Reduction Formulae for Euclidean Lattice Theories

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Abstract. LSZ reduction formulae for Euclidean Lattice Theories are presented.

1. Introduction

There is an increasing interest in the physics literature in the study of scattering
processes in Euclidean Lattice Field Theories (see f.i. [4, 5]). The existence of multi-
particle states and of a well defined S-matrix for these theories has been established
in [1] (see also [10]). An important question which rises in this context is the
question of the existence of LSZ reduction formulae for the computation of
S-matrix elements. This work is devoted to the rigorous derivation of these
formulae. For relativistic quantum field theories they were first obtained by
Lehmann, Symanzik, and Zimmermann in [3] under special assumptions
concerning weak asymptotic limits of the fields. The first rigorous proof in the
context of Wightman Quantum Field Theories was given by Hepp in [2]. Here we
will follow essentially Hepp's ideas but some adaptations to our case are necessary
due to the following two facts. First, [2] makes use of locality (Einstein causality) in
a strong sense, a property which is generally not available for Lattice Theories.
Second, the clustering of the Wightman functions obtained in [1] (see Theorem 2
below), which is an essential property for the construction of the scattering theory,
is not uniform in the whole region of space-like separated points.

In Sect. 2 the notation used and the results of [1] which will be needed are
introduced. Section 3 presents a smoothness theorem on suitably smeared
expectations of time-ordered products of fields which leads directly to the
reduction formulae in Sect. 4. The problems mentioned in the last paragraph
manifest in the proof of this smoothness theorem, given in Subsects. 3.1 and 3.2.
In this last subsection we show how to compensate the lack of locality with the use
of the clustering property.

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2. Definitions and Previous Results

The results of this work are valid in the framework of Euclidean Lattice Theories described in [1]. We consider a classical statistical mechanical spin system on a $d+1$-dimensional Euclidean cubic lattice $\mathbb{Z}^{d+1}$ ($d \geq 1$) provided with a translation invariant and reflection positive equilibrium state, which we call the vacuum state. As in [1] we denote by $\mathcal{H}$ the Hilbert space of finite energy states and by $\mathcal{H}^{(1)}$ the closed one-particle subspace of $\mathcal{H}$ (whose existence is assumed), on which the relation $(e^{-H} - e^{-\omega(\underline{P})})\restriction \mathcal{H}^{(1)} = 0$ holds. Here $\underline{P}$ is the momentum operator, i.e., the infinitesimal generator of space translations, $e^{i\underline{P} \cdot \underline{x}} = U(\underline{x})$, $\mathrm{sp}(\underline{P}) \subset (-\pi, \pi]^d$, where $U(\underline{x})$ is the unitary generator of space translations by $\underline{x} \in \mathbb{Z}^d$ and $H = -\ln T$ is the Hamilton operator, T being the infinite volume transfer matrix. Above ω is the dispersion relation of the particle, which will be assumed to be smooth with $\omega(p) \geq m$, $p \in (-\pi, \pi]^d$, m being the mass gap. We assume $m > 0$. We will also denote by $\mathcal{A}$ the *-algebra generated by the quasi-local observables with finite energy transfer (see [1]). As discussed in [1] a dense subspace $\mathcal{D}^{(1)}$ of $\mathcal{H}^{(1)}$ is created from the vacuum by almost local operators $A \in \mathcal{A}$ for which we may also require $A^*\Omega = 0$, Ω being the vacuum vector.

We call $f(\underline{x}, t)$: $\mathbb{Z}^d \times \mathbb{R} \to \mathbb{C}$ a wave function if for its Fourier transform

$$\tilde{f}(p, p_0) : (-\pi, \pi]^d \times \mathbb{R} \to \mathbb{C}$$

one has

$$\mathrm{supp}\, \tilde{f} \cap \mathrm{sp}(H, \underline{P}) = \mathrm{supp}\, \tilde{f} \cap W \neq \emptyset,$$

where $W = \{(p, \omega(p)),\ p \in (-\pi, \pi]^d\}$ is the "one-particle shell."

Like Hepp [2] consider for a wave function f with $\tilde{f} \in \mathcal{D}((-\pi, \pi]^d \times \mathbb{R})$ the wave packet $f^{(s)}(\underline{x}, t)$, parametrized by $s \in \mathbb{R}$, defined as the Fourier transform of $e^{i(p_0 - \omega(p))s} \tilde{f}(p, p_0)$, with $(p, p_0) \in (-\pi, \pi]^d \times \mathbb{R}$.

Set, for $B \in \mathcal{A}$,

$$B_f = \sum_{\underline{x} \in \mathbb{Z}^d} \int_R dt f(\underline{x}, t) B(t, \underline{x}), \qquad (1)$$

and

$$B_f(s) = \sum_{\underline{x} \in \mathbb{Z}^d} \int_R dt f^{(s)}(\underline{x}, t) B(t, \underline{x}), \qquad (2)$$

for the space-time translated operators $B(t, \underline{x}) = e^{iHt} U(\underline{x}) B e^{-iHt} U(-\underline{x})$. If $A \in \mathcal{A}$ is such that $A_f \Omega \in \mathcal{D}^{(1)}$, then $A_f(s)\Omega = A_f \Omega$.

Let Δ be a closed set in $(-\pi, \pi]^d$. Then the velocity content $V(\Delta)$ of the set of momenta Δ is

$$V(\Delta) = \{\mathrm{grad}\, \omega(p) | p \in \Delta\}. \qquad (3)$$

We define also for a wave function f,

$$\Delta(f) = \{p \in (-\pi, \pi]^d | (p, \omega(p)) \in \mathrm{supp}\, \tilde{f} \cap W\}. \qquad (4)$$

The following proposition summarises the relevant behavior of the functions $f^{(s)}(\underline{x}, t)$. For a proof see [6].

