag{11}$$

Each element of the generators contains three Grassmann even and two odd parameters and the condition (11) is invariant under $A_i \to M A_i M^{-1}$, $B_i \to M B_i M^{-1}$, $M \in SPL(2, \mathbb{R})$. Then the set of generators actually depends on $6h - 6$ Grassmann even and $4h - 4$ odd parameters. $S\Gamma$ acts properly discontinuously on SH and all its elements are hyperbolic, i.e., the reduced subgroups, where odd parameters are put to zero, consists of hyperbolic elements.

The Beltrami equation for the ordinary Riemann surfaces is given by,

$$w_{\bar{z}} = \mu w_z, \quad z \in H, \tag{12}$$

where μ is a Beltrami differential defining a $(-1, 1)$-type tensor on the Riemann surface. The super Beltrami equations proposed in [5] are given by,

$$\begin{aligned} Dw - \eta D\eta &= \nu\left\{\partial w + \eta\partial\eta - \frac{i\theta - \bar{\theta}}{2Y}(Dw - \eta D\eta)\right\}, \\ \bar{D}w - \eta\bar{D}\eta &= -\sigma\left\{\partial w + \eta\partial\eta - \frac{i\theta - \bar{\theta}}{2Y}(Dw - \eta D\eta)\right\}, \end{aligned} \tag{13}$$

[1] We adopt such a convention of complex conjugation as,

$$\overline{X + Y} = \bar{X} + \bar{Y}, \quad \overline{XY} = \bar{Y}\bar{X}$$

[2] The real part Re_s and imaginary part Im_s are defined by

$$\mathrm{Re}_s X = \begin{cases} \dfrac{1}{2}(X + \bar{X}), & \text{for even } X, \\ \dfrac{1}{2}(X - i\bar{X}), & \text{for odd } X, \end{cases} \qquad \mathrm{Im}_s X = \begin{cases} \dfrac{1}{2i}(X - \bar{X}), & \text{for even } X, \\ \dfrac{1}{2i}(X + i\bar{X}), & \text{for odd } X \end{cases}$$

where D is given in Eq. (10) and

$$\bar{D} = -\frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\frac{\partial}{\partial\bar{z}}, \quad \partial = \frac{\partial}{\partial z}, \quad \left(\bar{\partial} = \frac{\partial}{\partial\bar{z}}\right), \tag{14}$$

$$Y = \mathrm{Im}\, z + \tfrac{1}{2}\theta\bar{\theta}, \tag{15}$$

and $(\nu, \sigma) \equiv \vec{\mu}$ are our super Beltrami differentials. They are Grassmann odd tensors under $SPL(2, \mathbb{R})$ of weight $(-\frac{1}{2}, 0)$ and $(-1, \frac{1}{2})$, respectively;

$$\widetilde{\nu(\vec{z})} = \Omega(\vec{z})\nu(\vec{z}), \quad \widetilde{\sigma(\vec{z})} = [\Omega^2(\vec{z})\overline{\Omega(\vec{z})}^{-1}]\sigma(\vec{z}), \quad \Omega(\vec{z}) \equiv (D\tilde{\theta}). \tag{16}$$

We shall explain the derivation of Eq. (13). It does not seem easy to get the super-extended version of the Beltrami equation (12) as it stands. So we rewrite the Beltrami equation as

$$\begin{aligned} dw &= dz h_+^+ + d\bar{z} h_-^+, \\ d\bar{w} &= dz h_+^- + d\bar{z} h_-^-, \end{aligned} \tag{17}$$

where complex-valued coefficients h_a^b $(a, b = \pm)$ satisfy integrability conditions,

$$\bar{\partial} h_+^+ = \partial h_-^+, \quad \bar{\partial} h_+^- = \partial h_-^-, \quad (h_\pm^+ = \overline{h_\mp^-}). \tag{18}$$

Hence only h_-^+ (or, $h_+^- = \overline{h_-^+}$) may be regarded as an independent variable and it involves the degree of freedom of the Beltrami differential μ. In fact,

$$\mu = h_-^+ / h_+^+. \tag{19}$$

Equation (17) actually represents the relation between the set of basis 1-forms in both coordinate systems. The super Beltrami equations should be similar to that. In considering the question, we should bear in mind that the basis 1-form dz is a tensor under the Möbius transformations, the automorphism of the complex upper half plane, while the flat basis 1-forms $\mathring{E}^+ = d\theta$ and $\mathring{E}^- = d\bar{\theta}$ are not tensors under the automorphism $SPL(2, \mathbb{R})$ of SH, although the rests, $\mathring{E}^{++} = dz + \theta d\theta$ and $\mathring{E}^{--} = d\bar{z} - \bar{\theta} d\bar{\theta}$, are tensors of weight $(-1, 0)$ and $(0, -1)$, respectively. Hence we shall seek for tensors corresponding to $d\theta$ and $d\bar{\theta}$. The proper basis 1-forms having tensorial transformation laws under $SPL(2, \mathbb{R})$ are found to be,

$$\begin{aligned} E^{++}(\vec{z}) &= dz + \theta d\theta, \\ E^+(\vec{z}) &= d\theta + \frac{i\theta - \bar{\theta}}{2Y}(dz + \theta d\theta), \\ E^{--}(\vec{z}) &= \overline{E^{++}(\vec{z})}, \quad E^-(\vec{z}) = \overline{E^+(\vec{z})}, \end{aligned} \tag{20}$$

where Y is a $(-\frac{1}{2}, -\frac{1}{2})$-tensor (15) and E^{++} and E^+ are tensors of weight $(-1, 0)$ and $(-\frac{1}{2}, 0)$, respectively.

The super Beltrami equations corresponding to (17) are

$$E^A(\vec{w}) = E^B(\vec{z}) H_B^A, \tag{21}$$

with H_B^A satisfying the integrability conditions. The analysis of the integrability condition for the flat basis 1-forms $\mathring{E}^A$ can be done and the corresponding coefficients $\overset{\circ}{H}{}^A_B$ should take the form [5],

$$(\overset{\circ}{H}{}^A_B) = \begin{pmatrix} D\xi + h^2 & D\bar{\psi} - \bar{k}^2 & Dh & D\bar{k} \\ \bar{D}\psi - k^2 & \bar{D}\bar{\xi} + \bar{h}^2 & \bar{D}k & \bar{D}\bar{h} \\ \xi & \bar{\psi} & h & \bar{k} \\ \psi & \bar{\xi} & k & \bar{h} \end{pmatrix}, \tag{22}$$

with

$$\begin{aligned} \bar{D}h &= DK, \quad 2hk + D\psi - \bar{D}\xi = 0, \\ D\bar{h} &= \bar{D}\bar{k}, \quad 2\bar{h}\bar{k} + \bar{D}\bar{\psi} - D\bar{\xi} = 0. \end{aligned} \tag{23}$$

Equation (23) implies that $\xi = \overset{\circ}{H}{}^{++}_{+}$ and $\psi = \overset{\circ}{H}{}^{++}_{-}$ are independent Grassmann odd superfields and hence involve the super Beltrami differentials. E's (20) and $\overset{\circ}{E}$'s are related,

$$E^A(\vec{z}) = \overset{\circ}{E}{}^B(\vec{z}) M^A_B(\vec{z}),$$

$$(M^A_B(\vec{z})) = \begin{pmatrix} 1 & 0 & -\dfrac{DY}{Y} & 0 \\ 0 & 1 & 0 & -\dfrac{\bar{D}Y}{Y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

and hence so are H^A_B and $\overset{\circ}{H}{}^A_B$,

$$H^A_B(\vec{z}) = (M^{-1})^C_B(\vec{z}) \overset{\circ}{H}{}^D_C(\vec{z}) M^A_D(\vec{w}). \tag{25}$$

We find that $H^{++}_{\pm} = \overset{\circ}{H}{}^{++}_{\pm}$ and then we get the super Beltrami differentials defined by,

$$\begin{aligned} v &= H^{++}_{+}/H^{++}_{++}, \\ \sigma &= H^{++}_{-}/H^{++}_{++}. \end{aligned} \tag{26}$$

And the super Beltrami equations are given by (13). The result of the analysis of the integrability conditions agree with the fact from the 2D supergravity theory: Howe [10] showed that the independent components of $E^M_A \delta E^A_M$, where E^A_M are supervielbeins, are H^{+}_{+}, H^{++}_{+}, H^{++}_{-} and their c.c. and the degrees of freedom of the super Weyl and local Lorentz transformations are involved in H^{+}_{+} and its c.c.

Next we shall solve the super Beltrami equations. In fact we examine the linearized equations (cf. Refs. [11, 12]). We extend the Beltrami differentials to the super lower half-plane SL by reflection. According to our convention of the "super" real axis (8), the super Beltrami differentials on SL should be

$$\overline{i\sigma(\bar{z}, -i\bar{\theta})}, \quad \overline{iv(\bar{z}, -i\bar{\theta})} \quad \vec{z} \in SL. \tag{27}$$

We normalize the mapping $\vec{z} \to \vec{w}$ at $\vec{z} = (0,0)$, $(1,0)$ and $(\infty, 0)$ as

$$\begin{aligned} \vec{w} &= (0,0) && \text{at } \vec{z} = (0,0), \\ \vec{w} &= (1,*) && \text{at } \vec{z} = (1,0), \\ \vec{w} &= (\infty, \text{finite}) && \text{at } \vec{z} = (\infty, 0). \end{aligned} \tag{28}$$

The third condition means that $w/z^2 \to 0$ and $\eta/z \to 0$ for $z \to \infty$ at $\theta = 0$. We can take such a normalization due to the symmetry of the solution $\vec{w}(\vec{z})$, i.e., if $\vec{w}$ is a solution, then $\tilde{\vec{w}} = A\vec{w}$, $A \in SPL(2, \mathbb{R})$, is also a solution.

Rewriting the variables $\vec{w}, v$ and σ as

$$w = z + \dot{w}, \quad \eta = \theta + \dot{\eta}, \quad v = \dot{v}, \quad \sigma = \dot{\sigma}, \tag{29}$$

we obtain the linearized super Beltrami equations,

$$\begin{aligned} &D\dot{w} - \dot{\eta} - \theta D\dot{\eta} = \dot{v}, && \bar{D}\dot{w} - \theta\bar{D}\dot{\eta} = -\dot{\sigma} && z \in SH, \\ &\overline{D\dot{w} - \dot{\eta} - \theta D\dot{\eta}} = i\dot{v}(\bar{z}, -i\bar{\theta}), && \overline{\bar{D}\dot{w} - \theta\bar{D}\dot{\eta}} = -i\dot{\sigma}(\bar{z}, -i\bar{\theta}) && z \in SL, \end{aligned} \tag{30}$$

with the normalizations,

$$\begin{aligned} &\dot{w} = 0 && \text{for} \quad \vec{z} = (0,0), (1,0), \\ &\dot{\eta} = 0 && \text{for} \quad \vec{z} = (0,0), \\ &\left.\begin{aligned} &\dot{w}/z^2 \to 0 \\ &\dot{\eta}/z \to 0 \end{aligned}\right\} && \text{for} \quad z \to \infty, \theta = 0. \end{aligned} \tag{31}$$

We shall solve the equations as follows: expanding each variable in θ,

$$\dot{w}(\vec{z}) = \dot{w}^0(z) + \theta\dot{w}^\theta(z) + \bar{\theta}\dot{w}^{\bar{\theta}}(z) + \theta\bar{\theta}\dot{w}^{\theta\bar{\theta}}(z), \qquad \text{etc.}, \tag{32}$$

we rewrite Eq. (30) in components,

$$\left\{\begin{aligned} &\dot{w}^\theta - \dot{\eta}^0 = \dot{v}^0, \\ &\partial\dot{w}^0 - 2\dot{\eta}^\theta = \dot{v}^\theta, \\ &\dot{w}^{\theta\bar{\theta}} - \dot{\eta}^{\bar{\theta}} = \dot{v}^{\bar{\theta}}, \\ &\partial\dot{w}^{\bar{\theta}} - 2\dot{\eta}^{\theta\bar{\theta}} = \dot{v}^{\theta\bar{\theta}}, \end{aligned}\right. \qquad \left\{\begin{aligned} &\dot{w}^{\bar{\theta}} = \dot{\sigma}^0, \\ &\dot{w}^{\theta\bar{\theta}} + \dot{\eta}^{\bar{\theta}} = \dot{\sigma}^\theta, \\ &\bar{\partial}\dot{w}^0 = -\dot{\sigma}^{\bar{\theta}}, \\ &\bar{\partial}\dot{w}^\theta + \bar{\partial}\dot{\eta}^0 = \dot{\sigma}^{\theta\bar{\theta}}, \qquad \text{etc.} \end{aligned}\right. \tag{33}$$

These equations can be solved by the use of potential integrals: for example,

$$\dot{w}^0(z) = -\frac{1}{\pi}\int_H d^2t \frac{\dot{\sigma}^{\bar{\theta}}(t)}{z - t} + f(z), \quad z \in H, \tag{34}$$

where $f(z)$ is an arbitrary analytic function, which is determined through the normalization conditions (31). The solution to Eq. (30) with the boundary conditions of Eq. (31) is found to be

$$\begin{aligned} \dot{w}[\vec{\mu}](\vec{z}) &= \tfrac{1}{2}\theta\dot{v}(\vec{z}) + \bar{\theta}[\dot{\sigma}(\vec{z}) - \tfrac{1}{2}\theta D\dot{\sigma}(\vec{z})] \\ &\quad + \frac{1}{\pi}\int_H d^2t [R^{(0,1)}(t,z)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},z)\overline{\dot{\sigma}^{\bar{\theta}}(t)}] \end{aligned}$$

$$-\frac{\theta}{2\pi}\int_H d^2t[R^{(0)}(t,z)\dot{\sigma}^{\theta\bar{\theta}}(t) - iR^{(0)}(\bar{t},z)\overline{\dot{\sigma}^{\theta\bar{\theta}}(t)}], \tag{35}$$

$$\dot{\eta}[\vec{\mu}](\vec{z}) = \tfrac{1}{2}\dot{v}(\vec{z}) - \tfrac{1}{2}\bar{\theta}D\dot{\sigma}(\vec{z})]$$

$$-\frac{1}{2\pi}\int_H d^2t[R^{(0)}(t,z)\dot{\sigma}^{\theta\bar{\theta}}(t) - iR^{(0)}(\bar{t},z)\overline{\dot{\sigma}^{\theta\bar{\theta}}(t)}],$$

$$+\frac{\theta}{2\pi}\partial\int_H d^2t[R^{(0,1)}(t,z)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},z)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]$$

where

$$R^{(0,1)}(t,z) = \frac{1}{t-z} - \frac{1-z}{t} + \frac{z}{1-t},$$

$$R^{(0)}(t,z) = \frac{1}{t-z} - \frac{1}{t}. \tag{36}$$

Here use has been made of an assumption that

$$v(\vec{z}) = \sigma(\vec{z}) = 0 \quad \text{for} \quad \vec{z}\in\mathbb{R}_s. \tag{37}$$

This is based on the following analysis of the transformation laws of the super Beltrami differentials (see Eq. (16)): the transition function Ω takes the value of $\chi N^{\pm 1/2}(\neq 1)$ on the fixed points, with χ and N being the sign factor and the norm function, respectively [13]. Hence $v(\vec{z})$ and $\sigma(\vec{z})$ vanish on every fixed point. Since the bodies of fixed points are dense on $\mathbb{R}$, we assume that the fixed points are "dense" on $\mathbb{R}_s$, and we get Eq. (3.7). [Unfortunately we have not given a rigorous meaning of "dense" on $\mathbb{R}_s$ so far, and hence we just assume (37).] Note that Eq. (37) does not mean that each coefficient in the expansion with respect to θ vanishes. Especially, the values of $\sigma^{\theta\bar{\theta}}(z)$ and $\sigma^{\bar{\theta}}(z)$ (or $\sigma^{\theta}(z)$) on $\mathbb{R}$ can remain arbitrary. Equation (37) implies (cf. 32))

$$v^0(z) = v^\theta(z) + iv^{\bar{\theta}}(z) = \sigma^0(z) = \sigma^\theta(z) + i\sigma^{\bar{\theta}}(z) = 0 \quad \text{for} \quad \bar{z} = z. \tag{38}$$

