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On the Determinant of Elliptic Differential and Finite Difference Operators in Vector Bundles over S^1

D. Burghelea[1,*], L. Friedlander[2,*], and T. Kappeler[3,*]

[1] Department of Mathematics, Ohio State University, Columbus, OH 43210-1174, USA
[2] Department of Mathematics, University of California, Los Angles, CA 90024, USA
[3] Department of Mathematics, Brown University, Providence, RI 02912, USA

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Abstract. For an elliptic differential operator A over S^1, $A=\sum_{k=0}^{n} A_k(x)D^k$, with $A_k(x)$ in $\mathrm{END}(\mathbb{C}^r)$ and θ as a principal angle, the ζ-regularized determinant $\mathrm{Det}_\theta A$ is computed in terms of the monodromy map P_A, associated to A and some invariant expressed in terms of A_n and A_{n-1}. A similar formula holds for finite difference operators. A number of applications and implications are given. In particular we present a formula for the signature of A when A is self adjoint and show that the determinant of A is the limit of a sequence of computable expressions involving determinants of difference approximation of A.

1. Introduction and Summary of the Results

In this paper we study the determinant of elliptic differential operators on a complex vector bundle $E \xrightarrow{p} M$ of rank N over a compact oriented connected manifold M of dimension 1, as well as the determinants of its finite difference approximations. For this purpose we introduce a new invariant S_θ which, in the case of odd order self adjoint operators, calculates the η-invariant (Corollary 5.4).

In order to state the first main theorem we have to introduce the following notions for elliptic differential operators.

(1) The monodromy map P_A: Denote by $\Gamma(E)$ the smooth sections of $E \xrightarrow{p} M$. For an elliptic differential operator $A:\Gamma(E)\to\Gamma(E)$ of order $n\geqq 1$ consider the lift $\tilde{A}:\Gamma(\tilde{E})\to\Gamma(\tilde{E})$, where $\tilde{E}\xrightarrow{\tilde{p}}\tilde{M}$ is the pullback of $E\xrightarrow{p} M$ by the universal covering $\tilde{M}\to M$. Due to the ellipticity of A, the nullspace $\mathrm{Null}(\tilde{A})$ has dimension nN. The fundamental group $\pi_1(M,*)=\mathbb{Z}$ (with 1 corresponding to the orientation of M)

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acts on $\mathrm{Null}(\tilde{A})$ and the isomorphism corresponding to $1 \in \mathbb{Z}$ is called the monodromy map and is denoted by P_A.

(2) Invariant $R(A)$: Observe that M is diffeomorphic to S^1 and E is always trivial. A parametrization of p is a pair of maps (φ, η), where $\varphi: S^1 \to M$ is an orientation preserving diffeomorphism and $\eta: S^1 \times \mathbb{C}^N \to E$ is a diffeomorphism, linear in each fiber such that $p \cdot \eta = \varphi \cdot p_{S^1}$, where $p_{S^1}: S^1 \times \mathbb{C}^N \to S^1$ denotes the projection of $S^1 \times \mathbb{C}^N$ on S^1. With respect to a given parametrization (φ, η) the operator A can be written as $A = \sum_{k=0}^{n} A_k(x) D^k \; x \in S^1 \; \left(D = \frac{1}{i} \frac{d}{dx}\right)$, where $A_k(x)$ is in $\mathrm{End}(\mathbb{C}^N)$ $(0 \leqq k \leqq n)$ and, due to the ellipticity of A, $A_n(x)$ is in $GL_N(\mathbb{C})$. Define the local expression $R(A) = \exp\left(\frac{i}{2} \int_{S^1} \operatorname{tr} A_n^{-1}(x) A_{n-1}(x) dx\right)$. It is straightforward to verify that $R(A)$ does not depend on the chosen parametrization (φ, η), thus it is an invariant of A.

(3) ζ-regularized determinant $\mathrm{Det}_\theta(A)$: For an angle θ (in $\mathbb{R}$) denote by R_θ the ray $R_\theta = \{\rho e^{i\theta}: 0 \leqq \rho < \infty\}$. The elliptic operator A is said to have θ as a principal angle if $\mathrm{spec}(\sigma_L(A)(x,\xi)) \cap R_\theta \neq \emptyset$ $(x \in M, \xi \in T_x^* M \setminus \{0\})$, where $\sigma_L(A)(x,\xi)$ denotes the leading symbol of A and $\mathrm{spec}(\sigma_L(A)(x,\xi))$ the spectrum of the endomorphism $\sigma_L(A)(x,\xi)$ $(= A_n(x)\xi^n)$ with respect to parametrization of E.

By extending slightly results of Seeley ([Se]) one can define the ζ-regularized determinant $\mathrm{Det}_\theta(A)$ for an injective elliptic differential operator A, having θ as a principal angle (cf. Sect. 2). In case A is not injective one defines $\mathrm{Det}_\theta(A) = 0$.