Proposition 1. *For $f^{(s)}$ as above with $\tilde{f} \in \mathscr{D}(\mathbb{R}^{d+1})$ we have:*
a) *For all $N \in \mathbb{N}$ there are constants $C_N > 0$ so that*

$$|f^{(s)}(\underline{x}, t)| \le C_N (1 + |s|)^{-d/2} (1 + |t - s|)^{-N}, \tag{5}$$

uniformly in $\underline{x}$.
b) *There exists a positive constant C so that for every s,*

$$\sum_{\underline{x} \in \mathbb{Z}^d} \int_{-\infty}^{\infty} dt |f^{(s)}(\underline{x}, t)| \le C(1 + |s|)^{d/2}. \tag{6}$$

c) *For all $L, N \in \mathbb{N}$ there is a positive constant $C_{L,N}$ so that,*

$$|f^{(s)}(\underline{x}, t)| \le C_{L,N} (1 + |s - t|)^{-L} (1 + \text{dist}(\underline{x}, sV(\Delta(f))))^{-N}. \tag{7}$$

Now let $\phi_1, \dots, \phi_n \in \mathscr{D}^{(1)}$ such that $V(\text{sp}_U \phi_i) \cap V(\text{sp}_U \phi_j) = \emptyset$ for $i \ne j$, $\text{sp}_U \phi$ is the momentum spectrum of $\phi \in \mathscr{D}^{(1)}$. Choose $A \in \mathscr{A}$ with $A_{f_i} \Omega = \phi_i$, $A_{f_i}^* \Omega = 0$ and $\tilde{f}_i \in \mathscr{D}((-\pi, \pi]^d \times \mathbb{R})$ with non-overlapping velocities:

$$V(\Delta(f_i)) \cap V(\Delta(f_j)) = \emptyset \tag{8}$$

for $i \ne j$.

To simplify we will often denote A_{f_i} by A_i.

The Haag-Ruelle approximant on the scattering state of particles with single particle states $\phi_1, \dots, \phi_n$ is defined by

$$\phi_{1, \dots, n}(s) = A_1(s) \dots A_n(s) \Omega. \tag{9}$$

The following theorem on the existence of asymptotic multi-particle states [here denoted by $(\phi_1 \times \dots \times \phi_n)_{\text{in}}^{\text{out}}$] and their statistics has been proven in [1][1], according to the methods of the Haag-Ruelle Scattering Theory:

Theorem 1. (i) *The Haag-Ruelle approximants $\phi_{1, \dots, n}(s)$ in (9) converge for $t \to \pm \infty$. The limits $(\phi_1 \times \dots \times \phi_n)_{\text{in}}^{\text{out}}$ depend only on the single particle states $\phi_i \in \mathscr{D}^{(1)}$, and for all $N \in \mathbb{N}$,*

$$\|\phi_{1, \dots, n}(s) - (\phi_1 \times \dots \times \phi_n)_{\text{in}}^{\text{out}}\| s^N \to 0, \quad s \to \pm \infty. \tag{10}$$

(ii) *Let $\psi_1, \dots, \psi_k \in \mathscr{D}^{(1)}$ with $V(\text{sp}_U \psi_i) \cap V(\text{sp}_U \psi_j) = \emptyset$, $i \ne j$. Then*

$$((\psi_1 \times \dots \times \psi_k)_{\text{in}}^{\text{out}}, (\phi_1 \times \dots \times \phi_n)_{\text{in}}^{\text{out}}) = \delta_{n,k} \sum_{\sigma} \prod_i (\psi_i, \phi_{\sigma(i)}), \tag{11}$$

where the sum is over elements σ of the permutation group of $\{1, \dots, n\}$.

This is a consequence of the sufficiently strong clustering property for the Wightman functions that has been established in [1] for the systems in question in the following form:

Theorem 2. *Let $B_1, \dots, B_n \in \mathscr{A}$, $n \ge 2$. Then for the Wightman functions*

$$\mathscr{W}^{(n)}(\underline{x}, t_1; \dots; \underline{x}_n, t_n) := (\Omega, B_1(t_1, \underline{x}) \dots B_n(t_n, \underline{x}_n) \Omega) \tag{12}$$

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[1] The fields $A_f(s)$ used in the Haag-Ruelle approximants in [1] actually differ slightly from (2). This does not change the result

the following clustering property holds: for each ε, $0 < \varepsilon < 1$ and for each $q \in \mathbb{N}$, there is a positive constant $C_{q,\varepsilon}$, depending on the B_i's, so that

$$|\mathscr{W}_T^{(n)}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n)| \leqq C_{q,\varepsilon} \frac{(1 + \|t\|)^q}{(1 + \|\underline{x}\|)^{(q-1)-\varepsilon}}, \tag{13}$$

where $\|t\| := \max\limits_{1 \leqq i \leqq n-1} |t_{i+1} - t_i|$ and $\|\underline{x}\| := \max\limits_{i,j \in \{1, \ldots, n\}} |\underline{x}_i - \underline{x}_j|$. $\mathscr{W}_T^{(n)}$ are the truncated n-point Wightman functions.

We will also make use of the following result:

Proposition 2. *Let $s \in \mathbb{R}$, $s < -1$. Hence for all $N \in \mathbb{N}$ there exists a positive constant C_M so that*

$$\|A_1(s)(\phi_2 \times \ldots \times \phi_n)_{\text{in}} - (\phi_1 \times \ldots \times \phi_n)_{\text{in}}\| \leqq C_M |s|^{-M}. \tag{14}$$

An analogous result holds for out *states for $s > 1$.*

Proof. According with (6) $\|A(s)\| \leqq C(1 + |s|)^{d/2}$. We have

$$A_1(s)(\phi_2 \times \ldots \times \phi_n)_{\text{in}} - (\phi_1 \times \ldots \times \phi_n)_{\text{in}}$$
$$= A_1(s)((\phi_2 \times \ldots \times \phi_n)_{\text{in}} - \phi_{2,\ldots,n}(s)) - ((\phi_1 \times \ldots \times \phi_n)_{\text{in}} - \phi_{1,\ldots,n}(s)), \tag{15}$$

where $\phi_{2,\ldots,n}(s) = A_2(s)\ldots A_n(s)\Omega$. Hence,

$$\|A_1(s)(\phi_2 \times \ldots \times \phi_n)_{\text{in}} - (\phi_1 \times \ldots \times \phi_n)_{\text{in}}\|$$
$$\leqq \|A_1(s)\| \|(\phi_2 \times \ldots \times \phi_n)_{\text{in}} - \phi_{2,\ldots,n}(s)\|$$
$$+ \|(\phi_1 \times \ldots \times \phi_n)_{\text{in}} - \phi_{1,\ldots,n}(s)\| \leqq C_M |s|^{-M} \tag{16}$$

for $s < -1$ for all $M \in \mathbb{N}$, according with Theorem 1. $\square$

We will use the following definition for the time-ordering of products of fields.[2]