Due to the properties of the super Beltrami differentials (37) we see that the solution $\vec{w}$ (35) maps the "super" real axis $\mathbb{R}_s$ onto itself,

$$\dot{w}(\vec{z}) = \overline{\dot{w}(\vec{z})}, \quad i\dot{\eta}(\vec{z}) = \overline{\dot{\eta}(\vec{z})}, \quad \vec{z}\in\mathbb{R}_s. \tag{39}$$

In fact $\vec{w}(\vec{x})$, $\vec{x} = (x,\alpha)\in\mathbb{R}_s$, is given by

$$\dot{w}(\vec{x}) = \frac{1}{\pi}\int_H d^2t[R^{(0,1)}(t,x)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},x)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]$$

$$-\frac{\alpha}{2\pi}\int_H d^2t[R^{(0)}(t,x)\dot{\sigma}^{\theta\bar{\theta}}(t) - iR^{(0)}(\bar{t},x)\overline{\dot{\sigma}^{\theta\bar{\theta}}(t)}],$$

$$\dot{\eta}(\vec{x}) = -\frac{1}{2\pi}\int_H d^2t[R^{(0)}(t,x)\dot{\sigma}^{\theta\bar{\theta}}(t) - iR^{(0)}(\bar{t},z)\overline{\dot{\sigma}^{\theta\bar{\theta}}(t)}],$$

$$+\frac{\alpha}{2\pi}\partial_x\int_H d^2t[R^{(0,1)}(t,x)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},x)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]. \tag{40}$$

Note that we should be careful in calculating $\partial R^{(0,1)}(t,z)$ and $\partial R^{(0,1)}(\bar{t},z)$ of $\dot{\eta}(\vec{z})$ in (35). They are evaluated as,

$$\begin{aligned}
&\frac{\theta}{2\pi}\partial \int_H d^2t[R^{(0,1)}(t,z)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},z)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]\bigg|_{\vec{z}\in\mathbb{R}_s} \\
&= \frac{\theta}{2\pi}(\partial + \bar{\partial})\int_H d^2t[R^{(0,1)}(t,z)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},z)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]\bigg|_{\vec{z}\in\mathbb{R}_s} \\
&\quad + \frac{\theta}{2}\int_H d^2t[\delta^2(t-z)\dot{\sigma}^{\bar{\theta}}(t) + \delta^2(\bar{t}-z)\overline{\dot{\sigma}^{\bar{\theta}}(t)}]\bigg|_{\vec{z}\in\mathbb{R}_s} \\
&= \frac{\alpha}{2\pi}\partial_x \int_H d^2t[R^{(0,1)}(t,x)\dot{\sigma}^{\bar{\theta}}(t) + R^{(0,1)}(\bar{t},x)\overline{\dot{\sigma}^{\bar{\theta}}(t)}] + \frac{\alpha}{2}\dot{\sigma}^{\bar{\theta}}(x).
\end{aligned} \tag{41}$$

In Eq. (40), we find that $\vec{w}[\vec{\mu}](\vec{z})$, $\vec{z}\in\mathbb{R}_s$, is determined by only $\sigma(\vec{z})$, more precisely by $\sigma^{\bar{\theta}}(z)$ and $\sigma^{\theta\bar{\theta}}(z)$ [11], and hence we may write $\vec{w}[\sigma](\vec{z})$, $\vec{z}\in\mathbb{R}_s$. Those equations are used in the next section.

3. The Super Teichmüller Space

The Teichmüller deformation of SRSs is characterized by the isomorphism of the super Fuchsian groups. The super Teichmüller space $ST(S\Gamma)$ is the set of equivalence classes in the isomorphism under $SPL(2,\mathbb{R})$. The super Teichmüller space $ST_h(S\Gamma)$ of compact SRSs of genus $h(\geqq 2)$ is a $6h-6$ even and $4h-4$ odd dimensional supermanifold[3]. The body of $ST_h(S\Gamma)$ is the ordinary Teichmüller space $T_h(\Gamma)$ with spin structure. $ST_h(S\Gamma)$ is described more definitely as follows: let $\vec{w}^{\mu}$ be the solution of the super Beltrami equations. Consider the isomorphism $j[\vec{\mu}]$ from $S\Gamma$ to a deformed super Fuchsian group $S\Gamma^{\vec{\mu}}$;

$$\begin{aligned}
&j[\vec{\mu}]: A \to A[\vec{\mu}], \\
&A[\vec{\mu}] = \vec{w}^{\mu}\circ A \circ (\vec{w}^{\mu})^{-1}, \quad A\in S\Gamma.
\end{aligned} \tag{42}$$

The relation $j[\vec{\mu}_1] \sim j[\vec{\mu}_2]$ means that there exists a $SPL(2,\mathbb{R})$ transformation M such that

$$A[\vec{\mu}_2] = M\circ A[\vec{\mu}_1]\circ M^{-1} \quad \text{for} \quad \forall A\in S\Gamma. \tag{43}$$

Then $ST_h(S\Gamma)$ is a set of the equivalence classes of the isomorphism $j[\vec{\mu}]$.

We shall be interested in the local structure of $ST_h(S\Gamma)$. We consider the deformation corresponding to an infinitesimal super Beltrami differential $\varepsilon\vec{\mu}$;

$$j[\varepsilon\vec{\mu}]: A \to A + \varepsilon\dot{A}[\vec{\mu}], \tag{44}$$

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[3] The super Teichmüller space should be regarded as a superorbifold due to the Z_2 identification of the odd coordinates [11]

and

$$\dot{w}[\vec{\mu}](A\vec{z}) = \dot{w}[\vec{\mu}](\vec{z})\frac{\partial A(z)}{\partial z} + \dot{\eta}[\vec{\mu}](\vec{z})\frac{\partial A(z)}{\partial \theta} + \dot{A}[\vec{\mu}](z),$$
$$\dot{\eta}[\vec{\mu}](A\vec{z}) = \dot{\eta}[\vec{\mu}](\vec{z})\frac{\partial A(\theta)}{\partial \theta} + \dot{w}[\vec{\mu}](\vec{z})\frac{\partial A(\theta)}{\partial z} + \dot{A}[\vec{\mu}](\theta), \tag{45}$$

where $A\vec{z} = (A(z), A(\theta))$ takes the same form given by (6). Using Eqs. (35) and (45) we obtain

(i) $\dot{A}[\vec{\mu}] = 0$ for $\vec{\mu} = (\nu, 0)$.
(ii) $\dot{A}[\vec{\mu}]$ is superholomorphic, i.e., $\bar{D}\dot{A}[\vec{\mu}](z) = 0$ and $\bar{D}\dot{A}[\vec{\mu}](\theta) = 0$.

The first result together with Eq. (44) implies that only the super Beltrami differential of $\vec{\mu} = (0, \sigma)$-type is relevant for our purposes here. Let $SB(S\Gamma)$ and $SQ(S\Gamma)$ be the complex supervector spaces of $(-1, \frac{1}{2})$ differentials and superholomorphic $\frac{3}{2}$-differentials, respectively. We shall define the complex (anti) linear mapping Λ from $SB(S\Gamma)$ to $SQ(S\Gamma)$: the solution $\dot{\eta}[\vec{\mu}](\vec{z})$ in (35) is a linear functional of the super Beltrami differential $\vec{\mu} = (\nu, \sigma)$ in the real sense. A complex linear functional can be obtained by

$$\begin{aligned}\Phi[\sigma](\vec{z}) &= \dot{\eta}[\vec{\mu}](\vec{z}) + i\dot{\eta}[i\vec{\mu}](\vec{z}) \\ &= \frac{i}{\pi}\int_H d^2t R^{(0)}(\bar{t}, z)\overline{\sigma^{\theta\bar{\theta}}(t)} + \frac{\theta}{\pi}\partial \int_H d^2t R^{(0,1)}(\bar{t}, z)\overline{\sigma^{\bar{\theta}}(t)},\end{aligned} \tag{46}$$

which is a functional of $\sigma(\vec{z})$, or $\sigma^{\bar{\theta}}(z)$ and $\sigma^{\theta\bar{\theta}}(z)$, and superanalytic for $\vec{z} \in SH$. Operating D^4 on Φ we get

$$\begin{aligned}D^4\Phi[\sigma](\vec{z}) &= \frac{2i}{\pi}\int_H d^2t\left[\frac{\overline{\sigma^{\theta\bar{\theta}}(t)}}{(\bar{t}-z)^3} - 3i\theta\frac{\overline{\sigma^{\bar{\theta}}(t)}}{(\bar{t}-z)^4}\right] \\ &= -\frac{2i}{\pi}\int_{SH} dt d\bar{t} d\xi d\bar{\xi}\frac{\overline{\sigma(\vec{t})}}{(\bar{t}-z+i\bar{\xi}\theta)^3}.\end{aligned} \tag{47}$$

One can see that the right-hand side is a superholomorphic $\frac{3}{2}$-differential from the transformation low under $SPL(2,\mathbb{R}) \ni A$ for each quantity in the above equation;

$$\begin{aligned}\overline{\sigma(A\vec{t})} &= [\overline{\Omega_A(\vec{t})^2 \Omega_A(\vec{t})^{-1}}]\overline{\sigma(\vec{t})}, \\ A(\bar{t}-z+i\bar{\xi}\theta)^{-3} &= [\overline{\Omega_A(\vec{t})}\Omega_A(\vec{z})]^{-3}(\bar{t}-z+i\bar{\xi}\theta)^{-3}, \\ A dt d\bar{t} d\xi d\bar{\xi} &= [\Omega_A(\vec{t})\overline{\Omega_A(\vec{t})}] dt d\bar{t} d\xi d\bar{\xi}, \quad \Omega_A(\vec{z}) \equiv D(A\theta).\end{aligned} \tag{48}$$

Then the mapping Λ is defined by,

$$\Lambda[\sigma](\vec{z}) = -\frac{4i}{\pi}\int_{SH} dt d\bar{t} d\xi d\bar{\xi}\frac{\overline{\sigma(\vec{t})}}{(\bar{t}-z+i\bar{\xi}\theta)^3}, \quad \sigma \in SB(S\Gamma). \tag{49}$$

Note that $(-1, \frac{1}{2})$-differential σ in (49) is an element of $SB(S\Gamma)$ and hence it is either Grassmann even or odd. We find, for any element S in $SQ(S\Gamma)$, that $Y^2\bar{S} \in SB(S\Gamma)$ and

$$\Lambda[Y^2\bar{S}] = S. \tag{50}$$

Proof.

$$\Lambda[Y^2\bar{S}](\vec{z}) - \frac{4i}{\pi}\int_{SH} d^2t d^2\xi \frac{S(\vec{t})Y_t^2}{(\bar{t}-z+i\bar{\xi}\theta)^3}$$

$$= -\frac{4i}{\pi}\int_{SH} d^2t d^2\xi \frac{(\operatorname{Im} t)\{S^0(t)+\xi S^\theta(t)\}}{(\bar{t}-z)^3}\left[(\operatorname{Im} t) - 3i\frac{\operatorname{Im} t}{(\bar{t}-z)}\bar{\xi}\theta + \xi\bar{\xi}\right]$$

$$= -\frac{4i}{\pi}\int_H d^2t\left[-\frac{(\operatorname{Im} t)S^0(t)}{(\bar{t}-z)^3} + 3i\frac{(\operatorname{Im} t)^2}{(\bar{t}-z)^2}S^\theta(t)\theta\right]$$

$$= \frac{2}{\pi}\int_H d^2t\partial\bar{t}\left[\left(\frac{1}{(\bar{t}-z)} - \frac{t-z}{2(\bar{t}-z)^2}\right)S^0(t)\right.$$

$$\left. + \frac{3}{2}\left(\frac{1}{(\bar{t}-z)} - \frac{t-z}{(\bar{t}-z)^2} + \frac{(t-z)^2}{3(\bar{t}-z)^3}\right)S^\theta(t)\theta\right]$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\frac{S^0(x)+\theta S^\theta(x)}{x-z} = S(\vec{z}). \quad \text{Q.E.D.} \tag{51}$$

It is useful to introduce a complex linear mapping $H: SB(S\Gamma) \to SB(S\Gamma)$ defined by

$$H[\sigma] = Y^2\overline{\Lambda[\sigma]}, \quad \sigma \in SB(S\Gamma). \tag{52}$$

We will write as

$$\operatorname{Im} H = SH(S\Gamma) \quad \text{(space of harmonic super differential)},$$
$$\operatorname{Ker} H = N(S\Gamma). \tag{53}$$

Equation (5) implies that $H^2 = H$ and hence

$$SB(S\Gamma) = SH(S\Gamma) \oplus N(S\Gamma). \tag{54}$$

Now we show that the following four conditions are all equivalent [14]:[4]

(a) $\dot{A}[\vec{\mu}] = 0$ in SH for $\forall A \in S\Gamma$.
(b) $\dot{\vec{w}}[\sigma] = 0$ on $\mathbb{R}_s$.
(c) $\Lambda[\sigma] = 0$ in SH.
(d) $H[\sigma] = 0$ in SH.

Proof. (c)⇔(d). This is due to (52).

(c)⇔(b).

$$\Lambda[\sigma](\vec{z}) = 2D^4\Phi[\sigma](\vec{z}) = 2\partial_z^2\Phi[\sigma](\vec{z}) = 0. \tag{55}$$

From the definition of Φ (46) and the boundary conditions for $\dot{\eta}(\vec{z})$ in (31), the above equation yields that $\Phi[\sigma](\vec{z})$ is a polynomial in $\vec{z}$ as,

$$\Phi[\sigma](\vec{z}) = \theta(a_1 + a_2 z), \tag{56}$$

[4] The proof of (a)⇒(b) is based on the analysis below Eq. (37) and hence is not rigorous

where a's are complex constant parameters. And this implies

$$\int_H d^2t R^{(0)}(\bar{t}, z)\overline{\sigma^{\theta\bar{\theta}}(t)} = 0,$$
$$\frac{1}{\pi}\partial \int_H d^2t R^{(0,1)}(\bar{t}, z)\overline{\sigma^{\bar{\theta}}(t)} = a_1 + a_2 z, \quad \vec{z} \in SH, \tag{57}$$

and hence $\int_H d^2t R^{(0,1)}(\bar{t}, z)\overline{\sigma^{\bar{\theta}}(t)}$ is a polynomial in z. Then from Eq. (40) we get

$$\dot{w}[\sigma](\vec{x}) = \tfrac{1}{2}(a_2 + \bar{a}_2)x^2 + (a_1 + \bar{a}_1)x + (\text{real const.}),$$
$$\dot{\eta}[\sigma](\vec{x}) = \frac{\alpha}{2}(a_2 + \bar{a}_2)x + \frac{\alpha}{2}(a_1 + \bar{a}_1). \tag{58}$$

Due to the boundary condition for $\dot{w}(\vec{z})$ in (31), all the (real) coefficients on the right-hand side should vanish and we get to the condition (b).