(4) Invariant $S_\theta(A)$: Let A be 1-1 and have the property that both θ and $-\theta$ are principal angles. (Observe that if A is of odd order and θ is a principal angle for A, then $-\theta$ is also a principal angle.) Then the argument β of an eigenvalue of $\sigma_L(A)(x,\xi)$, $\xi > 0$, satisfies either $\theta < \beta < \theta + \pi$ or $\theta + \pi < \beta < \theta + 2\pi$. Denote by Π_θ^+ and Π_θ^- the projections of E_x into the subspaces generated by the generalized eigenvectors corresponding to the eigenvalues with arguments in the interval $(\theta, \theta + \pi)$ and $(\theta + \pi, \theta + 2\pi)$ respectively. Define $\Gamma_\theta(x)$ to be the involution $\Pi_\theta^+ - \Pi_\theta^-$. Actually, $E \to M$ decomposes as a direct sum of two bundles $E^+ \to M$ and $E^- \to M$ of dimension N^+ and $N^- = N - N^+$ respectively where the fibers are given by $\Pi_\theta^+(x)(E_x)$ and $\Pi_\theta^-(x)(E_x)$ respectively. Observe that $\det \Gamma_\theta(x)$ does not depend on x. A parametrization (φ, η) of $E \to M$ is called admissible if η splits, i.e. $\eta = \eta^+ + \eta^-$, where $\eta_+: S^1 \times \mathbb{C}^{N^+} \to E^+$ and $\eta_-: S^1 \times \mathbb{C}^{N^-} \to E^-$. An admissible parametrization will be also denoted by $(\varphi, \eta^+, \eta^-)$. With respect to an admissible parametrization $\Gamma_\theta(x)$ is given by the diagonal matrix $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. With respect to an admissible parametrization, one defines

$$S_\theta(A) = \det \Gamma_\theta(x) \exp\left\{\frac{i}{2n} \int_{S^1} dx \operatorname{tr} \Gamma_\theta(x) A_n^{-1}(x) A_{n-1}(x)\right\}.$$

Again one can verify that $S_\theta(A)$ is independent of the chosen admissible parametrization thus an invariant of A. Of course, one can express $S_\theta(A)$ with respect to a parametrization, not necessarily admissible, but the formula is more complicated.

Theorem 1. *Let A be an elliptic differential operator of order n, having θ as a principal angle.*

(A) *If n is even, then*

$$\operatorname{Det}_\theta(A) = (i)^{Nn} R(A) \det(\mathrm{Id} - P_A).$$

(B) *If n is odd, then*

$$\operatorname{Det}_\theta(A) = (i)^{N(n+1)} S_\theta(A) R(A) \det(\mathrm{Id} - P_A).$$

The next result concerns the determinants of finite difference approximations of an elliptic operator A.

Let V be a complex vector space of dimension N, (m,n) a pair of positive integers and T an integer with $m+n \leqq T$. A periodic finite difference operator of type (m,n) and period T, $\tilde{A}: V^{\mathbb{Z}} \to V^{\mathbb{Z}}$, is given by $\tilde{A}y(k) = \sum_{j=-m}^{n} A_j(k) y(k+j)$, where $A_j(k) \in \operatorname{End}(V)$, $A_j(k+T) = A_j(k)$ and $y \in V^{\mathbb{Z}}$. $\tilde{A}$ and T are completely determined by the operator $A: V^{\mathbb{Z}/T\mathbb{Z}} \to V^{\mathbb{Z}/T\mathbb{Z}}$, defined in a straightforward way. A (as well as $\tilde{A}$) is called elliptic if $A_n(k)$ and $A_{-m}(k)$ are in $GL_N(\mathbb{C})$ for all k. The ellipticity implies that the nullspace $\operatorname{Null}\tilde{A}$ of $\tilde{A}$ is a finite dimensional subspace of $V^{\mathbb{Z}}$ of dimension $(n+m)N$. The translation by T induces a linear map $P_A: \operatorname{Null}\tilde{A} \to \operatorname{Null}\tilde{A}$. One defines $R_d(A) = \prod_{k=1}^{T} \det A_n(k)$. Note that A can be equally well viewed as a $(m-1, n+1)$ type operator in which case $\det A$ changes by multiplication with $(-1)^{(T-1)N}$. The second main result is the following version of Theorem 1 for finite difference operators:

Theorem 2. *Let A be a finite difference operator as above of type (m,n) and period T, then*

$$\det A = (-1)^{nN(T-1)} R_d(A) \det(Id - P_A).$$

The proofs of Theorems 1 and 2 are very similar, so let us explain the main ideas of the proof of Theorem 1:

(A) By studying various appropriate variations of the terms $R(A), S_\theta(A), \det_\theta(A)$ one reduces the proof of Theorem 1 to the case of scalar elliptic differential operators (i.e. $N=1$).

(B) Using that $R(A), S_\theta(A)$ and $\operatorname{Det}_\theta(A)$ are holomorphic functions of A and again using similar variations as in (A), one reduces the proof of Theorem 1 to the case of scalar elliptic operators of the form $(D+\lambda)^n$, where λ is a constant. For $(D+\lambda)^n$ the identities in Theorem 1 are proved by calculation.

(C) An important ingredient for studying variations of $\operatorname{Det}_\theta(A)$ is the extension of results of Seeley, due to Guillemin [G] and Wodzicky [W], concerning the meromorphic continuation of $\operatorname{Tr}(QA^{-s})$ in the whole complex s-plane. Here Q and A are pseudo-differential operators with A elliptic having an angle which satisfies the Agmon conditions. Given two elliptic pseudo-differential operators A_1 and A_2 of the same positive order and with the same principal angle, the meromorphic function $\operatorname{Tr} Q(A_1^{-s} - A_2^{-s})$ has $s=0$ as a regular point and the value is given by a local formula.

