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Scattering States of Charged Particles in the $\mathbb{Z}_2$ Gauge Theories

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Abstract. Scattering states of charged particles in a massive Euclidean lattice gauge model are constructed.

1 Introduction

The particle spectrum of Euclidean Quantum Field Theories on the lattice has been the object of extensive studies in various models (see [2, 10] for references). Recently, under general assumptions (essentially existence of a transfer matrix and mass gap), a full construction of the scattering states for particles of the vacuum sector of those theories has been performed [3] following the ideas of Haag and Ruelle [7, 8]. This work extends the main result of [3], namely the construction of multiparticle states, to the charged particles of the $\mathbb{Z}_2$ Higgs model whose existence has been shown in [2]. The construction presented here depends in some details on particularities of this model but they might certainly be adapted in its essential tools to other massive models involving charged particles. In the general framework of relativistic quantum fields the construction of the scattering states of charged particles in massive theories was performed in [9].

As in [3], the main problem to be overcome is the lack of locality (Einstein causality) of the real-time evolution. Following [3] we by-pass this problem by making use of the exponential decay of certain Euclidean correlations, a fact related to the existence of a mass gap in the spectrum of the Hamiltonian operator.

1.1. The Model and Previous Results

The $\mathbb{Z}_2$ gauge-Higgs lattice model is particularly interesting for testing structural properties of gauge theories. Detailed results on the superselection sectors'

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structure of its associated quantum spin system in the "free charges" region of its phase diagram have been obtained in [1] (see also [5]). That work established for that region of the phase diagram the existence of two inequivalent sectors, the vacuum sector and a charged one (with associated Hilbert spaces here denoted by $\mathscr{H}_0$ and $\mathscr{H}_+$, respectively). These sectors are believed to be the only existing ones in this model (in $d+1=3$ there is also a magnetically charged sector). In [1] charged states with finite energy have been explicitly constructed and in [2] it has been shown that corresponding charged particle state exist in $\mathscr{H}_+$. The present work completes the next step of showing the existence of multiparticle states with even (in $\mathscr{H}_0$) and odd (in $\mathscr{H}_+$) charged particles. Our notation follows [1] and [2] closely.

The $\mathbb{Z}_2$ gauge-Higgs model has the action

$$S=\sum_p \beta_g \delta\tau(p)+\sum_b \beta_h \tau(b)\delta\sigma(b), \tag{1}$$

where σ and τ are Ising fields living on sites and bonds respectively of $\mathbb{Z}^{d+1}$ ($d \geqq 2$), representing Higgs fields and gauge fields respectively, and where β_g and β_h are positive coupling constants. Above δ denotes the lattice exterior derivative:

$$\delta\tau(p)=\prod_{b\in\partial p}\tau(b), \qquad \delta\sigma(b)=\prod_{x\in\partial b}\sigma(x), \tag{2}$$

where ∂p is the set of bonds contained in the plaquette p and ∂b is the set of sites contained in the bound b.

The results of [1] and [2] have been obtained for $g:=e^{-2\beta_g}$ and $h:=\tanh\beta_h$ sufficiently small, a restriction maintained here to provide the necessary convergence of the expansions.

For the quantum spin system associated to this model the time-zero field algebra is generated by hermitian operators associated to the sites of $\mathbb{Z}^d$, $\sigma_1(\underline{x})$, $\sigma_3(\underline{x})$, and hermitian operators associated to the bonds of $\mathbb{Z}^d$, $\tau_1(\underline{b})$, $\tau_3(\underline{b})$, satisfying the algebra of Pauli matrices and commuting at different points (the σ-operators also commute with the τ-operators). The operators σ_3 and σ_1 are analogues of the Higgs field and its canonically conjugated momentum, and the τ_3 and τ_1 operators correspond to the gauge field and electric field respectively. These operators generate local and global algebras of fields and gauge invariant observables and in [1] a translation invariant vacuum state and translation covariant charged states have been constructed, to which two inequivalent representations of the algebras, one in space $\mathscr{H}_0$ and the other in $\mathscr{H}_+$, are associated.

In algebraic level the euclidean dynamics is generated by an automorphism defined as the strong limit of local automorphisms implemented by local transfer matrices, and is interpreted as the action of discrete euclidean time translations. It is implemented in $\mathscr{H}_0$ and in $\mathscr{H}_+$ by two inequivalent global transfer matrices with densely defined inverses [1, 4].

To simplify the notation we shall denote both transfer matrices by the same symbol, T, and shall not distinguish the representatives of σ_i and τ_j, irrespective to which they are acting in $\mathscr{H}_0$ or in $\mathscr{H}_+$, and shall denote then again by $\sigma_3(\underline{x})$, $\tau_3(\underline{b})$, etc. The action of the space displacements by $\underline{x}\in\mathbb{Z}^d$ is implemented by unitaries denoted in both cases by $U(\underline{x})$.

Real-time translations are then defined in $\mathscr{B}(\mathscr{H}_{0,+})$ by

$$\alpha_t(\cdot)=T^{-it}\cdot T^{+it}, \qquad t\in\mathbb{R}. \tag{3}$$

The following important result ([1], Theorem 6.4) has to be mentioned:

For any set of distinct points $\{\underline{x}_1,\ldots,\underline{x}_n\}\subset\mathbb{Z}^d$ there are eigenvectors of the transfer matrix $\phi_{\underline{x}_1,\ldots,\underline{x}_n}\in\mathscr{H}_0$ or $\mathscr{H}_+$ (according whether n is even or odd, *respectively) inducing ground states (in the sense of* [1]) *with a configuration of external charges in the points* $\{\underline{x}_1, \ldots, \underline{x}_n\}$. *The vectors* $\phi_{\underline{x}_1, \ldots, \underline{x}_n}$ *are covariant under space translations:* $U(\underline{y})\phi_{\underline{x}_1, \ldots, \underline{x}_n} = \phi_{\underline{x}_1 - \underline{y}, \ldots, \underline{x}_n - \underline{y}}$, $\forall \underline{y} \in \mathbb{Z}^d$.

The eigenvalues of $\phi_{\underline{x}_1, \ldots, \underline{x}_n}$ are denoted here by $\beta_{\underline{x}_1, \ldots, \underline{x}_n}$. For $n=1$ we call $\beta_{\underline{x}} = \beta$, for any $\underline{x}$, by translation invariance.

The importance for us of the vectors $\phi_{\underline{x}_1, \ldots, \underline{x}_n}$ is the following. The gauge invariant vectors $\prod_{i=1}^{n} \sigma_3(\underline{x}_i)\phi_{\underline{x}_1, \ldots, \underline{x}_n}$ can be interpreted as states of n dynamical charges located at the points $\{\underline{x}_1, \ldots, \underline{x}_n\}$. This suggests the use of vectors of the form $\prod_{i=1}^{n} \alpha_{t_i}(\sigma_3(\underline{x}_i))\phi_{\underline{x}_1, \ldots, \underline{x}_n}$, after adequate smearings, as approximants for the multiparticle states, replacing the vectors like $\varphi(x_1)\ldots\varphi(x_n)\Omega$, where $\varphi(x)$ are charged fields, used in the standard Haag-Ruelle construction. Charged fields connecting $\mathcal{H}_0$ and $\mathcal{H}_+$ are for the model presently not available (but there is an announced result by Szlachanyi [6]) and here we show how to proceed in this case. Otherwise the methods of [3] could in principle be used.