(b)⇔(c). The condition (b) implies

$$\frac{\alpha}{2\pi}\int_H d^2t R^{(0)}(t, x)\dot{\sigma}^{\theta\bar{\theta}}(t) = iR^{(0)}(\bar{t}, x)\overline{\dot{\sigma}^{\theta\bar{\theta}}(t)},$$
$$\partial_x \int_H d^2t R^{(0,1)}(t, x)\dot{\sigma}^{\bar{\theta}}(t) = -R^{(0,1)}(\bar{t}, x)\overline{\dot{\sigma}^{\bar{\theta}}(t)}, \tag{59}$$

Then $\Phi[\sigma](\vec{z})$ can be superanalytically continued in the whole supercomplex plane $S\mathbb{C}$,

$$\Phi[\sigma](\vec{z}) = \begin{cases} \dfrac{i}{\pi}\displaystyle\int_H d^2t R^{(0)}(\bar{t}, z)\overline{\sigma^{\theta\bar{\theta}}(t)} + \dfrac{\theta}{\pi}\partial \int_H d^2t R^{(0,1)}(\bar{t}, z)\overline{\sigma^{\bar{\theta}}(t)} & \text{for} \quad z \in H, \\ \dfrac{i}{\pi}\displaystyle\int_H d^2t R^{(0)}(t, z)\sigma^{\theta\bar{\theta}}(t) - \dfrac{\theta}{\pi}\partial \int_H d^2t E^{(0,1)}(t, z)\sigma^{\bar{\theta}}(t) & \text{for} \quad z \in L. \end{cases} \tag{60}$$

Similarly, $\Psi[\sigma](\vec{z})$, defined by

$$\Psi[\sigma](\vec{z}) \equiv \dot{w}[\vec{\mu}](\vec{z}) + i\dot{w}[i\vec{\mu}](\vec{z})$$
$$= \frac{1}{\pi}\int_H d^2t R^{(0,1)}(t, z)\dot{\sigma}^{\bar{\theta}}(t) - \frac{\theta}{2\pi}\int_H d^2t R^{(0)}(t, z)\dot{\sigma}^{\theta\bar{\theta}}(t), \tag{61}$$

can also be superanalytically continued in the whole supercomplex plane $S\mathbb{C}$. According to the boundary conditions (31), we get $\Phi[\sigma](\vec{z}) = \theta \times (\text{const})$ and hence $\Lambda[\sigma](\vec{z}) = 0$.

(a)⇒(b). By the use of (45), one can see that $\dot{\vec{w}}[\sigma]$ vanishes on every fixed point, and hence it vanishes on $\mathbb{R}_s$.

(b)⇒(a). This is a direct result of (45) and the condition (ii). Q.E.D.

A natural pairing of $SB(S\Gamma)$ and $SQ(S\Gamma)$ is given by the integral,

$$(\sigma, S) = \int_{SH/S\Gamma} \sigma S, \quad \sigma \in SB(S\Gamma), \quad S \in SQ(S\Gamma). \tag{62}$$

Then we find

$$\sigma \in N(S\Gamma) \Rightarrow (\sigma, S) = 0 \quad \text{for} \quad \forall S \in SQ(S\Gamma). \tag{63}$$

Proof. For any $\sigma \in SB(S\Gamma)$ and $S \in SQ(S\Gamma)$, we shall show an equality,

$$\int_{SH/S\Gamma} \sigma S = \int_{SH/S\Gamma} \overline{\Lambda[\sigma]SY^2}. \tag{64}$$

Let $K(\vec{t},\vec{z}) = \frac{4i}{\pi} \frac{1}{(t - \bar{z} + i\xi\bar{\theta})^3}$. Then the right-hand side of (64) is evaluated as

$$\begin{aligned}
&\int_{SH/S\Gamma} d^2z d^2\theta \left[\int_{SH} d^2t d^2\xi K(\vec{t},\vec{z})\sigma(\vec{t}) \right] S(\vec{z}) Y_z^2 \\
&= \sum_{A \in S\Gamma} \int_{SH/S\Gamma} d^2z d^2\theta \left[\int_{A(SH/S\Gamma)} d^2t d^2\xi K(\vec{t},\vec{z})\sigma(\vec{t}) \right] S(\vec{z}) Y_z^2 \\
&= \sum_{A \in S\Gamma} \int_{A^{-1}(SH/S\Gamma)} d^2(Az) d^2(A\theta) \left[\int_{SH/S\Gamma} d^2(At) d^2(A\xi) K(A\vec{t},A\vec{z})\sigma(A\vec{t}) \right] S(A\vec{z}) Y_{Az}^2 \\
&= \int_{SH/S\Gamma} d^2t d^2\xi \sigma(\vec{t}) \left[\int_{SH} d^2z d^2\theta K(\vec{t},\vec{z}) S(\vec{z}) Y_z^2 \right] \\
&= \int_{SH/S\Gamma} \sigma \Lambda[Y^2\bar{S}] = \int_{SH/S\Gamma} \sigma S,
\end{aligned} \tag{65}$$

where use has been made of Eq. (50). Then (63) has been proved. Q.E.D.

4. $SPL(2,\mathbb{R})$-Invariant Coordinate Functions

First we consider a local super coordinates for the super Teichmüller space $ST_h(S\Gamma)$. We shall see that the Grassmann even coordinates can be essentially the same as those for the ordinary Teichmüller space $T_h(\Gamma)$ which is the body of $ST_h(S\Gamma)$. Fricke and Klein established that local coordinates for $T_h(\Gamma)$ are given by an appropriate set of geodesic length functions. The (non-super) geodesic length functions are invariant under the Möbius transformations, however, they are not under $SPL(2,\mathbb{R})$. We shall give the $SPL(2,\mathbb{R})$-invariant length functions. (The action of $SPL(2,\mathbb{R})$ extends to the boundary $\mathbb{R}_s \cup \{\infty\}$.) Let $(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4)$ be the super cross ratio defined by,

$$(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \frac{z_{13}z_{24}}{z_{14}z_{23}}, \quad z_{ij} = z_i - z_j - \theta_i\theta_j. \tag{66}$$

This is invariant under $SPL(2\mathbb{R})$ and its body is actually the ordinary cross ratio invariant under the Möbius transformations. Let α be a geodesic on $SH/S\Gamma$ corresponding to an element $A \in S\Gamma$ and $\vec{r}_A = (r_A, v_A)$ and $\vec{a}_A = (a_A, \mu_A)$ be repelling and attracting fixed points for the element A, respectively [13]. Then the super length l_α is give by,

$$l_\alpha = \log(A\vec{s}, \vec{s}, \vec{r}_A, \vec{a}_A) \quad \text{for} \quad \forall \vec{s} \in \mathbb{R}_s - \{\vec{r}_A, \vec{a}_A\}. \tag{67}$$

Note that the body of l_α is precisely the (non-super) length for the element of the Fuchsian group corresponding to $A \in S\Gamma$. An appropriate set of those length functions l_α provides the Grassmann even coordinates for $ST_h(S\Gamma)$.

Next we consider Grassmann odd coordinates. Let λ_z be a $SPL(2,\mathbb{R})$-invariant odd quantity defined by,

$$\lambda_z \equiv [\vec{z}_1, \vec{z}_2, \vec{z}_3] = \frac{\theta_{123}}{(z_{12}z_{23}z_{31})^{1/2}}, \quad \theta_{123} = \theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} + \theta_1\theta_2\theta_3. \tag{68}$$

Note that the ordering of the three points in each λ_z is fixed (up to cyclic permutations) by demanding that $(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) > 0$. Using the invariant we can provide $4h-4$ odd coordinates. The Teichmüller deformation of the SRS is described by the isomorphism of the super Fuchsian groups and any matrix element of a super Fuchsian group is a function of $6h-6$ even and $4h-4$ odd parameters, so that the deformation of the SRS is caused by the change of the $6h-6$ even and $4h-4$ odd parameters. The $4h-4$ odd parameters in the generators can be odd coordinates for the super Teichmüller space $ST_h(S\Gamma)$, however, each odd parameter, in general, is not $SPL(2,\mathbb{R})$-invariant and then it will be difficult to get a $SPL(2,\mathbb{R})$-invariant volume form. We present the $SPL(2,\mathbb{R})$-invariant $4h-4$ odd coordinates [6]. Let $S\Gamma$ be the super Fuchsian group of a reference super Riemann surface $SH/S\Gamma$, $\{A_i, B_i\}$ $(i = 1,\ldots,h)$ generators of $S\Gamma$, and $\{\vec{u}_i^A, \vec{v}_i^A\}$ and $\{\vec{u}_i^B, \vec{v}_i^B\}$ the fixed points of the generators A_i and B_i, respectively. Then a set of the odd coordinates $\{\lambda_\kappa\}(\kappa = 1,\ldots,4h-4)$ for $ST_h(S\Gamma)$ are given by,

$$\{\lambda_\kappa\} = \{[\vec{u}_1^A, \vec{v}_1^A, \vec{u}_j^A], [\vec{u}_1^A, \vec{v}_1^A, \vec{v}_j^A], [\vec{u}_1^A, \vec{v}_1^A, \vec{u}_j^B], [\vec{u}_1^A, \vec{v}_1^A, \vec{v}_j^B]\}, \quad (j = 2,3,\ldots,h). \tag{69}$$

Since the condition (11) on the generators is invariant under conjugation, one may regard A_1 as "diagonal," i.e., the fixed points of A_1 can be put to $\{(0,0), (\infty,0)\}$. Then the condition reveals that the parameters in B_1 is written by the parameters in $\{A_j, B_j\}$ $(j = 2,\ldots,h)$. The $\{\lambda_\kappa\}$ represent essentially all the odd parameters in the remaining generators, $\{A_j, B_j\}$ $(j = 2,\ldots,h)$. Then $\{l_i, \lambda_\kappa\}$ $(i = 1,\ldots,n; \kappa = 1,\ldots,4h-4)$ provide the local coordinate system for the super Teichmüller space $ST_h(S\Gamma)$ (see the first footnote in the previous section). In the following we consider the differentials of those coordinates.

For distinct points on $\mathbb{R}_s$, $\vec{p} = (p,\alpha)$, $\vec{q} = (q,\beta)$, $\vec{r} = (r,\gamma)$ and $\vec{s} = (s,\delta)$, we define the superholomorphic odd function $\Phi(\vec{z};\vec{p},\vec{q},\vec{r},\vec{s})$ and even function $\Psi(\vec{z};\vec{p},\vec{q},\vec{r})$ on SH;

$$\begin{aligned}
\Phi(\vec{z};\vec{p},\vec{q},\vec{r},\vec{s}) &= L^{(1)}(\vec{z};\vec{p},\vec{q})L^{(0)}(\vec{z};\vec{r},\vec{s}) + L^{(1)}(\vec{z};\vec{r},\vec{s})L^{(0)}(\vec{z};\vec{p},\vec{q}) \\
&\quad + L^{(1)}(\vec{z};\vec{p},\vec{q})L^{(1)}(\vec{z};\vec{q},\vec{s})L^{(1)}(\vec{z};\vec{r},\vec{s}), \qquad (70)\\
\Psi(\vec{z};\vec{p},\vec{q},\vec{r}) &= [L^{(0)}(\vec{z};\vec{p},\vec{q})L^{(0)}(\vec{z};\vec{q},\vec{r})L^{(0)}(\vec{z};\vec{r},\vec{p})]^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
L^{(0)}(\vec{z};\vec{p},\vec{q}) &\equiv \frac{z_{pq}}{z_{pz}z_{qz}} = \frac{1}{q - z - \beta\theta} - \frac{1}{p - z - \alpha\theta} + \frac{\beta\theta + \alpha\theta - \alpha\beta}{(p-q)(q-z)}, \\
L^{(1)}(\vec{z};\vec{p},\vec{q}) &\equiv \frac{\theta_{pqz}}{z_{pz}z_{qz}} = \frac{\alpha - \theta}{p - z - \alpha\theta} - \frac{\beta - \theta}{q - z - \beta\theta}.
\end{aligned} \tag{71}$$

They have the following properties for $A \in SPL(2,\mathbb{R})$,

$$\begin{aligned}
\Phi(A\vec{z};\vec{p},\vec{q},\vec{r},\vec{s}) &= \Phi(\vec{z};A^{-1}\vec{p},A^{-1}\vec{q},A^{-1}\vec{r},A^{-1}\vec{s})\Omega_A^{-3}, \\
\Psi(A\vec{z};\vec{p},\vec{q},\vec{r}) &= \Psi(\vec{z};A^{-1}\vec{p},A^{-1}\vec{q},A^{-1}\vec{r})\Omega_A^{-3}.
\end{aligned} \tag{72}$$

The Poincaré series of Φ and Ψ are given by

$$P\Phi(\vec{z};\vec{p},\vec{q},\vec{r},\vec{s})=\sum_{A\in S\Gamma}\Phi(A\vec{z};\vec{p},\vec{q},\vec{r},\vec{s})\Omega_A^3,$$
$$P\Psi(\vec{z};\vec{p},\vec{q},\vec{r})=\sum_{A\in S\Gamma}\Psi(A\vec{z};\vec{p},\vec{q},\vec{r})\Omega_A^3. \tag{73}$$

Both of them are of weight $(\frac{3}{2},0)$ and hence $P\Phi$ $(P\Psi)$ is an Grassmannian odd (even) element of $SQ(S\Gamma)$. [Note that $SQ(S\Gamma)$ was defined by the vector space of superholomorphic $\frac{3}{2}$-differentials, and it involves both Grassmannian even and odd quantities.] As for the convergence of the Poincaré series, we have not given a rigorous proof. If we put the Grassmann odd parameters zero, we can show that both of them convergence absolutely on compacts by the standard analysis. Since both of them are superconformal tensors of weight $(\frac{3}{2},0)$ and both the body part of $P\Phi$ and the odd parameter independent part of $P\Psi$ are convergent, we can expect that the Poincaré series converge [15].

Let $\vec{w}^{\varepsilon\vec{\mu}}\equiv(z+\varepsilon\dot{w}[\vec{\mu}](\vec{z}),\ \theta+\varepsilon\dot{\eta}[\vec{\mu}](\vec{z}))$ be the linearized solution of the super Beltrami equation. Then using (40), we get

$$\begin{aligned}&\frac{d}{d\varepsilon}(\vec{w}^{\varepsilon\sigma}(\vec{p}),\vec{w}^{\varepsilon\sigma}(\vec{q}),\vec{w}^{\varepsilon\sigma}(\vec{r}),\vec{w}^{\varepsilon\sigma}(\vec{s}))\Big|_{\varepsilon=0}\\
&=(\vec{p},\vec{q},\vec{r},\vec{s})\left\{\frac{\dot{w}[\sigma](\vec{q})-\dot{w}[\sigma](\vec{s})-\dot{\eta}[\sigma](\vec{q})\delta-\beta\dot{\eta}[\sigma](\vec{s})}{q-s-\beta\delta}\right.\\
&\quad+\frac{\dot{w}[\sigma](\vec{p})-\dot{w}[\sigma](\vec{r})-\dot{\eta}[\sigma](\vec{p})\gamma-\alpha\dot{\eta}[\sigma](\vec{r})}{p-r-\alpha\gamma}\\
&\quad-\frac{\dot{w}[\sigma](\vec{p})-\dot{w}[\sigma](\vec{s})-\dot{\eta}[\sigma](\vec{q})\delta-\alpha\dot{\eta}[\sigma](\vec{s})}{p-s-\alpha\delta}\\
&\quad\left.-\frac{\dot{w}[\sigma](\vec{q})-\dot{w}[\sigma](\vec{r})-\dot{\eta}[\sigma](\vec{q})\gamma-\beta\dot{\eta}[\sigma](\vec{r})}{q-r-\beta\gamma}\right\}\\
&=\frac{(\vec{p},\vec{q},\vec{r},\vec{s})}{\pi}\mathrm{Re}_s\int_{SH}d^2zd^2\theta\,\sigma(\vec{z})\Phi(\vec{z};\vec{p},\vec{q},\vec{r},\vec{s}).\end{aligned}\tag{74}$$

Similar calculation leads to

$$\frac{d}{d\varepsilon}[\vec{w}^{\varepsilon\sigma}(\vec{p}),\vec{w}^{\varepsilon\sigma}(\vec{q}),\vec{w}^{\varepsilon\sigma}(\vec{r})]\Big|_{\varepsilon=0}=\frac{1}{\pi}\mathrm{Re}_s\int_{SH}d^2zd^2\theta\,\sigma(\vec{z})\Psi(\vec{z};\vec{p},\vec{q},\vec{r}). \tag{75}$$