Definition. Let $\sigma_{a_1, \ldots, a_b}$ be the permutation group of $\{a_1, \ldots, a_b\} \subset \{1, \ldots, n\}$, $n \geqq 1$. Define the time-ordering operation on products of fields A by

$$T_\beta \left[\prod_{i=1}^{b} A(t_{a_i}, \underline{x}_{a_i}) \right] := \sum_{\pi \in \sigma_{a_1, \ldots, a_b}} \prod_{i=1}^{b-1} \theta_\beta(t_{\pi(a_i)} - t_{\pi(a_{i+1})}) \prod_{j=1}^{b} A(t_{\pi(a_j)}, \underline{x}_{\pi(a_j)}), \tag{17}$$

with $\theta_\beta = \theta * \beta$, where $\beta \in \mathscr{D}(\mathbb{R})$,[3] with $\int_{-\infty}^{\infty} \beta(s')ds' = 1$ and $\beta(x) = \beta(-x)$ and where

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqq 0. \end{cases} \tag{18}$$

Another definition that will be used is:

Definition 2. For $a, b \in \mathbb{N}$, $b \geqq a$, define

$$\Gamma_{a,b} = \sum_{\underline{x}_a, \ldots, \underline{x}_b \in \mathbb{Z}^d} \int_{\mathbb{R}^{b-a+1}} \{dt_i\}_{i=a}^{b} \prod_{j=a}^{b} \dot{f}_j^{(s_j)}(\underline{x}_j, t_j) T_\beta \left[\prod_{k=a}^{b} A(t_k, \underline{x}_k) \right], \tag{19}$$

with $\{dt_i\}_{i=a}^{b} = dt_a \ldots dt_b$. Above $\dot{f}^{(s)} = \frac{d}{ds} f^{(s)}$. Note that $\Gamma_{a,b}$ depends on $s_a, \ldots, s_b$.

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[2] The same results can be obtained with more general time-ordering prescriptions [11]
[3] The case $\beta \in \mathscr{S}(\mathbb{R})$ can also be treated but involves irrelevant technical complications

3. The Main Theorem

Following Hepp [2] (see also [7] for a brief account) we will establish the following smoothness theorem, which is the central tool for the derivation of the LSZ reduction formulae performed in Sect. 4. Below $\{\tilde{f}_i\}_{i=1}^n$, for $n > 1$, will be a set of wave functions in $\mathcal{D}((-\pi, \pi]^d \times \mathbb{R})$ satisfying (8) (non-overlapping velocities) with $A_{f_i}\Omega = (\phi_i) \in \mathcal{H}^{(1)}$.

Theorem 3. *For $\{\tilde{f}_i\}_{i=1}^n$ as above and $0 \leq k < m \leq n$,*

$$((\phi_1 \times \ldots \times \phi_k)_{\text{out}}, \Gamma_{k+1,m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}) \tag{20}$$

belongs to $\mathcal{S}(\mathbb{R}^{m-k})$ as function of $s_{k+1}, \ldots, s_m$.

Remark. Above, for $k = 0, (\phi_1 \times \ldots \times \phi_k)_{\text{out}}$ has to be replaced by Ω, and the same for $(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}$ if $m = n$.

The next two subsections present the proof of this theorem, which will be finished with the proof of Propositions 5 and 6 below. The LSZ reduction formulae follow directly from Theorem 3 in Sect. 4.

3.1. Proof of Theorem 3. The First Step

First we note that we just have to prove the fast decay of (20) since the same argumentation given below holds for the derivatives, because $\dfrac{d^n}{ds^n} f^{(s)}$ satisfy Proposition 1 for all n.

Due to symmetry, we may restrict to the sector

$$s_m \leqq s_{m-1} \leqq \ldots \leqq s_{k+2} \leqq s_{k+1}. \tag{21}$$

We also restrict to the case in which $s := -s_m \geqq |s_{k+1}|$. The argumentation in the case $s_{k+1} \geqq |s_m|$ is analogous.

For some fixed $\delta, \varepsilon > 0, \gamma : 0 \leqq \gamma \leqq 1$, two successive variables $s_i, s_{i-1}, m \geqq i \geqq k+2$ have to satisfy one of the two conditions below:
1. $s_{i-1} - s_i \leqq \varepsilon s^\gamma + \delta$,
2. $s_{i-1} - s_i > \varepsilon s^\gamma + \delta$.

In his analysis [2] Hepp takes $\gamma = 1$. In our case, due to the non-uniform clustering of the Wightman functions for space-like separated points we shall need $\gamma < 1$, as will be clear at the end of the proof of Theorem 3. This is the reason to consider this more general situation from the beginning.

We write $s_{i-1} \not\sim s_i$ if 1 is valid and $s_{i-1} \sim s_i$ if 2 is valid. In the first case we say that s_{i-1} and s_i are "connected" and in the second case "disconnected."

There are two possible situations we have to consider:
A) *For some p, $1 \leqq p \leqq m-k-1$, $s_m \sim s_{m-1} \sim \ldots \sim s_{m-p}$, i.e., $s_{j-1} - s_j > \varepsilon s^\gamma + \delta$ for $m-p+1 \leqq j \leqq m$.*
B) *For some q, $1 \leqq q \leqq m-k-1$, $\{s_m, \ldots, s_{m-q}\}$ is an isolated cluster of connected variables, i.e., $s_{i-1} - s_i \leqq \varepsilon s^\gamma + \delta$ for $m \geqq i \geqq m-q+1$ and $s_{m-q-1} \sim s_{m-q}$, i.e., $s_{m-q-1} - s_{m-q} > \varepsilon s^\gamma + \delta$.*

The next two propositions show two convenient approximations for (20) under the cases A and B above.

Proposition 3. *Under condition* A, *expression* (20) *may be approximated by*

$$\left((\phi_1 \times \ldots \times \phi_k)_{\text{out}}, \, (\Gamma_{k+1, m-1}) \frac{d}{ds_m} A_{f_m}(s_m)(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}} \right), \qquad (22)$$

with an error which falls faster than any power in s.

Proposition 4. *Under condition* B, *holding for some* q, $1 \leq q \leq m-k-1$, (20) *may be approximated by*

$$((\phi_1 \times \ldots \times \phi_k)_{\text{out}}, \, \Gamma_{k+1, m-q-1} \Gamma_{m-q, m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}), \qquad (23)$$

with an error which falls faster than any power in s.

Remark. Using Proposition 3 we can show that Theorem 3 follows for case A since by Proposition 2, $\dfrac{d}{ds_m} A_{f_m}(s_m)(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}$ approximates

$$\left(\left(\frac{d}{ds_m} \phi_m \right) \times \phi_{m+1} \times \ldots \times \phi_n \right)_{\text{in}} = 0$$

faster than any power in $|s_n| = s$. The case B is more involving and will be treated in Subsect. 3.2.

