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On the Algebraic Structure of Link-Diagrams on a 2-Dimensional Surface*

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Abstract. Following and generalizing a paper by Turaev, we consider some algebraic structures on the set of (generalized) link-diagrams, meant simply as collections of immersed loops on a two dimensional surface, with the specification of an over/undercrossing prescription at each double point. This definition is general enough to be relevant not only to traditional knot-theory, but possibly to some statistical mechanical models. A coalgebra structure is introduced on some modules (over polynomial rings), generated by these diagrams. The compatibility of this coalgebra structure with the skein invariance and the invariance under Reidemeister moves is discussed. A Hopf algebra structure results only in some special cases, which are thoroughly examined. It is shown that a special choice of the ground (polynomial) ring over which the diagram module is defined, allows us to define link-invariants for links (in the ordinary sense) in $\Sigma \times [0,1]$, where Σ is a (closed or open) two-dimensional surface. These invariants generalize in a non-trivial way the Jones polynomials and the Homfly polynomials (at least when the last ones are computed for some special values of the variables). In a sequel paper the relation between the algebraic structures of link-diagrams, some special types of quantum groups and the quantum holonomy will be discussed.

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1. Introduction

The study of link-invariants has always attracted the interest of many mathematicians and physicists; in recent times this interest increased dramatically due to the Jones' revolution, which brought into the game new powerful invariants [1]. Shortly after Jones' paper, these new invariants were related to some exactly solvable models in statistical mechanics (see. e.g. [2]).

Later on, Witten, following also some suggestions by Atiyah [3] proposed, in a seminal paper [4], a connection between Jones polynomials, on one side, and quantum field theories with Chern-Simons action, on the other side. In the same paper Witten discussed the relation between Jones polynomials, (3-dimensional) Chern-Simons theories and (2-dimensional) conformal field theories; this relation, also in connection with quantum groups, has been further discussed by Alvarez-Gaumé, Gomez and Sierra (see e.g. [5]).

Quantum groups [6] entered the picture of link-invariants also independently of conformal field theories. They were directly related to the Jones polynomials or to the two-variables Homfly polynomials [7] by Turaev [8] and Reshetikhin [9].

At the same time that Witten was relating the Jones polynomials with the Quantum Chern-Simons theory, Turaev wrote two papers [10, 11] in which he constructed "skein algebras" of link-diagrams which can be considered as a quantized version of the Poisson algebras of loops on a two-dimensional surface, which is more precisely defined as (a deformation of) the symmetric algebra of the Goldman Lie algebra of free homotopy classes of loops [12]. Modulo the non-trivial differences between the word "quantization" used in quantum field theory and the same word which appears in "quantization of Poisson Algebras" one could claim that Witten's and Turaev's approach are somehow related. The key observation is that there is a Poisson map between the symmetric algebra of the Goldman Lie Algebra deformed with parameter $1/kN$ and the Poisson algebra relevant to the symplectic manifold $\frac{\mathscr{A}^{\text{flat}}}{\mathscr{G}}$ of $SU(N)$-gauge orbits of flat connections over a closed two-dimensional surface, where the standard symplectic form is multiplied by the factor k (see for more details [10, 12] and also [13] and [14]). This symplectic manifold is in turn related to a 3-dimensional Chern-Simons theory with level k [15, 16].

In this paper we follow and generalize the approach suggested by Turaev [10]. Turaev's ideas, in turn, are based very much on the ideas proposed by Jaeger [17] and also on the ideas proposed by Jones [18] and by Kauffman [19].

The paper is organized as follows: First we define a link-diagram (in a generalized sense) as a collection of generic immersed oriented loops on an oriented two dimensional surface, where at each (transversal) double point an under/over crossing specification is attached. The only equivalence relation which is initially taken into account is the equivalence under ambient isotopies of the surface. Then, following [10], we define a labelling map as a map assigning an integer in $\{1, \ldots, n\}$ [1] to each edge (connecting two vertices – or double points – of the diagram). Such labelling maps are required to satisfy a Kirchhoff's law. Consider now the four (oriented) edges meeting at a given vertex; to each of the 6 possible choices of labels ($n=2$) we attach an indeterminate variable. We then consider the module over the polynomial ring in such variables (and possibly their inverses), generated by link-diagrams.

Any attempt to introduce a coalgebra structure in such a module (i.e. to introduce a coassociative comultiplication with counit) will require that two of the above 6 variables are equal. So only 5 variables are left.

Furthermore if we want to introduce a generalized skein relation on the diagram-module and we want the comultiplication to be compatible with it, then the number of variables has to be reduced to 4, in a suitable way. At this point the framework is general enough to include "link-invariants" (or better pseudo-link invariants) which may be related to the algebra $A_{g,t}$ considered by Drinfel'd in the framework of Quasi-Hopf Algebras [20] or to some specific models in statistical mechanics (e.g. 6-vertex model [21]. See [13] for further discussion.

But if we want to consider true link invariants in the space $\Sigma \times [0,1]$, then we have to require also the invariance under the three Reidemeister moves. This will force us to redefine the comultiplication by including some "rotation factors" in order to take into account the Reidemeister move I. Moreover invariance under the three Reidemeister moves will require that one of the remaining 4 variables is set equal to 0, or, in other words, that a specific limitation is forced on the set of labelling maps. This last condition is the assumption made since the very beginning in [10].

But differently from [10], we have, after having considered the skein relation and the invariance under Reidemeister moves, an extra variable left. And this variable (let us call it z) will allow us to exhibit link invariants of $\Sigma \times [0,1]$ which are generalizations of the Jones polynomials [2]; the exponent of the variable z being different from zero only when Σ is non-contractible. Moreover when Σ is non-contractible, then the variable z plays an essential rôle in the definition of link-invariants. It is in fact shown that there exist two different links which have a different invariant only when $z \neq 1$. In order to understand why the variable z can detect the non-contractibility of Σ, let us mention only the fact that the exponent of the variable z is expressed in terms of the intersection numbers of some specified sub-diagrams of a given diagram D; these subdiagrams are in turn obtained by collecting together the edges of D which share a common value of the label.

In the Appendix we examine the very special situation when the diagram-module is an Hopf algebra and prove explicitly the existence of an antipode map for any surface where link-diagrams are considered. This proves a conjecture by Turaev [10].

In a subsequent paper [13] we will relate link invariants of $\Sigma \times [0,1]$ to a new quantum group given by a two-parameter deformation of the universal enveloping algebra of $sl(n, \mathbf{C})$. The relation with Quantum Holonomy and Quasi-Hopf algebras will also be discussed.

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[1] In most cases we will consider $n=2$

[2] They are also a generalization of the Homfly polynomials, provided that among the two variables of these polynomials, we have the same relation which is considered in [4 and 17]

2. Coalgebra Structure on Link-Diagrams

Throughout this paper Σ will denote an oriented compact connected two-dimensional Riemannian manifold [3]. We will consider on Σ collections of generic immersed loops (all the crossing points being transverse double points). A link-diagram is, by definition, any such collection of loops together with the specification at each double point of an over/under crossing symbol. The projection of a link-diagram will be simply the collection of loops in Σ obtained by forgetting the over/under crossing information.

The number of components of a link-diagram is, by definition, the number of loops in it. If this number is 1, then we speak of a knot-diagram.

In this paper we will always consider *oriented* loops and *oriented* link diagrams so the word oriented will be omitted from now on. With an abuse of notation, and unless the contrary is specified, we will use the term link-diagrams also to denote the equivalence classes of link-diagrams, meaning that two diagrams are equivalent if they have the same over/under crossings at the corresponding double points and their projections are related by an ambient isotopy of Σ, namely a diffeomorphism of Σ which is connected to the identity. Link-diagrams, as defined above, can be thought of as diagrams of links in $\Sigma \times [0,1]$, but it should be emphasized that, in the initial setting we are considering, the only equivalence relation taken into account is the one mentioned above (e.g. we do *not* consider, at the beginning, Reidemeister moves).

We define the vertices of a link diagram to be the double points of its projection; the edges of a link-diagram are defined as the lines joining two vertices. With k vertices we have obviously $2k$ edges. The (finite) set of all the vertices of a link-diagram D will be denoted by the symbol $V(D)$ or simply by V, when no confusion may arise.

An n-labelling map f (in symbols $f \in Lbl_n$) of a link diagram will be a map (see [10]):

$$f : \text{Edges} \to \{1, \ldots, n\}$$

satisfying the following requirement [4]:

For each vertex $v \in V$, we denote by a_v and b_v the incoming edges at v and by c_v and d_v the outgoing edges at v; then we should have either

$$f(a_v) = f(c_v) \quad \text{and} \quad f(b_v) = f(d_v)$$
$$\text{or}$$
$$f(a_v) = f(d_v) \quad \text{and} \quad f(b_v) = f(c_v),$$

for each $v \in V$.

This requirement is just a form of Kirchhoff's law; it is equivalent to requiring that, given any integer number k in $[1, n]$ and any n-labelling map f, then the edges which belong to the inverse image $f^{-1}(k)$ constitute a new link-diagram, such that the orientation of all the edges is the same as in the original diagram. We always assume that at the double points of $f^{-1}(k)$ the over/under information is the one inherited by the original diagram.

[3] Σ can be either with or without boundary. When Σ has a boundary, then we will generally require that $\partial\Sigma$ has one component

[4] Notice that the requirements on the labelling maps, which are considered in [10], are stricter than ours

To each vertex v we assign a number $w(v) = \pm 1$ according to whether the type of the crossing is L_+ or L_-. Here $L_\pm$ are the standard configuration in knot theory, namely L_- is the configuration where by turning the upper outgoing edge counterclockwise we meet the lower incoming edge, whereas the configuration L_+ is the configuration where by turning the upper outgoing edge counterclockwise we meet the lower outgoing edge.

For any link-diagram we denote by the symbol $w(D)$ (the "writhe" in Kauffman's terminology [22]) the sum of the numbers $w(v_i)$ extended to all the vertices $v_i \in V(D)$. More generally if W is any subset of the set $V(D)$ then we will denote by $w(W)$ the sum of the numbers $w(v_j)$ extended to all the vertices $v_j \in W$.

If we are given a set of commuting variables splitted in two subsets: $\{x, y, z, \dots\}$ and $\{h, k, l, \dots\}$ we can consider $\mathscr{D} \equiv \mathscr{D}(x, y, z, \dots \mid h, k, l, \dots)$, namely the free module generated by the link-diagrams over the polynomial ring $\mathbf{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}, \dots, h, k, l, \dots]$.

Our next aim will be to define a family of comultiplications in such a module; more precisely we will consider maps: $V : \mathscr{D} \to \mathscr{D} \underset{K}{\otimes} \mathscr{D}$, where K is a given subring of $\mathbf{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}, \dots, h, k, l, \dots]$.

Our strategy will be to put different requirements on these comultiplications (e.g. coassociativity, the compatibility with some equivalence relations like the one obtained by considering link diagrams modulo Reidemeister moves and/or some kind of skein-relation, etc.). These requirements will put in turn some constraints on the indeterminate variables and on the definition of V itself.

Let D be any link-diagram and let f be any 2-labelling map. Consider a vertex v in D of a given type and assume, for the sake of definitiveness, that it is of type L_+. The possible values of the labelling map f on the edges meeting at v, allow six possible configurations, and to each of these configurations we associate a different indeterminate variable as follows:

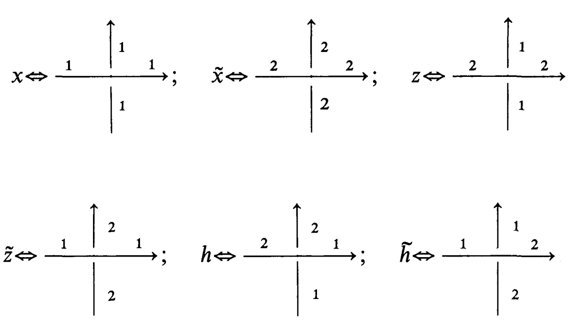

Fig. 1

One can see the pairing of the above configurations with the above set of variables as something like attaching a weight (or probability) to each possible "interaction" between vertices and labelling maps.

Following the general arguments leading to the construction of the Homfly polynomials [10, 17], we will associate to any n-labelling map f the subdiagrams $D_{f,k} \equiv f^{-1}(k)$, $k = 1, \dots, n$. And again, following and generalizing [10], our comultiplication will map any link-diagram D into the sum, over all the possible

2-labelling maps f, of the tensor products of two factors, the first one being proportional to the subdiagram $D_{f,1}$ and the second one being proportional to subdiagram $D_{f,2}$.

It is clear that the indeterminate variables involved here are $x, \tilde{x}, z, \tilde{z}, h, \tilde{h}$ but it is also clear that they play different rôles. The variables x and $\tilde{x}$ are associated respectively to the self-crossing of $D_{f,1}$ and $D_{f,2}$, the variables z and $\tilde{z}$ are associated to the crossing of $D_{f,2}$ over $D_{f,1}$ and, respectively, to the crossings of $D_{f,1}$ over $D_{f,2}$; finally both the variables h and $\tilde{h}$ are associated to the splitting of the original diagram at the given vertex, in the only orientation-preserving way.

We now assume that if instead of a vertex of type L_+ in the original diagram, we had considered a vertex of type L_- then we would have had to replace the variables $x, \tilde{x}, z, \tilde{z}$ with their inverses. As far as the variables h and $\tilde{h}$ are concerned we notice that they correspond to configurations where the crossing points are eliminated irrespectively of whether the original crossing points are of type L_+ or L_-. Hence we will not consider the inverses of these last variables.

The above arguments lead us to consider the polynomial ring $\mathbf{C}[x, x^{-1}, \tilde{x}, \tilde{x}^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ and hence the module $\mathscr{D} \equiv \mathscr{D}(x, \tilde{x}, z, \tilde{z} \mid h, \tilde{h})$.

Here and below we always assume that $x, \tilde{x}, z, \tilde{z}$ are different from zero. On the contrary h or $\tilde{h}$ can be set equal to zero. In this case we mean that we want to consider only labelling maps f which satisfy respectively the following additional condition at any vertex of a link diagram:

$$\tilde{h} = 0 \Leftrightarrow f(d) \geqq f(a); \qquad h = 0 \Leftrightarrow f(d) \leqq f(a), \tag{2.1}$$

where a and d denote respectively the lower incoming and the lower outgoing edge.

The next question we have to decide is over what subring K we should consider tensor products. Since the configurations corresponding to the variables x and $\tilde{x}$ are separately associated to the subdiagrams $D_{f,1}$ and $D_{f,2}$, we find it reasonable to assume that K should *not* contain $\mathbf{C}[x, x^{-1}, \tilde{x}, \tilde{x}^{-1}]$.

It is also clear that on the contrary K should contain $\mathbf{C}[h, \tilde{h}]$ since these last two variables are not associated separately to $D_{f,1}$ or $D_{f,2}$ and should therefore be allowed to pass freely from one factor of the tensor product to the other. The question of whether K should include the variables $z, \tilde{z}$ and their inverses is at this point debatable, so we will consider two options: either K will be given by $K_1 \equiv \mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ or K will be given by $K_2 \equiv \mathbf{C}[h, \tilde{h}]$. Eventually we will consider only $K = K_1$ but, for the moment, we will keep open both options.