Using (72) and (73), we can rewrite the above equations as the invariant pairings on a $S\Gamma$ fundamental domain,

$$\begin{aligned}&\frac{d}{d\varepsilon}(\vec{w}^{\varepsilon\sigma}(\vec{p}),\vec{w}^{\varepsilon\sigma}(\vec{q}),\vec{w}^{\varepsilon\sigma}(\vec{r}),\vec{w}^{\varepsilon\sigma}(\vec{s}))\Big|_{\varepsilon=0}\\
&=\frac{(\vec{p},\vec{q},\vec{r},\vec{s})}{\pi}\mathrm{Re}_s\int_{SH/S\Gamma}d^2zd^2\theta\,\sigma(\vec{z})P\Phi(\vec{z};\vec{p},\vec{q},\vec{r},\vec{s}).\\
&\frac{d}{d\varepsilon}[\vec{w}^{\varepsilon\sigma}(\vec{p}),\vec{w}^{\varepsilon\sigma}(\vec{q}),\vec{w}^{\varepsilon\sigma}(\vec{r})]\Big|_{\varepsilon=0}=\frac{1}{\pi}\mathrm{Re}_s\int_{SH/S\Gamma}d^2zd^2\theta\,\sigma(\vec{z})P\Psi(\vec{z};\vec{p},\vec{q},\vec{r}).\end{aligned}\tag{76}$$

Let t_σ denote a derivative operator (tangent vector) acting on the functions, $(\vec{p},\vec{q},\vec{r},\vec{s})$ and $[\vec{p},\vec{q},\vec{r}]$. The Grassmann even-odd "parity" p_G of t_σ is assigned to be opposite to that of the Beltrami differential σ: $p_G[t_\sigma]=p_G[\sigma]+1 \pmod 2$. Operating t_σ on the length l_α of the closed geodesic α associated with $A\in S\Gamma$, we obtain

$$t_\sigma l_\alpha = t_\sigma \log(A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A) = \frac{1}{\pi}\mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta\, \sigma(\vec{z}) P\Phi(\vec{z};A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A). \quad (77)$$

Let Θ_α be a superholomorphic $\frac{3}{2}$-differential defined by a Poincaré series,

$$\begin{aligned}\Theta_\alpha(\vec{z}) &= \sum_{C\in\langle A\rangle\backslash S\Gamma} \Theta(C\vec{z},\vec{r}_A,\vec{a}_A)\Omega_C^3,\\ \Theta(\vec{z},\vec{p},\vec{q}) &\equiv -L^{(0)}(\vec{z};\vec{p},\vec{q})L^{(1)}(\vec{z};\vec{p},\vec{q}),\end{aligned} \quad (78)$$

where $\langle A\rangle$ is the cyclic group generated by A. We find

$$P\Phi(\vec{z};A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A) = 2\Theta_\alpha(\vec{z}). \quad (79)$$

Proof. Using (7) ~ (72), we calculate the right-hand side of the equation,

$$\begin{aligned}
&P\Phi(\vec{z};A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A)\\
&= \sum_{B\in S\Gamma} \Phi(B\vec{z};A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A)\Omega_B^3\\
&= \sum_{C\in\langle A\rangle\backslash S\Gamma}\sum_{n=-\infty}^{\infty} \Phi(A^nC\vec{z};A\vec{s},\vec{s},\vec{r}_A,\vec{a}_A)\Omega_{A^nC}^3\\
&= \sum_{C\in\langle A\rangle\backslash S\Gamma}\sum_{n=-\infty}^{\infty} \{L^{(1)}(C\vec{z};A^{1-n}\vec{s},A^{-n}\vec{s})L^{(0)}(C\vec{z};\vec{r}_A,\vec{a}_A)\\
&\quad + L^{(1)}(C\vec{z};\vec{r}_A,\vec{a}_A)L^{(0)}(C\vec{z};A^{1-n}\vec{s},A^{-n}\vec{s})\\
&\quad + L^{(1)}(C\vec{z};A^{1-n}\vec{s},A^{-n}\vec{s})L^{(1)}(C\vec{z};A^{-n}\vec{s},\vec{a}_A)L^{(1)}(C\vec{z};\vec{r}_A,\vec{a}_A)\}\Omega_C^3\\
&= \sum_{C\in\langle A\rangle\backslash S\Gamma}\Bigg[L^{(0)}(C\vec{z};\vec{r}_A,\vec{a}_A)\sum_{n=-\infty}^{\infty}\{F(1-n)-F(-n)\}\\
&\quad + L^{(1)}(C\vec{z};\vec{r}_A,\vec{a}_A)\sum_{n=-\infty}^{\infty}\Bigg\{G(-n)-G(1-n)\\
&\quad + (F(-n)-F(1-n))\frac{\mu_A - C(\theta)}{a_A - C(z) - \mu_A C(\theta)}\Bigg\}\Bigg]\Omega_C^3,
\end{aligned} \quad (80)$$

where

$$F(n) = \frac{A^n(\delta)-C(\theta)}{A^n(s)-C(z)-A^n(\delta)C(\theta)}, \quad G(n) = \frac{1}{A^n(s)-C(z)-A^n(\delta)C(\theta)}, \quad (81)$$

and they have the following limits;

$$\lim_n F(n) = \begin{cases} \dfrac{\mu_A - C(\theta)}{a_A - C(z) - \mu_A C(\theta)} & (n\to\infty),\\[2mm] \dfrac{\nu_A - C(\theta)}{r_A - C(z) - \nu_A C(\theta)} & (n\to-\infty),\end{cases}$$

$$\lim_n G(n) = \begin{cases} \dfrac{1}{a_A - C(z) - \mu_A C(\theta)} & (n \to \infty), \\ \dfrac{1}{r_A - C(z) - \nu_A C(\theta)} & (n \to -\infty). \end{cases} \tag{82}$$

Thus we get

$$\begin{aligned} &P\Phi(\vec{z}; A\vec{s}, \vec{s}, \vec{r}_A, \vec{a}_A) \\ &= \sum_{C \in \langle A \rangle \backslash S\Gamma} \Bigg[L^{(0)}(C\vec{z}; \vec{r}_A, \vec{a}_A) \left\{ \frac{\mu_A - C(\theta)}{a_A - C(z) - \mu_A C(\theta)} - \frac{\nu_A - C(\theta)}{r_A - C(z) - \nu_A C(\theta)} \right\} \\ &+ L^{(1)}(C\vec{z}; \vec{r}_A, \vec{a}_A) \Bigg\{ \left(\frac{1}{r_A - C(z) - \nu_A C(\theta)} - \frac{1}{a_A - C(z) - \mu_A C(\theta)} \right) \\ &+ \left(\frac{\nu_A - C(\theta)}{r_A - C(z) - \nu_A C(\theta)} - \frac{\mu_A - C(\theta)}{a_A - C(z) - \mu_A C(\theta)} \right) \frac{\mu_A - C(\theta)}{a_A - C(z) - \mu_A C(\theta)} \Bigg\} \Bigg] \Omega_C^3. \end{aligned} \tag{83}$$

We see that the right-hand side is $2\Theta_\alpha$. (See Eq. (71).) Q.E.D.

For the coordinate functions $\{l_i, \lambda_\kappa\}$, $(i = 1 \sim n;\ \kappa = 1 \sim 4h-4)$, we get

$$\begin{aligned} t_\sigma l_i &= \frac{2}{\pi} \mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta \sigma \Theta_i, \\ t_\sigma \lambda_\kappa &= \frac{2}{\pi} \mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta \sigma \Xi_\kappa, \end{aligned} \tag{84}$$

where

$$\{\Xi_\kappa(\vec{z})\} = \{\tfrac{1}{2} P\Psi(\vec{z}, \vec{u}_1^A, \vec{v}_1^A, \vec{x}_j) | \vec{x}_j = \vec{u}_j^A, \vec{v}_j^A, \vec{u}_j^B, \vec{v}_j^B, (j = 2, 3, \ldots, h)\}. \tag{85}$$

Equation (84) implies that the cotangent vectors, dl_i and $d\lambda_\kappa$, correspond to the superholomorphic $\frac{3}{2}$-differentials Θ_i and Ξ_κ, respectively. The Grassmann even-odd "parity" p_G of dl_i and $d\lambda_\kappa$ are opposite to those of Θ_i and Ξ_κ respectively, similarly to the case of t_σ and σ.

5. The Super Fenchel–Nielsen Deformation

In the theory of the ordinary Riemann surfaces, the Beltrami differential

$$t_\alpha = \frac{i}{\pi} (\mathrm{Im}\, z)^2 \bar{\phi}_\alpha, \tag{86}$$

where ϕ_α is a holomorphic quadratic differential, which corresponds to the θ-component of Θ_α, is identified with the tangent vector to the Fenchel–Nielsen (FN) deformation about the closed geodesic α [2]. One generalization of the FN deformation for the super Riemann surface would be that $t_\alpha(\tau_a)$ is the tangent vector to the "even (odd) FN deformation" if t_α and τ_a are derivative operators associated with the $(-1,\frac{1}{2})$-differentials[5],

$$t_\alpha \to \frac{i}{\pi} Y^2 \bar{\Theta}_\alpha,$$

$$\tau_a \to \frac{1}{\pi} Y^2 \bar{\Xi}_a. \tag{87}$$

The even FN deformation is essentially the same as the ordinary one, however, the geometrical description of odd FN deformation is obscure so far. Equation (87) gives the correspondence between the elements, t_α and τ_a, of the tangent space to the super Teichmüller space and the elements, $\frac{i}{\pi} Y^2 \bar{\Theta}_\alpha$ and $\frac{1}{\pi} Y^2 \bar{\Xi}_a^6$, of $SB(S\Gamma)$.

Then we find the linear reciprocity identity of the FN deformation;

$$t_A l_B + (-)^{AB} t_B l_A = 0, \tag{88}$$

where we have used the convention that $t_A = (t_\alpha, \tau_a)$, $l_A = (l_\alpha, \lambda_a)$ and the sign factor $(-)^{AB}$ is an abbreviation of $(-)^{PG[t_A]PG[l_B]} = (-)^{PG[l_A]PG[t_B]}$.

Proof. From (84) and (87), we get

$$t_\alpha l_\beta = \frac{2}{\pi^2} \mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta i Y^2 \bar{\Theta}_\alpha \Theta_\beta = -t_\beta l_\alpha, \tag{89}$$

$$t_\alpha \lambda_a = \frac{2}{\pi^2} \mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta i Y^2 \bar{\Theta}_\alpha \Xi_a = -\tau_a l_\alpha, \tag{90}$$

$$\tau_a \lambda_b = \frac{2}{\pi^2} \mathrm{Re}_s \int_{SH/S\Gamma} d^2z d^2\theta Y^2 \bar{\Xi}_a \Xi_b = \tau_b \lambda_a. \quad \text{Q.E.D.} \tag{91}$$

In order to evaluate the integrals we rewrite the above equations as,

$$t_\alpha l_\beta = \sum_{C \in \langle A \rangle \backslash S\Gamma} t(C^{-1}\vec{r}_A, C^{-1}\vec{a}_A) \log(B\vec{t}, \vec{t}, \vec{r}_B, \vec{a}_B), \tag{92}$$

$$t_\alpha \lambda_b = \sum_{C \in \langle A \rangle \backslash S\Gamma} t(C^{-1}\vec{r}_A, C^{-1}\vec{a}_A)[\vec{b}_1, \vec{b}_2, \vec{b}_3], \tag{93}$$

$$\tau_\alpha \lambda_b = \sum_{C \in \langle A \rangle \backslash S\Gamma} \tau(C^{-1}\vec{a}_1, C^{-1}\vec{a}_2, C^{-1}\vec{a}_3)[\vec{b}_1 \, \vec{b}_2, \vec{b}_3], \tag{94}$$

where we have introduced the derivative operators $t(\vec{s}_1, \vec{s}_2)$ and $\tau(\vec{s}_1, \vec{s}_2, \vec{s}_3)$, $\vec{s}_i \in \mathbb{R}_s$, acting on $(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4)$ and $[\vec{z}_1, \vec{z}_2, \vec{z}_3]$, $\vec{z}_i \in \mathbb{R}_s$, as (recall (70) and (78) for definitions)

$$t(\vec{s}_1, \vec{s}_2)(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4)$$
$$= \frac{(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4)}{\pi^2} \mathrm{Re}_s \int_{SH} d^2z d^2\theta i Y^2 \bar{\Theta}(\vec{z}; \vec{s}_1, \vec{s}_2) \Phi(\vec{z}; \vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4), \tag{95}$$

---

[5] We use a convention that Ξ_a is a superholomorphic $\frac{3}{2}$-differential corresponding to a cotangent vector $d\lambda_a = d[\vec{a}_1, \vec{a}_2, \vec{a}_3]$

[6] Both of them satisfy the assumption (37)

$$\tau(\vec{s}_1,\vec{s}_2,\vec{s}_3)(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)$$
$$=\frac{(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)}{\pi^2}\operatorname{Re}_s\int_{SH} d^2zd^2\theta Y^2\bar{\Psi}(\vec{z};\vec{s}_1,\vec{s}_2,\vec{s}_3)\Phi(\vec{z};\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4), \tag{96}$$

$$t(\vec{s}_1,\vec{s}_2)[\vec{z}_1,\vec{z}_2,\vec{z}_3]$$
$$=\frac{1}{\pi^2}\operatorname{Re}_s\int_{SH} d^2zd^2\theta iY^2\bar{\Theta}(\vec{z};\vec{s}_1,\vec{s}_2)\Psi(\vec{z};\vec{z}_1,\vec{z}_2,\vec{z}_3), \tag{97}$$

$$\tau(\vec{s}_1,\vec{s}_2,\vec{s}_3)[\vec{z}_1,\vec{z}_2,\vec{z}_3]$$
$$=\frac{1}{\pi^2}\operatorname{Re}_s\int_{SH} d^2zd^2\theta Y^2\bar{\Psi}(\vec{z};\vec{s}_1,\vec{s}_2,\vec{s}_3)\Psi(\vec{z};\vec{z}_1,\vec{z}_2,\vec{z}_3). \tag{98}$$

Note that these equations are invariant under the cyclic permutations of $\{\vec{s}_i\}$. Let $\widehat{s_is_j}$ be a geodesic with the endpoints $\vec{s}_i,\vec{s}_j$, separating SH into two regions. [The ordering of s_i and s_j defines the left and right regions.] We have the following lemma.