3.1.1. Proof of Proposition 3. First we restrict the integration $\int\limits_{-\infty}^{\infty} (\cdot) dt_m$ in (20) to an integration on the interval $[-\infty, b(s)]$ with $b(s) := s_m + \varepsilon' s^{\gamma'} + \delta'$, for ε', $\delta' > 0$, $0 \leq \gamma' \leq 1$. This introduces an error which is majorized using Proposition 1 by

$$C_M \left[\prod_{i=k+1}^{m} (1 + |s_i|)^{d/2} \right] \int\limits_{\varepsilon' s^{\gamma'} + \delta'}^{\infty} \frac{dy}{(1 + |y|)^M} \leq C_P' |s|^{-P} \left[\prod_{i=k+1}^{m} (1 + |s_i|)^{d/2} \right], \qquad (24)$$

for any $M \in \mathbb{N}$ for some $P \in \mathbb{N}$ depending on M, for $s \geq 1$. Since $|s_m| = s \geq |s_i|$, $\forall i$, the last expression has $C_Q'' |s|^{-Q}$ as upper bound for any $Q \in \mathbb{N}$, $s > 1$, by choosing P large enough.

Now, for all i, $i \neq m$ we restrict the integrations $\int\limits_{-\infty}^{\infty} (\cdot) dt_i$ to the interval $[c(s), \infty]$, where $c(s) := s_m + \varepsilon'' s^{\gamma'} + \delta''$ for ε'', $\delta'' > 0$, $0 \leq \gamma' \leq 1$. This introduces an error which is bounded by

$$C_M \left[\int\limits_{-\infty}^{c(s)} \frac{dt_i}{(1 + |t_i - s_i|)^M} \right] \prod_{j=k+1}^{m} (1 + |s_j|)^{d/2}, \qquad \forall M \in \mathbb{N}. \qquad (25)$$

Above the integral can be written as $\int\limits_{-\infty}^{c(s) - s_i} dy(1 + |y|)^{-M}$. Now, since $s_i \geq s_{m-1}$ $(i \neq m)$ we have

$$c(s) - s_i \leq s_m - s_{m-1} + \varepsilon'' s^{\gamma'} + \delta'' < -\varepsilon s^{\gamma} - \delta + \varepsilon'' s^{\gamma'} + \delta'', \qquad (26)$$

where in the last inequality we made use of condition A. We see that, choosing $\gamma' < \gamma$ or $\gamma' = \gamma$ but $\varepsilon > \varepsilon''$ (25) becomes bounded by

$$C_M' s^{-M} \left(\prod_{j=k+1}^{m} (1 + |s_j|)^{d/2} \right) \leq C_Q'' s^{-Q}, \qquad (27)$$

for any $Q \in \mathbb{N}$ for some positive constant C_Q'', by choosing M large enough.

Now choose $\varepsilon'' \geqq \varepsilon'$ and $\delta'' - \delta' > d_\beta$, where d_β is the diameter of supp β. With this choice, under the restrictions on the integrations given above, we have for the integration variables $t_i - t_m \geqq c(s) - b(s) > d_\beta$, $i \neq m$. Due to this inequality we may split $A(t_m, \underline{x}_m)$ to the right side in the time-ordered product of (20) and replace it by

$$\sum_{\{\underline{x}_a\}_{a=k+1}^m} \int_{c(s)}^\infty \cdots \int_{c(s)}^\infty dt_{k+1} \ldots dt_{m-1} \int_{-\infty}^{b(s)} dt_m \prod_{i=k+1}^m \hat{f}_i^{(s_i)}(\underline{x}_i, t)$$

$$\times \left((\phi_1 \times \ldots \times \phi_k)_{\text{out}}, T_\beta \left[\prod_{j=k+1}^{m-1} A(t_j, \underline{x}_j) \right] A(t_m, \underline{x}_m)(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}} \right) \qquad (28)$$

with an error which falls faster than any power in s. Returning to full integrations, what costs again an error falling faster than any power in s, we complete the proof of Proposition 3. $\quad\square$

3.1.2. Proof of Proposition 4. The proof is analogous to the proof of Proposition 3. We first consider the integrations $\int_{-\infty}^\infty (\cdot) dt_i$, $m - q \leqq i \leqq m$. We restrict them to the interval $[-\infty, d(s)]$, where $d(s) := s_{m-q} + \varepsilon' s^{\gamma'} + \delta'$. As before, the error is majorized by $C_M \int_{d(s)-s_i}^\infty dy(1 + |y|)^{-M}$, for all $M \in \mathbb{N}$, for constants C_M. Now, since $s_i \leqq s_{m-q}$ (for $m - q \leqq i \leqq m$) we have $d(s) - s_i = s_{m-q} - s_i + \varepsilon' s^{\gamma'} + \delta' \geqq \varepsilon' s^{\gamma'} + \delta'$ and so the last integral has

$$\int_{\varepsilon' s^{\gamma'} + \delta'}^\infty \frac{dy}{(1 + |y|)^M} \leqq C_N s^{-N}, \qquad N \in \mathbb{N}, \qquad (29)$$

as upper bound for a constant $C_N > 0$, where N can be made arbitrarily large by making M large.

On the other hand, we restrict the integrations $\int_{-\infty}^\infty (\cdot) dt_j$, $k + 1 \leqq j \leqq m - q - 1$ to the interval $[e(s), \infty]$, where $e(s) := s_{m-q} + \varepsilon' s^{\gamma'} + \delta''$ with an error which is bounded by

$$\int_{-\infty}^{e(s)} \frac{dt_j}{(1 + |t_j - s_j|)^M} = \int_{-\infty}^{e(s)-s_j} \frac{dy}{(1 + |y|)^M}. \qquad (30)$$

Now, note that, since $s_j \geqq s_{m-q-1}$ for $k + 1 \leqq j \leqq m - q - 1$, we have

$$e(s) - s_j = s_{m-q} - s_j + \varepsilon' s^{\gamma'} + \delta''$$

$$\leqq s_{m-q} - s_{m-q-1} + \varepsilon' s^{\gamma'} + \delta'' < -\varepsilon s^\gamma + \varepsilon' s^{\gamma'} + \delta'' - \delta \qquad (31)$$

(the last inequality coming from condition B) and taking $\gamma' < \gamma$ or $\gamma' = \gamma$ but $\varepsilon > \varepsilon'$ we get for (30) a bound like (29) for s large enough.