We are now led to considering the following comultiplication in $\mathscr{D}$:

$$\begin{aligned} \nabla(D) = \sum_{f \in Lbl_2} & (-1)^{|(S_f)_-|} (-1)^{|(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} x^{w(D_{f,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} D_{f,1} \\ & \bigotimes_K \tilde{x}^{w(D_{f,2}) - w(D)} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} D_{f,2} \,. \end{aligned} \tag{2.2}$$

In the above formula we used the following notation: for any diagram D, $w(D)$ denotes as usual its writhe; for any pair of diagrams D and D', $w(D \Downarrow D')$ denotes the total writhe of all the vertices where D crosses over D'; for any labelling map f, S_f and $\tilde{S}_f$ denote respectively the set of vertices where a splitting occurs in the original diagram corresponding to the configuration associated to h or, respectively, to $\tilde{h}$ (see Fig. 1); $(S_f)_\pm$ and $(\tilde{S}_f)_\pm$ denote respectively the subsets of S_f and $\tilde{S}_f$ which correspond to vertices in the original diagram of type $L_\pm$; finally for any finite set X, here and in the future, we denote by $|X|$ the number of its elements. In particular, for any diagram D, $|D|$ will denote the number of components, while for any set of vertices W, $|W|$ will denote the number of vertices in W.

Notice that in the above definition of the comultiplication, all the terms which we introduced are justified on the basis of the assocation of the different variables to the various configurations, with the exception of the factor ± 1 in front of everything (which is not needed in order to define a comultiplication but will be convenient later on, when we will consider the invariance under the skein relation) and of the normalization factor $-w(D)$ which appears at the exponent of both x and $\tilde{x}$ (indeed this factor is required in order to have a coassociative comultiplication – see below).

To complete the definition of the comultiplication we have to define $\nabla(x)$, $\nabla(x^{-1})$, $\nabla(z)$, etc. In order to have a meaningful object we require that for any link diagram D and for any pair of elements $a, b \in \mathbf{C}[x, x^{-1}, \tilde{x}, \tilde{x}^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}]$ we have $\nabla(abD) = \nabla(a)\nabla(b)\nabla(D)$. Furthermore we assume, for simplicity, that the following relations hold:

$$\begin{gathered}\nabla(x) = x \underset{K}{\otimes} x, \quad \nabla(\tilde{x}) = \tilde{x} \underset{K}{\otimes} \tilde{x}, \quad \text{for} \quad K = K_1 \quad \text{and} \quad K = K_2 \\ \nabla(z) = z \underset{K}{\otimes} z, \quad \nabla(\tilde{z}) = \tilde{z} \underset{K}{\otimes} \tilde{z} \quad \text{for} \quad K = K_2 .\end{gathered} \tag{2.3}$$

It is obvious that if we consider $K = K_1$ then the definitions involving $\nabla(z)$, $\nabla(\tilde{z})$, $\nabla(z^{-1})$, $\nabla(\tilde{z}^{-1})$ should (as they eventually will) be disregarded[5].

We recall that the comultiplication ∇ is, by definition, coassociative if we have:

$$\left(\nabla \underset{K}{\otimes} \mathrm{id}\right) \nabla = \left(\mathrm{id} \underset{K}{\otimes} \nabla\right) \nabla .$$

We have now the following:

2.1 Theorem. *The comultiplication defined in* (2.2) *and* (2.3) *is coassociative if and only if* $x = \tilde{x}$ *and* $K = K_1 \equiv \mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$.

Proof. Let us first assume that $K \equiv K_2$ and let us consider the following equation:

$$\begin{aligned}&\left(\nabla \underset{K_2}{\otimes} \mathrm{id}\right) \nabla(D) \\ &\quad = \sum_{f \in Lbl_2(D)} (-1)^{|(S_f)_-| + |(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} \nabla\left(x^{w(D_{f,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} D_{f,1}\right) \\ &\qquad \underset{K_2}{\otimes} \tilde{x}^{w(D_{f,2}) - w(D)} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} D_{f,2} \\ &\quad = \sum_{f \in Lbl_2(D)} \sum_{g \in Lbl_2(D_{f,1})} (-1)^{|(S_f)_-| + |(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} (-1)^{|(S_g)_-| + |(\tilde{S}_g)_-|} h^{|S_g|} \\ &\qquad \tilde{h}^{|\tilde{S}_g|} x^{w((D_{f,1})_{g,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} z^{w((D_{f,1})_{g,2} \Downarrow (D_{f,1})_{g,1})} (D_{f,1})_{g,1} \\ &\qquad \underset{K_2}{\otimes} \tilde{x}^{w((D_{f,1})_{g,2}) - w(D_{f,1})} x^{w(D_{f,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} \tilde{z}^{w((D_{f,1})_{g,1} \Downarrow (D_{f,1})_{g,2})} (D_{f,1})_{g,2} \\ &\qquad \underset{K_2}{\otimes} \tilde{x}^{w(D_{f,2}) - w(D)} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} D_{f,2} .\end{aligned}$$

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[5] One could in fact figure out a more general situation where $\nabla(x) = x_1 \underset{K}{\otimes} x_2$ for some polynomial functions x_1 and x_2 depending on the other variables x, $\tilde{x}$, z, $\tilde{z}$ and their inverses and similar relations for $\nabla\tilde{x}$, ∇z, etc. Computations become a little more cumbersome, but one can easily prove that all these more complicated relations, when different from (2.3), are incompatible with the coassociativity of the comultiplication ∇

Now the pairs of labelling maps (f, g) with $f \in Lbl_2(D)$ and $g \in Lbl_2(D_{f,1})$ are in a one to one correspondence with the 3-labelling maps $p \in Lbl_3(D)$. In fact, as in [10], we can define:

$$p(e) \equiv \begin{cases} 1 & \text{if} \quad g(e)=1, \quad f(e)=1; \\ 2 & \text{if} \quad g(e)=2, \quad f(e)=1; \\ 3 & \text{if} \quad f(e)=2. \end{cases}$$

Hence $\left(\nabla \bigotimes_{K_2} \mathrm{id}\right) \nabla(D)$ can be written as

$$\sum_{p \in Lbl_3 D} (-1)^{|(S_p)_-| + |(\tilde{S}_p)_-|} h^{|S_p|} \tilde{h}^{|\tilde{S}_p|} x^{w(D_{p,1}) - w(D)}$$
$$\times z^{w(D_{p,3} \Downarrow (D_{p,1} \cup D_{p,2}))} z^{w(D_{p,2} \Downarrow D_{p,1})} D_{p,1} \bigotimes_{K_2} \tilde{x}^{-w(D_{p,1}) - w(D_{p,1} \# D_{p,2})}$$
$$\times x^{w(D_{p,1}) + w(D_{p,2}) + w(D_{p,1} \# D_{p,2}) - w(D)} z^{w(D_{p,3} \Downarrow (D_{p,1} \cup D_{p,2}))} \tilde{z}^{w(D_{p,1} \Downarrow D_{p,2})} D_{p,2}$$
$$\bigotimes_{K_2} \tilde{x}^{w(D_{p,3}) - w(D)} \tilde{z}^{w((D_{p,1} \cup D_{p,2}) \Downarrow D_{p,3})} D_{p,3}\,.$$

Here S_p and $\tilde{S}_p$ denote, as before, the set of vertices where a splitting occurs with $p(a) < p(d)$ and respectively $p(a) > p(d)$, where a and d denote respectively the lower incoming and the lower outgoing edge. Moreover $D_{p,1} \# D_{p,2}$ denotes the set of all the common vertices of $D_{p,1}$ and $D_{p,2}$.

On the other side we have:

$$\left(\mathrm{id} \bigotimes_{K_2} \nabla\right) \nabla(D)$$
$$= \sum_{f \in Lbl_2(D)} (-1)^{|(S_f)_-| + |(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} x^{w(D_{f,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} D_{f,1}$$
$$\bigotimes_{K_2} \nabla\left(\tilde{x}^{w(D_{f,2}) - w(D)} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} D_{f,2}\right)$$
$$= \sum_{f \in Lbl_2(D)} \sum_{g \in Lbl_2(D_{f,1})} (-1)^{|(S_f)_-| + |(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} (-1)^{|(S_g)_-| + |(\tilde{S}_g)_-|} h^{|S_g|} \tilde{h}^{|\tilde{S}_g|}$$
$$\times x^{w(D_{f,1}) - w(D)} z^{w(D_{f,2} \Downarrow D_{f,1})} D_{f,1} \bigotimes_{K_2} \tilde{x}^{w(D_{f,2}) - w(D)} x^{w((D_{f,2})_{g,1}) - w(D_{f,2})}$$
$$\times \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} z^{w((D_{f,2})_{g,2} \Downarrow (D_{f,2})_{g,1})} (D_{f,2})_{g,1} \bigotimes_{K_2} \tilde{z}^{w((D_{f,2})_{g,1} \Downarrow (D_{f,2})_{g,2})}$$
$$\times \tilde{x}^{w(D_{f,2}) - w(D)} \tilde{x}^{w((D_{f,2})_{g,2}) - w(D_{f,2})} \tilde{z}^{w(D_{f,1} \Downarrow D_{f,2})} (D_{f,2})_{g,2}\,.$$

We consider again a 3-labelling map $p \in Lbl_3(D)$ defined as:

$$p(e) \equiv \begin{cases} 1 & \text{if} \quad f(e)=1; \\ 2 & \text{if} \quad g(e)=1, \quad f(e)=2; \\ 3 & \text{if} \quad g(e)=2, \quad f(e)=2. \end{cases}$$

So $\left(\mathrm{id} \bigotimes_{K_2} \nabla\right) \nabla(D)$ can be written as

$$\sum_{p \in Lbl_3(D)} (-1)^{|(S_p)_-| + |(\tilde{S}_p)_-|} h^{|S_p|} \tilde{h}^{|\tilde{S}_p|} x^{w(D_{p,1}) - w(D)} z^{w((D_{p,2} \cup D_{p,3}) \Downarrow D_{p,1})} D_{p,1}$$
$$\bigotimes_{K_2} \tilde{x}^{w(D_{p,2}) + w(D_{p,3}) + w(D_{p,2} \# D_{p,3}) - w(D)} x^{-w(D_{p,3}) - w(D_{p,2} \# D_{p,3})}$$
$$\times \tilde{z}^{w(D_{p,1} \Downarrow (D_{p,2} \cup D_{p,3}))} z^{w(D_{p,3} \Downarrow D_{p,2})} D_{p,2} \bigotimes_{K_2} \tilde{z}^{w(D_{p,2} \Downarrow D_{p,3})}$$
$$\times \tilde{x}^{w(D_{p,3}) - w(D)} \tilde{z}^{w(D_{p,1} \Downarrow (D_{p,2} \cup D_{p,3}))} D_{p,3}\,.$$

If we compare the two expressions we check immediately that the coassociativity is guaranteed only if the following conditions hold:

$$x=\tilde{x}; \qquad z=\tilde{z}=1.$$

Hence we are forced to consider tensor products over K_1. In this case we have obviously that $\nabla(z)=z(1\otimes 1)$ and $\nabla(\tilde{z})=\tilde{z}(1\otimes 1)$. If we redo the calculation by replacing K_2 with K_1, then we see immediately that the coassociativity holds if and only if $x=\tilde{x}$, for any z and $\tilde{z}$. In this case the coassociative comultiplication reads[6]:

$$\nabla(D)=\sum_{f\in Lbl_2(D)}(-1)^{|(S_f)_-|}(-1)^{|(\tilde{S}_f)_-|}h^{|S_f|}\tilde{h}^{|\tilde{S}_f|}z^{w(D_{f,2}\Downarrow D_{f,1})}\tilde{z}^{w(D_{f,1}\Downarrow D_{f,2})}$$
$$\times x^{w(D_{f,1})-w(D)}D_{f,1}\bigotimes_{K_2}x^{w(D_{f,2})-w(D)}D_{f,2}. \quad \square \tag{2.4}$$

From now on we will consider tensor products of $\mathscr{D}$ over $K\equiv K_1$ so we will omit the specification of the ground ring. In order to simplify the notation we will write (2.4) also as follows:

$$\nabla(D)\equiv\sum_{f\in Lbl_2(D)}\sigma(D,f)x^{\varrho_1(D,f)}D_{f,1}\otimes x^{\varrho_2(D,f)}D_{f,2}\equiv\sum_{f\in Lbl_2(D)}\nabla(D,f), \tag{2.5}$$

where the identification of (2.5) with (2.4) provides the values of the scalar functions σ, ϱ_1, ϱ_2 and of $\nabla(D,f)\in\mathscr{D}\otimes\mathscr{D}$.

Remark. The coassociativity of the comultiplication implies that if we consider the following operator for each $N\geqq 2$:

$$\nabla^N\equiv(\nabla\otimes\underbrace{\mathrm{id}\otimes\ldots\otimes\mathrm{id}}_{N\text{-}2\text{ times}})\circ(\nabla\otimes\underbrace{\mathrm{id}\otimes\ldots\otimes\mathrm{id}}_{N\text{-}3\text{ times}})\circ\ldots\circ\nabla, \tag{2.6}$$

then we have the following equality for each i with $0\leqq i\leqq N-1$:

$$\nabla^{N+1}(D)=\underbrace{(\mathrm{id}\otimes\ldots\otimes\mathrm{id})}_{i\text{ times}}\otimes\nabla\otimes\underbrace{(\mathrm{id}\otimes\ldots\otimes\mathrm{id})}_{N-i-1\text{ times}}\nabla^N(D).$$

Moreover one can easily prove that:

$$\nabla^N(D)=\sum_{f\in Lbl_N(D)}\sigma(D,f)x^{\varrho_1(D,f)}D_{f,1}\otimes\ldots\otimes x^{\varrho_N(D,f)}D_{f,N}, \tag{2.7}$$

---

[6] Following [10] we could distinguish, in the two configurations corresponding to the variables h and $\tilde{h}$, the two cases when the splitting of the given diagram a) increases or b) decreases the number of the components of the diagram itself. In these two cases we may consider respectively the variables:
a) h_- and $\tilde{h}_-$
b) h_+ and $\tilde{h}_+$
and the corresponding coassociative comultiplication would look as follows:

$$\nabla(D)=\sum_{f\in Lbl_2(D)}(-1)^{|(S_f)_-|}(-1)^{|(\tilde{S}_f)_-|}h_+^{(|D|-|D_{S_f}|+|S_f|)/2}$$
$$\times h_-^{(-|D|+|D_{S_f}|+|S_f|)/2}\tilde{h}_+^{(|D|-|D_{\tilde{S}_f}|+|\tilde{S}_f|)/2}\tilde{h}_-^{(-|D|+|D_{\tilde{S}_f}|+|\tilde{S}_f|)/2}$$
$$\times z^{w(D_{f,2}\Downarrow D_{f,1})}\tilde{z}^{w(D_{f,1}\Downarrow D_{f,2})}$$
$$\times x^{w(D_{f,1})-w(D)}D_{f,1}\bigotimes_{K_2}x^{w(D_{f,2})-w(D)}D_{f,2}.$$

Here, and in the future, for any $W\subset V(D)$, D_W denotes the diagram obtained by splitting D at the vertices in W, in the only orientation-preserving way where $\varrho_i(D,f) \equiv w(D_{f,i}) - w(D)$ and

$$\sigma(D,f) \equiv (-1)^{|(S_f)_-|}(-1)^{|(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} \times z^{\sum_{i>j} w(D_{f,i} \Downarrow D_{f,j})} \tilde{z}^{\sum_{i>j} w(D_{f,j} \Downarrow D_{f,i})};$$

where in turn S_f and $\tilde{S}_f$ denote respectively the set of vertices such that $f(a) < f(d)$ and $f(a) > f(d)$, when a and d are respectively the lower incoming and the lower outgoing edge, while $S(_f)_\pm$ and $\tilde{S}(_f)_\pm$ denote respectively the set of vertices in S_f and $\tilde{S}_f$ of type $L_\pm$.