The following result on the existence of one-particle charged states has been established in [2]:

The Fourier transform of the 2-point function

$$G(x_0, \underline{x}) = (\sigma_3(\underline{0})\phi_{\underline{0}}, U(\underline{x})T^{|x_0|}\sigma_3(\underline{0})\phi_{\underline{0}}) \tag{4}$$

can be analytically extended for each $\underline{p} \in (-\pi, \pi]^d$ *to a meromorphic function of* p_0 *in the region* $\operatorname{Im} p_0 < \hat{\omega}(\underline{p})$ *with an isolated simple pole at* $p_0 = i\omega(\underline{p})$, *where* $\omega(\underline{p})$, *the energy-momentum relation of the particle, is smooth and* $\hat{\omega}(\underline{p})$ *is continuous with* $\hat{\omega}(\underline{p}) > \omega(\underline{p}) \geqq m$, *m being the mass gap. The velocity* $v(\underline{p}) = \operatorname{grad}\omega(\underline{p})$ *is nowhere constant.*

This implies that there is a closed subspace $\mathcal{H}_+^{(1)}$ of $\mathcal{H}_+$ (the single particle subspace) on which the relation

$$(T - e^{-\omega(\underline{P})}) \,|\, \mathcal{H}_+^{(1)} = 0 \tag{5}$$

holds. Here $\underline{P}$ is the momentum operator, i.e. the infinitesimal generator of spatial translations,

$$e^{i\underline{P}\cdot\underline{x}} = U(\underline{x}), \qquad sp(\underline{P}) \subset (-\pi, \pi]^d, \tag{6}$$

and $\mathcal{H}_+^{(1)}$ is the closure of the linear space

$$\mathcal{D}^{(1)} = \Big\{\Psi_f, \Psi_f = \sum_{\underline{x}} \int dt \beta^{-it} f(\underline{x}, t)\alpha_t(\sigma_3(\underline{x}))\phi_{\underline{x}},$$

$$\operatorname{supp}\tilde{f} \cap sp(H, \underline{P}) \subset \{(\omega(\underline{p}), \underline{p}), \underline{p} \in (-\pi, \pi)^d\}, \tilde{f} \in \mathcal{D}(\mathbb{R}^{d+1})\Big\}, \tag{7}$$

H being the Hamiltonian defined as $H = -\ln T$.

The results of [1] (and of [2]) have been obtained with the use of polymer and cluster expansions for the "free charges" region of the phase diagram of the model. We resume here the most important ingredients of those expansions, since the results of Sect. 4 make strongly use of them. For details see [1, 2]. The polymers are pairs $\gamma = \{P_\gamma, N_\gamma\}$, where P_γ is a coclosed set of plaquettes, N_γ a closed set of bonds, γ being closed as a graph, where the graphs in question are constructed in the following way: The vertices are the co-connected components P_i of P_γ and the connected components N_j of N_γ and the edges are pairs $\{P_i, N_j\}$, where N_j winds an odd number of times $\omega(P_i, N_j)$ around P_i. For general P_i, N_j define

$$(P_i, N_j) = (-1)^{\omega(P_i, N_j)}. \tag{8}$$

For the definition of a polymer model one needs a definition of compatibility between polymers. Two polymers γ_1, γ_2 are compatible, $\gamma_1 \sim \gamma_2$, if no elementary 3-cube has plaquettes in P_{γ_1} and P_{γ_2} as faces, if no point is a boundary point of bonds in N_{γ_1} and N_{γ_2} and if co-connected component of P_{γ_1} has a odd winding number with a connected component of N_{γ_2} and vice-versa. They are called incompatible $\gamma_1 \nsim \gamma_2$ otherwise.

The activity of a polymer γ is

$$\mu(\gamma) = h^{|N_\gamma|} g^{|P_\gamma|} (P_\gamma, N_\gamma) \tag{9}$$

with $(P_\gamma, N_\gamma) = \prod_{i,j} (P_i, N_j)$, $|N_\gamma|$ being the number of bonds in N_γ and $|P_\gamma|$ being the number of plaquettes in P_γ.

Let χ_L denote the product of link variables for a set L of bonds. Then one gets for its vacuum expectation value the expression

$$\langle \chi_L \rangle = \sum_{M \in \operatorname{Conn}(L)} h^M \exp\left\{ \sum_{\Gamma} c_\Gamma \mu^\Gamma (a^\Gamma_{L,M} - 1) \right\}. \tag{10}$$

Above $\operatorname{Conn}(L)$ denotes the set of all sets of bonds M with $\partial M = \partial K$; $h^M = h^{|M|}$; Γ are clusters of polymers, i.e., nonnegative integer valued functions with finite support on the set of all polymers, $\mu^\Gamma = \prod_\gamma \mu(\gamma)^{\Gamma(\gamma)}$ (multi-index notation). The coefficients c_Γ are the Ursell functions, are of purely combinatorial nature, and $a_{L,M}$ is defined by

$$a_{L,M}(\gamma) = \begin{cases} 0, & \text{if } N_\gamma \text{ is connected with } M, \\ (P_\gamma, L \Delta M), & \text{otherwise.} \end{cases} \tag{11}$$

If $M = \emptyset$ we write $a_{L,\emptyset} = a_L$.

The following results are often used. One is a remarkable property of the Ursell functions: if $\Gamma = \Gamma_1 + \Gamma_2$ with $\gamma_1 \sim \gamma_2$ for all $\gamma_1 \in \operatorname{supp} \Gamma_1$, $\gamma_2 \in \operatorname{supp} \Gamma_2$ then $c_\Gamma = 0$. The other are the following estimates:

$$\sum_{\Gamma \nsim \gamma} |c_\Gamma| \, |\mu^\Gamma| \leqq F_1(-\ln \|\mu\|) \, |\gamma| \,, \tag{12}$$

$$\sum_{\Gamma \nsim \gamma, \, \|\Gamma\| \geqq n} |c_\Gamma| \, |\mu^\Gamma| \leqq \left(\frac{\|\mu\|}{\|\mu_c\|} \right)^n F_1(-\ln \|\mu_c\|) \, |\gamma| \,, \tag{13}$$

for $\|\mu\| = \sup_\gamma |\mu(\gamma)|^{1/|\gamma|} = \max\{h, g\}$; where $\|\mu_c\|$ is a fixed constant with $F_1(-\ln \|\mu_c\|) < \infty$; where $\|\Gamma\| = \sum \Gamma(\gamma) |\gamma|$ for $|\gamma| = |P_\gamma| + |N_\gamma|$, and where F_1 is a monotonically decreasing function (see [1], Appendix). The convergence of sums like (10) follows from (12) together with $|\{M : M \in \operatorname{Conn}(L), |M| \leqq n\}| \leqq |L| (2d+1)^n$, for h small.

2. The Construction of the Scattering States

Let E_λ denote the Spectral projections of T and define $h(T) = \int_{(0,1]} h(\lambda) dE_\lambda$, where $h \in C^\infty([0,1])$, $0 \leqq h(x) \leqq 1$, $x \in [0,1]$, with $h(e^{-y}) = 1$ for $0 \leqq y \leqq y_1$, $h(e^{-y}) = 0$ for $y \geqq y_2$ for $y_2 > y_1$. For the construction of n-particle states we shall need $y_1 > n \cdot \sup\{\omega(p), p \in (-\pi, \pi]^d\}$, the Maximal energy of a n-particle state. Below we shall mostly use $g(x) := h(x)^2$.

We consider the one-particle states written in the form

$$\Psi_f = \sum_{\underline{x}} \int dt\, f^{(s)}(\underline{x}, t) h(T) \alpha_t(\sigma_3(\underline{x}))\phi_{\underline{x}} \tag{14}$$

with

$$f^{(s)}(\underline{x}, t) = \frac{\beta^{-it}}{(2\pi)^{(d+1)/2}} \int dp \tilde{f}(\underline{p}, p_0) e^{-ip_0 t - i(\omega(\underline{p}) - p_0)s + i\underline{p}\underline{x}}, \qquad s \in \mathbb{R}. \tag{15}$$

For $\tilde{f} \in \mathscr{D}(\mathbb{R}^{d+1})$ we define the velocity content of $\operatorname{supp} \tilde{f}$ as the set of values of the group velocity associated to it:

$$V(\tilde{f}) = \{\operatorname{grad} \omega(\underline{p}),\ (\omega(\underline{p}), \underline{p}) \in \operatorname{supp} \tilde{f} \cap sp(H, \underline{P})\}. \tag{16}$$

The functions $f^{(s)}$ have the following decay properties (see [12]):