Lemma 1. *For* $z_i\in\mathbb{R}_s$ $(i=1,2,3,4)$, *we have*

$$t(\vec{s}_1,\vec{s}_2)(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)=\frac{(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)}{2}\sum_{i=1}^{4}\chi_L^{\widehat{s_1s_2}}(\vec{z}_i)[(\vec{z}_{\sigma(i)},\vec{s}_1,\vec{s}_2,\vec{z}_i)$$
$$-(\vec{z}_i,\vec{s}_1,\vec{s}_2,\vec{z}_{\sigma(i)})-(\vec{z}_{\tau(i)},\vec{s}_1,\vec{s}_2,\vec{z}_i)+(\vec{z}_i,\vec{s}_1,\vec{s}_2,\vec{z}_{\tau(i)})], \tag{99}$$

$$t(\vec{s}_1,\vec{s}_2)[\vec{z}_1,\vec{z}_2,\vec{z}_3]=\frac{1}{4}\sum_{i=1}^{3}\chi_L^{\widehat{s_1s_2}}(\vec{z}_i)\frac{z_{jk}}{z_{s_1s_2}(z_{12}z_{23}z_{31})^{1/2}}$$
$$\cdot\left[\frac{z_{s_1i}}{z_{ij}}\theta_{s_2ij}+\frac{z_{s_1i}}{z_{ki}}\theta_{s_2ki}+\frac{z_{s_2i}}{z_{ij}}\theta_{s_1ij}+\frac{z_{s_2i}}{z_{ki}}\theta_{s_1ki}\right], \tag{100}$$

$$\tau(\vec{s}_1,\vec{s}_2,\vec{s}_3)(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)$$
$$=\frac{(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)}{4}\left[\sum_{i=1}^{4}\chi_L^{\widehat{s_1s_3}}(\vec{z}_i)\frac{z_{is_1}z_{s_2s_3}}{(z_{s_1s_2}z_{s_2s_3}z_{s_3s_1})^{1/2}}\right.$$
$$\cdot\left\{\frac{\theta_{i\sigma(i)\tau(i)}}{z_{i\sigma(i)}z_{i\tau(i)}}+\frac{\theta_{s_1s_2\sigma(i)}}{z_{i\sigma(i)}z_{s_1s_2}}+\frac{\theta_{s_3s_1\sigma(i)}}{z_{i\sigma(i)}z_{s_3s_1}}-\frac{\theta_{s_1s_2\tau(i)}}{z_{i\tau(i)}z_{s_1s_2}}-\frac{\theta_{s_3s_1\tau(i)}}{z_{i\tau(i)}z_{s_3s_1}}\right\}$$
$$+\sum_{i=1}^{4}\chi_L^{\widehat{s_2s_3}}(\vec{z}_i)\frac{z_{is_2}z_{s_3s_1}}{(z_{s_1s_2}z_{s_2s_3}z_{s_3s_1})^{1/2}}$$
$$\left.\cdot\left\{\frac{\theta_{i\sigma(i)\tau(i)}}{z_{i\sigma(i)}z_{i\tau(i)}}+\frac{\theta_{s_2s_3\sigma(i)}}{z_{i\sigma(i)}z_{s_2s_3}}+\frac{\theta_{s_1s_2\sigma(i)}}{z_{i\sigma(i)}z_{s_1s_2}}-\frac{\theta_{s_2s_3\tau(i)}}{z_{i\tau(i)}z_{s_2s_3}}-\frac{\theta_{s_1s_2\tau(i)}}{z_{i\tau(i)}z_{s_1s_2}}\right\}\right], \tag{101}$$

$$\tau(\vec{s}_1,\vec{s}_2,\vec{s}_3)[\vec{z}_1,\vec{z}_2,\vec{z}_3]=\frac{1}{8}\sum_{i=1}^{3}\chi_L^{\widehat{s_1s_3}}(\vec{z}_i)\frac{z_{s_1i}z_{s_2s_3}z_{jk}}{(z_{12}z_{23}z_{31})^{1/2}(z_{s_1s_2}z_{s_2s_3}z_{s_3s_1})^{1/2}}$$
$$\cdot[6-(\vec{s}_2,\vec{z}_j,\vec{z}_i,\vec{s}_1)-(\vec{s}_1,\vec{z}_j,\vec{z}_i,\vec{s}_2)-(\vec{s}_3,\vec{z}_j,\vec{z}_i,\vec{s}_1)$$
$$-(\vec{s}_1,\vec{z}_j,\vec{z}_i,\vec{s}_3)-(\vec{s}_2,\vec{z}_k,\vec{z}_i,\vec{s}_1)-(\vec{s}_1,\vec{z}_k,\vec{z}_i,\vec{s}_2)$$
$$-(\vec{s}_3,\vec{z}_k,\vec{z}_i,\vec{s}_1)-(\vec{s}_1,\vec{z}_k,\vec{z}_i,\vec{s}_3)]$$

$$=\frac{1}{8}\sum_{i=1}^{3}\chi_L^{\widehat{s_2s_3}}(\vec{z}_i)\frac{z_{s_2i}z_{s_3s_1}z_{jk}}{(z_{12}z_{23}z_{31})^{1/2}(z_{s_1s_2}z_{s_2s_3}z_{s_3s_1})^{1/2}}$$
$$\cdot[6-(\vec{s}_3,\vec{z}_j,\vec{z}_i,\vec{s}_2)-(\vec{s}_2,\vec{z}_j,\vec{z}_i,\vec{s}_3)-(\vec{s}_1,\vec{z}_j,\vec{z}_i,\vec{s}_2)$$
$$-(\vec{s}_2,\vec{z}_j,\vec{z}_i,\vec{s}_1)-(\vec{s}_3,\vec{z}_k,\vec{z}_i,\vec{s}_2)-(\vec{s}_2,\vec{z}_k,\vec{z}_i,\vec{s}_3)$$
$$-(\vec{s}_1,\vec{z}_k,\vec{z}_i,\vec{s}_2)-(\vec{s}_2,\vec{z}_k,\vec{z}_i,\vec{s}_1)], \tag{102}$$

where

(i) *The suffices* $\{i,j,k\}$ *take* $\{1,2,3\}$ *in cyclic order.*

(ii) σ *and* τ *are the elements of the permutation group* S_4 *given by*

$$\sigma=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\quad \tau=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}. \tag{103}$$

(iii) $\chi_L^{\widehat{s_is_j}}(\vec{z}_k)$ *is the left characteristic function;*

$$\chi_L^{\widehat{s_is_j}}(\vec{z}_k)=\begin{cases}1 & \text{if } \vec{z}_k\in\{\text{the left region of } \widehat{s_is_j}\},\\ 0 & \text{otherwise.}\end{cases} \tag{104}$$

(iv) $\chi_L^{\widehat{s_is_j}}(\vec{z}_k)$ *can be replaced by* $-\chi_R^{\widehat{s_is_j}}(\vec{z}_k)$, *where* $\chi_R^{\widehat{s_is_j}}(\vec{z}_k)$ *is the right characteristic function.*

Proof. Equations (95) ~ (98) are evaluated cumbersomely but straightforwardly by the use of contour integrals, yielding (99) ~ (102), respectively. Q.E.D.

As an explicit example, let us consider a configuration of $\{\vec{s}_i,\vec{z}_j\}$ illustrated in Fig. 1. [Here and hereafter, in the figures the "super" real axis is represented as a circle in order to give an information of only the ordering of the points.] Then Eq. (100), for example, becomes

$$t(\vec{s}_1,\vec{s}_2)[\vec{z}_1,\vec{z}_2,\vec{z}_3]$$
$$=\frac{z_{23}}{4z_{s_1s_2}(z_{12}z_{23}z_{31})^{1/2}}\left[\frac{z_{s_11}}{z_{12}}\theta_{s_212}+\frac{z_{s_11}}{z_{31}}\theta_{s_231}+\frac{z_{s_21}}{z_{12}}\theta_{s_112}+\frac{z_{s_21}}{z_{31}}\theta_{s_131}\right]$$
$$=\frac{\mathrm{sign}\left[\dfrac{z_{23}z_{s_11}}{z_{s_1s_2}z_{12}}\right]}{4}\{(\vec{z}_3,\vec{z}_1,\vec{z}_2,\vec{s}_1)(\vec{s}_1,\vec{z}_2,\vec{z}_1,\vec{s}_2)(\vec{s}_1,\vec{z}_1,\vec{z}_3,\vec{s}_2)\}^{1/2}[\vec{s}_2,\vec{z}_1,\vec{z}_2]$$
$$+\frac{\mathrm{sign}\left[\dfrac{z_{23}z_{s_11}}{z_{s_1s_2}z_{13}}\right]}{4}\{(-)(\vec{z}_2,\vec{z}_1,\vec{z}_3,\vec{s}_1)(\vec{s}_1,\vec{z}_3,\vec{z}_1,\vec{s}_2)(\vec{s}_1,\vec{z}_1,\vec{z}_2,\vec{s}_2)\}^{1/2}[\vec{s}_2,\vec{z}_1,\vec{z}_3]$$
$$+\frac{\mathrm{sign}\left[\dfrac{z_{23}z_{s_21}}{z_{s_1s_2}z_{21}}\right]}{4}\{(-)(\vec{z}_3,\vec{z}_1,\vec{z}_2,\vec{s}_2)(\vec{s}_1,\vec{z}_1,\vec{z}_2,\vec{s}_2)(\vec{s}_1,\vec{z}_3,\vec{z}_1,\vec{s}_2)\}^{1/2}[\vec{s}_1,\vec{z}_2,\vec{z}_1]$$
$$+\frac{\mathrm{sign}\left[\dfrac{z_{23}z_{s_21}}{z_{s_1s_2}z_{31}}\right]}{4}\{(\vec{z}_2,\vec{z}_1,\vec{z}_3,\vec{s}_2)(\vec{s}_1,\vec{z}_1,\vec{z}_3,\vec{s}_2)(\vec{s}_1,\vec{z}_2,\vec{z}_1,\vec{s}_2)\}^{1/2}[\vec{s}_1,\vec{z}_3,\vec{z}_1]. \tag{105}$$

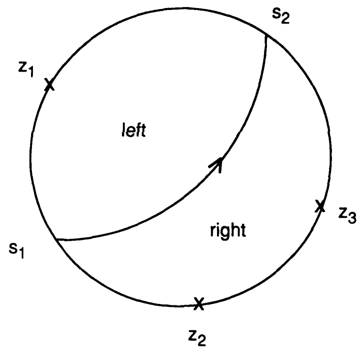

Fig. 1. A configuration of five points on $\mathbb{R}_s$ which is represented schematically as a circle. The left and right regions are for the geodesic $\widehat{s_1 s_2}$

In general, the right-hand sides of Eqs. (99) $\sim$ (102) are functions of the $SPL(2,\mathbb{R})$-invariants, $(*,*,*,*)$ and $[*,*,*]$, so that we can compute derivatives of all orders. It is now clear that Lemma 1 gives the right-hand sides of Eqs. (92) $\sim$ (94) explicitly. In particular, the derivative of the length function reduces to

$$\begin{aligned} t_\alpha l_\beta = \sum_{D\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} & [\chi_L^{\widehat{r_D a_D}}(\vec{r}_B)\{(\vec{a}_B,\vec{r}_D,\vec{a}_D,\vec{r}_B)-(\vec{r}_B,\vec{r}_D,\vec{a}_D,\vec{a}_B)\} \\ & +\chi_L^{\widehat{r_D a_D}}(\vec{a}_B)\{(\vec{r}_B,\vec{r}_D,\vec{a}_D,\vec{a}_B)-(\vec{a}_B,\vec{r}_D,\vec{a}_D,\vec{r}_B)\}], \end{aligned} \tag{106}$$

where $\vec{r}_D$ ($\vec{a}_D$) denote the repelling (attracting) fixed point of $D^{-1}AD$.

Proof. Equation (92) is evaluated as follows:

$$\begin{aligned} t_\alpha l_\beta &= \sum_{C\in\langle A\rangle\backslash S\Gamma} t(C^{-1}\vec{r}_A, C^{-1}\vec{a}_A)\log(B\vec{t},\vec{t},\vec{r}_B,\vec{a}_B) \\ &= \sum_{D\in\langle A\rangle\backslash S\Gamma/\langle B\rangle}\sum_{n=-\infty}^{\infty} t(B^{-n}D^{-1}\vec{r}_A, B^{-n}D^{-1}\vec{a}_A)\log(B\vec{t},\vec{t},\vec{r}_B,\vec{a}_B). \end{aligned} \tag{107}$$

Due to Lemma 1 we find

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} t(B^{-n}\vec{r}_D, B^{-n}\vec{a}_D)\log(B\vec{t},\vec{t},\vec{r}_B,\vec{a}_B) \\ &= \sum_{n=-\infty}^{\infty} t(\vec{r}_D,\vec{a}_D)\log(B^{n+1}\vec{t},B^n\vec{t},\vec{r}_B,\vec{a}_B) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\Big[\chi_L^{\widehat{r_D a_D}}(B^{n+1}\vec{t})\{(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^{n+1}\vec{t})-(B^{n+1}\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B) \\ &\quad -(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^{n+1}\vec{t})-(B^{n+1}\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B)\} \\ &\quad +\chi_L^{\widehat{r_D a_D}}(B^n\vec{t})\{(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^n\vec{t})-(B^n\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B) \\ &\quad -(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^n\vec{t})+(B^n\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)\} \\ &\quad +\chi_L^{\widehat{r_D a_D}}(\vec{r}_B)\{(B^{n+1}\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)-(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^{n+1}\vec{t}) \end{aligned}$$

$$-(B^n\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)+(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^n\vec{t})\}$$
$$+\chi_L^{\widehat{\vec{r}_D\vec{a}_D}}(\vec{a}_B)\{(B^n\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)-(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^n\vec{t})$$
$$-(B^{n+1}\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B)-(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^{n+1}\vec{t})\}]. \tag{108}$$

The summation with respect to n yields

$$\sum_{n=-\infty}^{\infty} t(\vec{r}_D,\vec{a}_D)\log(B^{n+1}\vec{t},B^n\vec{t},\vec{r}_B,\vec{a}_B)$$
$$=\frac{1}{2}\lim_{N\to\infty}[\chi_L^{\widehat{\vec{r}_D\vec{a}_D}}(B^N\vec{t})\{(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^N\vec{t})-(B^N\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)$$
$$-(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^N\vec{t})-(B^N\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B)\}$$
$$+\chi_L^{\widehat{\vec{r}_D\vec{a}_D}}(B^{-N}\vec{t})\{(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^{-N}\vec{t})-(B^{-N}\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B)$$
$$-(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^{-N}\vec{t})+(B^{-N}\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)\}$$
$$+\chi_L^{\widehat{\vec{r}_D\vec{a}_D}}(\vec{r}_B)\{(B^N\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)-(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^N\vec{t})$$
$$-(B^{-N}\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)+(\vec{r}_B,\vec{r}_D,\vec{a}_D,B^{-N}\vec{t})\}$$
$$+\chi_L^{\widehat{\vec{r}_D\vec{a}_D}}(\vec{a}_B)\{(B^{-N}\vec{t},\vec{r}_D,\vec{a}_D,\vec{r}_B)-(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^{-N}\vec{t})$$
$$-(B^N\vec{t},\vec{r}_D,\vec{a}_D,\vec{a}_B)-(\vec{a}_B,\vec{r}_D,\vec{a}_D,B^N\vec{t})\}]. \tag{109}$$

Due to the equation, $\lim_N[B^N\vec{t}]=\begin{cases}\vec{a}_B & \text{for } N\to\infty,\\ \vec{r}_B & \text{for } N\to-\infty\end{cases}$, we get (106). Q.E.D.

Now we proceed to the second derivatives of invariant functions. We give a theorem here, whose proof will be given in the next section.

Theorem. *Let t_A, t_B and t_C be Fenchel–Nielsen tangent vectors corresponding to the invariant functions l_A, l_B and l_C, respectively. Then*

$$t_A t_B l_C+(-)^{A(B+C)}t_B t_C l_A+(-)^{C(A+B)}t_C t_A l_B=0. \tag{110}$$

6. Proof of the Theorem

We shall prove the theorem in the previous section. We may show the following equations written in components,

$$\text{(a)}\quad t_\alpha t_\beta l_\gamma+t_\beta t_\gamma l_\alpha+t_\gamma t_\alpha l_\beta=0, \tag{111}$$
$$\text{(b)}\quad t_\alpha t_\beta \lambda_c+t_\beta \tau_c l_\alpha+\tau_c t_\alpha l_\beta=0, \tag{112}$$
$$\text{(c)}\quad t_\alpha \tau_b \lambda_c+\tau_b \tau_c l_\alpha-\tau_c t_\alpha \lambda_b=0, \tag{113}$$
$$\text{(d)}\quad \tau_a \tau_b \lambda_c+\tau_b \tau_c \lambda_a+\tau_c \tau_a \lambda_b=0. \tag{114}$$

Our method is based on a direct computation with Lemma 1.