Let P be the permutation operator in $\mathscr{D} \otimes \mathscr{D}$. We recall that the comultiplication ∇ is, by definition, cocommutative if and only if $P \circ \nabla = \nabla$. We have now the following:

2.2 Theorem. *The comultiplication* (2.5) *is cocommutative if and only if* $h = \tilde{h}$, $z = \tilde{z}$.

Proof.

$$\nabla(D) - (P \circ \nabla)(D) = \sum_{f \in Lbl_2(D)} \sigma(D,f) x^{\varrho_1(D,f)} D_{f,1} \otimes x^{\varrho_2(D,f)} D_{f,2} - \sum_{f \in Lbl_2(D)} \sigma(D,f) x^{\varrho_2(D,f)} D_{f,2} \otimes x^{\varrho_1(D,f)} D_{f,1}.$$

Now if h and $\tilde{h}$ are either both 0, or both different from 0, then to any labelling f we can associate a labelling $\tilde{f}$ by interchanging the values 1 and 2. Obviously the possibility of

$$\sigma(D,f) - \sigma(D,\tilde{f})$$

being different from zero, is the only obstruction to the cocommutativity. But $\sigma(D,\tilde{f})$ is obtained from $\sigma(D,f)$ by interchanging h and $\tilde{h}$ and z with $\tilde{z}$. So the comultiplication is cocommutative if and only if $h = \tilde{h}$ and $z = \tilde{z}$. □

In the set of link diagrams we can attempt to define a product of two diagrams D and D' (to be denoted by the symbol $D \circ D'$) as the diagram obtained by the union of the two diagrams D and D', with the additional prescription that at all the intersections of D with D', D crosses *over* D'. The problem with this product is that it is not well defined if we consider equivalence classes of diagrams, i.e. diagrams modulo ambient 2-dimensional isotopies, without considering the additional equivalence relation determined by Reidemeister moves (see below). So, when considering the product of link-diagrams, we look at them, provisionally, as *rigid* diagrams and not as equivalence classes modulo ambient 2-dimensional isotopies. The product of two diagrams is considered only when the resulting diagram has at most isolated double points. Moreover in order to have the comultiplication well defined on the module of rigid link-diagrams, we either assume that we can smooth the crossing points, or, instead, that our loops are piecewise smooth (with only double points). The unit with respect to the product of rigid link-diagrams is the *empty knot-diagram* which will be denoted by the symbol $\emptyset$ or simply by 1.

Now we want to discuss the conditions under which the comultiplication (2.4) is a morphism with respect to the product. Generalizing [10], we modify the product $D \circ D'$ as follows:

$$D * D' = (z^{-1}x)^{w(D \downarrow D')} D \circ D', \tag{2.8}$$

where $D{\downarrow}D'$ denotes the set of all the vertices in the diagram $D \circ D'$ corresponding to the intersection points of the projection of D with the projection of D'. The number $w(D{\downarrow}D')$ coincides with the intersection number of $\pi(D)$ with $\pi(D')$.

We will use the symbol $\mathscr{D}_*$ for the module of (rigid) link-diagrams with the modified product (2.8) (and comultiplication (2.4)). We have now the following:

2.3 Theorem. *The comultiplication* (2.4) *in* $\mathscr{D}_*$ *is a morphism with respect to the product if and only if* $z=\tilde{z}$ *and either* h *or* $\tilde{h}$ *is* 0.

Proof. This proof is a generalization of the proof in [10], so we refer that paper for the omitted details.

To each pair of 2-labelling maps f, f' defined respectively on the edges of the diagrams D and D', we can naturally associate a 2-labelling map $f \vee f'$ defined on the edges of $D \circ D'$ as:

$$(f \vee f')(a) = \text{either } f(a) \quad \text{or} \quad f'(a),$$

depending on whether a is an edge of D or of D'.

It is easy to prove that each 2-labelling map in $D \circ D'$ can be written as $f \vee f'$ if and only if either h or $\tilde{h}$ is 0. In fact in this case it is not possible to have a splitting of the diagram $D \circ D'$ which "mixes up" D and D'. It is also easy to realize that unless the previous condition is fulfilled, then the comultiplication cannot possibly be a morphism with respect to the product. So from now on, in this proof, we assume (for the sake of definitiveness) that $\tilde{h}$ is 0.

$$\begin{aligned}
&\nabla(D * D')\\
&\quad = \nabla((z^{-1}x)^{w(D\downarrow D')}) \sum_{\substack{f \in Lbl_2(D)\\ f' \in Lbl_2(D')}} (-1)^{|(S_f)_-| + |(S_{f'})_-|} h^{|S_f| + |S_{f'}|}\\
&\qquad \times z^{w((D\circ D')_{f\vee f',2} \Downarrow (D\circ D')_{f\vee f',1})} \tilde{z}^{w((D\circ D')_{f\vee f',1} \Downarrow (D\circ D')_{f\vee f',2})}\\
&\qquad \times x^{w((D\circ D')_{f\vee f',1}) - w(D\circ D')} (D \circ D')_{f\vee f',1} \otimes x^{w(D\circ D'_{f\vee f',2}) - w(D\circ D')} (D \circ D')_{f\vee f',2}\\
&\quad = \nabla((z^{-1}x)^{w(D\downarrow D')}) \sum_{\substack{f \in Lbl_2(D)\\ f' \in Lbl_2(D')}} (-1)^{|(S_f)_-| + |(S_{f'})_-|} h^{|S_f| + |S_{f'}|}\\
&\qquad \times z^{w((D_{f,2}\circ D'_{f',2}) \Downarrow (D_{f,1}\circ D'_{f',1}))} \tilde{z}^{w((D_{f,1}\circ D'_{f',1}) \Downarrow (D_{f,2}\circ D'_{f',2}))}\\
&\qquad \times x^{w(D_{f,1}) + w(D'_{f',1}) + w(D_{f,1}\downarrow D'_{f',1}) - w(D\circ D')} (x^{-1}z)^{w(D_{f,1}\downarrow D'_{f',1})} (D * D')_{f\vee f',1}\\
&\qquad\quad \otimes x^{w(D_{f,2}) + w(D'_{f',2}) + w(D_{f,2}\downarrow D'_{f',2}) - w(D\circ D')} (x^{-1}z)^{w(D_{f,2}\downarrow D'_{f',2})} (D * D')_{f\vee f',2}\\
&\quad = \sum_{\substack{f \in Lbl_2(D)\\ f' \in Lbl_2(D')}} z^{w(D_{f,2}\downarrow D'_{f',1}) + w(D_{f,1}\downarrow D'_{f',1}) + w(D_{f,2}\downarrow D'_{f',2}) - w(D\downarrow D')}\\
&\qquad \times \tilde{z}^{w(D_{f,1} \Downarrow D'_{f',2})} \nabla(D, f) * \nabla(D', f')\\
&\quad = \sum_{\substack{f \in Lbl_2(D)\\ f' \in Lbl_2(D')}} z^{-w(D_{f,1}\downarrow D'_{f',2})} \tilde{z}^{w(D_{f,1}\downarrow D'_{f',2})} \nabla(D, f) * \nabla(D', f').
\end{aligned}$$

Hence, once we assumed that either h or $\tilde{h}$ is zero, then the comultiplication becomes a morphism with respect to the product if and only if $z=\tilde{z}$. □

We can summarize in the following table the different properties of the comultiplication (2.2) $\left(\nabla : \mathscr{D} \to \mathscr{D} \underset{\mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]}{\otimes} \mathscr{D}\right)$ and the relevant constraints on the variables which were previously displayed in Fig. 1:

Table 1

Properties of the comultiplication		
coassociativity	cocommutativity	morphism with respect to the product
$x=\tilde{x}$;	$x=\tilde{x}$; $z=\tilde{z}$; $h=\tilde{h}$	$x=\tilde{x}$; $z=\tilde{z}$; $h=0$ or $\tilde{h}=0$

In the above table it is meant that we should consider $\mathscr{D}_*$ instead of $\mathscr{D}$ when the comultiplication is required to be an morphism with respect to the product.

As it was mentioned before, we will always assume that the minimum requirement we impose on the comultiplication is the coassociativity. Notice that when $z=\tilde{z}$ then the exponent of z becomes the linking number of the relevant subdiagrams. And this already suggests (as it will be shown later on) that the condition $z=\tilde{z}+1$ is not compatible with the construction of link invariants.

When $x=\tilde{x}$ we can also define a counit in $\mathscr{D}$, given by the unique $\mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h})$-module homomorphism $\eta:\mathscr{D}\to\mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$, satisfying the following requirements:

$$\eta(\emptyset)=1; \quad \eta(D)=0 \quad \text{for} \quad D\neq\emptyset; \quad \eta(x^k D)=\eta(D), \quad \forall D, \quad \forall k\in\mathbf{Z}; \tag{2.9}$$

here D denotes a generic link diagram and $\emptyset$ denotes the empty knot-diagram. It is immediate to check that in this way $\mathscr{D}$ becomes a coalgebra with counit.

Next we want to study the compatibility of the (coassociative) comultiplication with some special kinds of invariance, namely skein and Reidemeister invariance.

3. Skein Invariance

We define now as a (generalized) skein relation any equivalence relation in $\mathscr{D}$ of the following type:

$$\gamma D_+ - \delta D_- = \beta D_0, \tag{3.1}$$

where $\beta\in\mathbf{C}[z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$, $\gamma,\delta\in\mathbf{C}[x,x^{-1},z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ and $\{D_+,D_-,D_0\}$ is a Conway triple, namely is a triple of link-diagrams which differ only around one point where D_+ and D_- display respectively a vertex of type L_+ and a vertex of type L_-, while in D_0 the vertex is eliminated in the only orientation preserving way.

We would like to check now whether the comultiplication is compatible with such a skein relation. More precisely we want to prove the following:

3.1 Theorem. *Let us consider a skein relation of the form:*

$$xD_+ - x^{-1}D_- = (h+\tilde{h})D_0 \tag{3.2}$$

and let us quotient out $\mathscr{D}$ by this relation. The comultiplication (2.4) *gives a comultiplication on the quotient module if and only if $\tilde{z}=z^{-1}$. Moreover any skein relation of the type* (3.1), *different from the above one, is incompatible with the comultiplication* (2.4).[7]

[7] If we distinguish between Conway triples where the splitting of the diagram a) increases or b) decreases the number of components (see the previous footnote #6), then the relevant skein relation should read [10]:

$$xD_+ - x^{-1}D_- = (h_\varepsilon+\tilde{h}_\varepsilon)D_0,$$

where $\varepsilon=+$ for case a) and $\varepsilon=-$ for case b)

Proof. Given a Conway triple D_ε with $\varepsilon \in \{+,-,0\}$, let us consider the possible values of a 2-labelling on the edges in the region where these three diagrams differ from each other. The possible values are described by the six configurations a_i, $i=1,\ldots,6$ described below:

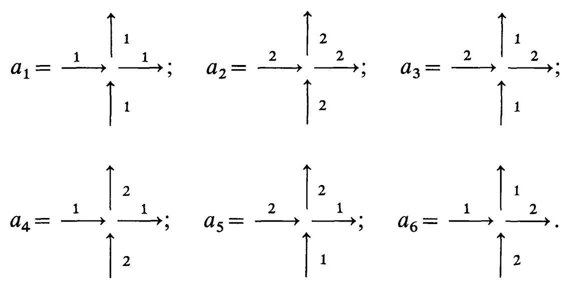


Fig. 2

In the above figure, it is understood that the arrows are prolonged in such a way as to create a crossing of type $L_\pm$ for $D_\pm$ and an orientation-preserving splitting for D_0.

For each $\varepsilon \in \{+,-,0\}$ we denote by f_ε^i the 2-labelling of the diagram D_ε which, when it exists[8], assumes the value f on all the edges which are common to all the three diagrams D_+, D_-, D_0 and otherwise assumes the value described by the configuration a_i. In order to check the skein-invariance of the comultiplication we have only to verify the following relation:

$$\nabla(\gamma)\nabla(D_+, f_+^i) - \nabla(\delta)\nabla(D_-, f_-^i) = \beta\nabla(D_0, f_0^i), \quad \forall_i = 1,\ldots,6, \quad \forall f. \tag{3.3}$$

In order to simplify the notation in this proof, we will set:

$$D_\varepsilon^j \equiv (D_\varepsilon)_{f_\varepsilon^i, j}; \quad \varrho_j(f_\varepsilon^i) \equiv \varrho_j(D_\varepsilon, f_\varepsilon^i); \quad \text{for} \quad j=1,2; \quad i=1,\ldots,6.$$

Otherwise, for the notation, see (2.5). We want now to verify Eq. (3.3) in the different configurations a_i (Fig. 2):

Case a_1

We have $D_+^2 = D_-^2 = D_0^2$ and $\{D_+^1, D_-^1, D_0^1\}$ is a Conway triple, moreover $\sigma(D_+, f_+^1) = \sigma(D_-, f_-^1) = \sigma(D_0, f_0^1)$ and $\varrho_1(f_+^1) = \varrho_1(f_-^1) = \varrho_1(f_0^1)$, $\varrho_2(f_+^1) = \varrho_2(f_-^1) - 2 = \varrho_2(f_0^1) - 1$; hence

$$\nabla(D_+, f_+^1) = \sigma(D_0, f_0^1) x^{\varrho_1(f_0^1)} D_+^1 \otimes x^{\varrho_2(f_0^1)-1} D_0^2$$

and

$$\nabla(D_-, f_-^1) = \sigma(D_0, f_0^1) x^{\varrho_1(f_0^1)} D_-^1 \otimes x^{\varrho_2(f_0^1)+1} D_0^2 .$$

So the only skein relation which is compatible with the comultiplication must be of the type

$$xD_+ - x^{-1}D_- = \beta D_0$$

with $\beta \in \mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$.