Proposition 1. *For $f^{(s)}$ as in* (15) *with $\tilde{f} \in \mathscr{D}(\mathbb{R}^{d+1})$ we have:*
a) *For all $N \in \mathbb{N}$ there are constants $C_N > 0$ so that*

$$|f^{(s)}(\underline{x}, t)| \leqq C_N (1 + |s|)^{-d/2} (1 + |t - s|)^{-N} \tag{17}$$

uniformly in $\underline{x}$.
b) *There exists a positive constant C so that for every s,*

$$\sum_{\underline{x} \in \mathbb{Z}^d} \int_{-\infty}^{\infty} dt |f^{(s)}(\underline{x}, t)| \leqq C(1 + |s|)^{d/2}. \tag{18}$$

c) *For all $L, M, N \in \mathbb{N}$ there is a positive constant $C_{L,M,N}$ so that, if for all s with $|s| > 1$,*

$$\operatorname{dist}\left(\frac{\underline{x}}{|s|}, V(\tilde{f})\right) > \delta \tag{19}$$

for some constant $\delta > 0$, then

$$|f^{(s)}(\underline{x}, t)| \leqq C_{L,M,N} (1 + |s - t|)^{-L} (1 + |s|)^{-M} (1 + |\underline{x}|)^{-N}. \tag{20}$$

To follow the Haag-Ruelle construction we propose to use the vectors

$$\begin{aligned} \Psi_{f_1, \ldots, f_n}(s) := \sum_{\underline{x}_1, \ldots, \underline{x}_n} \int dt_1 \ldots dt_n \prod_{i=1}^{n} f_i^{(s)}(\underline{x}_i, t_i) (\beta_{\underline{x}_1, \ldots, \underline{x}_n})^{-it_n} \\ \times h(T) \alpha_{t_1}(\sigma_3(\underline{x}_1)) \prod_{j=2}^{n} h(T)^2 \alpha_{t_j}(\sigma_3(\underline{x}_j)) \phi_{\underline{x}_1, \ldots, \underline{x}_n} \end{aligned} \tag{21}$$

as approximants for the scattering states for $s \to \pm\infty$. We have

$$\begin{aligned} \left\| \frac{d}{ds} \Psi_{f_1, \ldots, f_n}(s) \right\|^2 = \sum_{k, k' = 1}^{n} \sum_{\underline{x}_1, \ldots, \underline{x}_n} \sum_{\underline{x}'_1, \ldots, \underline{x}'_n} \int dt_1 \cdots dt_n dt'_1 \cdots dt'_n \\ \times \prod_{i=1}^{n} f_{i,k}^{(s)}(\underline{x}_i, t_i) \prod_{i=1}^{n} \overline{f_{j,k'}^{(s)}(\underline{x}'_j, t'_j)} F(\underline{x}, x', t, t'), \end{aligned} \tag{22}$$

where

$$f_{j,k}^{(s)} := \begin{cases} f_j^{(s)}, & j \neq k \\ \dfrac{d}{ds} f_j^{(s)}, & j = k, \end{cases} \tag{23}$$

and

$$F(\underline{x},\underline{x}',t,t'):=(\beta_{\underline{x}'_1,\ldots,\underline{x}'_n})^{it'_n}(\beta_{\underline{x}_1,\ldots,\underline{x}_n})^{-it_n}$$
$$\times\left(\alpha_{t'_1}(\sigma_3(\underline{x}'_1))\left\{\prod_{j=2}^{n} g(T)\alpha_{t'_j}(\sigma_3(\underline{x}'_j))\right\}\phi_{\underline{x}'_1,\ldots,\underline{x}'_n},\left\{\prod_{i=1}^{n} g(T)\alpha_{t_i}(\sigma_3(\underline{x}_i))\right\}\phi_{\underline{x}_1,\ldots,\underline{x}_n}\right). \tag{24}$$

The main result is the following Theorem:

Theorem 1. *Let* $\{\tilde{f}_i\}_{i=1}^n\subset\mathscr{D}(\mathbb{R}^{d+1})$ *with* $\{\Psi_{f_i}\}_{i=1}^n\subset\mathscr{D}^{(1)}$, $\Psi_{f_i}\neq 0$, *and with non-overlapping velocities:* $V(\tilde{f}_i)\cap V(\tilde{f}_j)$, $i\neq j$. *Then:*

i) *For each* $n\in\mathbb{N}$ *there is a positive constant* C_n *so that for all* $s\in\mathbb{R}$,

$$\left\|\frac{d}{ds}\Psi_{f_1,\ldots,f_n}(s)\right\|\leqq C_n(1+|s|)^{-n}. \tag{25}$$

ii) *The strong limits*

$$\lim_{s'\to\pm\infty}\Psi_{f_1,\ldots,f_n}(s') \tag{26}$$

exist in $\mathscr{H}_0$ *or* $\mathscr{H}_+$ *(according with n being even or odd respectively) and the convergence to the limit vectors, denoted respectively by* $\Psi^{\text{out/in}}_{f_1,\ldots,f_n}$ *is faster than any power in s for* $s\to\pm\infty$ *respectively.*

iii) *For* $\psi_{f_1,\ldots,f_k}$, $\psi_{g_1,\ldots,g_n}$ *as given above then*

$$(\psi^{\text{out/in}}_{f_1,\ldots,f_k},\psi^{\text{out, in}}_{g_1,\ldots,g_n})=\delta_{n,k}\sum_{\eta}\prod_{i}(\psi_{f_i},\psi_{g_{\eta(i)}}), \tag{27}$$

where the sum is over elements η *of the permutation group of* $\{1,\ldots,n\}$.

Remarks. Above, ii is an immediate consequence of i. The proof of part iii will not be given here since, as in the relativistic case, it follows the ideas of the proof of part ii. Parts ii and iii establish the existence of asymptotic particle states and the statistics of the particles (bosonic in this case).

Definition 1. To simplify the notation we introduce the ordered sets

$$(\mu_1,\ldots,\mu_n,\mu_{n+1},\ldots,\mu_{2n}):=(t'_n,\ldots,t'_1,t_1,\ldots,t_n) \tag{28}$$

and

$$(\underline{z}_1,\ldots,\underline{z}_n,\underline{z}_{n+1},\ldots,\underline{z}_{2n}):=(\underline{x}'_n,\ldots,\underline{x}'_1,\underline{x}_1,\ldots,\underline{x}_n), \tag{29}$$

and write

$$F(\underline{x},\underline{x}',t,t'):=\left(\phi_{\underline{z}_1,\ldots,\underline{z}_n},\left[\prod_{i=1}^{2n-1}\sigma_3(\underline{z}_i)g(T)T^{i(\mu_i-\mu_{i+1})}\right]\sigma_3(\underline{z}_{2n})\phi_{\underline{z}_{n+1},\ldots,\underline{z}_{2n}}\right). \tag{30}$$

Definition 2. We denote by σ set of all partitions of $\{\underline{z}_1,\ldots,\underline{z}_{2n}\}$ into ordered pairs such that

$$i\in\sigma\Rightarrow i=\{(\underline{z}_{i_1},\underline{z}_{i_{n+1}}),(\underline{z}_{i_2},\underline{z}_{i_{n+2}}),\ldots,(\underline{z}_{i_n},\underline{z}_{i_{2n}})\} \tag{31}$$

with $i_a\in\{1,\ldots,n\}$ and $i_{n+a}\in\{n,\ldots,2n\}$ for all a, $1\leqq a\leqq n$.

Proof of Theorem 1. One starts with the following result on clustering properties of $F(\underline{x},\underline{x}',t,t')$. The proof is given in the next subsection.