Proof of case (a). Let $l_\alpha, l_\beta, l_\gamma$ be given by

$$l_\alpha=\log(A\vec{t},\vec{t},\vec{a}_1,\vec{a}_2),\quad l_\beta=\log(B\vec{t},\vec{t},\vec{b}_1,\vec{b}_2),\quad l_\gamma=\log(C\vec{t},\vec{t},\vec{c}_1,\vec{c}_2), \tag{115}$$

and we introduce a function F for three pairs of points $\vec{p}_i, \vec{q}_j, \vec{r}_k \in \mathbb{R}_s$ $(i,j,k=1,2)$,

$$F(\vec{p}_i|\vec{q}_j|\vec{r}_k) = t(\vec{p}_1,\vec{p}_2)\chi_L^{\widehat{q_1q_2}}(\vec{r}_1)\{(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)-(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)\} + t(\vec{p}_1,\vec{p}_2)\chi_L^{\widehat{q_1q_2}}(\vec{r}_2)\{(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)-(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)\}. \tag{116}$$

From Eq. (106) we obtain

$$t_\gamma t_\alpha l_\beta = \sum_{X\in\langle C\rangle\backslash S\Gamma} \sum_{Y\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} F(X^{-1}\vec{c}_i|Y^{-1}\vec{a}_j|\vec{b}_k), \tag{117}$$

$$t_\beta t_\gamma l_\alpha = \sum_{P\in\langle B\rangle\backslash S\Gamma} \sum_{Q\in\langle C\rangle\backslash S\Gamma/\langle A\rangle} F(P^{-1}\vec{b}_i|Q^{-1}\vec{c}_j|\vec{a}_k), \tag{118}$$

$$t_\alpha t_\beta l_\gamma = \sum_{R\in\langle A\rangle\backslash S\Gamma} \sum_{S\in\langle B\rangle\backslash S\Gamma/\langle C\rangle} F(R^{-1}\vec{a}_i|S^{-1}\vec{b}_j|\vec{c}_k), \tag{119}$$

Due to $SPL(2,\mathbb{R})$-invariance, Eq. (118) is rewritten by

$$\begin{aligned} t_\beta t_\gamma l_\alpha &= \sum_{P\in\langle B\rangle\backslash S\Gamma} \sum_{Q\in\langle C\rangle\backslash S\Gamma/\langle A\rangle} F(\vec{b}_i|PQ^{-1}\vec{c}_j|P\vec{a}_k) \\ &= \sum_{P\in S\Gamma/\langle B\rangle} \sum_{Q\in\langle C\rangle\backslash S\Gamma/\langle A\rangle} F(\vec{b}_i|(QP)^{-1}\vec{c}_j|P^{-1}\vec{a}_k) \\ &= \sum_{Y\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} \sum_{Q\in\langle C\rangle\backslash S\Gamma/\langle A\rangle} \sum_{n=-\infty}^{\infty} F(\vec{b}_i|(QA^nY)^{-1}\vec{c}_j|Y^{-1}\vec{a}_k) \\ &= \sum_{Y\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} \sum_{X\in\langle C\rangle\backslash S\Gamma} F(\vec{b}_i|X^{-1}\vec{c}_j|Y^{-1}\vec{a}_k). \end{aligned} \tag{120}$$

Similarly we find

$$t_\alpha t_\beta l_\gamma = \sum_{X\in\langle C\rangle\backslash S\Gamma} \sum_{Y\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} F(Y^{-1}\vec{a}_i|\vec{b}_j|X^{-1}\vec{c}_k). \tag{121}$$

Thus Eq. (111) becomes

$$\sum_{X\in\langle C\rangle\backslash S\Gamma} \sum_{Y\in\langle A\rangle\backslash S\Gamma/\langle B\rangle} \{F(Y^{-1}\vec{a}_i|\vec{b}_j|X^{-1}\vec{c}_k) + F(\vec{b}_i|X^{-1}\vec{c}_j|Y^{-1}\vec{a}_k) + F(X^{-1}\vec{c}_i|Y^{-1}\vec{a}_j|\vec{b}_k)\} = 0. \tag{122}$$

This equation comes from the following lemma:

Lemma 2. *F satisfies a cyclic identity,*

$$F(\vec{p}_i|\vec{q}_j|\vec{r}_k) + F(\vec{r}_k|\vec{p}_i|\vec{q}_j) + F(\vec{q}_j|\vec{r}_k|\vec{p}_i) = 0 \quad \text{for} \quad \vec{p}_i,\vec{q}_j,\vec{r}_k\in\mathbb{R}_s, (i,j,k=1,2). \tag{123}$$

Proof. The configurations of the three pairs of points on $\mathbb{R}_s$ are classified according to the configurations of the geodesics, $\widehat{p_1p_2}$, $\widehat{q_1q_2}$ and $\widehat{r_1r_2}$;

(I) A geodesic does not intersect the remaining geodesics.
(II) A geodesic intersects the remaining separated geodesics.
(III) Three geodesics intersect each other.

We begin with case (I). Due to the symmetry we may assume the configurations in Fig. 2. Then Lemma 1 readily leads to

$$F(\vec{p}_i|\vec{q}_j|\vec{r}_k) = F(\vec{r}_k|\vec{p}_i|\vec{q}_j) = F(\vec{q}_j|\vec{r}_k|\vec{p}_i) = 0. \tag{124}$$

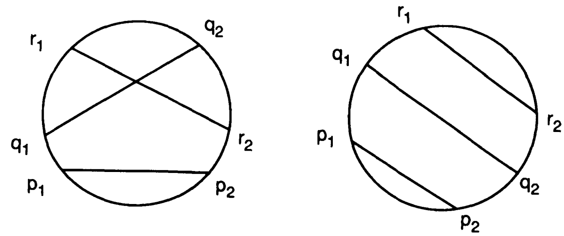

Fig. 2. Two kinds of representative configurations of three pairs of points where one of the geodesics, represented schematically by lines, does not intersect the others

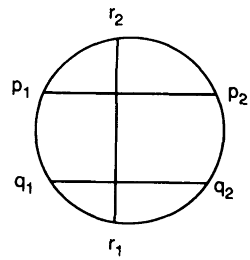

Fig. 3. A representative configuration of three pairs of points where one of the geodesics intersects the remaining separated geodesics

Thus Eq. (123) holds in case (I).

For case (II) we get from Fig. 3,

$$
\begin{aligned}
F(\vec{r}_k|\vec{p}_i|\vec{q}_j) &= 0,\\
F(\vec{p}_i|\vec{q}_j|\vec{r}_k) &= t(\vec{p}_1,\vec{p}_2)\{(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)-(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)\}\\
&= (\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)\{(\vec{q}_1,\vec{p}_1,\vec{p}_2,\vec{r}_2)-(\vec{r}_2,\vec{p}_1,\vec{p}_2,\vec{q}_1)\\
&\quad -(\vec{r}_1,\vec{p}_1,\vec{p}_2,\vec{r}_2)+(\vec{r}_2,\vec{p}_1,\vec{p}_2,\vec{r}_1)\}\\
&\quad -(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)\{(\vec{q}_2,\vec{p}_1,\vec{p}_2,\vec{r}_2)-(\vec{r}_2,\vec{p}_1,\vec{p}_2,\vec{q}_2)\\
&\quad -(\vec{r}_1,\vec{p}_1,\vec{p}_2,\vec{r}_2)+(\vec{r}_2,\vec{p}_1,\vec{p}_2,\vec{r}_1)\}, \qquad (125)\\
F(\vec{q}_j|\vec{r}_k|\vec{p}_i) &= t(\vec{q}_1,\vec{q}_2)\{(\vec{p}_2,\vec{r}_1,\vec{r}_2,\vec{p}_1)-(\vec{p}_1,\vec{r}_1,\vec{r}_2,\vec{p}_2)\}\\
&= -(\vec{p}_2,\vec{r}_1,\vec{r}_2,\vec{p}_1)\{(\vec{p}_1,\vec{q}_1,\vec{q}_2,\vec{r}_1)-(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{p}_1)\\
&\quad -(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)+(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)\}\\
&\quad +(\vec{p}_1,\vec{r}_1,\vec{r}_2,\vec{p}_2)\{(\vec{p}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)-(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{p}_2)\\
&\quad -(\vec{r}_2,\vec{q}_1,\vec{q}_2,\vec{r}_1)+(\vec{r}_1,\vec{q}_1,\vec{q}_2,\vec{r}_2)\}.
\end{aligned}
$$

Due to some identities of the (super) cross ratios,

$$
\begin{aligned}
&(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)=(\vec{z}_2,\vec{z}_1,\vec{z}_3,\vec{z}_3)=(\vec{z}_4,\vec{z}_3,\vec{z}_2,\vec{z}_1),\\
&(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4)(\vec{z}_2,\vec{z}_5,\vec{z}_6,\vec{z}_4)=(\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_6)(\vec{z}_1,\vec{z}_5,\vec{z}_6,\vec{z}_4), \qquad (126)
\end{aligned}
$$

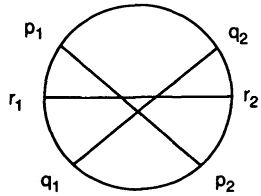

Fig. 4. A representative configuration of three pairs of points where three geodesics intersect each other

we find

$$F(\vec{p}_i|\vec{q}_j|\vec{r}_k) + F(\vec{q}_j|\vec{r}_k|\vec{p}_i) = 0. \tag{127}$$

The case (II) has been proved.

Case (III) reduces to case (II). In fact the direct calculations of Fig. 4 yields

$$\begin{aligned} F(\vec{p}_i|\vec{q}_j|\vec{r}_k) &= t(\vec{p}_1,\vec{p}_2)\chi_L^{\widehat{q_1q_2}}(\vec{r}_1)* = \{\chi_L^{\widehat{p_1p_2}}(\vec{q}_2) + \chi_L^{\widehat{p_1p_2}}(\vec{r}_2)\}\chi_L^{\widehat{q_1q_2}}(\vec{r}_1)*, \\ F(\vec{q}_j|\vec{r}_k|\vec{p}_i) &= t(\vec{q}_1,\vec{q}_2)\chi_L^{\widehat{r_1r_2}}(\vec{p}_1)* = \{\chi_L^{\widehat{q_1q_2}}(\vec{r}_1) + \chi_L^{\widehat{q_1q_2}}(\vec{p}_1)\}\chi_L^{\widehat{r_1r_2}}(\vec{p}_1)*, \\ F(\vec{r}_k|\vec{p}_i|\vec{q}_j) &= t(\vec{r}_1,\vec{r}_2)\chi_L^{\widehat{p_1p_2}}(\vec{q}_2)* = \{\chi_L^{\widehat{r_1r_2}}(\vec{p}_1) + \chi_L^{\widehat{r_1r_2}}(\vec{q}_2)\}\chi_L^{\widehat{p_1p_2}}(\vec{q}_2)*, \end{aligned} \tag{128}$$

where we have written only the relevant characteristic functions explicitly to simplify the expressions. Then the cyclic sum of F's becomes

$$\sum_{\text{cyclic}} F = \sum_{\text{cyclic}} F^{(1)} + \sum_{\text{cyclic}} F^{(2)} + \sum_{\text{cyclic}} F^{(3)}, \tag{129}$$

$$F^{(1)}(\vec{p}_i|\vec{q}_j|\vec{r}_k) = 0, \tag{130}$$

$$F^{(1)}(\vec{r}_k|\vec{p}_i|\vec{q}_j) = \chi_L^{\widehat{r_1r_2}}(\vec{p}_1)\chi_L^{\widehat{p_1p_2}}(\vec{q}_2)*, \tag{131}$$

$$F^{(1)}(\vec{q}_j|\vec{r}_k|\vec{p}_i) = \{\chi_L^{\widehat{q_1q_2}}(\vec{r}_1) + \chi_L^{\widehat{q_1q_2}}(\vec{p}_2) + \chi_L^{\widehat{q_1q_2}}(\vec{r}_2)\}\chi_L^{\widehat{r_1r_2}}(\vec{p}_1)*, \tag{132}$$

$$F^{(2)}(\vec{p}_i|\vec{q}_j|\vec{r}_k) = \{\chi_L^{\widehat{p_1p_2}}(\vec{r}_1) + \chi_L^{\widehat{p_1p_2}}(\vec{r}_2) + \chi_L^{\widehat{p_1p_2}}(\vec{q}_2)\}\chi_L^{\widehat{q_1q_2}}(\vec{r}_1)*, \tag{133}$$

$$F^{(2)}(\vec{r}_k|\vec{p}_i|\vec{q}_j) = \chi_L^{\widehat{r_1r_2}}(\vec{q}_2)\chi_L^{\widehat{p_1p_2}}(\vec{q}_2)*, \tag{134}$$

$$F^{(2)}(\vec{q}_j|\vec{r}_k|\vec{p}_i) = 0, \tag{135}$$

$$F^{(3)}(\vec{p}_i|\vec{q}_j|\vec{r}_k) = \chi_R^{\widehat{p_1p_2}}(\vec{r}_1)\chi_L^{\widehat{q_1q_2}}(\vec{r}_1)*, \tag{136}$$

$$F^{(3)}(\vec{r}_k|\vec{p}_i|\vec{q}_j) = 0, \tag{137}$$

$$F^{(3)}(\vec{q}_j|\vec{r}_k|\vec{p}_i) = \chi_R^{\widehat{q_1q_2}}(\vec{r}_2)\chi_L^{\widehat{r_1r_2}}(\vec{q}_1)*, \tag{138}$$

where use has been made of the identity,

$$\{\chi_L^{\widehat{z_1z_2}}(\vec{z}_3) + \chi_R^{\widehat{z_1z_2}}(\vec{z}_3)\}\chi_L^{\widehat{z_*z_*}}(*) = 0. \tag{139}$$

Each cyclic sum, $\sum_{\text{cyclic}} F^{(i)}$ $(i = 1, 2, 3)$, corresponds to a configuration in case (II) and it vanishes. Thus Eq. (123) holds in case (III). Lemma 2 has been proved. Q.E.D.

Then case (a) has been proved. Note that the decomposition of $\sum_{\text{cyclic}} F$ in (129) is represented diagrammatically as

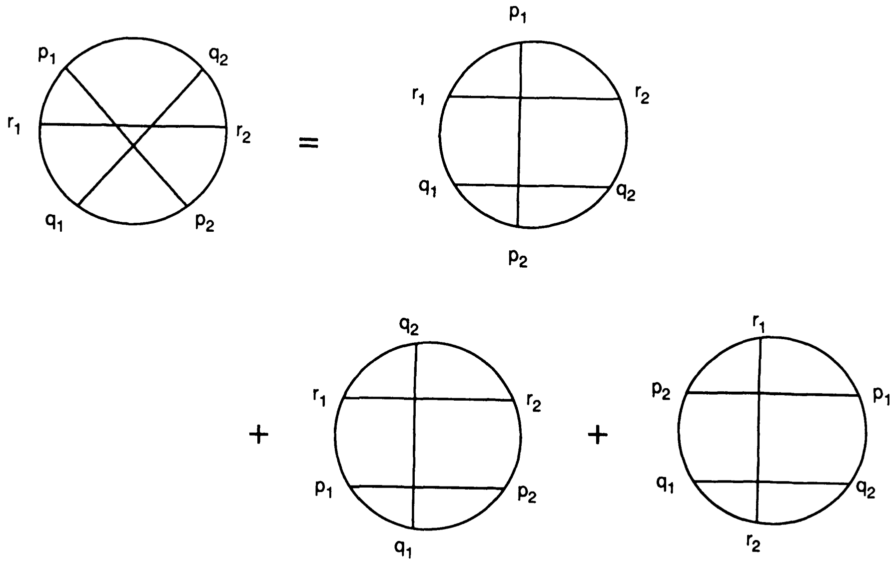

Fig. 5. A diagrammatic representation of the decomposition law in Eq. (129)

Proof of case (b): For $\lambda_c = [\vec{c}_1, \vec{c}_2, \vec{c}_3]$, we have

$$t_\alpha t_\beta \lambda_c = \sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in \langle B \rangle \backslash S\Gamma} t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2) t(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2)[\vec{c}_1, \vec{c}_2, \vec{c}_3]. \quad (140)$$

The linear reciprocity (90) leads to

$$\begin{aligned} t_\beta \tau_c l_\alpha &= -t_\beta t_\alpha \lambda_c \\ &= -\sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in \langle B \rangle \backslash S\Gamma} t(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2) t(X^{-1}\vec{a}_1, \vec{x}^{-1}\vec{a}_2)[\vec{c}_1, \vec{c}_2, \vec{c}_3]. \quad (141) \end{aligned}$$

We define a $SPL(2,\mathbb{R})$-invariant function for $\vec{s}_i, \vec{p}_j, \vec{q}_k \in \mathbb{R}_s$, $(i = 1,2,3;\ j,k = 1,2)$,

$$\begin{aligned} G(\vec{s}_i | \vec{p}_j | \vec{q}_k) = {} & \tau(\vec{s}_1, \vec{s}_2, \vec{s}_3) \chi_L^{\widehat{p_1 p_2}}(\vec{q}_1)\{(\vec{q}_2, \vec{p}_1, \vec{p}_2, \vec{q}_1) - (\vec{q}_1, \vec{p}_1, \vec{p}_2, \vec{q}_2)\} \\ & + \tau(\vec{s}_1, \vec{s}_2, \vec{s}_3) \chi_L^{\widehat{p_1 p_2}}(\vec{q}_2)\{(\vec{q}_1, \vec{p}_1, \vec{p}_2, \vec{q}_2) - (\vec{q}_2, \vec{p}_1, \vec{p}_2, \vec{q}_1)\}. \quad (142) \end{aligned}$$

Similarly to Eq. (120) we get

$$\begin{aligned} \tau_c t_\alpha l_\beta &= \sum_{P \in S\Gamma} \sum_{Q \in \langle A \rangle \backslash S\Gamma / \langle B \rangle} G(P^{-1}\vec{c}_i | Q^{-1}\vec{a}_j | \vec{b}_k) \\ &= \sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in \langle B \rangle \backslash S\Gamma} G(\vec{c}_i | X^{-1}\vec{a}_j | Y^{-1}\vec{b}_k). \quad (143) \end{aligned}$$

Then Eq. (112) holds due to the following Lemma:

Lemma 3. *For any seven points,* $\vec{s}_i, \vec{p}_j$ *and* $\vec{q}_k$ $(i = 1,2,3;\ j,k = 1,2)$, *on* $\mathbb{R}_s$, *an equation holds,*

$$t(\vec{p}_1, \vec{p}_2) t(\vec{q}_1, \vec{q}_2)[\vec{s}_1, \vec{s}_2, \vec{s}_3] - t(\vec{q}_1, \vec{q}_2) t(\vec{p}_1, \vec{p}_2)[\vec{s}_1, \vec{s}_2, \vec{s}_3] + G(\vec{s}_i | \vec{p}_j | \vec{q}_k) = 0. \quad (144)$$

Proof. The configurations of $\vec{s}_i, \vec{p}_j$, and $\vec{q}_k$ on $\mathbb{R}_s$ are classified into two graphs;

(I) $G=0$. ($\widehat{p_1p_2}$ does not intersect $\widehat{q_1q_2}$.)
(II) $G\neq 0$.