Let us point out the following consequences of Theorems 1 and 2.

1) It is well known that for a finite difference approximation A_T of an elliptic

differential operator A, one has $\det(\mathrm{Id}-P_A)=\lim_{T\to\infty}\det(\mathrm{Id}-P_{A_T})$. Using Theorems 1 and 2 one then obtains $\mathrm{Det}_\theta(A)$ (Corollary 4.7) up to factors as a limit

$$\lim_{T\to\infty}(-1)^{nN(T-1)}\frac{\det(A_T)}{R_d(A)}.$$

To the best of our knowledge corresponding results for elliptic operators acting on manifolds of higher dimension are not known.

2) Theorem 1 implies that for even order elliptic differential operators A, satisfying the Agmon condition for an angle $\theta, \mathrm{Det}_\theta(A)$ does *not* depend on θ and behaves multiplicative with respect to the composition of even order elliptic operators (cf. Corollary 5.1). For operators of add order we obtain a formula Corollary 5.4 for the η-invariant which turns out to be a local invariant. As pointed out to us by P. Gilkey this formula can be also recovered from [Gi]. It is well known that the analogous results for elliptic operators on higher dimensions cannot be true.

3) Our interest in Theorem 1 comes from the differential geometry of the space of closed curves (strings) of a Riemannian manifold where determinants of Hessians of smooth functions and Pfaffians of symplectic structures can be interpreted as determinants of self-adjoint and skew-adjoint elliptic differential operators in vector bundles over S^1. Theorem 1 gives a geometric interpretation for them. More important it removes an unpleasant indeterminancy for the regularized determinant through the choice of the angle, dispersing doubts about the correctness of the regularization by the ζ-function method.

4) Theorem 2 provides a formula which expresses the determinant of an $NT\times NT$ matrix in terms of a determinant of a $(n+m)N\times(n+m)N$ matrix. Despite its elementary flavor we were unable to give it a direct elementary proof.

5) The formulas given in Theorem 1 have a similar structure as the ones for the residues at 0 of the ζ-function of algebraic number fields. They involve a local invariant like $R(A)$ and a transcendental invariant like $\det(I-P_A)$.

6) If one restricts the attention to the algebra of smooth functions of a compact manifold then the algebraic K-theory of such an algebra describes homological complexity of the groups of elliptic operators of order zero. It is natural to consider an algebraic K-theory for these algebras which involves the monoid of all elliptic differential operators instead of the differential operator of order 0 only. From the point of view of such an algebraic K-theory, the results of Theorem 1 are analogous to the well known decomposition

$$K_1(A[t,t^{-1}])=K_0(A)\oplus K_1(A)\oplus\mathrm{Nil}_0(A).$$

7) First results of the type described in Theorem 1 are due to Forman. By different methods, he has proved in [F] that the quotient of the ζ-regularized determinants of elliptic differential operators A and B of even order and with identical principal symbol, expressed in our notation, is given by $R(A)\det(\mathrm{Id}-P_A)/R(B)\det(\mathrm{Id}-P_B)$. His formula corresponding to the odd case, contains a small error, the correct one can be easily read off from Theorem 1.

8) The results of this paper, in an earlier version, have been obtained by different

methods. The version presented here is more suited for generalization to pseudo-differential operators. In a subsequent paper we will treat the determinant of an elliptic pseudo-differential operator and write it as a product of local invariants with a Fredholm determinant of a pseudo-differential operator of determinant class, canonically associated to A; the Fredholm determinant corresponds to $\det(\mathrm{Id} - P_A)$ in the case when A is a differential operator.

2. Auxiliary Results

In this section we collect some auxiliary results needed for the proof of Theorem 1. Let $E \xrightarrow{p} M$ be a complex vector bundle over M of rank N, where M is diffeomorphic to S^1. Denote by $\mathrm{EDO}_n \equiv \mathrm{EDO}_{n,N}$ the set of all elliptic differential operators on E of order n. Denote by $\mathrm{EDO}_{n;\theta}$ the subset of all elliptic differential operators of order n, having $\theta(\in R)$ as a principal angle. A parametrization (φ, η) of $E \xrightarrow{p} M$, as defined in the introduction, induces an identification of EDO_n to the space $C^\infty(S^1; GL_N(\mathbb{C})) \times C^\infty(S^1; \mathrm{End}\,\mathbb{C}^n)^n$ by assigning to an operator A in EDO_n its coefficients $(A_n, \dots, A_0)$ with respect to the given parametrization. In this way EDO_n becomes an open set in the complex Fréchet space $C^\infty(S^1, \mathrm{End}(\mathbb{C}^N))^{n+1}$. Clearly $\mathrm{EDO}_{n;\theta}$ is an open subset of EDO_n. Further denote by $\widehat{\mathrm{EDO}}_n$ the subset of $\mathrm{EDO}_n \times S^1$ consisting of pairs (A, θ) with A in $\mathrm{EDO}_{n;\theta}$. Clearly $R(A)$ and $\det(\mathrm{Id} - P_A)$ can be considered as complex valued functions on EDO_n as well as $\widehat{\mathrm{EDO}}_n$, where $S_\theta(A)$ is defined on $\widehat{\mathrm{EDO}}_n$. Given the parametrization (φ, η) and A in EDO_n one defines the degree $\deg A$ to be the degree of $\det(\sigma_L(A)(\cdot, 1)) = \det A_n(\cdot)$, considered as a function on S^1 with values in $\mathbb{C}\backslash\{0\}$. It is an invariant of the operator independent of the parametrization.