Theorem 2. *For $F(\underline{x}, \underline{x}', t, t')$ as above there are, for each $q \in \mathbb{N}$, positive constants $c'_{a,q}$, $1 \leqq a \leqq n$ so that*

$$|F(\underline{x}, \underline{x}', t, t') - G(\underline{x}, \underline{x}', t, t')| \leqq \sum_{a=1}^{n-1} c'_{a,q} \frac{(1 + \|t\|)^{qa}}{(1 + D_1^{(2)}(\underline{z}))^{(q-1)a}}, \tag{32}$$

with

$$G(\underline{x}, \underline{x}', t, t') := \left(\bigotimes_{a=1}^{n} T^{-it_a} \sigma_3(\underline{x}'_a) \phi_{\underline{x}'_a}, \prod_{b=1}^{2n-1} g(\mathscr{T}_{|n-b|}) \bigotimes_{c=1}^{n} T^{-it_a} \sigma_3(\underline{x}_c) \phi_{\underline{x}_c} \right) \tag{33}$$

for

$$\mathscr{T}_a := 1^{\otimes a} \otimes T^{\otimes (n-a)}, \qquad 0 \leqq a \leqq n, \tag{34}$$

where $\|t\| := \max_{i \neq j} |\mu_i - \mu_j|$ *and where*

$$D_1^{(2)}(\underline{z}) \equiv D_1^{(2)}(\underline{z}_1, \ldots, \underline{z}_{2n}) := \min \{ D_2^{(2)}(\underline{z}_1, \ldots, \underline{z}_{2n}), D_3^{(2)}(\underline{z}_1, \ldots, \underline{z}_{2n}) \}, \tag{35}$$

with

$$D_2^{(2)}(\underline{z}) \equiv D_2^{(2)}(\underline{z}_1, \ldots, \underline{z}_{2n}) := \min_{a, b \in \{1, \ldots, n\}, a \neq b} \{ |\underline{x}'_a - \underline{x}_b| \}, \tag{36}$$

and

$$D_3^{(2)}(\underline{z}) \equiv D_3^{(2)}(\underline{z}_1, \ldots, \underline{z}_{2n}) := \min \left\{ \min_{a \neq b} \{ |\underline{x}_a - \underline{x}_b| \}, \min_{c \neq d} \{ |\underline{x}'_c - \underline{x}'_d| \} \right\}. \tag{37}$$

Replacing (32) into (22) and using that $\sum_{k,k'=1}^{n} \prod_{i=1}^{n} (\psi_{f_{i,k}}, \psi_{f_{i,k'}}) = 0$, since the ψ_f are independent s, we get

$$\begin{aligned} \left\| \frac{d}{ds} \Psi_{f_1, \ldots, f_n}(s) \right\| \leqq \text{const.} \sum_{k,k'=1}^{n} \sum_{\underline{x}_1, \ldots, \underline{x}_n} \sum_{\underline{x}'_1, \ldots, \underline{x}'_n} \int dt_1 \cdots dt_n dt'_1 \cdots dt'_n \\ \times \prod_{i=1}^{n} |f_{i,k}^{(s)}(\underline{x}_i, t_i)| \prod_{j=1}^{n} |f_{j,k'}^{(s)}(\underline{x}'_j, t'_j)| \sum_{a=1}^{n-1} c'_a (1 + \|t\|)^{qa} (1 + D_1^{(2)}(\underline{z}))^{-(q-1)a} \end{aligned} \tag{38}$$

and from this the proof of Theorem 1 is completed by making use of the decay properties of the functions $f^{(s)}(\underline{x}, t)$ (Proposition 1) and the fact that they represent a set of wave functions with non-overlapping velocities, in complete analogy with the relativistic case (see [3, 8, 10]). □

2.1. *Proof of Theorem 2*

The first step to the proof of Theorem 2 is to approximate $g(T)T^{i\mu}$ by polynomials on the transfer matrix, following ideas of [3, 10]. This is possible since $g(T)T^{i\mu}$, in contrast to $T^{i\mu}$, is norm continuous in T. Using Chebishev polynomials for the approximation we write for $\mu \in \mathbb{R}$,

$$g(T)T^{i\mu} = E^{(N)}(\mu) + R^{(N)}(\mu), \tag{39}$$

where

$$E^{(N)}(\mu) = \sum_{m=0}^{N} a_m^{(N)}(\mu) T^m = \sum_{n=0}^{N} b_n(\mu) T_n(2T - 1) \tag{40}$$

is the approximating polynomial of degree N and the rest $R^{(N)}(\mu)$ is given by

$$R^{(N)}(\mu)=\int_{(0,1]}\mathscr{R}^{(N)}(\lambda,\mu)dE_\lambda, \quad \text{with} \quad \mathscr{R}^{(N)}(\lambda,\mu)=\sum_{n>N}b_n(\mu)T_n(2\lambda-1). \tag{41}$$

Above $T_n(\cdot)$, $n\in\mathbb{N}$, are Chebishev polynomials $T_n(x):=\cos(n\arccos x)$, $x\in[-1,1]$ and

$$b_n(\mu)=\pi^{-1}(2-\delta_{n,0})\int_{-\pi/2}^{\pi/2}g((\cos\alpha)^2)e^{2i\mu\ln\cos\alpha+2in\alpha}d\alpha. \tag{42}$$

Relation (42) comes from the fact that Chebishev polynomials form a complete orthogonal basis in $L^2([-1,1],(1-x^2)^{-1/2}dx)$ (see e.g. [11]) and the second equality in (40) is the defining relation of the $a_n^{(N)}$'s in terms of the b_n's (see also the Appendix). The two following lemmas hold:

Lemma 1. *For the $a_n^{(N)}$'s given in (40) there are functions $A(\alpha)>0$ and $C(\alpha)$ so that for $\alpha\geqq 0$,*

$$\gamma_N(\mu):=\sum_{n=0}^{N}|a_n^{(N)}(\mu)|e^{-\alpha n} \tag{43}$$

has the bound $\gamma_N(\mu)\leqq C(\alpha)e^{A(\alpha)N}$ uniformly in $\mu\in\mathbb{R}$, for all $N\in\mathbb{N}$. We can choose $A(\alpha)=2\operatorname{argsinh}(e^{-\alpha/2})$ and $C(\alpha)=2(e^{A(\alpha)})/(e^{A(\alpha)}-1)$.

Lemma 2. *For $\mathscr{R}^{(N)}(\lambda,\mu)$, $N\geqq 1$, $\mu\in\mathbb{R}$ defined in (41) there is for each $q\in\mathbb{N}$ a positive constant C_q so that $|\mathscr{R}^{(N)}(\lambda,\mu)|\leqq C_qN^{-(q-1)}(1+|\mu|)^q$.*

The proofs are given in the Appendix, see also [10, 3]. Replacing (39) into (30) we get $F(\underline{x},\underline{x}',t,t')=F_E(\underline{x},\underline{x}',t,t')+F_R(\underline{x},\underline{x}',t,t')$, where

$$F_E(\underline{x},\underline{x}',t,t')=\sum_{n_1,\ldots,n_{2n-1}=0}^{N}\prod_{a=1}^{2n-1}a_{n_a}^{(N)}(\mu_a-\mu_{a+1})I(\underline{z},\{n_j\}) \tag{44}$$

for

$$I(\underline{z},\{n_j\}):=\left(\phi_{\underline{z}_1,\ldots,\underline{z}_n},\left[\prod_{i=1}^{2n-1}\sigma_3(\underline{z}_i)T^{n_i}\right]\sigma_3(\underline{z}_{2n})\phi_{\underline{z}_{n+1},\ldots,\underline{z}_{2n}}\right), \tag{45}$$

and

$$F_R(\underline{x},\underline{x}',t,t')$$
$$=\sum_{\substack{B\subset\{1,\ldots,2n-1\}\\ B\neq\emptyset}}\left(\phi_{\underline{z}_1,\ldots,\underline{z}_n},\left[\prod_{i=1}^{2n-1}\sigma_3(\underline{z}_i)\mathscr{O}_{i,B}(\mu_i-\mu_{i+1})\right]\sigma_3(\underline{x}_{2n})\phi_{\underline{z}_{n+1},\ldots,\underline{z}_{2n}}\right), \tag{46}$$

where

$$\mathscr{O}_{a,B}(y)=\begin{cases}R^{(N)}(y) & \text{if} \quad a\in B,\\ E^{(N)}(y) & \text{if} \quad a\notin B.\end{cases} \tag{47}$$

for any $B\subset\{1,\ldots,2n-1\}$. The terms in the sum in (46) are called rest terms since they contain at least one factor $R^{(N)}(y)$ in the scalar product. The right-hand side of (44) will be called Euclidean term and will be object of an detailed analysis in the next sections.

The proof of Theorem 2 follows after the two following lemmas:

Lemma 3. *For each $q\in\mathbb{N}$ there are positive constants $c_{a,q}$ so that*

$$|F_R(\underline{x},\underline{x}',t,t')|\leqq\sum_{a=1}^{n-1}c_{a,q}\left[\frac{(1+\|t\|)^{qa}}{N^{(q-1)a}}\right]. \tag{48}$$

Lemma 4. *The approximation of $F_E(\underline{x},\underline{x}',t,t')$ by $G(\underline{x},\underline{x}',t,t')$ is governed by the following estimate:*

$$|F_E(\underline{x},\underline{x}',t,t')-G(\underline{x},\underline{x}',t,t')|\leqq ke^{AN-\alpha D_1^{(2)}(\underline{z})}+\sum_{a=1}^{n-1} c_{a,q}\left[\frac{(1+\|t\|)^{qa}}{N^{(q-1)a}}\right], \tag{49}$$

k, A, α and the $c_{a,q}$'s being positive constants ($A=2(2n-1)\operatorname{argsinh}(1)$).