Under the restrictions on the integrations given above, for $k + 1 \leqq j \leqq m - q - 1$ and $m - q \leqq i \leqq m$ we have

$$t_j - t_i > e(s) - d(s) \geqq 2b_\beta \qquad (32)$$

by choosing $\varepsilon' > \varepsilon$ and, as before, $\delta'' - \delta' > d_\beta$. So we get for this case

$$T_\beta \left[\prod_{a=k+1}^m A(t_a, \underline{x}_a) \right] = T_\beta \left[\prod_{a=k+1}^{m-q-1} A(t_a, \underline{x}_a) \right] T_\beta \left[\prod_{b=m-q}^m A(t_b, \underline{x}_b) \right]. \qquad (33)$$

The reduction of $\sigma_{k+1,\ldots,m}$ to $\sigma_{k+1,\ldots,m-q-1} \times \sigma_{m-q,\ldots,m}$ is due to the inequality (32) which implies the vanishing of the theta functions involving permutations mixing the groups of variables $\{t_{k+1}, \ldots, t_{m-q-1}\}$ and $\{t_{m-q}, \ldots, t_m\}$.

Returning to full integrations, what costs an error which again falls faster than any power in s, we conclude the proof of the Proposition 4. $\square$

3.2. Proof of Theorem 3. The Last Step

Now we complete the proof of Theorem 3 by treating (20) for case B using Proposition 4. Our analysis will differ from that of [2] since Hepp makes there use of locality (Einstein locality) to get rid of the time-ordering operation on the fields associated with the connected s variables. Since this property is not available in the systems considered here we have to make use of the clustering of the Wightman functions in a suitable way.

First, one has

$$|((\phi_1 \times \ldots \times \phi_k)_{\text{out}}, \Gamma_{k+1,m-1}\Gamma_{m-q,m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}})|$$

$$\leqq C\|\Gamma_{m-q,m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}\| \left[\prod_{i=k+1}^{m-q-1} (1+|s_i|)^{d/2} \right], \tag{34}$$

by the Schwarz inequality and Proposition 1. Next we need an upper bound for $\|\Gamma_{m-q,m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}\|$. Then we replace $(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}$ by its Haag-Ruelle approximant at time s_m and define

$$\Lambda = \Gamma_{m-q,m}A_{m+1}(s_m) \ldots A_n(s_m). \tag{35}$$

By Theorem 1, $\|\Gamma_{m-q,m}(\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}\| \leqq \|\Lambda\Omega\| + R(s)$, where $R(s)$ falls faster than any power in s. We concentrate our attention on $\|\Lambda\Omega\|$ whose fast decay will follow directly from the clustering property. After writing $\|\Lambda\Omega\|^2$ explicitly and making convenient changes of variables (see below) it can be written as

$$\sum_{\{\underline{z}_a\}_{a=1}^{2r}} \int_{R^{2r}} \{d\mu_b\}_{b=1}^{2r} \prod_{a=1}^{2r} g_a^{(v_a)}(\underline{z}_a, \mu_a) \sum_{\substack{\pi' \in \sigma_{n-m+1,\ldots,r} \\ \pi \in \sigma_{r+1,\ldots,r+q+1}}} \prod_{k=n-m+1}^{r-1} \theta_\beta(\mu_{\pi'(\varrho(k))} - \mu_{\pi'(\varrho(k+1))})$$

$$\times \prod_{l=r+1}^{r+q} \theta_\beta(\mu_{\pi(l)} - \mu_{\pi(l+1)}) \mathscr{W}^{(2r)}(\underline{z}_{\hat{\pi}(1)}, \mu_{\hat{\pi}(1)}; \ldots; \underline{z}_{\hat{\pi}(2r)}, \mu_{\hat{\pi}(2r)}) \tag{36}$$

with $r := n - m + q + 1$, where $\varrho \in \sigma_{n-m+1,\ldots,r}$ is the permutation inverting the order of $(n-m+1, \ldots, r)$: $\varrho(i) = r + (n-m+1) - i$. Above the Wightman functions are defined by

$$\mathscr{W}^{(2r)}(\underline{z}_1, \mu_1; \ldots; \underline{z}_{2r}, \mu_{2r}) := \left(\Omega, \prod_{a=1}^{r} A(\mu_a, \underline{z}_a)^* \prod_{b=r+1}^{2r} A(\mu_b, \underline{z}_b)\Omega \right) \tag{37}$$

and there we used following definitions: for $\pi' \in \sigma_{n-m+1,\ldots,r}$ and $\pi \in \sigma_{r+1,\ldots,r+q+1}$,

$$\hat{\pi}(a) = \begin{cases} a, & \text{if } 1 \leqq a \leqq n-m, \\ \pi'(a), & \text{if } n-m+1 \leqq a \leqq r, \\ \pi(a), & \text{if } r+1 \leqq a \leqq r+q+1, \\ a, & \text{if } r+q+2 \leqq a \leqq 2r, \end{cases} \tag{38}$$

and

$$(v_1, \ldots, v_{n-m}, v_{n-m+1}, \ldots, v_r, v_{r+1}, \ldots, v_{r+q+1}, v_{r+q+2}, \ldots, v_{2r}):$$
$$(s_m, \ldots, s_m, s_m, \ldots, s_{m-q}, s_{m-q}, \ldots, s_m, s_m, \ldots, s_m) \tag{39}$$

and also

$$(g_1, \ldots, g_{n-m}, g_{n-m+1}, \ldots, g_r, g_{r+1}, \ldots, g_{r+q+1}, g_{r+q+2}, \ldots, g_{2r}):$$
$$= (\bar{f}_n, \ldots, \bar{f}_{m+1}, \bar{f}_m, \ldots, \bar{f}_{m-q}, \dot{f}_{m-q}, \ldots, \dot{f}_m, f_{m+1}, \ldots, f_n). \tag{40}$$

Our objective is to show that $\|\Lambda\Omega\|$ falls faster than any power in s. To do this we apply the results on the clustering properties of the Wightman functions to expression (36). First, we write the Wightman functions of (36) as a sum over products of truncated vacuum expectation values. In order to consider each relevant case separately one has to split the sum over partitions $\mathscr{P}$ of $\{1, \ldots, 2r\}$ into two sums:

1. *A sum S_1 over partitions of $\{1, \ldots, 2r\}$ in sub-sets of two elements,*
and
2. *A sum S_2 over partitions where at least one sub-set contains more than two elements.*

Note that the set $\{1, \ldots, 2r\}$ contains an even number of elements. The sum S_1 is given by

$$\sum_{\mathscr{P}_2} \prod_{J \in \mathscr{P}_2} \mathscr{W}_T^{(2)}(\underline{z}_{j_1}, \mu_{j_1}; \underline{z}_{j_2}, \mu_{j_2}), \tag{41}$$

where $\mathscr{P}_2 = \{J_1, \ldots, J_r\}$ is a partition of $\{\hat{\pi}(1), \ldots, \hat{\pi}(2r)\}$ in sets of two elements in increasing order, i.e., for each $J \in \mathscr{P}_2$ we have $J = (j_1, j_2) = (\hat{\pi}(a), \hat{\pi}(b))$ with $a < b$. We split the sum S_1 in two pieces, which we call, respectively, S_1' and S_1'':

1a. *A sum S_1' over all partitions $\mathscr{P}_2' = \{J_1, \ldots, J_r\}$ so that each $J \in \mathscr{P}_2$ is of the form $J = (j_1, j_2) = (\hat{\pi}(a), \hat{\pi}(b))$ with $1 \leq a \leq r$ and $r+1 \leq b \leq 2r$,*
and
1b. *A sum S_1'' over the rest.*

With the next two propositions we complete the proof of Theorem 3.

Proposition 5. *The contribution of S_1' to (36) is zero.*

Proposition 6. *The contributions of S_1'' and S_2 to (36) fall faster than any power in s.*

3.2.1. Proof of Proposition 5. The sum S_1' is explicitly given by

$$\sum_{\pi^\dagger \in \sigma_{r+1, \ldots, 2r}} \prod_{a=1}^r \mathscr{W}_T^{(2)}(\underline{z}_{\hat{\pi}(a)}, \mu_{\hat{\pi}(a)}; \underline{z}_{\pi^\dagger(\hat{\pi}(a+r))}, \mu_{\pi^\dagger(\hat{\pi}(a+r))}). \tag{42}$$

After the transformation $\pi^\dagger \rightarrow \pi^\dagger \circ (\hat{\pi}^{-1}|(r+1, \ldots, 2r))$ expression (42) becomes independent of $\pi \in \sigma_{r+1, \ldots, r+q+1}$ [see (38)]. Hence, inserting S_1' into (36), the expectation values factorize from the sum over $\pi \in \sigma_{r+1, \ldots, r+q+1}$ and one can isolate the factor

$$\sum_{\pi \in \sigma_{r+1, \ldots, r+q+1}} \prod_{l=r+1}^{r+q} \theta_\beta(\mu_{\pi(l)} - \mu_{\pi(l+1)}) \tag{43}$$

which turns out to be independent of the variables $\{\mu_j\}_{j=r+1}^{r+q+1}$ due to the symmetry of the function β.

To complete the proof of Proposition 5 we perform the sums and integrations over $\{z_j, \mu_j\}_{j=r+1}^{r+q+1}$, taking into account that (43) is constant. The expression to compute is therefore

$$\sum_{\{z_a\}_{a=r+1}^{r+q+1}} \int_{\mathbb{R}^{q+1}} \{d\mu_b\}_{b=r+1}^{r+q+1} \prod_{a=r+1}^{r+q+1} g_a^{(v_a)}(z_a, \mu_a)$$

$$\times \sum_{\pi^{\dagger} \in \sigma_{r+1,\dots,2r}} \prod_{a=1}^{r} \mathcal{W}_T^{(2)}(z_{\hat{\pi}(a)}, \mu_{\hat{\pi}(a)}; z_{\pi^{\dagger}(a+r)}, \mu_{\pi^{\dagger}(a+r)}), \tag{44}$$

which equals

$$\sum_{\pi^{\dagger} \in \sigma_{r+1,\dots,2r}} \prod_{a=1}^{r} (A(\mu_{\hat{\pi}(a)}, z_{\hat{\pi}(a)})\Omega, (\phi_{g_{\pi^{\dagger}(a+r)}})_{\text{out}}^{\text{in}}) = 0, \tag{45}$$

since, after (40), for $r+1 \leq j \leq r+q+1$ one has $(\phi_{g_j})_{\text{out}}^{\text{in}} = 0$ because these vectors are of the form $\dfrac{d}{ds_i}\phi_i = 0$, $m-q \leq i \leq m$. This completes the proof of Proposition 5. $\quad\square$

Remark. For the sums S_1'' and S_2 there will be for each term at least one factor, a truncated expectation value, smeared with wave functions $\{g_{a_i}^{(v_{a_i})}\}_{i=1}^{v}$, $v \geq 2$, with the property

$$\bigcap_{i=1}^{v} V(\Delta(g_{a_i})) = \emptyset, \tag{46}$$

i.e., there are at least two wave functions in $\{g_{a_i}^{(v_{a_i})}\}_{i=1}^{v}$ whose velocities do not overlap. Note that this is not the case for the sum S_1' since there we may have partitions like $\mathscr{P}_2' = \{(1, 2r), (2, 2r-1), \dots, (r, r+1)\}$ coupling pairs of wave functions $(\bar{f}_m, f_m), \dots, (\bar{f}_n, f_n)$ to the same 2-point truncated vacuum expectation value. This is the reason why Proposition 5 is needed.

3.2.2. Proof of Proposition 6. According with the last remark we consider for $v \geq 2$ expressions of the form

$$\sum_{z_{a_1}, \dots, z_{a_v}} \int_{\mathbb{R}^v} d\mu_{a_1} \dots d\mu_{a_v} \prod_{i=1}^{v} |g_{a_i}^{(v_{a_i})}(z_{a_i}, \mu_{a_i})| \, |\mathcal{W}_T^{(v)}(z_{a_1}, \mu_{a_1}; \dots; z_{a_v}, \mu_{a_v})|, \tag{47}$$

with $\{a_1, \dots, a_v\} \subset \{1, \dots, 2r\}$, $\{g_{a_i}\}_{i=1}^{v}$ satisfying (46). First, we restrict the sums $z_{a_i} \in \mathbb{Z}^d$ to sums over $z_{a_i} \in v_{a_i} V(\Delta(g_{a_i}))$. This introduces for each a_i an error which is majorized, according with Proposition 1, by

$$C_N(1 + |v_{a_i}|)^{-N} \prod_{\substack{j=1 \\ j \neq a_i}}^{2r} (1 + |v_j|)^{d/2}, \quad \forall N \in \mathbb{N}. \tag{48}$$