[8] It is clear, for instance, that f_0^3 and f_0^4 do not exist

Case a_5

We have $D^j_+ = D^j_- = D^j_0$, $j=1,2$, $\sigma(D_+, f^5_+) = h\sigma(D_0, f^5_0)$ and $\sigma(D_-, f^5_-) = -\tilde{h}\sigma(D_0, f^5_0)$. Moreover we have $\varrho_1(f^5_+) = \varrho_1(f^5_-) - 2 = \varrho_1(f^5_0) - 1$, and $\varrho_2(f^5_+) = \varrho_2(f^5_-) - 2 = \varrho_2(f^5_0) - 1$ and so we get

$$V(D_+, f^5_+) = h\sigma(D_0, f^5_0) x^{\varrho_1(f^5_0)-1} D^1_0 \otimes x^{\varrho_2(f^5_0)-1} D^2_0 ,$$

$$V(D_-, f^5_-) = -\tilde{h}\sigma(D_0, f^5_0) x^{\varrho_1(f^5_0)+1} D^1_0 \otimes x^{\varrho_2(f^5_0)+1} D^2_0 ,$$

and hence

$$(x \otimes x) V(D_+, f^5_+) - (x^{-1} \otimes x^{-1}) V(D_-, f^5_-) = (h + \tilde{h}) V(D_0, f^5_0)$$

which is only compatible with a skein relation of the type $xD_+ - x^{-1}D_- = (h+\tilde{h})D_0$.

Case a_3

We have $D^j_- = D^j_+$; $j=1,2$,

$$\tilde{z}^{-1}\sigma(D_+, f^3_+) = z\sigma(D_-, f^3_-), \qquad \varrho_i(f^3_+) = \varrho_i(f^3_-) - 2$$

which implies

$$(x \otimes x) V(D_+, f^3_+) = z\tilde{z}(x^{-1} \otimes x^{-1}) V(D\ \ , f^3_-),$$

and due to the fact that f^3_0 does not exist, the compatibility with the previous skein relation requires $z = \tilde{z}^{-1}$.

Case a_2 is completely analogous to case a_1, case a_4 is completely analogous to case a_3 and finally case a_6 yields the same conclusion as case a_5, due to the factor $(-1)^{|(S_f)-|}(-1)^{|(\tilde{S}_f)-|}$ which appears in the comultiplication. □

Notice that the usual skein relation for link-diagrams is obtained by setting $\tilde{h}=0$. As far as the relation $\tilde{z} = z^{-1}$ is concerned, we notice that this relation is incompatible with V being a morphism with respect to the product, unless $z=1$. If the surface Σ is the disk B^2, namely if we are considering ordinary diagrams for links in the euclidean 3-space, then the exponent of z is always the same as the exponent of $\tilde{z}$, so there is no loss of generality in setting $z=1$. In order to allow ourselves to be convinced that the above statement is true, we notice that the contribution of any labelling map f to the exponent of z is given by $w(D_{f,2} \downarrow D_{f,1})$, namely the intersection number of $D_{f,2}$ with $D_{f,1}$ which is 0 when Σ is contractible.

On the contrary, if Σ is a surface of higher genus, then the exponent of z needs not to be the same as the exponent of $\tilde{z}$ (for instance, when loops have an odd number of intersection points) and so one cannot set in general $z=1$.

In fact, it is exactly by considering the variable z, that one is able to find link-invariants for $\Sigma \times [0,1]$ *which are specific of surfaces of higher genus* (see below).

We set then $z = \tilde{z}^{-1}$ and denote, by the symbol $\mathscr{D}^S$, the module obtained from $\mathscr{D}$ by considering the equivalence classes with respect to the skein relation (3.2). Since the counit (2.9) is compatible with the skein relation (3.2), $\mathscr{D}^S$ is a coalgebra with counit.

4. Invariance of the Comultiplication under Reidemeister Moves

We would like now to discuss the invariance of the comultiplication (2.4) under Reidemeister moves. We restrict ourselves to the situation where the skein-invariance holds; in fact, as we will see shortly, the invariance under Reidemeister moves is proved by using the skein-relation. More precisely we want to show that, under suitable conditions, the comultiplication descends to a comultiplication on the quotient module where *both* the skein relation and the Reidemeister moves are taken into consideration. Hence from now we will set

$$\tilde{z}=z^{-1} \tag{4.1}$$

in (2.4.). Moreover, if we want to require the comultiplication to be invariant under the first Reidemeister move (see below) then we have to take into account the need of compensating the effect of adding a curl to a link diagram. So we are led to modify slightly the definition of the comultiplication by introducing an integer $r(D)$ called the winding number or the rotation factor of the diagram D. We define first the rotation factor for the diagram of an (oriented) knot and subsequently define the rotation factor of a link-diagram as the sum of the rotation factors of its (oriented) components. The rotation factor of a knot-diagram does not depend on the over/under crossings of its double points, so we are in fact only considering winding numbers (rotation factors) of loops.

We will consider from now on loops given by regular closed curves which are contained in the interior of Σ. By regular curve we mean a C^∞-curve which has a non-zero tangent vector at each point.

We distinguish two cases:

1. Σ is a parallelizable surface, e.g. the disk B^2, the torus or a surface obtained by removing the interior of a disk from a closed surface of genus $g \geqq 2$.
2. Σ is a closed surface of genus $g \geqq 2$.

In order to define the rotation factor in both cases, we recall briefly some facts from [23–26]. If Σ is parallelizable, then given a regular closed curve $\gamma: S^1 \to \Sigma$ we can choose a parallelization $X: \Sigma \to T\Sigma$ which at the base point x_0 of γ assigns a vector parallel (and oriented as) the tangent vector to γ itself.

Now one can consider the circle bundle of normalized tangent vectors in $T\Sigma$ and pull it back via γ to a circle bundle E over S^1. On this bundle one has two sections, one which associates to each point $t \in S^1$ the normalized tangent vector to γ at $\gamma(t)$ and the other one given by the pullback, via γ, of the section of the circle bundle over Σ represented by $\frac{X}{\|X\|}$, where X is the chosen parallelization. These two sections represent two elements of the fundamental group of $E \approx S^1 \times S^1$, and since they are sections, the homotopy exact sequence tells us that the quotient of these two elements is represented by an integer, which is called the winding number or the rotation factor of γ with respect to X.

This winding number does depend neither on the element inside a given homotopy class of X (with fixed base point) nor on the element inside a given regular homotopy class of γ (with fixed base point and tangent direction).

Regular homotopy classes of (regular) curves are in a one to one correspondence with the elements of $\pi_1(T\Sigma)$ and so they can be given a group structure.

The winding number (rotation factor) considered before is in fact an homomorphism from the group of regular homotopy classes with fixed base point, to the integers.[9] Moreover is it possible to find a parallelization X such that the relevant winding number satisfies the following two constraints:

i) has value 0, when computed over a system of regular simple curves which generate $\pi_1(\Sigma)$ and whose homology classes form a basis of $H_1(\Sigma, \mathbf{Z})$;

ii) has value 1, when computed over a contractible, simple, positively oriented (i.e. counterclockwise oriented) loop.

The case when Σ is a closed surface of genus $g \geqq 2$ needs some modification, due to the fact that we do not have a parallelization in this case. We can take off a point $v \in \Sigma$ (which does not belong to the given curve), and consider a non-vanishing vector field X on $\Sigma \backslash \{v\}$. We now repeat the construction as in the parallelizable case and we see that, due to the arbitrariness of the choice of the point v we are only able to define a winding number of a regular closed curve as an element of $Z/(2g-2)Z$.

Let us now take into account the winding number (rotation factor) in the comultiplication (2.4) (see also (2.5) for the notation).

We define:

$$\tau_i(D, f) \equiv \varrho_i(D, f) + \left(\sum_{j>i} - \sum_{j<i}\right) r(D_{f,j}), \tag{4.2}$$

where ϱ_i is given as in (2.5) and (2.7), namely $\varrho_i(D, f) \equiv w(D_{f,i}) - w(D)$. For any diagram D and for any labelling map f, we will use the symbols $\nabla_r(D)$, $\nabla_r^N(D)$ and $\nabla_r(D, f)$ to denote the elements obtained by substituting $\varrho_i(D, f)$ with $\tau_i(D, f)$ in $\nabla(D)$, $\nabla^N(D)$ and $\nabla(D, f)$.

If Σ is a parallelizable surface, then the above definitions are unambiguous. If on the contrary Σ is a closed surface of genus g, then we will have to restrict ourself to the case when x is a $(2g-2)^{\text{th}}$ root of 1; namely the ring $\mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ should be replaced as follows. We consider the abelian (multiplicative) group of the $(2g-2)^{\text{th}}$ roots of 1 that we denote by the symbol $R_{2g-2} \approx Z/(2g-2)Z$, we then consider the group algebra $\mathbf{C}[R_{2g-2}]$ and we replace $\mathbf{C}[x, x^{-1}, z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ with $\mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}, \tilde{z}, \tilde{z}^{-1}, h, \tilde{h}]$ and we do the same for the other rings containing the variables x and x^{-1}. First we have:

4.1 Theorem. *Let $\tilde{z} = z^{-1}$. Then the comultiplication ∇_r is coassociative and it is compatible with the skein relation*

$$xD_+ - x^{-1}D_- = (h + \tilde{h})D_0 .$$

Proof. The proof of the coassociativity is a matter of simple calculations, which are completely analogous to the ones made in Sect. 2. As far as the skein relation is concerned, it is obvious that the introduction of the rotation factor does not alter the results obtained in Theorem 3.1 of this paper. □

[9] Here one requires the choice of a fixed point. This does not imply that the definition of the winding number depends necessarily on the choice of a point in the given (regular) curve. In particular, it is shown in [25] that two freely homotopic closed curves (with a finite number of double intersection points), which do not contain any nullhomotopic loop, are also free regularly homotopic, and hence have equal winding number, with respect to any given parallelization

The previous theorem guarantees that, when we set $\tilde{z}=z^{-1}$, then the comultiplication ∇_r descends to a comultiplication on the quotient module $\mathscr{D}^S$ given by $\mathscr{D}$ modulo the skein relation. *Hence from now on we will consider the comultiplication ∇_r as defined on $\mathscr{D}^S$* $\left(\text{over the field } \mathbf{C}[x,x^{-1},z,z^{-1},h,\tilde{h}] \text{ or, respectively, } \mathbf{C}[R_{2g-2}] \otimes_{\mathbf{C}} \mathbf{C}[z,z^{-1},h,\tilde{h}]\right)$. The comultiplication ∇_r maps then $\mathscr{D}^S$ into $\mathscr{D}^S \otimes \mathscr{D}^S$, where the tensor product is taken over $\mathbf{C}[z,z^{-1},h,\tilde{h}]$.

Next we have to check the invariance of ∇_r under Reidemeister moves. The definition of the three Reidemeister moves will be recalled briefly below. But we have to recall that we are considering in general, a non-contractible two-dimensional surface; hence we have to point out, as a general caveat, that all the moves we are going to consider are meant to take place in a single *contractible* region of Σ.

We proceed as follows: for any link diagram D we denote by $D^{\#}$ the diagram obtained by applying to it the Reidemeister move under consideration. In order to show that the comultiplication is compatible with the given Reidemeister move we have to show that for any labelling map f, defined on all the edges of D not involved in the move under consideration, we have:

$$\sum_{f_i} \nabla_r(D, f_i) = \sum_{f_j^{\#}} \nabla_r(D^{\#}, f^{\#}),$$

where f_i and $f_j^{\#}$ are the different possible labelling maps, relevant to D and, respectively, to $D^{\#}$, which extend f. As a final result we have the following:

4.2 Theorem. *The comultiplication $\nabla_r: \mathscr{D}^S \to \mathscr{D}^S \otimes \mathscr{D}^S$ is compatible with the three Reidemeister moves if and only if $\tilde{h}=0$.*

The proof of the above theorem will take the rest of this section. Notice that in the above theorem the choice between the condition $\tilde{h}=0$ and the condition $h=0$ is due to the chosen orientation (and convention). Here, our convention is to consider a contractible, simple, counterclockwise oriented loop, as a loop with winding number $+1$.

Proof. We first consider the Reidemeister move I. This move consists in adding a curl to an edge of the link-diagram. We can add a positively or a negatively oriented curl and generate a vertex of type L_+ or L_-. Of these four possibilities, we will consider only the case of adding a negatively oriented curl with a L_+ vertex (see Fig. 3), the other cases being similar to this one.

When we consider the first Reidemeister move, the set $Lbl_2(D)$ (and $Lbl_2(D^{\#})$) splits into two subsets according to the values of the labels on the edge where the Reidemeister move takes place. Moreover to each labelling f for D we can associate two labelling maps for $D^{\#}$, call it $f_1^{\#}$, $f_2^{\#}$ which assign respectively the values 1 and 2 to the new edge, namely to the added curl.

We have to check that the following equation holds:

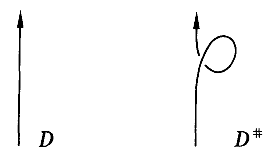


Fig. 3. Reidemeister move I

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \nabla_r(D, f), \quad \forall f.$$

Consider now the case in which f assigns label 1 to the edge in which we consider the Reidemeister move.

We have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D, f) = \tau_1(D^{\#}, f_2^{\#}) + 2,$$
$$\tau_2(D^{\#}, f_1^{\#}) = \tau_2(D, f) = \tau_2(D^{\#}, f_2^{\#}) + 1,$$

$D^{\#}_{f_1^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},2} = D_{f,2} \cup \bigcirc$, where $\bigcirc$ denotes the (contractible) unknot.

Moreover we have (up to the first Reidemeister move): $D^{\#}_{f_2^{\#},1} = D_{f,1} = D^{\#}_{f_1^{\#},1}$ and $\sigma(D^{\#}, f_1^{\#}) = \sigma(D, f)$; $\sigma(D^{\#}, f_2^{\#}) = \tilde{h}\sigma(D, f)$ and hence:

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \tilde{h}\sigma(D, f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)-1}(D_{f,2} \cup \bigcirc)$$
$$+ \sigma(D, f) x^{\tau_1(D,f)} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}.$$

In conclusion we have

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \nabla_r(D, f)$$

if and only if $\tilde{h} = 0$.

From now on, in this section, we will assume $\tilde{h} = 0$.