Proposition 2.1.

(1) *If n is even then* $\mathrm{EDO}_{n,\theta}$ *and* $\widehat{\mathrm{EDO}}_n$ *are connected.*

(2) *If n is odd then both* $\mathrm{EDO}_{n;\theta}$ *and* $\widehat{\mathrm{EDO}}_n$ *have precisely N connected components. Moreover two elements A and A' are in a same connected component of* $\mathrm{EDO}_{n;\theta}$ iff $\operatorname{tr} \Pi_\theta^+(A) = \operatorname{tr} \Pi_\theta^+(A')$.

Proof. (1) For $A = \sum_{k=0}^{n} A_k D^k$ in $\mathrm{EDO}_{n;\theta}$ the path

$$A(t) = (te^{i(\theta+\pi)}\,\mathrm{Id} + (1-t)A_n)D^n + (1-t)\sum_{k=0}^{n-1} A_k D^k \quad (0 \leqq t \leqq 1)$$

in $\mathrm{EDO}_{n;\theta}$ with $A(0) = A$ and $A(1) = e^{i(\theta+\pi)}D^n$ defines a retraction of $\mathrm{EDO}_{n;\theta}$ to $\{e^{i(\theta+\pi)}D^n\}$ and (1) follows.

(2) Let n be odd. For A in $\mathrm{EDO}_{n;\theta}$ the path

$$A(t) = \{te^{i(\theta+\pi/2)}(\Pi_\theta^+(A) - \Pi_\theta^-(A)) + (1-t)A_n\}D^n + (1-t)\sum_{k=1}^{n-1} A_k D^k$$

in $\mathrm{EDO}_{n;\theta}$ with $A(0) = A$ and $A(1) = e^{i(\theta+\pi/2)}(\Pi_\theta^+(A) - \Pi_\theta^-(A))$ defines a retraction of $\mathrm{EDO}_{n;\theta}$ onto the space of smooth maps from S^1 to involutions of C^N. The space of involutions of C^N identifies to the complex Grassmannian $\bigcup_{k=1}^{N} G_{k,N}$ and (2) follows from its simple connectivity.

Next observe that $\det(\mathrm{Id}-P_{A-\lambda})$ is an entire function in λ whose zeroes correspond to $\operatorname{spec} A$, the spectrum of A. This implies that $\{A\in \mathrm{EDO}_{n;\theta}: A$ is not injective$\}$ is a closed subset of $\mathrm{EDO}_{n;\theta}$ and we get the following

Corollary 2.2. *$\{A\in \mathrm{EDO}_{n,\theta}: A$ injective$\}$ is open and connected in $\mathrm{EDO}_{n,\theta}$.*

Proposition 2.3.

(1) *$R(A)$ and $\det(\mathrm{Id}-P_A)$ are holomorphic on EDO_n.*
(2) *S is locally constant in θ and is holomorphic when restricted to $\mathrm{EDO}_{n;\theta}$.*

Proof. The fact that $\det(\mathrm{Id}-P_A)$ is holomorphic is a consequence of the analyticity of P_A when considered as a map $\mathrm{EDO}_n \to GL_N(\mathbb{C})$. The remaining assertions are easily verified.

Proposition 2.4. *Let $A\in \mathrm{EDO}_n$ and $\alpha \in \mathrm{EDO}_0$ $(n \geqq 1)$.*

(1) *$R(\alpha A)=R(A)$; $\det(\mathrm{Id}-P_{\alpha A})=\det(\mathrm{Id}-P_A)$.*
(2) *Let $(A,\theta)\in \widehat{\mathrm{EDO}}_{2k+1}$ (i.e. $n=2k+1$ is odd). Then $(e^{i\theta'}A, \theta+\theta')\in \widehat{\mathrm{EDO}}_{2k+1}$ and $S_\theta(A)=S_{\theta+\theta'}(e^{i\theta}A)$.*
(3) $\dfrac{1}{R(A)^2}=\det P_A$.
(4) *If $B\in \mathrm{EDO}_m$ $(m\geqq 1)$ then $R(B\circ A)=R(B)R(A)(-1)^{\deg A}(-1)^{\operatorname{ord} A}$, where $\operatorname{ord} A$ is the order of A.*

Proof. (1) and (2) are straightforward and (3) is known as Liouville's theorem (cf. [A]).

(4) Let $A=\sum_{k=0}^{n} A_k D^k$ and $B=\sum_{k=0}^{m} B_k D^k$. Then $C=\sum_{k=0}^{m+n} C_k(x)D^k \equiv B\cdot A$ has coefficients $C_{m+n}(x)=B_m(x)A_n(x)$ and $C_{m+n-1}(x)=nB_m(x)DA_n(x)+B_m(x)A_{n-1}(x)+B_{m-1}(x)A_n(x)$. Therefore $\operatorname{tr} C_{m+n}(x)^{-1}C_{m+n-1}(x)=-ni\operatorname{tr} A_n^{-1}(x)\frac{d}{dx}A_n(x)+\operatorname{tr} A_n(x)^{-1}A_{n-1}(x)+\operatorname{tr} B_m(x)^{-1}B_{m-1}(x)$. Since $-i\int_{S^1}\operatorname{tr} A_n(x)^{-1}\frac{d}{dx}A_n(x)dx=-i\int_{S^1}\frac{d}{dx}\det A_n(x)\cdot(\det A_n(x))^{-1}dx=2\pi \deg A$, the result follows.