But from condition B and from (39) for any a_i we have $v_{a_i} - s_m \leq q(\varepsilon s^{\gamma} + \delta)$ and so for s large enough

$$|v_{a_i}| \geq |s - q(\varepsilon s^{\gamma} + \delta)|, \tag{49}$$

implying that (48) falls faster than any power in s, by taking ε small. Using Theorem 2 and Proposition 1, (47) becomes for any $p, N \in \mathbb{N}$,

$$C_{p,N} \left[\sum{}' (1 + \|z\|)^{-(p-1-\varepsilon)} \right] \prod_{i=1}^{v} (1 + |v_{a_i}|)^{-d/2}$$

$$\times \left[\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d\mu_{a_1} \dots d\mu_{a_v} \left(\prod_{i=1}^{v} (1 + |\mu_{a_i} - v_{a_i}|)^{-N} \right) (1 + \|\mu\|)^p \right], \tag{50}$$

where $\|\underline{z}\| = \max|\underline{z}_{a_i} - \underline{z}_{a_j}|$, $\|\mu\| = \max|\mu_{a_i} - \mu_{a_{i+1}}|$, $C_{p,N}$ is a constant and where $\sum'$ is a sum over $\underline{z}_{a_i} \in v_{a_i} V(\Delta(g_{a_i}))$, $1 \leq i \leq v$.

For each μ_{a_i} we proceed the change of variables $\mu_{a_i} \to \mu_{a_i} + v_{a_i}$ and use the inequality $(1 + |x - y|)^p \leq (1 + |x|)^p (1 + |y|)^p$, $\forall p \geq 0$, to get for the integrals above the bound $\text{const}(1 + \|v\|)^p$. Now, by condition B and (39)

$$\|v\| = \max|v_{a_i} - v_{a_{i+1}}| \leq q(\varepsilon s^\gamma + \delta). \tag{51}$$

To find an upper bound for the sums in (50) we use for $\underline{z}_{a_i} \in v_{a_i} V(\Delta(g_{a_i}))$, $\forall i$, the inequality $\|\underline{z}\| \geq \min|\underline{v}_{a_i} v_{a_i} - \underline{v}_{a_j} v_{a_j}|$, where the minimum is taken over all $\underline{v}_{a_i} \in V(\Delta(g_{a_i}))$, $\underline{v}_{a_j} \in V(\Delta(g_{a_j}))$, for some pair (g_{a_i}, g_{a_j}) with $V(\Delta(g_{a_i})) \cap V(\Delta(g_{a_j})) = \emptyset$. But

$$|\underline{v}_{a_i} v_{a_i} - \underline{v}_{a_j} v_{a_j}| = |\underline{v}_{a_i}(v_{a_i} - v_{a_j}) - (\underline{v}_{a_j} - \underline{v}_{a_i}) v_{a_j}|$$
$$\geq \|\underline{v}_{a_i}| |v_{a_i} - v_{a_j}| - |\underline{v}_{a_j} - \underline{v}_{a_i}| |v_{a_j}\|. \tag{52}$$

From condition B $|v_{a_i} - v_{a_j}| \leq q(\varepsilon s^\gamma + \delta)$ and using (49) we conclude that, choosing s large enough, we will get

$$\|\underline{v}_{a_i}| |v_{a_i} - v_{a_j}| - |\underline{v}_{a_j} - \underline{v}_{a_i}| |v_{a_j}\|$$
$$\geq (s - q(\varepsilon s^\gamma + \delta))|\underline{v}_{a_i} - \underline{v}_{a_j}| - |\underline{v}_{a_i}| q(\varepsilon s^\gamma + \delta) \geq K(1 + s) \tag{53}$$

for some constant K, by choosing ε small enough. The sums in (50) are then bounded by

$$\text{const}\left[\prod_{i=1}^{v}(1 + |v_{a_i}|)^{d/2}\right](1 + s)^{-(p-1-\varepsilon)}. \tag{54}$$

We conclude from (51) and (54) that (50) has for all $p \in \mathbb{N}$ the upper bound

$$C_p' s^{dv/2}(1 + s^\gamma)^p (1 + s)^{-p+1+\varepsilon}. \tag{55}$$

Taking $\gamma < 1$ we see that (50) falls faster than any power in s. This completes the proof of Proposition 6 and consequently of Theorem 3. $\quad\square$

Remark. Note that (55) is the point where the condition $\gamma < 1$ becomes necessary. Its need is a consequence of the fact that in Theorem 2 the clustering property of the Wightman functions was proven for a region smaller than the usual region of space-like separated points.

4. The Reduction Formulae

We are able now to derive the LSZ reduction formulae. Using Theorem 3 and the same kind of argument used in the proof of Proposition 3 one shows straightforwardly that

$$(2\pi)^{-(d+1)n/2} \int_{\mathbb{R}^n} ds \sum_{\underline{x}_1,\ldots,\underline{x}_n} \int_{\mathbb{R}^n} dt \prod_{i=1}^{n} \hat{f}_i^{(s_i)}(\underline{x}_i, t_i)\left(\Omega, T_\beta\left[\prod_{j=1}^{n} A(t_j, \underline{x}_j)\right]\Omega\right)$$
$$= \sum_{X \subset \{1,\ldots,n\}} (-1)^{|X|}\left(\prod_{a \in \{1,\ldots,n\}\setminus X} A^{\text{out}}(f_a)^*\Omega, \prod_{b \in X} A^{\text{in}}(f_b)\Omega\right), \tag{56}$$

where $ds = ds_1 \ldots ds_n$, $dt = dt_1 \ldots dt_n$ and where $A^{\text{in/out}}(f)$ are creation operators of in/out one-particle states with wave function f.