We consider now the case in which the edge on which we are applying the Reidemeister move has label 2. With the same notations as before we have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D, f) - 1 = \tau_1(D^{\#}, f_2^{\#}) + 1;$$
$$\tau_2(D^{\#}, f_1^{\#}) = \tau_2(D, f) = \tau_2(D^{\#}, f_2^{\#});$$

$D^{\#}_{f_1^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},2} = D_{f,2}$; $D^{\#}_{f_2^{\#},1} = D_{f,2}$; $D^{\#}_{f_1^{\#},1} = D_{f,1} \cup \bigcirc$; $\sigma(D^{\#}, f_1^{\#}) = h\sigma(D, f)$; $\sigma(D^{\#}, f_2^{\#}) = \sigma(D, f)$. In conclusion we have:

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = h\sigma(D, f) x^{\tau_1(D,f)-1}(D_{f,1} \cup \bigcirc) \otimes x^{\tau_2(D,f)} D_{f,2}$$
$$+ \sigma(D, f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}.$$

Now we use the skein relation (with $\tilde{h} = 0$) and obtain

$$\nabla_r(D^{\#}, f_1^{\#}) + \nabla_r(D^{\#}, f_2^{\#}) = \sigma(D, f) x^{\tau_1(D,f)-1}(x - x^{-1}) D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}$$
$$+ \sigma(D, f) x^{\tau_1(D,f)-2} D_{f,1} \otimes x^{\tau_2(D,f)} D_{f,2}$$
$$= \nabla_r(D, f).$$

This implies the required invariance under Reidemeister move I.

We will consider now the second Reidemeister move. There are essentially two types of Reidemeister moves: type A where we consider two arcs oriented in the same direction and type B where we consider two arcs oriented in the opposite direction (see Fig. 4).

In each case the set $Lbl_2(D)$ (and $Lbl_2(D^{\#})$) splits into five subsets, according to the values of the possible labels which are assigned to the edges exiting and

Fig. 4. Reidemeister moves II of type A and B

Fig. 5a

entering the region where the Reidemeister move takes place. The contributions to $\nabla_r(D^{\#})$ coming from each one of the five subsets of $Lbl_2(D^{\#})$ match separately with the contributions to $\nabla_r(D)$, coming from the corresponding subsets of $Lbl_2(D)$. We sketch now the proof of the invariance of the comultiplication under Reidemeister move II, type A. The region in which the Reidemeister move takes place is depicted as a rectangular box in Fig. 5a

Case $f^{\#}(a)=f^{\#}(b)=f^{\#}(c)=f^{\#}(d)=1$

The labelling maps $f^{\#}$ of $D^{\#}$ are in a one to one correspondence with the labelling maps f of D. Hence we have

$$D^{\#}_{f^{\#},2}=D_{f,2}$$

and, up to Reidemeister move II A, we have $D^{\#}_{f^{\#},1}=D_{f,1}$. Moreover one has $\sigma(D^{\#},f^{\#})=\sigma(D,f)$ and $\tau_1(D^{\#},f^{\#})=\tau_1(D,f)$, $\tau_2(D^{\#},f^{\#})=\tau_2(D,f)$, and so

$$\nabla_r(D^{\#},f^{\#})=\nabla_r(D,f).$$

Case $f^{\#}(a)=f^{\#}(c)=2$, $f^{\#}(b)=f^{\#}(d)=1$

Again the labelling maps $f^{\#}$ of $D^{\#}$ are in a one to one correspondence with the labelling maps f of D. We have

$$D^{\#}_{f^{\#},1}=D_{f,i}, \quad i=1,2$$

and $\sigma(D^{\#},f^{\#})=\sigma(D,f)$. Moreover one has

$$\tau_1(D^{\#},f^{\#})=\tau_1(D,f), \quad \tau_2(D^{\#},f^{\#})=\tau_2(D,f),$$

and so

$$\nabla_r(D^{\#},f^{\#})=\nabla_r(D,f).$$

Case $f^{\#}(a)=f^{\#}(d)=1$, $f^{\#}(b)=f^{\#}(c)=2$

In this case there is no corresponding labelling for D since the diagram D does not have any vertex in the region covered by the box. When we consider the diagram $D^{\#}$, on the contrary, we have two classes of labelling maps which satisfy the required condition, namely the ones which assign respectively the value 1 and 2 to the left edge created by the Reidemeister move. Denote these two kinds of labelling maps by the symbol $f_1^{\#}$ and $f_2^{\#}$. We have

$$D^{\#}_{f_1^{\#},i}=D^{\#}_{f_2^{\#},i}, \quad i=1,2$$

and, due to the relation $\tilde{z}=z^{-1}$, we have also:

$$\sigma(D^{\#}, f_1^{\#}) = -\sigma(D^{\#}, f_2^{\#}).$$

Moreover we have:

$$\tau_1(D^{\#}, f_1^{\#}) = \tau_1(D^{\#}, f_2^{\#}), \qquad \tau_2(D^{\#}, f_1^{\#}) = \tau_2(D^{\#}, f_2^{\#}),$$

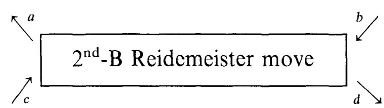
and so

$$V_r(D^{\#}, f_1^{\#}) + V_r(D^{\#}, f_2^{\#}) = 0.$$

The other cases are treated in a similar way.

We prove now the invariance under Reidemeister move II, type B. The region in which the Reidemeister move takes place is depicted in Fig. 5b.

Fig. 5b



The case $f^{\#}(a)=f^{\#}(b)=1$; $f^{\#}(c)=f^{\#}(d)=2$ is completely analogous to the case $f^{\#}(c)=f^{\#}(b)=2$, $f^{\#}(a)=f^{\#}(d)=1$ considered previously for the Reidemeister move of type II A. The other cases are trivial.

Let us now come to the Reidemeister move III

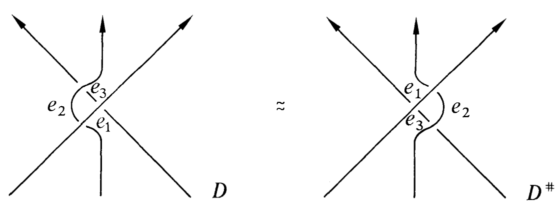


Fig. 6. Reidemeister move III

The set $Lbl_2(D)$ as well the set $Lbl_2(D^{\#})$ splits into twenty subsets[10] each one of them being characterized by the same label on the incoming and outgoing edges. In fact the possible labels of the three incoming edges are (up to permutations): $(1,1,1)$; $(1,1,2)$; $(1,2,2)$; $(2,2,2)$. The labels of the outgoing edges will be a permutation of the labels of the incoming edges. Hence the total number of possible labels are 20. Let now e_i, $i=1,2,3$ be as in Fig. 6.

Given any labelling $f^* \in Lbl\left(D \setminus \bigcup_{i=1}^{3} e_i\right)$ we can extend it to one or more labelling maps of D or of $D^{\#}$. Call f_i and $f_j^{\#}$ the relevant extensions. We want to prove that $\sum_{f_i} V_r(D, f_i) = \sum_{f_j^{\#}} V_r(D^{\#}, f_j^{\#})$. We observe now that the proof is trivial when f^* extends to only one labelling map of D and one labelling map of $D^{\#}$. The only non-trivial cases are the ones when a given labelling f^* extends to either two distinct labelling maps of D and one labelling map of $D^{\#}$ or vice versa.

This happens only if we have the following configurations: the incoming edges have label $(2,1,1)$ or $(2,2,1)$ and the same is true for the outgoing edges. These two

[10] Some of these subsets may possibly be empty

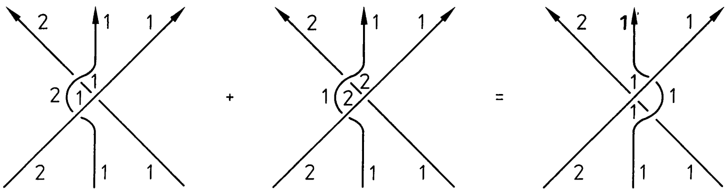


Fig. 7. Labels in Reidemeister move III

cases can be treated in exactly the same way. So we consider only the first one. In this case we have two labelling maps for D which extend f^*. Denote them by the symbol f_1 and respectively f_2. They give the following labels to the edges e_i: $f_1(e_1)=1$; $f_1(e_2)=2$; $f_1(e_3)=1$ and $f_2(e_1)=2$, $f_2(e_2)=1$, $f_2(e_3)=2$. On the contrary, on $D^\#$, f^* is extended as follows: $f^\#(e_1)=f^\#(e_2)=f^\#(e_3)=1$. We have

$$D_{f_1,2}=D_{f_2,2}=D^\#_{f^\#,2};\qquad \tau_2(D,f_1)=\tau_2(D,f_2)=\tau_2(D^\#,f^\#),$$
$$\tau_1(D,f_2)=\tau_1(D,f_1)-1=\tau_1(D^\#,f^\#)-2,$$
$$\sigma(D,f_1)=h\sigma(D,f_2)=h\sigma(D^\#,f^\#),$$

moreover

$$\{D^\#_{f^\#,1},D_{f_2,1},D_{f_1,1}\}$$

is a Conway triple, as can be seen by applying a Reidemeister move II A to $D_{f_2,1}$. Hence we have, by taking into account the skein relation:

$$\begin{aligned}\nabla_r(D,f_1)+\nabla_r(D,f_2)&=[\sigma(D^\#,f^\#)x^{\tau_1(D^\#,f^\#)-2}D_{f_2,1}\\&\quad+h\sigma(D^\#,f^\#)x^{\tau_1(D^\#,f^\#)-1}D_{f_1,1}]\otimes x^{\tau_2(D^\#,f^\#)}D^\#_{f^\#,2}\\&=\nabla_r(D^\#,f^\#).\ \square\end{aligned}$$

Now we can consider the quotient module given by $\mathscr{D}$ modulo (ambient 2-dimensional isotopies and) the skein relation *and* the Reidemeister moves. We denote this quotient module by the symbol $\mathscr{D}^{S,R}$. The results of this section imply that the comultiplication ∇_r descends to such a quotient module.

Notice that we used the fact that if we apply to any edge of a diagram D a Reidemeister move I and subsequently we split the diagram at the crossing point, then the skein relation tells us that:

$$xD-x^{-1}D=h(D\cup\bigcirc)\,\forall D.$$

In particular, as suggested by Turaev, by applying the above relation to the empty knot-diagram, we are led to assuming that:

$$x-x^{-1}=h\bigcirc. \tag{4.3}$$

We now consider again the two different products of two (rigid) link diagrams, that we defined before (i.e. $D\circ D'$ and $D*D'$). It turns out that these products are well defined on $\mathscr{D}^{S,R}$, namely they do not depend on the equivalence classes. Hence, depending on the choice of the product ($\circ$ vs. $*$), we have two algebras that we denote respectively by the symbols $\mathscr{D}^{S,R}$ and $\mathscr{D}^{S,R}_*$. By putting together the previous results we see that the comultiplication ∇_r, defined on the algebra $\mathscr{D}^{S,R}_*$, is an algebra homomorphism if and only if $z=1$.

When $z=1$ then $\mathscr{D}_*^{S,R}$ is a Hopf algebra according to the definition of [27]. Moreover one can define in $\mathscr{D}_*^{S,R}$ an antipode as follows. To each link-diagram D, we associate the diagram $\hat{D}$, obtained by changing in D every undercrossing into an overcrossing and vice versa. We define now the map

$$\gamma(D)\equiv(-1)^{|D|}\hat{D} \quad \text{and} \quad \gamma(x^{\pm 1})\equiv x^{\mp 1}. \tag{4.4}$$

Obviously $\gamma(D)$ does changes when we apply any Reidemeister move to D. Moreover, by applying γ to any Conway triple we obtain:

$$\gamma(x)\gamma(D_+)-\gamma(x^{-1})\gamma(D_-)=h\gamma(D_0)$$

or

$$(-1)^{|D_+|}(x\sigma(D_-)-x^{-1}\sigma(D_+))=-h(-1)^{|D_0|}\sigma(D_0)$$

or equivalently:

$$x\sigma(D_-)-x^{-1}\sigma(D_+)=h\sigma(D_0),$$

which shows that γ preserves the skein relation. In this way we showed that γ can be extended to a unique algebra anti-homomorphism (see [10]):

$$\gamma:\mathscr{D}_*^{S,R}\to\mathscr{D}_*^{S,R}.$$

It is possible to prove that, when $z=1$, (4.4) defines in fact an antipode; the relevant proof will be given in the appendix.

As a final remark on this section, we notice that by combining together the invariance of the projection of link-diagrams under ambient isotopies with the invariance under Reidemeister moves, we are in fact saying that two projections of link diagrams are equivalent if they differ by a homotopy of generic C^∞-immersions (see Lemma 5.6 in [12]).

5. Link Invariants for Links in $\Sigma\times[0,1]$

In the previous chapters we constructed a coassociative comultiplication ∇_r which is both invariant under the three Reidemeister moves and under the skein relation

$$xD_+-x^{-1}D_-=hD_0, \tag{5.1}$$

where $\{D_+,D_-,D_0\}$ is any Conway triple. Now we use this comultiplication to construct link-invariants, namely invariants of links in $\Sigma\times[0,1]$.

We recall that we started by considering a very general module of link-diagrams and that subsequently we

a) restricted the ring where this module is defined

b) divided the module itself modulo:

i) a skein relation

ii) the Reidemeister moves.

The main feature of this process has been the fact that not only the comultiplication descended to the various quotients which have been considered but also it "gained" properties while the ground ring was gradually restricted. Now (some) link invariants are in fact generated by a function which does *not* descend to the various quotients, as it will be shown below.

Let us assume first that Σ is parallelizable. We start by considering the module $\mathscr{D}$, *before* dealing with the skein relation and the Reidemeister moves. Nevertheless we restrict the ground ring by eliminating the "tilde variables" or, in other words, by setting $\tilde{z}=z^{-1}$, $\tilde{x}=x$, $\tilde{h}=0$. We then consider the homomorphism of modules[11]: $\hat{\psi}:\mathscr{D}^{\otimes k}\to\mathbf{C}[x,x^{-1},z,z^{-1},h]$ determined by the following conditions:

$$\begin{array}{ll} \text{i)}\ \hat{\psi}(aD)=a \quad \forall a\in\mathbf{C}[x,x^{-1},z,z^{-1},h] \text{ and for any diagram } D; \\ \text{ii)}\ \hat{\psi}(D_1\otimes D_2)=\hat{\psi}(D_1)\hat{\psi}(D_2) \quad \forall D_1,D_2\in\mathscr{D}. \end{array} \tag{5.2}$$

Recall that, in the above definition, the link-diagrams are the generators of $\mathscr{D}$; notice moreover that the map $\hat{\psi}$ is trivially invariant under Reidemeister moves, but it does not respect the skein relation.

In order to force the map $\hat{\psi}$ to be compatible with the skein relation, we have to "eliminate" the variable h by setting:

$$h=x-x^{-1}. \tag{5.3}$$

More precisely we proceed as follows: we introduce in the module $\mathscr{D}$ the following equivalence relation:

$$D_+=D_-=D_0, \tag{5.4}$$

where $\{D_+,D_-,D_0\}$ is any Conway triple. We then set $h=x-x^{-1}$ in this quotient module and, as a result, we obtain a module over the ring $\mathbf{C}[x,x^{-1},z,z^{-1}]$. We furthermore divide by the equivalence relation determined by Reidemeister moves and we obtain a module that we denote by the symbol $\mathscr{D}'$[12]. The map $\hat{\psi}$ determines a homomorphism[13]:

$$\psi:\mathscr{D}'\to\mathbf{C}[x,x^{-1},z,z^{-1}]. \tag{5.5}$$

More generally one can consider, instead of the map ψ, the map

$$\psi_H:\mathscr{D}'\to\mathbf{C}[x,x^{-1},z,z^{-1}]\bigotimes_{\mathbf{Z}}H_1(\Sigma,\mathbf{Z}), \tag{5.6}$$

which associates to each equivalence class of link-diagrams D the element given by the product of $\psi(D)$ times the product[14] of the homology classes of the components of (the projection of) D. Again the map ψ_H is a homomorphism.