The proof of Lemma 4 is much more involving and shall be the subject of the next section.

Proof of Lemma 3. We majorize the sum over the rest terms by

$$\sum_{\substack{B\subset\{1,\ldots,2n-1\}\\ B\neq\emptyset}}\left[\prod_{a\in B}\|R^{(N)}(\mu_a-\mu_{a+1})\|\right]\left[\prod_{b\in B^c}\|E^{(N)}(\mu_b-\mu_{b+1})\|\right], \tag{50}$$

where $B^c=\{1,\ldots,2n-1\}\backslash B$. Using the simple bound $\|E^{(N)}(\mu)\|=\|g(T)T^{i\mu}-R^{(N)}(\mu)\|\leqq 1+\|R^{(N)}(\mu)\|$ we majorize (50) by

$$\sum_{\substack{B\subset\{1,\ldots,2n-1\}\\ B\neq\emptyset}}(2^{|B|}-1)\prod_{a\in B}\|R^{(N)}(\mu_a-\mu_{a+1})\|$$

$$\leqq(2^{2n-1}-1)\sum_{\substack{B\subset\{1,\ldots,2n-1\}\\ B\neq\emptyset}}\prod_{a\in B}\frac{(1+|\mu_a-\mu_{a+1}|)^q}{N^{q-1}}\leqq\sum_{a=1}^{n-1}c_{a,q}\left[\frac{(1+\|t\|)^{qa}}{N^{(q-1)a}}\right], \tag{51}$$

$c_{a,q}$ being positive constants, where in the first inequality above we made use of Lemma 2. □

To complete the proof of the Theorem 2 take $N=[\varepsilon(1+D_1^{(2)}(\underline{z}))]$ for ε sufficiently small ($0<\varepsilon<\alpha/A$), where $[\cdot]$ is the lowest integer function. Then (32) follows straightforward from (48) and (49) for noew constants $c'_{a,q}$. □

3. Proof of the Lemma 4

The following theorem is the technically central result of this work.

Theorem 3. *For g and h sufficiently small one has the following Euclidean clustering property:*

$$|I(\underline{z},\{n_j\})-J(\underline{x},\underline{x}',\{n_j\})|\leqq ke^{-\alpha D_1^{(2)}(\underline{z})}, \tag{52}$$

where

$$J(\underline{x},\underline{x}',\{n_j\}):=\prod_{a=1}^{n}(\sigma_3(\underline{x}'_a)\phi_{\underline{x}'_a},T^{E(a)}\sigma_3(\underline{x}_a)\phi_{\underline{x}_a}) \tag{53}$$

with

$$E(a)=\sum_{b=n-a+1}^{n+a-1}n_b, \tag{54}$$

for some k, $\alpha>0$ (depending on n, β_g, and β_h).

The proof is given in the next section. A stronger decay than that implied by (52) can be obtained with more work, but (52) is enough for our purposes. Defining

$$H(\underline{x},\underline{x}',t,t'):=\sum_{n_1,\ldots,n_{2n-1}=0}^{N}\prod_{a=1}^{2n-1}a_{n_a}^{(N)}(\mu_a-\mu_{a+1})J(\underline{x},\underline{x}',\{n_j\}), \tag{55}$$

and using the result of Theorem 3 and (44) we get

$$|F_E(\underline{x}, \underline{x}', t, t') - H(\underline{x}, \underline{x}', t, t')| \\ \leqq k \sum_{n_1, \ldots, n_{2n-1}=0}^{N} \prod_{a=1}^{2n-1} |a_{n_a}^{(N)}(\mu_a - \mu_{a+1})| e^{-\alpha D_1^{(2)}(\underline{z})} \leqq k e^{AN - \alpha D_1^{(2)}(\underline{z})}, \tag{56}$$

where the last inequality follows from Lemma 1, taking $A = (2n-1)A(0)$. The proof of Lemma 4 is then completed with the following lemma, which together with (56) implies (49):

Lemma 5.

$$|H(\underline{x}, \underline{x}', t, t') - G(\underline{x}, \underline{x}', t, t')| \leqq \sum_{a=1}^{n-1} c'_{a,q} \left[\frac{(1 + \|t\|)^{qa}}{N^{(q-1)a}} \right]. \tag{57}$$

Proof of Lemma 5. We start from the identity:

$$J(\underline{x}, \underline{x}', \{n_j\}) = \left(\bigotimes_{a=1}^{n} \sigma_3(\underline{x}'_a) \phi_{\underline{x}'_a}, \bigotimes_{b=1}^{n} T^{E(b)} \bigotimes_{c=1}^{n} \sigma_3(\underline{x}_c) \phi_{\underline{x}_c} \right). \tag{58}$$

Now we write

$$\bigotimes_{b=1}^{n} T^{E(b)} = \prod_{c=1}^{n-1} (\mathscr{T}_{n-c})^{n_c} \prod_{d=1}^{n} (\mathscr{T}_{n-d})^{n_{2n-d}} = \prod_{a=1}^{2n-1} (\mathscr{T}_{|n-a|})^{n_a}, \tag{59}$$

where $\mathscr{T}_a$ is defined in Theorem 2. Hence, in analogy with (39) one has

$$\sum_{n_1, \ldots, n_{2n-1}=0}^{N} \prod_{a=1}^{2n-1} a_{n_a}^{(N)}(\mu_a - \mu_{a+1}) \bigotimes_{b=1}^{n} T^{E(b)} \\ = \prod_{a=1}^{2n-1} \sum_{n_a=0}^{N} a_{n_a}^{(N)}(\mu_a - \mu_{a+1}) (\mathscr{T}_{|n-a|})^{n_a} \\ = \prod_{a=1}^{2n-1} \{ g(\mathscr{T}_{|n-a|}) (\mathscr{T}_{|n-a|})^{i(\mu_a - \mu_{a+1})} - R^{(N)} \}, \tag{60}$$

$R^{(N)}$ representing the rest terms. Expanding the product in the right-hand side of (60) one gets $\prod_{a=1}^{2n-1} g(\mathscr{T}_{|n-a|}) (\mathscr{T}_{|n-a|})^{i(\mu_a - \mu_{a+1})}$ plus terms containing at least one factor $R^{(N)}$, which are bounded as (48) since $\|g(\mathscr{T}_a) \mathscr{T}_a^{i\mu}\| \leqq 1$. Finally note that

$$\prod_{a=1}^{2n-1} (\mathscr{T}_{|n-a|})^{i(\mu_a - \mu_{a+1})} = \bigotimes_{b=1}^{n} T^{(\sum_{j=n-b+1}^{n+b-1} i(\mu_j - \mu_{j+1}))} = \bigotimes_{b=1}^{n} T^{i(t'_b - t_b)}. \tag{61}$$

This proves Lemma 5. $\square$

4. Proof of Theorem 3. The Euclidean Clustering

The first step is to express $I(\underline{z}, \{n_j\})$ in terms of cluster expansions (see [1]). There are two cases to consider: $n = $ even and $n = $ odd.