Since supp $\widetilde{f}_i$ contain only the one-particle shell of sp(H, P) and since the f_i's are assumed to have non-overlapping velocities, what implies non-overlapping momenta, we conclude the following identity:

$$((\phi_1 \times \ldots \times \phi_m)_{\text{out}}, (\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}}) = (2\pi)^{-(d+1)n/2}(-1)^{n-m}$$

$$\times \int_{\mathbb{R}^n} ds \sum_{\underline{x}_1, \ldots, \underline{x}_n} \int_{\mathbb{R}^n} dt \left\{ \prod_{i=1}^m \dot{\widetilde{f}}_i^{(s_i)}(\underline{x}_i, t_i) \right\} \left\{ \prod_{j=m+1}^n \dot{f}_j^{(s_j)}(\underline{x}_j, t_j) \right\}$$

$$\times \left(\Omega, T_\beta \left[\prod_{k=1}^m A(t_k, \underline{x}_k)^* \prod_{l=m+1}^n A(t_l, \underline{x}_l) \right] \Omega \right). \tag{57}$$

The integrand in s may be written after Fourier transformation as

$$\int_{\mathbb{R}^{n(d+1)}} d^{n(d+1)}p \left\{ \prod_{i=1}^m e^{i(\omega(p_i)-p_i^0)s_i} \overline{\widetilde{f}}_i(p_i) \right\} \left\{ \prod_{j=m+1}^n e^{-i(\omega(p_j)-p_j^0)s_j} \widetilde{f}_j(p_j) \right\}$$

$$\times (2\pi)^{-(d+1)n/2}(i)^n \left[\prod_{k=1}^n (\omega(\underline{p}_k)-p_k^0) \right] \tau_\beta(-p_1, \ldots, -p_m, p_{m+1}, \ldots, p_n), \tag{58}$$

where $d^{n(d+1)}p = d^{d+1}p_1 \ldots d^{d+1}p_n$, and where τ_β is the distributional Fourier transform of

$$\left(\Omega, T_\beta \left[\prod_{k=1}^m A(t_k, \underline{x}_k)^* \prod_{l=m+1}^n A(t_l, \underline{x}_l) \right] \Omega \right). \tag{59}$$

Note that Theorem 3 tells that

$$\prod_{k=1}^n [\omega(\underline{p}_k)-p_k^0]\tau_\beta(-p_1, \ldots, -p_m, p_{m+1}, \ldots, p_n) \tag{60}$$

is C^∞ around the mass shell. Interpreting (58) as a Fourier transformation on the variables $q_i^0 := p_i^0 - \omega(p_i)$ the integrations on ds give

$$((\phi_1 \times \ldots \times \phi_m)_{\text{out}}, (\phi_{m+1} \times \ldots \times \phi_n)_{\text{in}})$$

$$= i^n(2\pi)^{-(d+1)n/2} \int_{\mathbb{R}^{n(d+1)}} d^{n(d+1)}p \left\{ \prod_{i=1}^m \overline{\widetilde{f}}_i(p_i) \right\} \left\{ \prod_{j=m+1}^n \widetilde{f}_j(p_j) \right\}$$

$$\times \prod_{k=1}^n [(\omega(\underline{p}_k)-p_k^0)\delta(\omega(\underline{p}_k)-p_k^0)]\tau_\beta(-p_1, \ldots, -p_m, p_{m+1}, \ldots, p_n). \tag{61}$$

The validity of this relation is obvious in the case where the distribution (60) is an element of $\mathscr{S}(\mathbb{R}^{n(d+1)})$. The general case follows from this one since $\mathscr{S}$ is weakly dense in $\mathscr{S}'$, the dual of $\mathscr{S}$, and since the Fourier transformation is weakly continuous on $\mathscr{S}'$ (see also [8, Sect. IX.1]). Expressions (61) are the LSZ reduction formulae in momentum space.

Write $p_j^0 \widetilde{f}_j(p_j) = (\widetilde{i\partial_{t_j} f_j})(p_j)$ and $\omega(\underline{p}_j)\widetilde{f}_j(p_j) = (\widetilde{Ef_j})(p_j)$, where '$\widetilde{}$' denotes Fourier transform and the E is defined by

$$(Ef)(\underline{x}, t) := (2\pi)^{-n/2} \int_{(-\pi, \pi]^d \times \mathbb{R}} \omega(\underline{p})\widetilde{f}(p)e^{i(\underline{p}\cdot\underline{x}-p_0 t)}d^{d+1}p, \tag{62}$$

and define the distribution $\Delta(\underline{x}, t)$ so that $\widetilde{\Delta}(\underline{p}, p_0) = \delta(\omega(\underline{p})-p_0)$, explicitly:

$$\Delta(\underline{x}, t) = (2\pi)^{-(d+1)/2} \int_{(-\pi, \pi]^d} e^{i(\underline{p}\cdot\underline{x}-\omega(\underline{p})t)}d^d\underline{p}. \tag{63}$$

Then we get

$$
((\phi_1 \times \ldots \times \phi_m)_{\mathrm{out}}, (\phi_{m+1} \times \ldots \times \phi_n)_{\mathrm{in}}) = i^n(-1)^{m(d+1)}(2\pi)^{-(d+1)n/2}
$$
$$
\times \sum_{\underline{x}_1, \ldots, \underline{x}_n} \int_{\mathbb{R}^n} \{dt_a\}_{a=1}^n \left(\Omega, T_\beta \left[\prod_{k=1}^m A(t_k, \underline{x}_k)^* \prod_{l=m+1}^n A(t_l, \underline{x}_l) \right] \Omega \right)
$$
$$
\times \left[\prod_{i=1}^m \overline{(\Delta * ((E - i\partial_{t_i})f_i))(\underline{x}_i, t_i)} \right] \left[\prod_{j=m+1}^n (\Delta * ((E - i\partial_{t_j})f_j)(\underline{x}_j, t_j)) \right], \quad (64)
$$

which are the LSZ reduction formulae in coordinate space.

The main distinction between these formulae and the corresponding ones in relativistic quantum field theories is the use of almost local fields with finite energy transfer in the Wightman functions. This comes from the need to cut out the high energy component of the fields in order to get control on the clustering properties of Wightman functions, as described in [1]. Since in our basic setting no special assumption on the high energy behavior of the theory was made, it is probably not possible in general to eliminate this restriction on the high energy of the fields without violating our basic results. Additional assumptions on the locality properties of the high energy of the fields are then necessary.

An additional remark concerns the time-ordered functions occurring in (61) and (64). In continuum theories, time-ordered functions can be obtained, at least formally, through analytical continuation of Euclidean expectation values (for an analysis in the framework of the Osterwalder-Schrader axioms see [9]). This possibility was studied by Lüscher [4] in the context of Euclidean lattice models of scalar fields, with the result that, for these lattice models, the formally defined time-ordered functions and the functions obtained by analytical continuation of Euclidean expectation values differ at asymptotic times by exponentially falling factors. It is presently not clear, whether Lüscher's results may be reproduced for our reduction formulae, since the restriction on the high energy component of the fields is essential in our analysis.

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