The map ψ can be extended to a morphism:

$$\mathscr{D}'^{\otimes k}\to\mathbf{C}[x,x^{-1},z,z^{-1}]$$

and an analogous statement holds for ψ_H. The case when Σ is a closed surface of genus g is similar; we have only to replace $\mathbf{C}[x,x^{-1},z,z^{-1}]$ with $\mathbf{C}[R_{2g-2}]\bigotimes_{\mathbf{C}}\mathbf{C}[z,z^{-1}]$.

There exists an obvious homomorphism (projection):

$$\lambda:\mathscr{D}^{S,R}\to\mathscr{D}', \tag{5.7}$$

which induces in $\mathscr{D}'$ a structure of commutative algebra.

---

[11] The tensor product considered here is meant as the tensor product over the ring $\mathbf{C}[x,x^{-1},z,z^{-1},h]$ itself

[12] Notice that it would be enough to consider only the equivalence under Reidemeister move II B, since the equivalence under the other types of Reidemeister moves is implied by the relations (5.4) and (5.3) and by the equivalence under Reidemeister move II B.

[13] Notice that (5.3) establishes an homomorphism between $\mathbf{C}[x,x^{-1},z,z^{-1},h]$ and $\mathbf{C}[x,x^{-1},z,z^{-1}]$ and between $\mathbf{C}[R_{2g-2}]\bigotimes_{\mathbf{C}}\mathbf{C}[z,z^{-1},h]$ and $\mathbf{C}[R_{2g-2}]\bigotimes_{\mathbf{C}}\mathbf{C}[z,z^{-1}]$.

[14] In what follows we think of the first homology group $H_1(\Sigma,\mathbf{Z})$ as an Abelian group in multiplicative form

The comultiplication ∇_r does *not* descend to $\mathscr{D}'$. This is very fortunate since it allows us to construct non-trivial link-invariants in $\Sigma \times [0,1]$ (see below). In fact we can consider for any link-diagram and any integer N,

$$\psi^N(D) \equiv \psi(\lambda^{\otimes N}(\nabla_r^N(D))) \in \mathbf{C}[x, x^{-1}, z, z^{-1}] \tag{5.8}$$

and

$$\psi_H^N(D) \equiv \psi_H(\lambda^{\otimes N}(\nabla_r^N(D))) \in \mathbf{C}[x, x^{-1}, z, z^{-1}] \bigotimes_{\mathbf{Z}} H_1(\Sigma, \mathbf{Z}), \tag{5.9}$$

where ∇_r^N is defined as in (2.6) with the rotation factors included. For Σ closed, the corresponding quantities are obtained by replacing $\mathbf{C}[x, x^{-1}, z, z^{-1}]$ with $\mathbf{C}[R_{2g-2}] \bigotimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]$. We have now the following:

5.1 Theorem. *Let D be any link-diagram on Σ. For any $N \in \mathbf{Z}$, $\psi^N(D)$ and $\psi_H^N(D)$ are link-invariants for links in $\Sigma \times [0,1]$. When $\Sigma = B^2$, then $\psi^N(D) = \psi_H^N(D)$ does not depend on the variables $\{z, z^{-1}\}$ and is related to the two variables Homfly polynomial $P_{l,m}(D)$ [7, 28] by the relation:*

$$\psi^N(D) = \frac{x^N - x^{-N}}{x - x^{-1}} P_{x^N, x - x^{-1}}(D).$$

When Σ is a non-contractible parallelizable surface, then in general $\psi^N(D)$ depends non-trivially on the variables $\{z, z^{-1}\}$. When Σ is a closed surface of genus $g \geqq 2$, then $\psi^N(D)$ is only defined when x is a $(2g-2)^{\text{th}}$ root of 1 and in general depends non-trivially on the variables $\{z, z^{-1}\}$. Finally for any Σ, $\psi_H^N(D)$ is proportional to $\psi^N(D)$, the relevant coefficient being the element of $H_1(\Sigma, \mathbf{Z})$ given by the products of the homology classes of the components of D.

Proof. The comultiplication ∇_r^N satisfies the following skein relation:

$$x^{\otimes N} \nabla_r^N(D_+) - (x^{-1})^{\otimes N} \nabla_r^N(D_-) = h \nabla_r^N(D_0).$$

Consequently, when we set $h = x - x^{-1}$ and we consider $\lambda^{\otimes N}(\nabla_r^N(D)) \in \mathscr{D}'^{\otimes N}$, then we see that the map ψ^N satisfies the following skein relation:

$$x^N \psi^N(D_+) - x^{-N} \psi^N(D_-) = (x - x^{-1}) \psi^N(D_0). \tag{5.10}$$

Moreover the correspondence $D \to \psi^N(D)$ behaves invariantly under the Reidemeister moves I, II, III, due to the invariance properties of the comultiplication. Hence $\psi^N(D)$ is a link-invariant; $\psi_H^N(D)$ is obviously proportional to $\psi^N(D)$.

Let us now consider the special case when Σ is the disk B^2. The exponent of z in $\psi^N(D)$ is zero since in this case, by the definition of the comultiplication, the exponent of z is the same as the exponent of $\tilde{z} = z^{-1}$. Moreover thanks to (5.10) one is able to express the value of $\psi^N(D)$ in terms of $\psi^N(\bigcirc)$. Now the relation (5.10), considered when both D_+ and D_- are the empty knot-diagram, tells us that we have: $\psi^N(\bigcirc) = \frac{x^N - x^{-N}}{x - x^{-1}}$ and, by taking into account the fact that the Homfly polynomials satisfy the skein relation $l P_{l,m}(D_+) - l^{-1} P_{l,m}(D_-) = m P_{l,m}(D_0)$ with $P(l,m)(\bigcirc) = 1$, we obtain the required relation between $\psi_N(D)$ and the Homfly polynomial of D.

If we now consider the case $N = 2$ and we denote the Jones polynomial by the symbol $V_D(t)$, then, by setting $t = x^2$, we have $\psi^2(D) = \frac{t - t^{-1}}{t^{1/2} - t^{-1/2}} V_D(t)$.

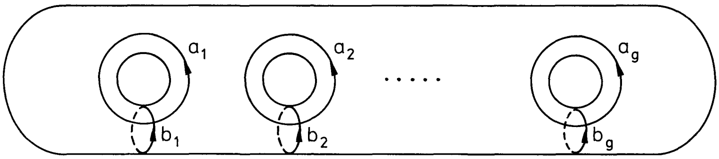


Fig. 8. Loops a_i and b_i

Finally let us consider the case when Σ is a (closed or open) surface of genus $g>0$ and show that there exist very simple link-diagrams whose invariants exhibit a non-trivial dependency on z. Let us consider for instance a link-diagram whose projection is given by two simple loops a_i and b_i whose homology classes are in the canonical basis of $H_1(\Sigma, \mathbf{Z})$ (see Fig. 8).

These two loops have only one crossing point; let us call D_+ and D_- the two link-diagrams which are obtained by assuming that the only vertex is of type L_+ and respectively L_-. Only four labelling maps exist for each diagram, moreover the rotation factors are all zero due to our definition of the winding number. A simple computation gives:

$$\psi^2(D_+)|_{x,z} = 2x^{-1} + x^{-2}(z^{-1}+z),$$
$$\psi^2(D_-)|_{x,z} = 2x + x^2(z^{-1}+z).$$

Hence we see that in general $\psi^N(D)$ is not trivial, meaning that it is not the product of $\psi^N(D)|_{x,1}$ times a polynomial in z. □

In the above description of link-invariants for $\Sigma \times [0,1]$ (with non-contractible Σ) the variable z is essential. In fact one has:

5.2 Theorem. *Let Σ be a (closed or open) surface with genus $g>0$. Then there exist two link-diagrams D and D' such that for any $N \geqq 2$ one has $\psi_H^N(D)|_{x,1} = \psi_H^N(D')|_{x,1}$ and $\psi^N(D) \neq \psi^N(D')$.*

Proof. Let us consider the two link-diagrams D and D' defined as follows:
i) D and D' have the same projection;
ii) D and D' have five components each, the projection of one component being the simple loop a_i as in Fig. 8. The projection of the other four components are simple loops with no intersection points among themselves, two of them are free homotopic to b_i, while the other two are free homotopic to b_i^{-1};
iii) the orientation of the components and the writhes of the vertices of D and D' are described in Fig. 9 below (where the loop a_i has been "opened up" for graphical reasons).

An easy calculation shows that we have:

$$\psi^N(D)|_{x,z} = \sum_{i=1}^{N} (x+(N-i)z+(i-1)z^{-1})^2 (x^{-1}+(N-i)z^{-1}+(i-1)z)^2$$

and:

$$\psi^N(D')|_{x,z} = \sum_{i=1}^{N} (x+(N-i)z+(i-1)z^{-1})(x^{-1}+(N-i)z^{-1}+(i-1)z)$$
$$\times (x^{-1}+(N-i)z+(i-1)z^{-1})(x+(N-i)z^{-1}+(i-1)z).$$

Hence we have $\psi_H^N(D)|_{x,1} = \psi_H^N(D')|_{x,1}$ and $\psi^N(D) \neq \psi^N(D')$. □

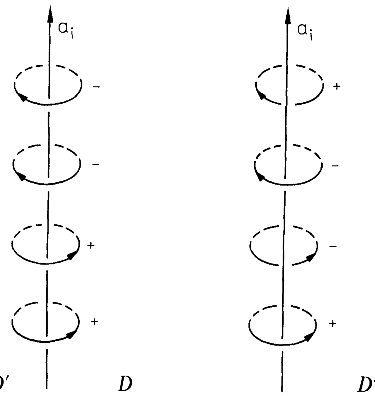


Fig. 9. Link-diagrams D and D'

We may also consider another link-invariant, which is obtained from $\psi^N(D)$ simply by a change of variables. Let $m^{\otimes N}$ denote the iterated multiplication defined on $(\mathscr{D}_*^{S,R})^{\otimes N}$ with values in $\mathscr{D}_*^{S,R}$ and let

$$\lambda : \mathscr{D}_*^{S,R} \to \mathscr{D}'$$

be the same morphism of modules constructed before[15].

We define now, for any link-diagram D, the following element in $\mathbf{C}[x, x^{-1}, z, z^{-1}]$ $\left(\text{or } \mathbf{C}[R_{2g-2}] \bigotimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]\right)$:

$$\chi^N(D) \equiv (\psi \circ \lambda)(m^{\otimes N}(V_r^N(D))). \tag{5.11}$$

The "new" invariant χ^N is simply a rescaling of ψ^N, since one has:

$$\chi^N(D)|_{x,z} = \psi^N(D)|_{x, x^{-1}z^2}.$$

In order to have a better appreciation of the role of the variable z, we would like to understand whether there exist two link-diagrams D and D' such that $\chi^N(D)|_{x,1} = \chi^N|_{x,1}$ (i.e. $\psi^N(D)|_{x,x^{-1}} = \psi^N(D')|_{x,x^{-1}}$) and $\psi^N(D) \neq \psi^N(D')$.

First of all let us point out that when Σ is a closed surface then the condition $x = z^{-1}$ forces also z to be a root of 1. So the question we are asking ourselves is relevant mainly in the case of open surfaces.

The previous example (Fig. 9) does not work any more here, since the function $\psi^N|_{x,x^{-1}}$ takes different values on D and D'. But notice that the two links of that example have all their components with the same (i.e. zero) winding number. Now, in order to construct examples of link diagrams which are not distinguished by $\psi^N|_{x,x^{-1}}$ (for some N), but are distinguished by ψ^N, one may look for links which have non-trivial rotation factors in some specific components. We consider here an example of this kind for $N=2$, while the situation for $N>2$ looks more complicated and will be discussed elsewhere.

We have now the following:

5.3 Theorem. *There exist two link-diagrams L and L' such that $\psi^2(L) \neq \psi^2(L')$, $\psi^2(L)|_{x,x^{-1}} = \psi^2(L')|_{x,x^{-1}}$ and the corresponding components of L and L' have the same projections.*

[15] We have only to remember that the multiplication in $\mathscr{D}_*^{S,R}$ is different from the one in $\mathscr{D}^{S,R}$

Proof. Let L and L' be the two link diagrams as in Fig. 10:

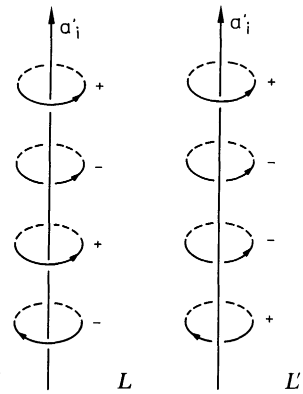


Fig. 10. Link-diagrams L and L'

Here the notation is as in Fig. 9, except that the simple loop a_i' is like the simple loop a_i only in the region where the intersections with the other components occur but, different from a_i, has winding number -1[16].

The calculations yield:

$$\psi^2(L)|_{x,z} = x(x+z)(x^{-1}+z)(x^{-1}+z^{-1})(x+z) + x^{-1}(x+z^{-1})(x^{-1}+z^{-1})(x^{-1}+z)(x+z^{-1})$$

and

$$\psi^2(L')|_{x,z} = x(x+z)(x^{-1}+z)(x^{-1}+z)(x+z^{-1}) + x^{-1}(x+z^{-1})(x^{-1}+z^{-1})(x^{-1}+z^{-1})(x+z).$$

Hence the difference $\psi^2(L) - \psi^2(L')$ is proportional to

$$x(x+z)(x^{-1}+z) - x^{-1}(x+z^{-1})(x^{-1}+z^{-1}),$$

which is zero when $x = z^{-1}$ but it is not zero for a general z. □

Finally one may wonder whether one could exhibit instead of $\psi_H^N(D)$, a more general link-invariant for links in $\Sigma \times [0,1]$ which restricts to $\psi^N(D)$ when we consider diagrams D on the disk B^2. We claim that the probability of finding such a more general invariant is rather slim.

The obvious choice would be to consider, instead of the map ψ_H defined in (5.9) a new map:

$$\tilde{\psi} : \mathscr{D}' \to \mathbf{C}[x, x^{-1}, z, z^{-1}] \bigotimes_{\mathbf{Z}} K, \tag{5.12}$$

which associates to each (equivalence class of) link-diagram(s) D the element given by the product of $\psi(D)$ times the product say of $\omega(D)$. Here ω is some function with values in some (multiplicative) abelian group K depending, for a knot-diagram, on the free homotopy class of its projection and, for a link-diagram, on the collection of the free homotopy classes of the projection of its components [17]. It is apparent

[16] We assume that the genus of Σ is great enough so that such a simple loop exists.