4.0.1. The Case $n = $ even. According to [1] (see the proof of its Theorem 6.4) the vector states $\phi_{\underline{z}_1, \ldots, \underline{z}_n}$ and $\phi_{\underline{z}_{n+1}, \ldots, \underline{z}_{2n}}$ can be strongly approximated by

$$\alpha_{ip}(\tau_3(L_{\{1, \ldots, n\}}))\Omega \|\alpha_{ip}(\tau_3(L_{\{1, \ldots, n\}}))\Omega\|^{-1} \tag{62}$$

($\Omega \in \mathscr{H}_0$ is the vacuum vector) and

$$\alpha_{iq}(\tau_3(L_{(n+1, \ldots, 2n\}}))\Omega \|\alpha_{iq}(\tau_3(L_{\{n+1, \ldots, 2n\}}))\Omega\|^{-1} \tag{63}$$

respectively, with $p, q \in \mathbb{N}$, where, as in [1], $\tau_3(B) := \prod_{\underline{b} \in B} \tau_3(\underline{b})$, for a set of bonds $B \in \mathbb{Z}^d$, and where $L_{\{1,\ldots,n\}}$ and $L_{\{n+1,\ldots,2n\}} \subset \mathbb{Z}^d$ are sets of time-zero bonds with $\partial L_{\{1,\ldots,n\}} = \{\underline{z}_1, \ldots, \underline{z}_n\}$ and $\partial L_{\{n+1,\ldots,2n\}} = \{\underline{z}_{n+1}, \ldots, \underline{z}_{2n}\}$. Then $I(\underline{z}, \{n_j\})$ may be expressed as

$$\lim_{p,q\to\infty} \frac{\left(\alpha_{ip}(\tau_3(L_{\{1,\ldots,n\}}))\Omega, \left[\prod_{j=1}^{2n-1} \sigma_3(\underline{z}_j) T^{n_j}\right] \sigma_3(\underline{z}_{2n}) \alpha_{iq}(\tau_3(L_{\{n+1,\ldots,2n\}}))\Omega\right)}{\|\alpha_{ip}(\tau_3(L_{\{1,\ldots,n\}}))\Omega\| \, \|\alpha_{iq}(\tau_3(L_{\{n+1,\ldots,2n\}}))\Omega\|}$$

$$= \lim_{p,q\to\infty} \langle \chi_{K^{p,q}} \rangle \langle \chi_{L^{p,p}} \rangle^{-1/2} \langle \chi_{L^{q,q}} \rangle^{-1/2}, \tag{64}$$

where

$$K^{p,q} = \{(-p, L_{\{1,\ldots,n\}})\} \cup \left[\bigcup_{i=1}^{n} T^{(\underline{z}_i)}\left(-p, \sum_{a=1}^{i-1} n_a\right)\right] \cup$$

$$\cup \left[\bigcup_{j=n+1}^{2n} T^{(\underline{z}_j)}\left(\sum_{a=1}^{j-1} n_a, q + \sum_{a=1}^{2n-1} n_a\right)\right] \cup \left\{\left(q + \sum_{a=1}^{2n-1} n_a, L_{\{n+1,\ldots,2n\}}\right)\right\} \tag{65}$$

and

$$L_1^{-p,p} = \{(-p, L_{\{1,\ldots,n\}})\} \cup \left[\bigcup_{i=1}^{n} T^{(\underline{z}_i)}(-p, p)\right] \cup \{(p, L_{\{1,\ldots,n\}})\} \tag{66}$$

and

$$L_2^{-q,q} = \{(-q, L_{\{n+1,\ldots,2n\}})\} \cup \left[\bigcup_{i=n+1}^{2n} T^{(\underline{z}_i)}(-q, q)\right] \cup \{(p, L_{\{n+1,\ldots,2n\}})\}, \tag{67}$$

where $\{(-p, L_{\{1,\ldots,n\}})\}$ is the set of bonds $L_{\{1,\ldots,n\}}$ placed on the Euclidean time hyperplane at time $-p$ and $T^{(\underline{x})}(a,b)$ is the temporal line joining the points $(a, \underline{x})$ and $(b, \underline{x})$ (for $a \leqq b$). The right-hand side of (64) is given in terms of the cluster expansion by

$$\sum_{M \in \mathrm{Conn}(K^{p,q})} h^M \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma (a^\Gamma_{K^{p,q},M} + a^\Gamma_{\theta K^{p,q},\theta M} - a^\Gamma_{L_1^{-p,p}} - a^\Gamma_{L_2^{-q,q}})\right\}, \tag{68}$$

where θ is the reflection on the $(x^0 = 0)$-hyperplane (see [1]), from which the limit $p, q \to \infty$ may be taken directly and is given by

$$\sum_{M \in \mathrm{Conn}(K)} h^M \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma (a^\Gamma_{K,M} + a^\Gamma_{\theta K,\theta M} - a^\Gamma_{L_1} - a^\Gamma_{L_2})\right\}, \tag{69}$$

where

$$K = \lim_{p,q\to\infty} K^{p,q} \quad \text{and} \quad L_i = \lim_{p\to\infty} L_i^{-p,p}, \quad i = 1, 2, \tag{70}$$

with

$$L_1 = \bigcup_{i=1}^{n} M_{\underline{z}_i}, \quad L_2 = \bigcup_{i=n+1}^{2n} M_{\underline{z}_i}, \quad M_{\underline{x}} = T^{(\underline{x})}(-\infty, \infty). \tag{71}$$

Note that $\partial K = \{z_1, \ldots, z_{2n}\} \subset \mathbb{Z}^{d+1}$, with

$$z_i = \left(\underline{z}_i, \sum_{j=1}^{i-1} n_j\right). \tag{72}$$

so $\mathrm{Conn}(K)$ depends only on $\{z_1, \ldots, z_{2n}\}$.

4.0.2. The Case $n=odd$. In this case, according with [1] (see the proof of its Theorem 6.4), we can approximate strongly $\phi_{z_1,\ldots,z_n}$ and $\phi_{z_{n+1},\ldots,z_{2n}}$ by

$$\alpha_{ip}(\tau_3(L^0_{\{1,\ldots,n\}}))\phi_0\|\alpha_{ip}(\tau_3(L^0_{\{1,\ldots,n\}}))\phi_0\|^{-1} \tag{73}$$

and

$$\alpha_{iq}(\tau_3(L^0_{\{n+1,\ldots,2n\}}))\phi_0\|\alpha_{iq}(\tau_3(L^0_{\{n+1,\ldots,2n\}}))\phi_0\|^{-1} \tag{74}$$

respectively, with p, $q\in\mathbb{N}$, $\partial L^0_{\{1,\ldots,n\}}=\{z_1,\ldots,z_n\}\Delta\underline{0}$ and $\partial L^0_{\{n+1,\ldots,2n\}}=\{z_{n+1},\ldots,z_{2n}\}\Delta 0$.

As in the previous case we express the scalar products in terms of cluster expansion and after taking the limits $p,q\to\infty$ we write $I(\underline{z},\{n_j\})$ as

$$\sum_{M\in\mathrm{Conn}(K)} h^M \exp\left\{1/2\sum_{\Gamma} c_\Gamma\mu^\Gamma(a^\Gamma_{K,M}+a^\Gamma_{\theta K,\theta M}-a^\Gamma_{L_{0,1}}-a^\Gamma_{L_{0,2}})a^\Gamma_{M_0}\right\}, \tag{75}$$

where $L_{0,i}=L_i\Delta M_0$.

Definition 3. The sets $\mathrm{Conn}(K)$ occurring in (69) and (75) can be decomposed into the disjoint union of three sets: $\mathrm{Conn}(K)=V^0+V^1+V^2$, where

$$V^0=\left\{M\,\middle|\,M\in\mathrm{Conn}(K),\ M=\bigcup_{i=1}^{n}M_i,\ M_i\sim M_j,\ i\neq j,\ \text{so that for all } i,\ 1\leqq i\leqq n: \partial M_i=\{z_{a_i},z_{b_i}\} \text{ for some } z_{a_i},z_{b_i}\in\{z_1,\ldots,z_{2n}\},\ a_i\neq b_i, \text{ with } \underline{z}_{a_i}\in\{\underline{x}_1,\ldots,\underline{x}_n\} \text{ and } \underline{z}_{b_i}\in\{\underline{x}'_1,\ldots,\underline{x}'_n\}\right\}, \tag{76}$$

$$V^1=\left\{M\,\middle|\,M\in\mathrm{Conn}(K),\ M=\bigcup_{i=1}^{n}M_i,\ M_i\sim M_j,\ i\neq j,\ \text{so that for all } i,\ 1\leqq i\leqq n: \partial M_i=\{z_{a_i},z_{b_i}\} \text{ for some } z_{a_i},z_{b_i}\in\{z_1,\ldots,z_{2n}\},\ a_i\neq b_i, \text{ but with } \{\underline{z}_{a_i},\underline{z}_{b_i}\}\subset\{\underline{x}_1,\ldots,\underline{x}_n\} \text{ or } \{\underline{z}_{a_i},\underline{z}_{b_i}\}\subset\{\underline{x}'_1,\ldots,\underline{x}'_n\}\right\} \tag{77}$$

and

$$V^2=\left\{M\,\middle|\,M\in\mathrm{Conn}(K),\ M=\bigcup_{i=1}^{f}M_i,\ M_i\sim M_j,\ i\neq j,\ f<n \text{ and } |\partial M_i|>2 \text{ for some } i,\ 1\leqq i\leqq f\right\}. \tag{78}$$