The next result is straightforward to verify.

Proposition 2.5. *Let $0\to E_1\to E_2\to E_3\to 0$ be a short exact sequence of vector bundles over M. Let A_i be in EDO_n $(1\leqq i \leqq 3)$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} \Gamma(E_1) & \to & \Gamma(E_2) & \to & \Gamma(E_3) \\ \downarrow{\scriptstyle A_1} & & \downarrow{\scriptstyle A_2} & & \downarrow{\scriptstyle A_3} \\ \Gamma(E_1) & \to & \Gamma(E_2) & \to & \Gamma(E_3) \end{array}.$$

Then

(1) *$R(A_2)=R(A_1)R(A_3)$; $\det(\mathrm{Id}-P_{A_2})=\det(\mathrm{Id}-P_{A_1})\det(\mathrm{Id}-P_{A_2})$.*
(2) *(A_2,θ) is in $\widehat{\mathrm{EDO}}_n$ iff (A_1,θ) and (A_3,θ) are in $\widehat{\mathrm{EDO}}_n$.*

In that case $S_\theta(A_2)=S_\theta(A_1)S_\theta(A_3)$.

Next, let us recall the notion of ζ-regularized determinant.

For an angle θ and $0 < \rho_0$ arbitrary small, define the contour $\Gamma = \Gamma_\theta$ in $\mathbb{C}$ consisting of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ as follows: $\Gamma_1 = \{\rho e^{i\theta} : \infty > \rho \geqq \rho_0\}$ is the ray at angle θ with direction to the origin, $\Gamma_2 = \{\rho_0 e^{i\theta'} : \theta \geqq \theta' \geqq \theta - 2\pi\}$ is the circle of radius ρ_0 with negative orientation and $\Gamma_3 = \{\rho e^{i(\theta + 2\pi)} : \rho_0 \leqq \rho < \infty\}$ is the ray at angle $\theta + 2\pi$, going to infinite. If A is elliptic and injective and has θ as a principal angle, one can choose $\rho_0 > 0$ sufficiently small such that $\operatorname{spec} A \cap \{z \in \mathbb{C} : |z| < 2\rho_0\} = \emptyset$. θ is called an Agmon angle for A if θ is a principal angle and in addition if there exists $\varepsilon > 0$ such that $\operatorname{spec} A \cap L_{[\theta - \varepsilon, \theta + \varepsilon]} = \emptyset$, where $L_{[a,b]}$ denotes the solid angle $\{\rho e^{i\theta} : 0 < \rho < \infty; a \leqq \theta \leqq b\}$. In the case where A has an Agmon angle θ, we may define complex powers of A: for any s in $\mathbb{C}$ with $\operatorname{Re} s > \frac{1}{n}$ the operator $A_\theta^{-s} = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^{-s}(A - \lambda)^{-1} d\lambda$ is a pseudo differential operator with smooth kernel $A_\theta^{-s}(x, y)$ (i.e. in $C^\infty(S^1 \times S^1; \operatorname{End} \mathbb{C}^N)$). One defines $\zeta_{A,\theta}(s) = \operatorname{tr} A^{-s} = \int_{S^1} \operatorname{tr} A_\theta^{-s}(x, x) dx$ and more general $\zeta_{\alpha,A,\theta}(s) = \int_{S^1} \operatorname{tr} \alpha(x) A_\theta^{-s}(x, x) dx$, where $\alpha(x)$ is in $C^\infty(S^1; G_N(\mathbb{C}))$. $\zeta_{\alpha,A,\theta}(s)$ is holomorphic in $\operatorname{Re} s > \frac{1}{n}$ and, according to Seeley [Se], has a meromorphic extension to the entire complex plane with only simple poles, all contained in the set $\left\{\frac{1-j}{n} : j \text{ non-negative integer}\right\}$. Moreover 0 is a regular point. The ζ-regularized determinant of A is defined by $\operatorname{Det}_\theta(A) = \exp - \frac{d}{ds}\Big|_{s=0} \zeta_{A,\theta}(s)$.

If θ is only a principal angle for A, there exists $\varepsilon > 0$ such that $\operatorname{spec} A \cap L_{[\theta - \varepsilon, \theta + \varepsilon]}$ is finite and $\operatorname{spec} \sigma_L(A)(x, \xi) \cap L_{[\theta - \varepsilon, \theta + \varepsilon]} = 0$ $(x \in M, \xi \in T_x M^* \backslash \{0\})$. Thus we can choose θ' and $0 < \varepsilon' < \varepsilon/2$ such that $|\theta - \theta'| < \varepsilon/2$ and $\operatorname{spec} A \cap L_{[\theta' - \varepsilon', \theta' + \varepsilon']} = \emptyset$, i.e. θ' is an Agmon angle. For two Agmon angles θ', θ'' with $|\theta - \theta'| < \varepsilon/2$ and $|\theta - \theta''| < \varepsilon/2$ a simple calculation shows that $\frac{d}{ds}\Big|_{s=0} \zeta_{A,\theta'}(s)$ and $\frac{d}{ds}\Big|_{s=0} \zeta_{A,\theta''}(s)$ differ by an integer multiple of $2\pi i$. Therefore $\operatorname{Det}_{\theta'}(A) = \operatorname{Det}_{\theta''}(A)$ and we define $\operatorname{Det}_\theta(A) = \operatorname{Det}_{\theta'}(A)$.