By the symmetry, the configurations of case(I) are represented diagrammatically in Fig. 6. [We have omitted the configurations where $t(\vec{p}_1,\vec{p}_2)t(\vec{q}_1,\vec{q}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3] = t(\vec{q}_1,\vec{q}_2)t(\vec{p}_1,\vec{p}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3]=0$.] For the left diagram in Fig. 6 we have

$$
\begin{aligned}
&t(\vec{p}_1,\vec{p}_2)t(\vec{q}_1,\vec{q}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3]\\
&= \chi_L^{\widehat{p_1p_2}}(\vec{s}_1)\chi_R^{\widehat{q_1q_2}}(\vec{s}_3)* = \frac{z_{s_1s_2}z_{q_1s_3}}{8z_{q_1q_2}z_{s_3s_1}}\left(\frac{z_{q_2s_1}z_{s_1s_3}z_{s_3q_2}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}[2t(\vec{p}_1,\vec{p}_2)[\vec{q}_2,\vec{s}_1,\vec{s}_3]\\
&\quad+\{t(\vec{p}_1,\vec{p}_2)\log[(\vec{q}_1,\vec{s}_1,\vec{s}_3,\vec{s}_2)(\vec{q}_1,\vec{s}_1,\vec{s}_3,\vec{q}_2)]\}[\vec{q}_2,\vec{s}_1,\vec{s}_3]]\\
&\quad-\frac{z_{s_1s_2}z_{q_1s_3}}{8z_{q_1q_2}z_{s_2s_3}}\left(\frac{z_{q_2s_2}z_{s_2s_3}z_{s_3q_2}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}\{t(\vec{p}_1,\vec{p}_2)\log(\vec{q}_1,\vec{s}_1,\vec{s}_3,\vec{s}_2)\}[\vec{q}_2,\vec{s}_2,\vec{s}_3]\\
&\quad-\frac{z_{s_1s_2}z_{q_2s_3}}{8z_{q_1q_2}z_{s_3s_1}}\left(\frac{z_{q_1s_3}z_{s_3s_1}z_{s_1q_1}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}[2t(\vec{p}_1,\vec{p}_2)[\vec{q}_1,\vec{s}_3,\vec{s}_1]\\
&\quad+\{t(\vec{p}_1,\vec{p}_2)\log[(\vec{q}_2,\vec{s}_1,\vec{s}_3,\vec{s}_2)(\vec{q}_2,\vec{s}_1,\vec{s}_3,\vec{q}_1)]\}[\vec{q}_1,\vec{s}_3,\vec{s}_1]]\\
&\quad+\frac{z_{s_1s_2}z_{q_2s_3}}{8z_{q_1q_2}z_{s_2s_3}}\left(\frac{z_{q_1s_3}z_{s_3s_2}z_{s_2q_1}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}\{t(\vec{p}_1,\vec{p}_2)\log(\vec{q}_2,\vec{s}_1,\vec{s}_3,\vec{s}_2)\}[\vec{q}_1,\vec{s}_3,\vec{s}_2],
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
&t(\vec{q}_1,\vec{q}_2)t(\vec{p}_1,\vec{p}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3]\\
&= \chi_R^{\widehat{q_1q_2}}(\vec{s}_3)\chi_L^{\widehat{p_1p_2}}(\vec{s}_1)*\\
&= \frac{z_{s_2s_3}z_{p_1s_1}}{8z_{p_1p_2}z_{s_1s_2}}\left(\frac{z_{p_2s_1}z_{s_1s_2}z_{s_2p_2}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}\{t(\vec{q}_1,\vec{q}_2)\log(\vec{p}_1,\vec{s}_3,\vec{s}_1,\vec{s}_2)\}[\vec{p}_2,\vec{s}_1,\vec{s}_2]\\
&\quad-\frac{z_{s_2s_3}z_{p_1s_1}}{8z_{p_1p_2}z_{s_3s_1}}\left(\frac{z_{p_2s_1}z_{s_1s_3}z_{s_3p_2}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}[2t(\vec{q}_1,\vec{q}_2)[\vec{p}_2,\vec{s}_1,\vec{s}_3]\\
&\quad+\{t(\vec{q}_1,\vec{q}_2)\log[(\vec{p}_1,\vec{s}_3,\vec{s}_1,\vec{s}_2)(\vec{p}_1,\vec{s}_3,\vec{s}_1,\vec{p}_2)]\}[\vec{p}_2,\vec{s}_1,\vec{s}_3]]\\
&\quad-\frac{z_{s_2s_3}z_{p_2s_1}}{8z_{p_1p_2}z_{s_1s_2}}\left(\frac{z_{p_1s_2}z_{s_2s_1}z_{s_1p_1}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}\{t(\vec{q}_1,\vec{q}_2)\log(\vec{p}_2,\vec{s}_3,\vec{s}_1,\vec{s}_2)\}[\vec{p}_1,\vec{s}_2,\vec{s}_1]\\
&\quad+\frac{z_{s_2s_3}z_{p_2s_1}}{8z_{p_1p_2}z_{s_3s_1}}\left(\frac{z_{p_1s_3}z_{s_3s_1}z_{s_1p_1}}{z_{s_1s_2}z_{s_2s_3}z_{s_3s_1}}\right)^{1/2}[2t(\vec{q}_1,\vec{q}_2)[\vec{p}_1,\vec{s}_3,\vec{s}_1]\\
&\quad+\{t(\vec{q}_1,\vec{q}_2)\log[(\vec{p}_2,\vec{s}_3,\vec{s}_1,\vec{s}_2)(\vec{p}_2,\vec{s}_3,\vec{s}_1,\vec{p}_1)]\}[\vec{p}_1,\vec{s}_3,\vec{s}_1]].
\end{aligned}
\tag{146}
$$

After tedious calculations, we obtain

$$t(\vec{p}_1,\vec{p}_2)t(\vec{q}_1,\vec{q}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3]-t(\vec{q}_1,\vec{q}_2)t(\vec{p}_1,\vec{p}_2)[\vec{s}_1,\vec{s}_2,\vec{s}_3]=0. \tag{147}$$

The above Eq. (147) holds also for the other diagram in Fig. 6. Hence in case(I), (144) has been shown to hold.

As for case(II), we have essentially four diagrams to consider (see Fig. 7), and explicit calculation for the first two diagrams in Fig. 7 yields the same equations

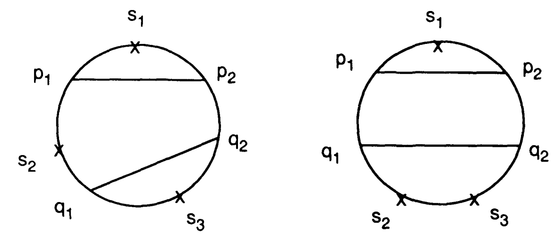

Fig. 6. Two kinds of representative configurations of seven points p_i, q_i and s_j $(i = 1, 2; j = 1, 2, 3)$ where $\widehat{p_1 p_2}$ does not intersect $\widehat{q_1 q_2}$

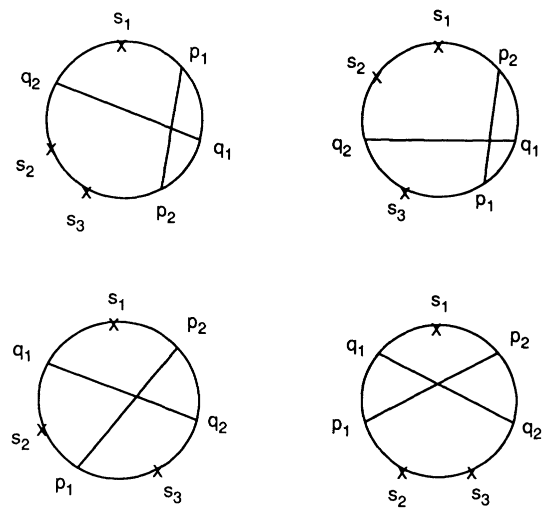

Fig. 7. Four kinds of representative configurations of seven points p_i, q_i and s_j $(i = 1, 2; j = 1, 2, 3)$ where $\widehat{p_1 p_2}$ intersects $\widehat{q_1 q_2}$

as

$$t(\vec{p}_1, \vec{p}_2)[\vec{s}_1, \vec{s}_2, \vec{s}_3] = 0,$$

$$t(\vec{p}_1, \vec{p}_2)t(\vec{q}_1, \vec{q}_2)[\vec{s}_1, \vec{s}_2, \vec{s}_3] + G(\vec{s}_i | \vec{p}_j | \vec{q}_k) = 0. \tag{148}$$

The remaining diagrams are decomposed into the sums of known diagrams:

Equation (144) holds for each diagram on the right-hand side of the above diagrammatic equations, and hence Eq. (144) holds also in case(II). The Lemma 3 has been proved. Q.E.D.

From Eqs. (140), (141) and (143), we find

$$\begin{aligned} & t_\alpha t_\beta \lambda_c + t_\beta \tau_c l_\alpha + \tau_c t_\alpha l_\beta \\ & = \sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in \langle B \rangle \backslash S\Gamma} \{ t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2) t(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2)[\vec{c}_1, \vec{c}_2, \vec{c}_3] \\ & \quad - t(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2) t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2)[\vec{c}_1, \vec{c}_2, \vec{c}_3] + G(\vec{c}_i | X^{-1}\vec{a}_j | Y^{-1}\vec{b}_k) \}. \end{aligned} \tag{149}$$

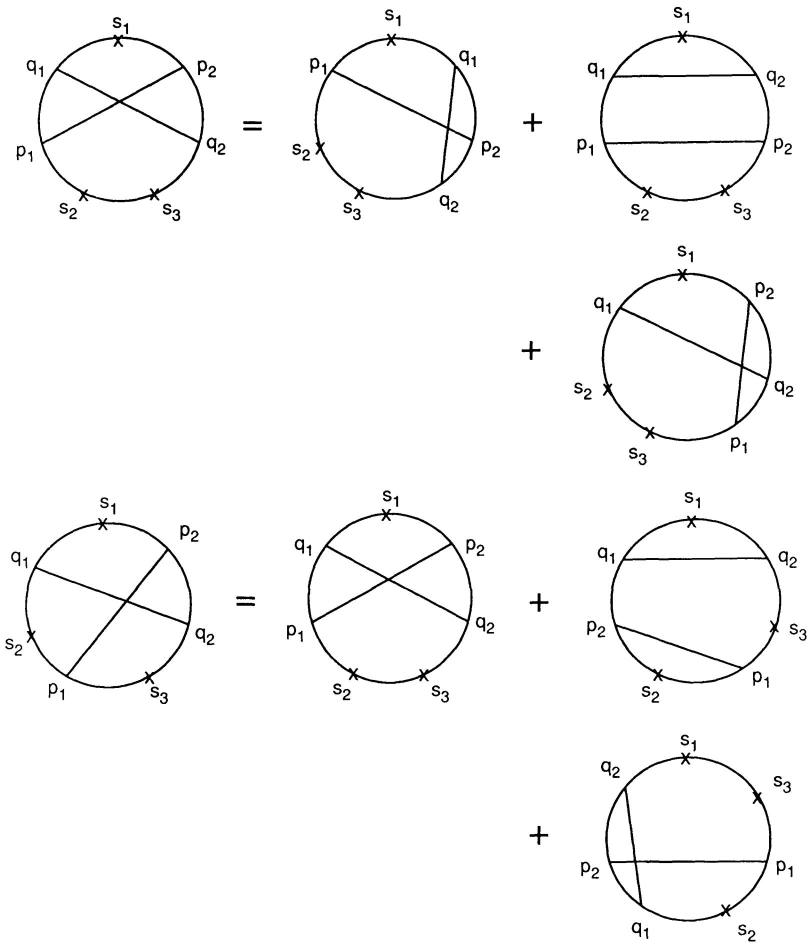

Fig. 8. Diagrammatic representations of the decomposition laws for the last two configurations in Fig. 7

This vanishes due to Lemma 3. Case (b) has been proved.

Case (c) and case (d) are shown similarly and we just give outlines of the proofs below.

Proof of case (c). Due to the linear reciprocity (88), or (90), Eq. (113) can be written by

$$t_\alpha \tau_b \lambda_c - \tau_b t_\alpha \lambda_c - \tau_c t_\alpha \lambda_b = 0. \tag{150}$$

Since

$$\begin{aligned}\tau_c t_\alpha \lambda_b &= \sum_{Q \in \langle AC \rangle \backslash S\Gamma} \sum_{Y \in S\Gamma} \tau(Y\vec{c}_1, Y\vec{c}_2, Y\vec{c}_3) t(Q^{-1}\vec{a}_1, Q^{-1}\vec{a}_2)[\vec{b}_1, \vec{b}_2, \vec{b}_3] \\ &= \sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in S\Gamma} \tau(\vec{c}_1, \vec{c}_2, \vec{c}_3) t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2)[Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2, Y^{-1}\vec{b}_3],\end{aligned} \tag{151}$$

we get

$$\begin{aligned} &t_\alpha \tau_b \lambda_c - \tau_b t_\alpha \lambda_c - \tau_c t_\alpha \lambda_b \\ &= \sum_{X \in \langle A \rangle \backslash S\Gamma} \sum_{Y \in S\Gamma} \{ t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2) \tau(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2, Y^{-1}\vec{b}_3)[\vec{c}_1, \vec{c}_2, \vec{c}_3] \\ &\quad - \tau(Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2, Y^{-1}\vec{b}_3) t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2)[\vec{c}_1, \vec{c}_2, \vec{c}_3] \\ &\qquad - \tau(\vec{c}_1, \vec{c}_2, \vec{c}_3) t(X^{-1}\vec{a}_1, X^{-1}\vec{a}_2)[Y^{-1}\vec{b}_1, Y^{-1}\vec{b}_2, Y^{-1}\vec{b}_3]\}. \end{aligned} \tag{152}$$

We have a lemma, which is proven similarly to Lemma 2 and/or Lemma 3:

Lemma 4. *For any eight points $\vec{x}_i$ $(i = 1, \ldots, 8)$ on $\mathbb{R}_s$, an identity holds,*

$$\begin{aligned} t(\vec{x}_1, \vec{x}_2)\tau(\vec{x}_3, \vec{x}_4, \vec{x}_5)[\vec{x}_6, \vec{x}_7, \vec{x}_8] - \tau(\vec{x}_3, \vec{x}_4, \vec{x}_5) t(\vec{x}_1, \vec{x}_2)[\vec{x}_6, \vec{x}_7, \vec{x}_8] \\ - \tau(\vec{x}_6, \vec{x}_7, \vec{x}_8) t(\vec{x}_1, \vec{x}_2)[\vec{x}_3, \vec{x}_4, \vec{x}_5] = 0. \end{aligned} \tag{153}$$

Due to this lemma, Eq. (152) vanishes. Hence case (c) has been proved.