Definition 4. To simplify (69) and (75) define

$$A(\Gamma,M,n):=\begin{cases} a^\Gamma_{K,M}+a^\Gamma_{\theta K,\theta M}-a^\Gamma_{L_1}-a^\Gamma_{L_2}, & n=\text{even}, \\ (a^\Gamma_{K,M}+a^\Gamma_{\theta K,\theta M}-a^\Gamma_{L_{0,1}}-a^\Gamma_{L_{0,2}})a^\Gamma_{M_0}, & n=\text{odd}, \end{cases} \tag{79}$$

and for later purposes, for $M_j\subset M\in V^0$,

$$A(\Gamma,M_j,1):=[a^\Gamma_{K^2_j,M_j}+a^\Gamma_{\theta K^2_j,\theta M_j}-a^\Gamma_{L^2_j}-1]a^\Gamma_{\underline{z}_{a_j}}, \tag{80}$$

with the simplifying notation $a^\Gamma_{\underline{x}}$ for $a^\Gamma_{M_{\underline{x}}}$ and with

$$L^2_j=M_{\underline{z}_{a_j}}\cup M_{\underline{z}_{b_j}}, \tag{81}$$

and

$$K^2_j=T^{(\underline{z}_{a_j})}\left(-\infty,\sum_{l=1}^{a_j-1}n_l\right)\cup T^{(\underline{z}_{b_j})}\left(\sum_{l=1}^{b_j-1}n_l,\infty\right), \tag{82}$$

where $\partial M_j=\{z_{a_j},z_{b_j}\}$ with $\underline{z}_{a_j}\in\{\underline{x}'_1,\ldots,\underline{x}'_n\}$, $\underline{z}_{b_j}\in\{\underline{x}_1,\ldots,\underline{x}_n\}$.

We have the following result, which holds for both n even or odd:

Proposition 2. *There are constants k, k', $\alpha > 0$ so that*

$$\left|\sum_{M \in V^1} h^M \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)\right\}\right| \leqq k e^{-\alpha D_3^{(2)}(z)} \tag{83}$$

and

$$\left|\sum_{M \in V^2} h^M \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)\right\}\right| \leqq k' e^{-\alpha D^{(4)}(z_1, \ldots, z_{2n})}, \tag{84}$$

where $D^{(4)}(z_1, \ldots, z_{2n})$ is the length of the smallest "minimal tree" of bonds joining four elements of $\{z_1, \ldots, z_{2n}\} \subset \mathbb{Z}^{d+1}$.

Proof. The proof uses standard techniques of cluster expansions, by showing with the use of (12) that both for (83) and for (84) one has

$$\left|\sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)\right| \leqq A|M|,$$

for a constant $A(g, h)$. For (83) we observe that the left-hand side decays exponentially with

$$\min\left\{\min_{a, b \in \{1, \ldots, n\}, a \neq b} \{|z_a - z_b|\}, \min_{c, d \in \{n+1, \ldots, 2n\}, c \neq d} \{|z_c - z_d|\}\right\}, \tag{85}$$

which has $D_3^{(2)}(z_1, \ldots, z_{2n})$ as lower bound. $\square$

Let us now concentrate on the sum over $M \in V^0$. We have to establish the exponential decay of the difference

$$\Upsilon \equiv \Upsilon(\{z_i\}_{i=1}^{2n}; \{n_j\}_{j=1}^{2n-1}) := \sum_{M \in V^0} h^M \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)\right\}$$

$$- \sum_{i \in \sigma} \prod_{a=1}^{n} (\sigma_3(z_{i_a}) \phi_{z_{i_a}}, T^{N(i,a)} \sigma_3(z_{i_{a+n}}) \phi_{z_{i_{a+n}}}) \tag{86}$$

with

$$N(i, a) := \sum_{j=i_a}^{i_{a+n}-1} n_j. \tag{87}$$

Theorem 4. *There are positive constants k and α such that*

$$|\Upsilon| \leqq k e^{-\alpha D^{(4)}(z_1, \ldots, z_{2n})}. \tag{88}$$

Proof. Each $M \in V^0$ is composed by a disjoint union $\bigcup_{j=1}^{n} M_j$. We write

$$\Upsilon = \sum_{M \in V^0} h^M \Xi(M, n) + R(z_1, \ldots, z_{2n}), \tag{89}$$

where

$$\Xi(M, n) = \exp\left\{1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)\right\} - \exp\left\{\sum_{j=1}^{n} 1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M_j, 1)\right\} \tag{90}$$

with

$$R(z_1, \ldots, z_{2n}) = (-1) \sum_{i \in \sigma} \sum_{M \in \mathscr{M}_i} \left(\prod_{c=1}^{n} h^{M_c}\right) \exp\left\{\sum_{j=1}^{n} 1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M_j, 1)\right\} \tag{91}$$

where, for $i \in \sigma$,

$$\mathscr{M}_i := \left\{M \,\middle|\, M = \bigcup_{a=1}^{n} M_a, \text{ with } \partial M_a = \{z_{i_a}, z_{i_{n+a}}\}, 1 \leqq a \leqq n, \text{ with } M_a \nsim M_b \text{ for some } (a, b),\ a \neq b\right\}. \tag{92}$$

In analogy with Proposition 2, $R(z_1, \ldots, z_{2n})$ has the bound

$$|R(z_1, \ldots, z_{2n})| \leqq k e^{-\alpha D^{(4)}(z_1, \ldots, z_{2n})}, \tag{93}$$

for constants k and α. Due to this fact we consider only the sum over $M \in V^0$ in (89). First define

$$\Theta(M, n) = 1/2 \sum_{\Gamma} c_\Gamma \mu^\Gamma \Delta(\Gamma, M, n), \tag{94}$$

where

$$\Delta(\Gamma, M, n) := A(\Gamma, M, n) - \left[\sum_{j=1}^{n} A(\Gamma, M_j, 1) \right]. \tag{95}$$

The following question has to be considered: what is, for each $M \in V^0$, a sufficient geometrical condition on Γ for $\Delta(\Gamma, M, n) \neq 0$? Write

$$A(\Gamma, M_j, 1) = a^\Gamma_{S^2_j, M_j} + a^\Gamma_{\theta S^2_j, \theta M_j} - a^\Gamma_{z_{a_j}} - a^\Gamma_{z_{b_j}}, \tag{96}$$

where $\partial M_j = \{z_{a_j}, z_{b_j}\}$, $S^2_j := K^2_j \Delta M_{z_{aj}}$. Then the clusters Γ of interest are of two types: a) *Γ is formed by polymers which wind around only one of the sets $S_j \cup M_j$, say $S_a \cup M_a$, (eventually with $\Gamma \nsim M_a$), so that*

$$a^\Gamma_{\theta S^2_k, \theta M_k} = a^\Gamma_{S^2_k, M_k} = a^\Gamma_{z_{\alpha_k}} = a^\Gamma_{z_{\beta_k}} = 1, \quad \forall k \neq a. \tag{97}$$

b) *Γ violates condition* a), *which means that Γ winds around at least two sets $S_j \cup M_j$, say $S_c \cup M_c$ and $S_d \cup M_d$, $c \neq d$. By a geometrical reasoning this implies that*

$$|M_c| + |M_d| + 1/2 \|\Gamma\| \geqq D^{(4)}(\partial M_c \cup \partial M_d). \tag{98}$$

It is easy to check that a) implies $\Delta(\Gamma, M, n) = 0$. So the clusters Γ contributing to (94) satisfy (98). By (13) this implies the following bound:

$$\begin{aligned} |\Theta(M, n)| \leqq \\ & k \exp\left\{ \ln\left(\frac{\|\mu\|}{\|\mu_c\|} \right) \left[2 \min_{M_c, M_d \subset M} \left(D^{(4)}(\partial M_c \cup \partial M_d) - |M_c| - |M_d| \right) \right] \right\} \\ & \leqq k \exp\left\{ 2 \ln\left(\frac{\|\mu\|}{\|\mu_c\|} \right) [D^{(4)}(z_1, \ldots, z_{2n}) - |M|] \right\}, \end{aligned} \tag{99}$$

since $\|\mu\| \leqq \|\mu_c\|$, $\min_{M_c, M_d \subset M} D^{(4)}(\partial M_c \cup \partial M_d) \geqq D^{(4)}(z_1, \ldots, z_{2n})$ and $|M| \geqq |M_c| + |M_d|$. Above k is a positive constant. Let us return to (90).