Observe that the above considerations also show that $\operatorname{Det}_\theta A$ is locally constant in θ. In case A is not injective we define $\operatorname{Det}_\theta(A) = 0$.

We now collect a few results concerning the ζ-regularized determinant.

Proposition 2.6.

(1) $\operatorname{Det}_\theta(A)$ *has a smooth extension to* $\widehat{\mathrm{EDO}}_n$ *and is locally constant in* θ.

(2) $\operatorname{Det}_\theta$ *is holomorphic when considered as a function on the open connected subset of injective operators in* $\mathrm{EDO}_{n,\theta}$.

Proof. (1) was already discussed above.

(2) This result is well known, but for the convenience of the reader we outline its proof as we could not find any reference in the literature. It suffices to verify the statement for the subset of operators in $\mathrm{EDO}_{n,\theta}$ having θ as an Agmon angle. The statement follows of a vector version with a complex parameter of Theorem 12.1 in [Sh]. To make notation easier, assume $\theta = \pi$. Consider a holomorphic 1-parameter family $A(z) = \sum_{k=0}^{n} A_k(x, z) D^k$ in $\mathrm{EDO}_{n;\pi}$ with z in $U = \{|z| < 1\}$. We

may assume that there exists $\varepsilon > 0$ and $\rho_0 > 0$ such that $(\operatorname{Spec} A(z)) \cap L_{[\pi-\varepsilon,\pi+\varepsilon]} = \emptyset$, $(\operatorname{Spec} A_n(x,z)) \cap L_{[\pi-\varepsilon,\pi+\varepsilon]} = \emptyset$ as well as $(\operatorname{Spec} A(z)) \cap \{|w| \leqq 2\rho_0\} = \emptyset$ for all z in $\mathbb{C}, |z| < 1$. For s with $\operatorname{Re} s > \frac{1}{n}$ denote by $A^{-s}(z,x,y)$ the smooth kernel of the operator $A^{-s}(z) = \frac{1}{2\pi i} \int_{\Gamma_\pi} \lambda^{-s}(\lambda - A)^{-1} d\lambda$. For any s fixed, this is a map in $\mathscr{H}(U, S^1 \times S^1; \operatorname{End} \mathbb{C}^N)$, and Fréchet space of continuous maps on $U \times S^1 \times S^1$ with values in $\operatorname{End}(\mathbb{C}^N)$ which are holomorphic in the first variable. The restriction to $U \times \Delta_{S^1}$ of the kernel $A^{-s}(\cdot,\cdot,\cdot)$ belongs to $\mathscr{H}(U, S^1; \operatorname{End}^N)$, where Δ_{S^1} denotes the diagonal in $S^1 \times S^1$. By the same arguments as in the proof of Theorem 12.1 in [Sh] one proves that this restriction $A^s(\cdot,\cdot,\cdot)$ is holomorphic for s in $\left\{ s \in \mathbb{C}; \operatorname{Re} s > \frac{1}{n} \right\}$ with values in $\mathscr{H}(U, S^1; \operatorname{End}(\mathbb{C}^n))$ and admits a meromorphic continuation in s, denoted by $\bar{A}^{-s}(z,x,x)$ with $s = 0$ a regular point. Clearly the composition $\mathscr{H}(U, S^1; \operatorname{End}(\mathbb{C}^N)) \xrightarrow{T} \mathscr{H}(U, S^1, \mathbb{C}) \xrightarrow{J} \mathscr{H}(U, \mathbb{C})$ of T, induced by the trace $\operatorname{tr}: \operatorname{End}(\mathbb{C}^N) \to \mathbb{C}$, with J denoting integration over S^1, is linear and continuous. Here $\mathscr{H}(U, \mathbb{C})$ denotes the Fréchet space of holomorphic functions on U. Thus $s \in \mathbb{C} \to \zeta_{A,\pi}(z,s) = \int_{S^1} \operatorname{tr} \bar{A}^{-s}(z,x,x) dx \in \mathscr{H}(U, \mathbb{C})$ is a meromorphic map with 0 as a regular point. Therefore $\left.\frac{d}{ds}\right|_{s=0} \zeta_{A,\pi}(z,s)$ is an element of $\mathscr{H}(U, \mathbb{C})$ and (2) follows.

Proposition 2.7. *Let* $A = \sum_{k=0}^{n} A_k D^k$ *be in* $\mathrm{EDO}_{n,\theta}$. *Assume that* $A_k(x) \in C^\infty(S^1; \operatorname{End} \mathbb{C}^N)$ *leaves an* N_1*-dimensional subspace* $F \subsetneqq \mathbb{C}^N$ *invariant (all k; all x), then the restriction* A_r *of A to F and the induced operator* A_{in} *of A on the quotient* $\mathbb{C}^N/F$ *are in* $\mathrm{EDO}_{n,N_1;\theta}$ *and* $\mathrm{EDO}_{n,N-N_1;\theta}$ *respectively. Moreover* $\operatorname{Det}_\theta A = \operatorname{Det}_\theta(A_r) \operatorname{Det}_\theta(A_{\text{in}})$.