[17] As an example, we can think of the product of the "traces of the holonomy" of the components of the diagram, computed with respect to some flat connection defined on Σ

that any such function ω would be invariant under the three Reidemeister moves. But this invariance is not the only property that ω should satisfy. In particular in order for (5.12) to be a consistent definition we should have the following identity:

$$\omega(D_+)=\omega(D_-)=\omega(D_0). \tag{5.13}$$

Now, if we assume as a reasonable condition that for any link diagram D with components D_1 and D_2 one should have:

$$\omega(D)=\omega(D_1)\omega(D_2), \tag{5.14}$$

then we would have necessarily:

$$\tilde{\psi}=\sigma\circ\psi_H$$

for some homomorphism $\sigma: H_1(\Sigma,\mathbf{Z})\to K$. In order to see this, let us use the notation $\omega(D)\equiv\omega([a_1],\ldots,[a_j])$ when the projection of the diagram D is composed by j loops whose free homotopy classes are respectively given by the conjugacy classes of $a_1,\ldots,a_j\in\pi_1(\Sigma)$. Equations (5.14) and (5.13) imply that $\forall a_1,a_2\in\pi_1(\Sigma)$ one has $\omega([a_1])\omega([a_2])=\omega([a_1a_2])$ which in turn implies that ω restricted to the conjugacy classes determined by the commutator subgroup of $\pi_1(\Sigma)$ is zero.

Appendix: The Hopf Algebra Structure of Link-Diagrams and the Antipode Map

In this appendix we want to consider the module $\mathscr{D}_*^S$ of rigid link-diagrams over $\mathbf{C}[x,x^{-1},h]$ obtained from the module of (rigid) link-diagrams over the polynomial ring $\mathbf{C}[x,x^{-1},z,z^{-1},\tilde{z},\tilde{z}^{-1},h,\tilde{h}]$ by

i) introducing the product (2.8),
ii) setting $z=\tilde{z}=1$ and $\tilde{h}=0$,
iii) dividing by the equivalence relation corresponding to the skein relation $xD_+-x^{-1}D_-=hD_0$, where $\{D_+,D_-,D_0\}$ is any Conway triple.

The module $\mathscr{D}_*^S$ is equipped with a comultiplication:

$$\nabla:\mathscr{D}_*^S\to\mathscr{D}_*^S\underset{\mathbf{C}[h]}{\otimes}\mathscr{D}_*^S$$

which is an morphism with respect to the product. Now we want to prove that the map (4.4), namely the anti-homomorphism with respect to the product $\gamma:\mathscr{D}_*^S\to\mathscr{D}_*^S$ given by: $\gamma(D)\equiv(-1)^{|D|}\hat{D}$ and $\gamma(x)\equiv x^{-1}$, satisfies the conditions $m(\gamma\otimes\mathrm{id})\nabla(\mathscr{D})=m(\mathrm{id}\otimes\gamma)\nabla(\mathscr{D})=0$. Here, as before, for any diagram D we denote by $\hat{D}$ the diagram where every undercrossing is turned into a overcrossing and vice versa, while $|D|$ denotes the number of components of D. Once we will have proved that γ satisfies the above property, it will be a matter of easy computations to show that γ descends to an antipode defined on $\mathscr{D}_*^{S,R}$, namely on the algebra given by $\mathscr{D}_*^S$ modulo 2-dimensional ambient isotopies and Reidemeister moves (with comultiplication given by ∇_r). We will comment, at the end of the appendix, about the changes which are needed in order to include the rotation factors in the computations.

We first prove some technical theorems and lemmas. For any diagram D we denote by $V=\{p_1,\ldots,p_n\}$ the set of all vertices of D. Any 2-labelling map f of D,

[18] This is essentially the content of a conjecture proposed by Turaev [10]. What follows is a complete proof. When this paper was already finished, we received the abstract of a paper by J.H. Przytycki, where the author announces a proof of the same Turaev conjecture

determines a partition of V into five subsets $X_f, \tilde{X}_f, Z_f, \tilde{Z}_f, S_f$ as shown in Fig. 11:

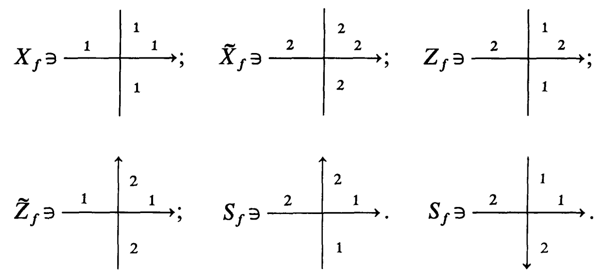


Fig. 11

We define moreover $Q_f \equiv \tilde{X}_f \cup Z_f \cup S_f$.

A.1 Theorem. *Let V denote the set of all the double points of a knot diagram D. Let us assume that V is non-empty and let $Lbl'(D)$ denote the set of all the 2-labelling maps f for which S_f is not empty. Then for any nonempty subset W of V, there exists an $f \in Lbl'(D)$ with $S_f \subset W \subset Q_f$.*

Proof. We consider any nonempty subset W of V, e.g. $W = \{p_1, p_2, \ldots, p_s\}$. We start at any given point p_i in W, and move in the backward direction along the knot, beginning with the upper incoming edge at p_i. Whenever we meet one of the other points in W, say p_j, we continue our (reverse) path along the upper incoming edge at p_i. If we meet twice a point in W then, the second time, we move backwards along the only incoming edge which has been not covered yet. We stop when we reach again our starting point and we assign the label 2 to all the edges we have covered in our path. We then start again from any point in W, that we have not met in the previous path, and we move backwards along the knot, repeating the above procedure. At the end we will have included all the points of W in a collection of circuits completely labelled by the label 2[19]. We assign the label 1 to each edge we did not cover. In this way we constructed a labelling map f such that $S_f \neq \emptyset$ and $S_f \subset W \subset Q_f$. In order to prove that S_f is in fact not empty, we consider the point $p_f \in W$ where, in the previous construction, we start covering the last circuit. Since this circuit is, by assumption, the last one, two of the four edges meeting in p_j will not be labelled by the label 2, namely $S_f \neq \emptyset$. □

Remark. Notice, first of all, that in the above theorem the specification that D is a *knot-diagram* as opposed to a generic link-diagram is quite essential.

With an abuse of notation, given any knot-diagram D and $W \subset V(D)$, we will denote any labelling map f such that $\emptyset \neq S_f \subset W \subset Q_f$ a W-labelling. We constructed in Theorem A.1 a labelling map which can be defined as a *W-maximal* labelling (i.e. a W-labelling with the maximum number of splitting vertices). More precisely we notice that, after having assigned the label 2 to the collection of edges

[19] One may notice that the set of the edges belonging to such collection of circuits, does not depend on the order in which the points of W have been selected in the previous construction. In fact the previous construction can also be described as follows: start by considering simultaneously the upper incoming edge at each point of W and accordingly move backwards along the knot, up to the moment in which you reach another point of W. Now you have a collection of edges and you complete it (in a unique way) as a link-diagram, by moving backwards along the knot. Then you assign the label 2 to each edge of such a link-diagram

determined by the theorem, we are left with a certain collection of knot-diagrams $\{K_1, \ldots, K_j\}$. In Theorem A.1 we labelled with label 1 all these knot-diagrams. This gives the maximum possible number of splittings. But we could have proceeded differently; namely we could have considered $2^j - 1$ different W-labelling maps simply by choosing any nonempty subset $\mathscr{K}$ of $\{K_1, \ldots, K_j\}$ and by assigning the value 1 to each knot-diagram $K_s \in \mathscr{K}$ and the value 2 to each knot-diagram $K_l \notin \mathscr{K}$. If $\mathscr{K}$ consists only of one knot-diagram then we say that the relevant labelling map is, by definition, a *W-minimal* labelling. It is easy to see that by considering all the possible (non-empty) subsets $\mathscr{K}$ of $\{K_1, \ldots, K_j\}$ we exhaust the class of W-labelling maps. In fact it is obvious that in order to have a W-labelling we have to assign the label 2 to the edges not belonging to $\{K_1, \ldots, K_j\}$ as indicated in the proof of Theorem A.1. Moreover no splitting point should appear in the link-diagram $\{K_1, \ldots, K_j\}$, because the vertices of this link-diagram do not belong to W.

As a final observation, we point out that, by varying W in the set of all the subsets of V, we obtain all the possible labelling maps of D, but the constant ones, as W-labellings.

A.2 Corollary. *Given a minimal W-labelling f and a general W-labelling g, either $S_f \cap S_g = \emptyset$ or $S_f \subset S_g$.*

Proof. This follows easily from the fact that W-labellings cannot have splittings in the link diagram with components $K_1, \ldots, K_j$ (see the previous remark for the notation). □

We can now consider some "operations" on the set of W-labellings as follows. For any labelling f we denote by $\mathscr{K}_f^1$ the set of knot-diagrams in $\mathscr{K}$ (see the previous remark) with label 1 with respect to f. For any pair (f, g) of W-labellings, we can define a new W-labelling $f \bigoplus_W g$ as the only W-labelling satisfying the condition

$$\mathscr{K}^1_{f \oplus_W g} \equiv \mathscr{K}^1_f \cup \mathscr{K}^1_g . \tag{A.1}$$

Moreover when $\mathscr{K}^1_f \subset \mathscr{K}^1_g$, $\mathscr{K}^1_f \neq \mathscr{K}^1_g$ we can define a new labelling $g \ominus_W f$ satisfying the condition

$$\mathscr{K}^1_{g \ominus_W f} \equiv \mathscr{K}^1_g \setminus \mathscr{K}^1_f . \tag{A.2}$$

The following relations hold:

$$f \bigoplus_W g = g \bigoplus_W f; \quad \left(g \ominus_W f\right) \bigoplus_W f = g; \quad Q_{f \oplus_W g} = Q_f \cap Q_g .$$

Now we define for each vertex v in a link-diagram D, the diagram $\sigma_v(D)$ as the link diagram obtained by changing the over/under crossing at v. The antipode map γ will be then given by:

$$\gamma(D) = (-1)^{|D|} \left(\prod_{v_i \in V} \sigma_{v_i} \right)(D) ,$$

where as usual V will denote the set of all double points (vertices) of D. Furthermore for each subset W of V (in symbols: $W \in \mathscr{P}(V)$), we denote (as usual) by $|W|$ the cardinaltiy of W, by D_W the diagram obtained from D by eliminating all the vertices in W in the only orientation-preserving way, by $W_\pm$ the set of all vertices in W of type $L_\pm$. Moreover for any subset P of the set V of all the vertices of D, we use the following notation:

$$\sigma_P(D) \equiv \Big(\prod_{v_i \in P} \sigma_{v_i}\Big)(D). \tag{A.3}$$

A.3 Lemma. *For any link diagram D and any subset P of the set V of all vertices of D, we have:*

$$\sigma_P(D) = \sum_{W \in \mathscr{P}(P)} (-1)^{|W_+|} x^{2w(D) - w(W)} h^{|W|} D_W .$$

In particular when $P = V$, we have:

$$(-1)^{|D|} \gamma(D) = \sum_{W \in \mathscr{P}(V)} (-1)^{|W_+|} x^{2w(D) - w(W)} h^{|W|} D_W .$$

Proof. The proof is by induction on the number of elements of P. So we will consider collection of vertices $\{v_1, v_2, \ldots, v_s\}$ of D and add an extra point v_{s+1}. Consequently W will be an element either of $\mathscr{P}_s \equiv \mathscr{P}(\{v_1, v_2, \ldots, v_s\})$ or of $\mathscr{P}_{s+1} \equiv \mathscr{P}(\{v_1, \ldots, v_{s+1}\})$. We will also use the following symbols: w_i denotes the writhe $w(v_i)$ and $\mathscr{P}'_{s+1} \equiv \mathscr{P}_{s+1} \backslash \mathscr{P}_s$.

What we would like to prove is that the following identity holds for any collection of points $\{v_1, \ldots, v_s\}$:

$$\Big(\prod_{i=1}^{s} \sigma_{v_i}\Big) D = \sum_{W \in \mathscr{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s} w_i - w(W)} h^{|W|} D_W .$$

The above expression is true for $s = 1$; let us now assume that it is also true for a given s and let us prove it for $s + 1$.

The skein relation implies:

$$\sigma_{v_{s+1}} D_W = x^{2w_{s+1}} D_W - x^{w_{s+1}} w_{s+1} h D_{W \cup v_{s+1}}$$

for $W \in \mathscr{P}_s$. Now

$$(\sigma_{v_{s+1}}) \Big(\sum_{W \in \mathscr{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s} w_i - w(W)} h^{|W|} D_W \Big)$$
$$= \sum_{W \in \mathscr{P}_s} (-1)^{|W_+|} x^{2\sum_{i=1}^{s+1} w_i - w(W)} h^{|W|} D_W$$
$$+ \sum_{W \in \mathscr{P}'_{s+1}} (-1)^{|W_+|} x^{2\sum_{i=1}^{s+1} w_i - w(W)} h^{|W|} D_W . \quad \square$$

A.4 Theorem. *Let m denote the multiplication in $\mathscr{D}^S_*$. For any non-empty knot-diagram D we have $m(\gamma \otimes \mathrm{id}) \nabla(D) = m(\mathrm{id} \otimes \gamma) \nabla(D) = 0$.*

Proof. Let us start by proving the equation $m(\mathrm{id} \otimes \gamma) \nabla(D) = 0$, for $D \neq \emptyset$.