Proposition 3. *For $a, b \in \mathbb{R}$ one has $|e^a - e^b| \leqq 8^{1/4} |b-a|^{1/4} e^{|\max(a,b)|}$.*

Proof. First, for $x \geqq 0$ one has $(1 - e^{-x}) \leqq 8^{1/4} x^{1/4}$. To see this note that $1 - e^{-x} \leqq x$. So

$$(1 - e^{-x})^2 = 1 - 2e^{-x} + e^{-2x} \leqq 2(1 - e^{-x}) \leqq 2x.$$

Now $(1 - e^{-x}) \leqq \sqrt{2} x^{1/2}$. Therefore

$$(1 - e^{-x})^2 \leqq 2(1 - e^{-x}) \leqq 2\sqrt{2} x^{1/2}.$$

So

$$|e^a - e^b| = e^{\max(a,b)}(1 - e^{-|b-a|}) \leqq 8^{1/4} |b-a|^{1/4} e^{|\max(a,b)|}. \quad \square$$

Taking $a=1/2\sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M, n)$, $b=1/2\sum_{j=1}^{n}\sum_{\Gamma} c_\Gamma \mu^\Gamma A(\Gamma, M_j, 1)$ with $a-b=\Theta(M,n)$ and using the fact that both $|a|$ and $|b|$ have $A|M|+B$ as upper bound, where A and B are positive and independent of M, we conclude from Proposition 3 and (99) that

$$|\Xi(M,n)| \leqq k \exp\left\{A|M| + \frac{2}{4}\ln\left(\frac{\|\mu\|}{\|\mu_c\|}\right)[D^{(4)}(z_1,\ldots,z_{2n}) - |M|]\right\} \tag{100}$$

for some constant k. Therefore

$$\left|\sum_{M\in V^0} h^{|M|}\Xi(M,n)\right| \leqq k'\left(\frac{\|\mu\|}{\|\mu_c\|}\right)^{D^{(4)}(z_1,\ldots,z_{2n})/2} \sum_{M\in V^0}\left(e^A h\left(\frac{\|\mu\|}{\|\mu_c\|}\right)^{-1/2}\right)^{|M|} \tag{101}$$

According with [1], $\|\mu\| = \max(g,h)$ and one has

$$h\|\mu\|^{-1/2} = h\min(g^{-1/2}, h^{-1/2}) \leqq h^{1/2}. \tag{102}$$

In (102) one sees the need for the exponent 1/4 in Proposition 3. Hence by choosing h small enough the sum in (101) becomes uniformly bounded on $\{z_1,\ldots,z_{2n}\}$ and we conclude

$$\left|\sum_{M\in V^0} h^{|M|}\Xi(M,n)\right| \leqq k'\left(\frac{\|\mu\|}{\|\mu_c\|}\right)^{D^{(4)}(z_1,\ldots,z_{2n})/2}, \tag{103}$$

which together with (93) proofs Theorem 4. □

Now we complete the proof of Theorem 3. Joining estimates (83), (84), and (88) we establish that

$$\left|I(\underline{z},\{n_j\}) - \sum_{i\in\sigma}\prod_{a=1}^{n}(\sigma_3(\underline{z}_{i_a})\phi_{z_{i_a}}, T^{N(i,a)}\sigma_3(\underline{z}_{i_{n+a}})\phi_{z_{i_{a+n}}})\right| \leqq k e^{-\alpha D_3^{(2)}(\underline{z})}, \tag{104}$$

since $D_3^{(2)}(\underline{z}_1,\ldots,\underline{z}_{2n}) \leqq D^{(4)}(z_1,\ldots,z_{2n})$. From the cluster expansions, or equivalently from the existence of a mass gap, one has

$$\left|\sum_{i\in\sigma}\prod_{a=1}^{n}(\sigma_3(\underline{z}_{i_a})\phi_{z_{i_a}}, T^{N(i,a)}\sigma_3(\underline{z}_{i_{n+a}})\phi_{z_{i_{a+n}}}) - \prod_{a=1}^{n}(\sigma_3(\underline{x}'_a)\phi_{x'_a}, T^{E(a)}\sigma_3(\underline{x}_a)\phi_{x_a})\right| \tag{105}$$
$$\leqq k'' e^{-\alpha D_2^{(2)}(\underline{z})},$$

which finally proofs Theorem 3. □

5. Appendix

Here we present the proofs of Lemmas 1 and 2.

Proof of Lemma 1. According with (40) we have

$$a_n^{(N)}(t) = (n!)^{-1}\sum_{m=n}^{N} b_m(t)\left[\frac{d^n}{dx^n}T_m(2x-1)\right]\Bigg|_{x=0}. \tag{106}$$

We use the fact that $T_m(2x-1) = T_{2m}(x^{1/2})$, which follows from the identity $T_m(T_n(x)) = T_{n,m}(x)$ and from $T_2(x) = 2x^2 - 1$. Applying the explicit polynomial form of $T_n(x)$

(see [11])

$$T_n(x) = \sum_{s=0}^{[n/2]} \binom{n}{2s} \sum_{q=0}^{s} \binom{s}{q} (-1)^q x^{n-2q}, \tag{107}$$

we get

$$a_n^{(N)}(t) = (-1)^n \sum_{m=n}^{N} b_m(t)(-1)^m \sum_{s=m-n}^{m} \binom{2m}{2s} \binom{s}{m-n}. \tag{108}$$

So we have

$$\gamma_{\mathrm{N}}(t) \leqq \sum_{n=0}^{N} e^{-\alpha n} \sum_{m=n}^{N} |b_m(t)| \sum_{s=m-n}^{m} \binom{2m}{2s} \binom{s}{m-n} = \sum_{m=0}^{N} |b_m(t)| \mu_m, \tag{109}$$

where

$$\mu_m = e^{-\alpha m} \sum_{k=0}^{m} e^{\alpha k} \sum_{s=k}^{m} \binom{2m}{2s} \binom{s}{k}. \tag{110}$$

The right-hand side equals

$$1/2\{(e^{-\alpha/2} + \sqrt{1+e^{-\alpha}})^{2m} + (e^{-\alpha/2} - \sqrt{1+e^{-\alpha}})^{2m}\} = \cosh(2mb) < e^{A(\alpha)m}, \tag{111}$$

for $b := \operatorname{argsinh}(e^{-\alpha/2})$ and $A(\alpha) = 2\operatorname{argsinh}(e^{-\alpha/2})$. Since $|b_m(t)| \leqq 2$ using (111) we get easily

$$\gamma_N(t) \leqq \left(\frac{2e^{A(\alpha)}}{e^{A(\alpha)} - 1} \right) e^{A(\alpha)N}. \quad \square \tag{112}$$

Proof of Lemma 2. Using $|T_n(x)| \leqq 1$, $\forall x \in [-1, 1]$, we have

$$|\mathscr{R}^{(N)}(\lambda, \mu)| \leqq \sum_{n>N} |b_n(\mu)|. \tag{113}$$

Taking (42), using the identity

$$e^{2in\alpha} = (2in)^{-q} \frac{d^q}{d\alpha^q} e^{2in\alpha}, \qquad \forall n \geqq 1,\ q \in \mathbb{N}, \tag{114}$$

integrating by parts and using smoothness of $g(\cdot)$ it is possible, for each $q \in \mathbb{N}$, to find a constant C_q, depending on the function $g(\cdot)$ but independent of n, so that $|b_n(\mu)| \leqq C_q n^{-q}(1 + |\mu|)^q$ holds [1]. The lemma follows from (113). $\square$

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[1] Using stationary phase methods a sharper estimate for $|b_n(\mu)|$ can be found, which is nevertheless not so useful for our purposes and more difficult to handle

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