Proof. It suffices to check the result in the case where θ is an Agmon angle. The claim then follows from $\zeta_{A,\theta}(s) = \zeta_{A_r,\theta}(s) + \zeta_{A_{\text{in}},\theta}(s)$.

Next we need to consider the value at 0 of $\zeta_{\alpha,A,\theta}(s)$, where $\alpha(x)$ is in $C^\infty(S^1; GL_N(\mathbb{C}))$. In contrast to its derivative at $0, \zeta_{\alpha,A,\theta}(0)$ can be computed by a local formula, since

$$\zeta_{\alpha;A;\theta}(0) = \frac{-e^{i\theta}}{2\pi n} \sum_{\xi = \pm 1} \int_{S^1} dx \int_0^\infty dr \operatorname{tr}(\alpha(x) B_{-n-1}(x, \xi, re^{i\theta})),$$

where $B_{-n-1}(x,\xi,\lambda) = I(x,\xi,\lambda) + II(x,\xi,\lambda)$ with $I(x,\xi,\lambda) = -\xi^{n-1} B_{-n} A_{n-1}(x) B_{-n}$, $II(x,\xi,\lambda) = -in\xi^{2n-1} B_{-n} A_n(x) B_{-n} d_x A_n(x) B_{-n}$ and $B_{-n} \equiv B_{-n}(x,\xi,\lambda) = (A_n(x)\xi^n - \lambda)^{-1}$. For the case $\alpha = \mathrm{Id}$ this formula is given in [Se] (corrected in [W] and [Sh]). The general case can be proven in exactly the same way.

Proposition 2.8. *Let A be in* $\mathrm{EDO}_{n,\theta}$ *with* θ *an Agmon angle for A and* $\alpha \in C^\infty(S^1, GL_N(\mathbb{C}))$.

(1) *If n is even, then* $\zeta_{\alpha,A,\theta}(0) = 0$.

(2) *If n is odd and* $\alpha(x)$ *commutes with* $A_n(x)$ *(all x) then* $\zeta_{\alpha,A,\theta}(0) = 0$. *As a consequence we obtain*

(3) *If* $w = re^{i\tau}$ $(r > 0, \tau \in \mathbb{R})$, *then* $wA \in \mathrm{EDO}_{n;\theta+\tau}$ *and* $\operatorname{Det}_{\theta+\tau}(wA) = \operatorname{Det}_\theta(A)$.

Proof. (1) follows from the observation that $B_{-n-1}(x,\xi,\lambda)$ is an odd function in ξ in the case n is even.

(2) In the case n is odd and $\alpha(x)$ commutes with $A_n(x)$, $\operatorname{tr}\alpha(x)B_{-n-1}(x,\xi,\lambda)$ is a total differential in λ and $\int_0^\infty dr \operatorname{tr}\alpha(x)B_{-n-1}(x,\xi,re^{i\theta})$ is an odd function of ξ.

(3) follows from the simple observation that $(wA)_{\theta+\tau}^{-s} = w^{-s}A_\theta^{-s}$. Thus $\zeta_{wA,\theta+\tau}(s) = w^{-s}\zeta_{A,\theta}(s)$. Applying (1) and (2) for $\alpha = w\,\mathrm{Id}$ one obtains (3).

Proposition 2.8 can be applied to the following situation. Let $A(t)$ be a smooth 1-parameter family in $\mathrm{EDO}_{n;\theta}$ of the form $A(t) = \alpha_t(x)\left(D^n + \sum_{k=0}^{n-1} A_k(x)D^k\right)$, where $\alpha_t(x)$ is in $GL_N(\mathbb{C})$ (all $x \in S^1, 0 \leqq t \leqq 1$).

Corollary 2.9. *If $\frac{d}{dt}\alpha_t(x)$ and $\alpha_t(x)$ commute (all x and t) then* $\mathrm{Det}_\theta A(t)$ *is independent of t.*

Proof. It is sufficient to prove the result when θ is an Agmon angle and then to show that $\frac{d}{dt}\log \mathrm{Det}_\theta A(t) = 0$. Observe that $-\frac{d}{dt}\frac{d}{ds}\zeta_{A(t),\theta}(s) = \operatorname{tr}\left(\frac{d}{dt}\alpha_t\right)\alpha_t^{-1}A(t)^{-s}$, where we used that $\left(\frac{d}{dt}A(t)\right)A(t)^{-1} = \left(\frac{d}{dt}\alpha_t\right)\alpha_t^{-1}$. The result now follows from Proposition 2.8.