Proof of case (d). We have a lemma:

Lemma 5. *For any nine points $\vec{x}_i$ $(i = 1, \ldots, 9)$ on $\mathbb{R}_s$,*

$$\begin{aligned} \tau(\vec{x}_1, \vec{x}_2, \vec{x}_3)\tau(\vec{x}_4, \vec{x}_5, \vec{x}_6)[\vec{x}_7, \vec{x}_8, \vec{x}_9] + \tau(\vec{x}_4, \vec{x}_5, \vec{x}_6)\tau(\vec{x}_7, \vec{x}_8, \vec{x}_9)[\vec{x}_1, \vec{x}_2, \vec{x}_3] \\ + \tau(\vec{x}_7, \vec{x}_8, \vec{x}_9)\tau(\vec{x}_1, \vec{x}_2, \vec{x}_3)[\vec{x}_4, \vec{x}_5, \vec{x}_6] = 0. \end{aligned} \tag{154}$$

Due to this lemma, Eq. (114) is proved.

We have proven the theorem in Sect. 4.

7. The Super Weil–Petersson Kähler Form

We begin with reviewing the Kähler form in the length functions on the ordinary (non-super) Teichmüller space for the compact Riemann surfaces H/Γ of genus $h \geqq 2$. Let $B(\Gamma)$ and $Q(\Gamma)$ be the vector spaces of Beltrami differentials and holomorphic quadratic differentials, respectively. A natural pairing of $B(\Gamma)$ and $Q(\Gamma)$ is given by the integral,

$$(\mu, \phi) = \int_{H/\Gamma} \mu\phi, \quad \mu \in B(\Gamma), \quad \phi \in Q(\Gamma). \tag{155}$$

A mapping $\Lambda: B(\Gamma) \to Q(\Gamma)$ is given by (cf. (49)),

$$\Lambda[\mu](z) = \frac{12}{\pi} \int_H \frac{\bar{\mu}(t)}{(z - \bar{t})^4} d\sigma(t), \quad \mu \in B(\Gamma), \tag{156}$$

where $d\sigma(t)$ is the Euclidean area form. The kernel of the mapping Λ is defined by $N(\Gamma)$. Then $B(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$ are identified with the holomorphic tangent and cotangent spaces of the Teichmüller space $T_h(\Gamma)$, respectively [14]. For any $\mu \in B(\Gamma)$ and $\phi \in Q(\Gamma)$ (cf. (64)),

$$\int_{H/\Gamma} \mu\phi = \int_{H/\Gamma} \bar{\Lambda}[\mu]\phi(\operatorname{Im} z)^2, \tag{157}$$

so that $N(\Gamma)$ is orthogonal to $Q(\Gamma)$. Multiplication by i is the automorphism J of $B(\Gamma)/N(\Gamma)$, which defines the complex structure of $T_h(\Gamma)$. The Hermitian product for the Weil–Petersson metric is

$$h(\mu, v) = \int_{H/\Gamma} \overline{\Lambda[\mu]}\Lambda[v](\operatorname{Im} z)^2, \quad \text{for} \quad \mu, v \in B(\Gamma). \tag{158}$$

Accordingly the Hermitian metric g, the real symmetric form, is given by,

$$g(\mu, v) = 2 \operatorname{Re} h(\mu, v), \quad \mu, v \in B(\Gamma). \tag{159}$$

and the Kähler form ω is

$$\omega(\mu, v) = g(J\mu, v) = -2 \operatorname{Im} h(\mu, v), \quad \mu, v \in B(\Gamma). \tag{160}$$

Let t_α be the tangent vector to the Fenchel–Nielsen deformation about α (cf. (86)). Due to Eq. (157), $\omega(t_\alpha, t_\beta)$ is calculated as

$$\begin{aligned}\omega(t_\alpha, t_\beta) &= -2 \operatorname{Im} \int_{H/\Gamma} \bar{\Lambda}[t_\alpha]\Lambda[t_\beta](\operatorname{Im} z)^2 \\ &= -2 \operatorname{Im} \int_{H/\Gamma} \bar{\Lambda}[t_\alpha]\left(-\frac{i}{\pi}\theta_\beta\right) \\ &= 2 \operatorname{Re} \int_{H/\Gamma} t_\alpha \theta_\beta = t_\alpha l_\beta,\end{aligned} \tag{161}$$

which leads to the cosine formula (2).

Since the Weil–Petersson metric is Hermitian, the linear reciprocity identity of the Fenchel–Nielsen deformation follows [2];

$$t_\alpha l_\beta + t_\beta l_\alpha = 2 \operatorname{Re} h(t_\alpha, -it_\beta) + 2 \operatorname{Re} h(t_\beta, -it_\alpha) = 0, \tag{162}$$

hence ω is, in fact, a 2-form;

$$\omega(t_\alpha, t_\beta) + \omega(t_\beta, t_\alpha) = 0. \tag{163}$$

The exterior derivative of the Kähler form ω is evaluated as

$$\begin{aligned}d\omega(t_\alpha, t_\beta, t_\gamma) &= t_\alpha\omega(t_\beta, t_\gamma) - t_\beta\omega(t_\alpha, t_\gamma) + t_\gamma\omega(t_\alpha, t_\beta) \\ &\quad - \omega([t_\alpha, t_\beta], t_\gamma) + \omega([t_\alpha, t_\gamma], t_\beta) - \omega([t_\beta, t_\gamma], t_\alpha) \\ &= t_\alpha t_\beta l_\gamma + t_\beta t_\gamma l_\alpha + t_\gamma t_\alpha l_\beta - [t_\alpha, t_\beta]l_\gamma + [t_\alpha, t_\gamma]l_\beta - [t_\beta, t_\gamma]l_\alpha \\ &= t_\beta t_\alpha l_\gamma + t_\alpha t_\gamma l_\beta + t_\gamma t_\beta l_\alpha,\end{aligned} \tag{164}$$

and the last expression vanishes due to the quadratic reciprocity relation [2]. Then ω on the Teichmüller space is closed. The Kähler form is expressed in terms of geodesic length functions. Let l_i $(i = 1, \ldots, n)$ provide local coordinates for the Teichmüller space $T_h(\Gamma)$. Let $\omega_{ij} = \omega(t_i, t_j)$ and (W_{ij}) be the inverse of (ω_{ij}). Then ω is given by [2],

$$\omega = -\sum_{j<k} W_{jk} dl_j \wedge dl_k. \tag{165}$$

Now we consider the Weil–Petersson Kähler form on the super Teichmüller space. We define the super Hermitian product h on $SB(S\Gamma)/N(S\Gamma)$ (super Weil–

Petersson product) using (49);

$$h(\sigma_1,\sigma_2)=\varepsilon(\sigma_2)\int_{SH/S\Gamma}\overline{\Lambda[\sigma_1]}\Lambda[\sigma_2]Y^2,\quad \sigma_1,\sigma_2\in SB(S\Gamma)/N(S\Gamma),\tag{166}$$

where the phase factor $\varepsilon(\sigma)$[7] is given by

$$\varepsilon(\sigma)=\begin{cases}-1, & \text{for}\quad \text{even}\,\sigma,\\ 1, & \text{for}\quad \text{odd}\,\sigma.\end{cases}\tag{167}$$

It satisfies

$$\begin{aligned}\overline{h(\sigma_1,\sigma_2)}&=\bar{\varepsilon}(\sigma_1)\bar{\varepsilon}(\sigma_2)h(\sigma_2,\sigma_1),\\ h(i\sigma_1,\sigma_2)&=ih(\sigma_1,\sigma_2).\end{aligned}\tag{168}$$

Let $t_A=(t_\alpha,\tau_a)$, $t_B=(t_\beta,\tau_b)$ be Fenchel–Nielsen tangent vectors associated with the super Beltrami differentials, $\sigma_A=\left(\frac{i}{\pi}Y^2\bar{\Theta}_\alpha,\frac{1}{\pi}Y^2\bar{\Xi}_a\right)$, $\sigma_B=\left(\frac{i}{\pi}Y^2\bar{\Theta}_\beta,\frac{1}{\pi}Y^2\bar{\Xi}_b\right)$, respectively. The Riemann metric g on the super Teichmüller space $ST_h(S\Gamma)$ induced by the super Hermitian product h is

$$\begin{aligned}g(t_A,t_B)&=\{h(\sigma_A,\sigma_B)+(-)^{AB}h(\sigma_B,\sigma_A)\}\\ &=2\,\mathrm{Re}_s\,h(\sigma_A,\sigma_B).\end{aligned}\tag{169}$$

Multiplication by i in $SB(S\Gamma)/N(S\Gamma)$ defines the complex structure J of $ST_h(S\Gamma)$. And the Kähler form ω is

$$\begin{aligned}\omega(t_A,t_B)&=g(Jt_A,t_B)=2\,\mathrm{Re}_s\,h(i\sigma_A,\sigma_B)\\ &=-2\,\mathrm{Im}_s\,h(\sigma_A,\sigma_B).\end{aligned}\tag{170}$$

Using (64) and (166) we find

$$\omega(t_\alpha,t_\beta)=\frac{2}{\pi^2}\mathrm{Re}_s\int_{SH/S\Gamma}d^2z\,d^2\theta\, iY^2\bar{\Theta}_\alpha\Theta_\beta,\tag{171}$$

$$\omega(t_\alpha,\tau_b)=\frac{2}{\pi^2}\mathrm{Re}_s\int_{SH/S\Gamma}d^2z\,d^2\theta\, iY^2\bar{\Theta}_\alpha\Xi_b,\tag{172}$$

$$\omega(\tau_a,\tau_b)=\frac{2}{\pi^2}\mathrm{Re}_s\int_{SH/S\Gamma}d^2z\,d^2\theta\, Y^2\bar{\Xi}_a\Xi_b,\tag{173}$$

Due to Eqs. (89) ~ (91) we get

$$\omega(t_A,t_B)=t_Al_B.\tag{174}$$

The linear reciprocity (88) of FN deformation shows that ω is actually an exterior 2-form on $ST_h(S\Gamma)$;

$$\omega(t_A,t_B)+(-)^{AB}\omega(t_B,t_A)=t_Al_B+(-)^{AB}t_Bl_A=0.\tag{175}$$

[7] This factor appears due to our convention of the complex conjugation and the definition of the real part of a Grassmannian odd quantity

The exterior derivative of ω is evaluated as

$$\begin{aligned}
& d\omega(t_A, t_B, t_C) \\
&= t_A\omega(t_B, t_C) + (-)^{A(B+C)} t_B\omega(t_C, t_A) + (-)^{C(A+B)} t_C\omega(t_A, t_B) \\
&\quad - \omega([t_A, t_B], t_C) - (-)^{A(B+C)}\omega([t_B, t_C], t_A) - (-)^{C(A+B)}\omega([t_C, t_A], t_B) \\
&= t_A t_B l_C + (-)^{A(B+C)} t_B t_C l_A + (-)^{C(A+B)} t_C t_A l_B \\
&\quad - [t_A, t_B] l_C - (-)^{A(B+C)}[t_B, t_C] l_A - (-)^{C(A+B)}[t_C, t_A] l_B. \qquad (176)
\end{aligned}$$

The last expression vanishes due to the theorem of the quadratic reciprocity in Sect. 5 and hence the Kähler form ω on the super Teichmüller space is closed,

$$d\omega = 0. \qquad (177)$$

Now we shall show that FN tangent vector t_A is a Hamiltonian vector for the Kähler form ω, i.e., the Lie derivative $L_{t_A}\omega$ vanishes. Let I_{t_A} be the inner product of a form with a vector t_A [16]. The 1-form $I_{t_A}\omega$ evaluated on t_B is

$$\begin{aligned}
I_{t_A}\omega(t_B) &= \omega(t_A, t_B) \\
&= t_A l_B \\
&= -(-)^{AB} t_B l_A \\
&= -dl_A(t_B). \qquad (178)
\end{aligned}$$

This implies that $I_{t_A}\omega$ is closed,

$$I_{t_A}\omega = -dl_A, \qquad (179)$$

or equivalently,

$$\omega(t_A,) = -dl_A. \qquad (180)$$

Then the Lie derivative $L_{t_A}\omega$ vanishes,

$$L_{t_A}\omega = dI_{t_A}\omega + I_{t_A}d\omega = 0. \qquad (181)$$

That is, the Kähler form ω is invariant under the local flow generated by the FN tangent vector t_A.

The Poisson bracket $\{\, ,\}_P$ is determined by the equation,

$$I_{[t_A, t_B]}\omega = -d\{l_A, l_B\}_P. \qquad (182)$$

Then we obtain

$$\{l_A, l_B\}_P = \omega(t_A, t_B). \qquad (183)$$

Proof. We evaluate the left-hand side of (183) on t_C using (178) and (110),

$$\begin{aligned}
I_{[t_A, t_B]}\omega(t_C) &= \omega([t_A, t_B], t_C) \\
&= [t_A, t_B] l_c \\
&= -(-)^{C(A+B)} t_C t_A l_B \\
&= -d(t_A l_B)(t_C) \\
&= -d(\omega(t_A, t_B))(t_C). \quad \text{Q.E.D.} \qquad (184)
\end{aligned}$$

8. Discussions

We have analyzed the symplectic geometry of the super Teichmüller space. We have shown that the Weil–Petersson Kähler form ω on the super Teichmüller space is a closed exterior 2-form. The Fenchel–Nielsen tangent vector has been shown to be a Hamiltonian vector for the Kähler form. Unfortunately we have not shown that the Kähler form has maximal rank at each tangent space; $\operatorname{sdet}\omega_{AB} \neq 0$. We have not shown that the super Teichmüller space has the complex structure; however, according to D'Hoker and Phong [17], they showed the integrability condition for the almost complex structure. Hence we have seen that the super Teichmüller space is a Kähler supermanifold.

As for the super Beltrami equations, we have not discussed in the global context the existence of the homeomorphic solutions. This problem was investigated by Hodgkin [12] with slightly different super Beltrami equations. He showed the global existence of the Bers embedding for the super Teichmüller space. The complex structure due to the Bers embedding is to be examined further.

In terms of the functions $\{l_I\} = \{l_i, \lambda_\kappa\}$ $(i = 1, 2, \ldots, n;\ \kappa = 1, 2, \ldots, 4h-4)$ providing the local coordinates of the super Teichmüller space, the Kähler form is

$$\omega = \sum_{A,B} W^{BA} dl_A \wedge dl_B, \tag{185}$$

where the matrix (W^{AB}) is the inverse of $((-)^B\omega_{AB}) = ((-)^B\omega(t_A, t_B))$. The Grassmann odd elements of the matrix (W^{AB}) do not vanish, in general, however, there should exist the $SPL(2,\mathbb{R})$-invariant canonical local coordinates $\{X_i, \Theta_\kappa\}$ $(i = 1, \ldots, 6h-6; k = 1, \ldots, 4h-4)$, in terms of which the Kähler form takes a simple expression,

$$\omega = \sum_{i=1}^{3h-3} dX_{2i-1} \wedge dX_{2i} + \sum_{\kappa=1}^{4h-4} d\Theta_\kappa \wedge d\Theta_\kappa. \tag{186}$$

The Grassmann even part will be some superanalog of the lengths of the closed geodesics and twisted angles, however, the Grassmann odd part is unclear so far. To give those coordinates explicitly is an interesting problem.

Acknowledgements. One of us (S.U.) is grateful to W. Harvey, L. Hodgkin and A. Rogers for discussions. We thank M. Takama for discussions and comments.

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Communicated by S.-T. Yau