For $D \neq \emptyset$ we have:

$$m(\mathrm{id} \otimes \gamma) \nabla(D)$$
$$= m(\mathrm{id} \otimes \gamma) \sum_{f \in Lbl_2(D)} \sigma(D, f) x^{\varrho_1(D,f)} D_{f,1} \otimes x^{\varrho_2(D,f)} D_{f,2}$$
$$= \sum_{f \in Lbl_2(D)} \sigma(D, f) (-1)^{|D_{f,2}|} x^{\varrho_1(D,f) - \varrho_2(D,f)} D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2})$$
$$= \sum_{f \in Lbl_2(D)} \sigma(D, f) (-1)^{|D_{f,2}|} x^{w(D_{f,1}) - w(D_{f,2})} D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2}), \tag{A.4}$$

and

$$x^{-w(D_{f1}\downarrow D_{f2})}D_{f,1} * \sigma_{\tilde{X}_f}(D_{f,2}) = \sigma_{\tilde{X}_f\cup Z_f}(D_{S_f}),$$

and hence

$$\begin{aligned}
&m(\mathrm{id}\otimes\gamma)V(D)\\
&= \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)_-|}(-1)^{|D_{f,2}|}h^{|S_f|}x^{w(D_{f1}\downarrow D_{f2})+w(D_{f,1})-w(D_{f,2})}\sigma_{\tilde{X}_f\cup Z_f}(D_{S_f})\\
&= \sum_{f\in Lbl_2(D)} (-1)^{|(S_f)_-|}(-1)^{|D_{f,2}|}h^{|S_f|}x^{w(D_{f1}\downarrow D_{f2})+w(D_{f,1})-w(D_{f,2})}\\
&= \sum_{E_f\in\mathscr{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|}x^{2w(\tilde{X}_f\cup Z_f)-w(E_f)}h^{|E_f|}D_{S_f\cup E_f}.
\end{aligned}$$

The above sum can be written as:

$$\begin{aligned}
&= \sum_{f\in Lbl_2(D)}\ \sum_{E_f\in\mathscr{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|}\\
&\quad\times x^{w(D_{f1}\downarrow D_{f2})+w(D_{f,1})-w(D_{f,2})+2w(\tilde{X}_f\cup Z_f)-w(E_f)}D_{S_f\cup E_f}\\
&= \sum_{f\in Lbl_2(D)}\ \sum_{E_f\in\mathscr{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|}\\
&\quad\times x^{w(\tilde{Z}_f)-w(Z_f)+w(X_f)-w(\tilde{X}_f)+2w(\tilde{X}_f\cup Z_f)+w(S_f)-w(S_f\cup E_f)}D_{S_f\cup E_f}\\
&= \sum_{f\in Lbl_2(D)}\ \sum_{E_f\in\mathscr{P}(\tilde{X}_f\cup Z_f)} (-1)^{|(E_f)_+|+|(S_f)_-|+|D_{f,2}|}h^{|S_f|+|E_f|} \qquad \text{(A.5)}\\
&\quad\times x^{w(D)-w(S_f\cup E_f)}D_{S_f\cup E_f}.
\end{aligned}$$

Now we extract from the above sum, the terms corresponding to the labelling maps which assign the same (constant) label to all the edges of D. The sum of these terms is given by

$$(x^{w(D)}D - x^{-w(D)}\sigma_V(D)) = -\sum_{W\in\mathscr{P}(V),\,W\neq\emptyset}(-1)^{|W_+|}x^{w(D)-w(W)}h^{|W|}D_W. \qquad \text{(A.6)}$$

Let us now consider a minimal W-labellings, for each non-empty $W\in\mathscr{P}(V)$. The relevant term in (A.5) exactly cancels the terms corresponding to the labellings without any splitting. In fact we have in general:

$$|D_{f,1}| + |D_{f,2}| = |S_f| + 1 \mod 2$$

and for a minimal W-labelling:

$$|D_{f,2}| = |S_f| \mod 2.$$

Suppose now that we are given another W-labelling g. Then, due to Corollary A.2, either $S_g\cap S_f=\emptyset$ or $S_f\subset S_g$. In both the above cases we can consider a third labelling p defined as $p\equiv f\underset{W}{\oplus} g$ in the first case and as $p\equiv g\underset{W}{\ominus} f$ in the second case. We are going to show that the relevant contributions of g and p cancel. We have in fact: $|D_{p,1}| = \pm|D_{f,1}| + |D_{g,1}|$ and hence:

$$\begin{aligned}
&(-1)^{|(S_g)_-|+|D_{g,2}|+|(E_g)_+|} + (-1)^{|(S_p)_-|+|D_{p,2}|+|(E_p)_+|}\\
&= (-1)^{|(S_g\cup E_g)_+|}[(-1)^{|S_g|+|D_{g,2}|} + (-1)^{|S_p|+|D_{p,2}|}]\\
&= -\{(-1)^{|(S_g\cup E_g)_+|}[(-1)^{|D_{g,1}|} + (-1)^{|D_{p,1}|}]\} = 0.
\end{aligned}$$

This completes the proof that $m(\mathrm{id}\otimes\gamma)\nabla(D)$ is zero for $D\neq\emptyset$. The proof that $m(\gamma\otimes\mathrm{id})\nabla(D)$ is also zero can be obtained in a completely similar way; it is enough to reverse the role of the labels 1 and 2 in all the previous considerations.

Now we can state the following theorem:

A.5 Theorem. *For any non-empty link diagram D we have* $m(\gamma\otimes\mathrm{id})\nabla(D) = m(\mathrm{id}\otimes\gamma)\nabla(D)=0$.

Proof. It is done by induction. We know that the theorem holds for knot-diagrams. We suppose it is true for link-diagrams with up to n components. Our aim is to show that it is true for link-diagrams with up to $n+1$ components. If a link-diagram D is a $\circ$-product of a link-diagram D' (with k components) by a link-diagram D'' (with $n+1-k$ components), then we can use the fact that the comultiplication is an morphism with respect to the product. In fact we have (remind that $\gamma(x)=x^{-1}$):

$$m(\mathrm{id}\otimes\gamma)\nabla(D'\circ D'')$$

$$=m(\mathrm{id}\otimes\gamma)\nabla(D' * D'')=m(\mathrm{id}\otimes\gamma)\Big\{\sum_{f\in Lbl_2(D'),\, g\in Lbl_2(D'')}\sigma(D',f)\sigma(D'',g)$$

$$\times x^{\varrho_1(D',f)+\varrho_1(D'',g)}D'_{f,1} * D''_{g,1}\otimes x^{\varrho_2(D',f)+\varrho_2(D'',g)}D'_{f,2} * D''_{g,2}\Big\}$$

$$=\sum_{f\in Lbl_2(D')}\sigma(D',f)x^{\varrho_1(D',f)-\varrho_2(D',f)}$$

$$\times\sum_{g\in Lbl_2(D'')}\sigma(D'',g)x^{\varrho_1(D'',g)-\varrho_2(D'',g)}D'_{f,1} * D''_{g,1} * \gamma(D''_{g,2}) * \gamma(D'_{g,2})$$

which is zero due to the fact that we assumed:

$$\mathrm{m}(\mathrm{id}\otimes\gamma)\nabla(D'')=0\,.$$

Let us now suppose that D is not the product of two link-diagrams. Then we have necessarily a vertex corresponding to an intersection point of (the projections of) two different components of D. Applying the skein relation to this vertex we easily see that the diagram D_0 must have n components. Hence, by assumption, $m(\mathrm{id}\otimes\gamma)\nabla(D_0)=0$ and the skein relation implies:

$$m(\mathrm{id}\otimes\gamma)\nabla(D_+)=m(\mathrm{id}\otimes\gamma)\nabla(D_-)\,. \tag{A.7}$$

Now we can always change the over/under crossing information at a set of vertices of D (corresponding to crossings of different components in D) in such a way that D is transformed into the product of two diagrams $\hat{D}'$ and $\hat{D}''$. The previous equation tells us that

$$m(\mathrm{id}\otimes\gamma)\nabla(D)=m(\mathrm{id}\otimes\gamma)\nabla(\hat{D}'\circ\hat{D}'')=0\,.$$

The proof that the equation $m(\gamma\otimes\mathrm{id})\nabla(D)=0$ holds for any (non-empty) link diagram D, is completely analogous to the previous proof. □

Remark. In the disk B^2 there is a much shorter proof: observe first that $m(\mathrm{id}\otimes\gamma)\nabla(x)=1$, and so for any skein triple $\{D_+,D_-,D_0\}$ we have

$$m(\mathrm{id}\otimes\gamma)\nabla(D_+)=m(\mathrm{id}\otimes\gamma)\nabla(D_-)+hm(\mathrm{id}\otimes\gamma)\nabla(D_0)\,.$$

In order to prove that, for any non-empty link-diagram S, one has $m(\mathrm{id}\otimes\gamma)\nabla(D)=0$, it is enough to prove that $m(\mathrm{id}\otimes\gamma)\nabla(\bigcirc^n)=0$, where $\bigcirc^n$ is the

n-component unlink. But this follows from the fact that

$$m(\mathrm{id}\otimes\gamma)\nabla(\bigcirc^n)=\sum_{X\in\mathscr{P}(\{1,\ldots,n\})}(-1)^{|X|}\bigcirc^n=0,$$

which in turn is due to the fact that there are exactly 2^n distinct terms, of which half have a plus sign and half have a minus sign. Similarly one can show that $m(\gamma\otimes\mathrm{id})\nabla(D)=0$, for any non-empty link-diagram D in B^2.

Finally we have:

A.6 Theorem. *For any link diagram D we have*

$$m(\gamma\otimes\mathrm{id})\nabla(D)=m(\mathrm{id}\otimes\gamma)\nabla(D)=(\varepsilon\circ\eta)(D),$$

where $\varepsilon:\mathbf{C}[h]\to\mathscr{D}^S_*$ *is the unit and* η *is the counit as defined in* (2.9).

Proof. After Theorem A.5 the only identity to be proved is the following one:

$$m(\gamma\otimes\mathrm{id})\nabla(\emptyset)=m(\mathrm{id}\otimes\gamma)\nabla(\emptyset)=\emptyset,$$

where $\emptyset$ is the empty knot-diagram. The above identity follows immediately from the fact that $\gamma(\emptyset)=\emptyset$. □

When we consider the algebra $\mathscr{D}^{S,R}_*$ with comultiplication ∇_r instead of ∇, it is easy to see that all the theorems proved in this appendix remain true. In particular Theorem A.1, Corollary A.2, and Lemma A.3 do not need any modification after the insertion of the rotation factor. As far as the proof of Theorem A.4 is concerned, the fact of considering $\tau_i(D,f)$ instead of $\varrho_i(D,f)$ implies that there is a multiplicative factor $x^{r(D)}$ in front of the second term (and in front of the following terms) of Eq. (A.4).

Hence we have:

A.7 Theorem. $\mathscr{D}^{S,R}_*$ *is a Hopf algebra (with antipode).*

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List of Principal Symbols

$\mathscr{D}=\mathscr{D}(x,y,z,\ldots\,|\,h,k,l,\ldots)\equiv$ Algebra over the polynomial ring $\mathbf{C}[x,x^{-1},y,y^{-1},z,z^{-1},\ldots,h,k,l,\ldots]$ generated by link-diagrams;

$Lbl_n(D)\equiv$ n-labelling maps for the diagram D;

$\nabla\equiv$ comultiplication in $\mathscr{D}$ (see formulas (2.2) and (2.3));

$V(D)\equiv$ set of all the vertices of the diagram D;

$W_+(W_-)\equiv$ set of the vertices belonging to $W\subset V(D)$ and having writhe $1(-1)$;

$w(W)\equiv$ total writhe of $W\subset V(D)$;

$w(D) \equiv w(V(D))$;

$D_{f,k} \equiv$ subdiagrams of D given by all the edges which have label k according to the labelling map f;

$\varrho_i(D,f) \equiv w(D_{f,i}) - w(D)$;

$|X| \equiv$ cardinality of X, for any set X;

$\mathscr{P}(X) \equiv$ set of all the subsets of X;

$S_f \equiv$ set of all the vertices with $f(a) < f(d)$, where a is the lower incoming edge and d is the lower outgoing edge;

$\tilde{S}_f \equiv$ set of all the vertices with $f(a) > f(d)$, where a is the lower incoming edge and d is the lower outgoing edge;

$D \# D' \equiv$ set of all the intersection points of D with D';

$D \Downarrow D' \equiv$ subset of $D \# D'$ given by the points where D crosses over D';

$$\sigma(D,f) \equiv (-1)^{|(S_f)_-| + |(\tilde{S}_f)_-|} h^{|S_f|} \tilde{h}^{|\tilde{S}_f|} \times z^{\sum_{i>j} w(D_{f,i} \Downarrow D_{f,j})} \tilde{z}^{\sum_{i>j} w(D_{f,j} \Downarrow D_{f,i})};$$

$D_W \equiv$ diagram obtained by splitting D at all the vertices in $W \subset V(D)$ in the only orientation-preserving way;

$V^N \equiv (V \otimes \mathrm{id}^{\otimes(N-2)}) \circ (V \otimes \mathrm{id}^{\otimes(N-3)}) \circ \ldots \circ V$;

$D \circ D' \equiv$ diagram obtained by putting the (rigid) diagram D over the (rigid) diagram D';

$w(D \downarrow D') \equiv$ intersection number of the projection of D with the projection of D';

$D * D' \equiv (z^{-1}x)^{w(D \downarrow D')} D \circ D'$;

$\mathscr{D}_* \equiv$ module of (rigid) link diagrams with the $*$-product defined above;

$\eta \equiv$ counit in $\mathscr{D}$ (see formula 2.9);

$\mathscr{D}^S \equiv$ module of link diagrams modulo the skein relation;

$r(D) \equiv$ rotation factor (winding number) of the projection of the diagram D;

$$\tau_i(D,f) \equiv \varrho_i(D,f) + \Big(\sum_{j>i} - \sum_{j<i}\Big) r(D_{f,j});$$

$V_r \equiv$ comultiplication obtained from V, by replacing the terms $\varrho_i(D,F)$ with $\tau_i(\mathrm{D}, \mathrm{f})$;

$R_{2g-2} \equiv$ group of the $(2g-2)$-roots of 1;

$D^\#, f^\# \equiv$ diagrams and corresponding labelling maps obtained by performing some Reidemeister move on the diagram D (see Sect. 4);

$\bigcirc \equiv$ (contractible) unknot;

$\mathscr{D}^{S,R} \equiv$ quotient module of $\mathscr{D}$ with respect to the skein relation and the Reidemeister moves;

$\mathscr{D}_*^{S,R} \equiv$ quotient algebra of $\mathscr{D}_*$ with respect to ambient 2-dimensional isotopies, the skein relation and the Reidemeister moves;

$\gamma \equiv$ antipode in $\mathscr{D}_*^{S,R}$ (see (4.4));

$\hat{\psi} \equiv$ homomorphism of modules $\mathscr{D}^{\otimes k} \to \mathbf{C}[x, x^{-1}, z, z^{-1}, h]$ (see (5.2));

$\mathscr{D}' \equiv$ quotient module of $\mathscr{D}$ with respect to the Reidemeister moves and the equivalence relation $D_+ = D_- = D_0$ (for any Conway triple $\{D_+, D_-, D_0\}$), where we set $h = x - x^{-1}$;

$\lambda \equiv$ projection: $\mathscr{D}^{S,R} \to \mathscr{D}'$;

$\psi \equiv$ homomorphism $\mathscr{D}' \to \mathbf{C}[x, x^{-1}, z, z^{-1}]$ induced by $\hat{\psi}$;

$\psi^N(D) \equiv \psi(\lambda(V_r^N(D)))$;

$P_{l,m}(D) \equiv$ Homfly polynomial in the variables l, m;

$\mathscr{D}_*^S \equiv$ quotient module of $\mathscr{D}_*$ with respect to the skein relation;

$X_f, \tilde{X}_f, Z_f, \tilde{Z}_f, Q_f \equiv$ see Fig. 11 in the appendix;

W-labelling $\equiv$ any 2-labelling such that $\emptyset \neq S_f \subset W \subset Q_f \subset V(D)$;

$\sigma_W(D) \equiv$ diagram obtained from D, by reversing all the over/under crossings at the vertices belonging to $W \subset V(D)$.

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