Finally we will need the following extension of $\zeta_{\alpha,A,\theta}$, due to Guillemin and Wodzicky (cf. [G, W]). Suppose θ is an Agmon angle for A. Let Q be a classical pseudodifferential operator of order $m(\in\mathbb{C})$. Then QA^{-s} is of trace class for $\operatorname{Re} s > \frac{1+m}{n}$. Define $\zeta_{Q,A,\theta}(s) = \operatorname{Tr} QA^{-s}$. Again $\zeta_{Q,A,\theta}(s)$ is holomorphic and has a meromorphic continuation to the entire complex s-plane with only simple poles, all contained in the set $\left\{\frac{1-j}{n} : n = 0,1,2\dots\right\}$. Denote by $\sigma(Q)(x,\xi) = \sum_{k\leqq m} q_k(x,\xi)$ the full symbol of Q. The noncommutative residue, introduced by Adler and Manin, is defined by $\frac{1}{2\pi}\int_{S^1} dx \sum_{\xi=\pm 1} q_{-1}(x,\xi)$. It was observed by Guillemin [G] and Wodzicki [W] that the residue of $\zeta_{Q;A;\theta}(s)$ at $s=0$ is equal to the noncommutative residue of Q multiplied by the order of A.

If A and A' are two elliptic pseudodifferential operators of the same order and θ is an Agmon angle for both of them, then the difference $\frac{d}{ds}(\zeta_{Q,A,\theta}(s) - \zeta_{Q,A',\theta}(s))|_{s=0}$ can be computed in the following way:

Proposition 2.10. ([Fr])

$$\frac{d}{ds}(\zeta_{Q,A,\theta}(s) - \zeta_{Q,A',\theta}(s))|_{s=0} = -\frac{1}{2\pi}\int_{S^1} dx \sum_{\xi=\pm 1} \operatorname{tr} c_{-1}(x,\xi),$$

where $c_{-1}(x,\xi)$ the homogeneous term of degree -1 in ξ of the full symbol of $Q\cdot H$ with H a pseudodifferential operator with full symbol $\sigma(H)(x,\xi) = \sum_{j\leqq 0} h_j(x,\xi)$ given by $h_{-j}(x,\xi) = \lim_{s\to 0}\frac{1}{2\pi i s}\int_{\Gamma_\theta} \lambda^{-s/n}(B_{-n-j}(\lambda,x,\xi,A) - B_{-n-j}(\lambda,x,\xi,A'))\,d\lambda$ and $B_{-n-j}(\lambda,x,\xi,A)$ being the homogeneous terms of degree $-n-j$ of the symbol of $(\lambda-A)^{-1}$. Thus, e.g. $B_{-n}(x,\xi,\lambda,A) = (\lambda - A_n(x)\xi^n)^{-1}$.

Remark 2.11. We will apply Proposition 2.10 in the case where Q is of order -1. Then

$$
\begin{aligned}
c_{-1}(x,\xi) &= q_{-1}(x,\xi)\cdot h_0(x,\xi) \\
&= \lim_{s\to 0}\frac{1}{2\pi i s}\, q_{-1}(x,\xi)\int_{\Gamma_\theta}\lambda^{-s/n}((\lambda - A_n(x)\xi^n)^{-1} - ((\lambda - A_n'(x)\xi^n)^{-1})d\lambda \\
&= \frac{e^{i\theta}}{in}\, q_{-1}(x,\xi)\int_0^\infty dr((r - e^{i\theta}A_n(x)\xi^n)^{-1} - (r - e^{-i\theta}A_n'(x)\xi^n)^{-1}).
\end{aligned}
$$

3. Proof of Theorem 1.

In this section we prove Theorem 1. One of the ingredients is the following

Proposition 3.1. *Suppose A and A' are two elliptic differential operators, $A=\sum_{k=0}^{n}A_k(x)D^k$ and $A'=\sum_{k=0}^{n}A_k'(x)D^k$, both injective with principal angle θ and with $A_n = A_n'$ and $A_{n-1}=A_{n-1}'$. Then*

$$\mathrm{Det}_\theta(A')\det(\mathrm{Id}-P_A) = \mathrm{Det}_\theta(A)\det(\mathrm{Id}-P_{A'}).$$

Proof. It suffices to consider the case where $n \geqq 2$ and θ is an Agmon angle. Consider the 1-parameter family $A(z) = A + zQ$ in $\mathrm{EDO}_{n;\theta}$, where $Q=\sum_{k=0}^{n-1}Q_kD^k$. For any z in the complex plane, $A(z)A^{-1} = \mathrm{Id} + zQA^{-1}$, where zQA^{-1} is a pseudodifferential operator of order $\leqq -2$ and hence of trace class. Thus $\mathrm{Id}+zQA^{-1}$ is of determinant class. Denote its Fredholm determinant by $\det(\mathrm{Id}+zQA^{-1})$. It is well known (cf. [Si]) that $\det(\mathrm{Id}+zQA^{-1})$ is an entire function in z of growth $< 1/2$. Further $\det(\mathrm{Id}+zQA^{-1})=0$ iff $A(z)=A+zQ$ is not injective and $\mathrm{Det}_\theta(A(z)) = \mathrm{Det}_\theta(A)\det(\mathrm{Id}+zQA^{-1})$ (cf. e.g. [Fr]).

Now consider $\dfrac{\mathrm{Det}_\theta A(z)}{\mathrm{Det}_\theta A}$ and $\dfrac{\det(\mathrm{Id}-P_{A(z)})}{\det(\mathrm{Id}-P_A)}$. They are entire functions in z, have the same zeroes and have the same value at $z=0$. The result now follows by Hadamard's theorem (cf. [T]) if both functions have growth <1. Thus is remains to show that $\det(\mathrm{Id}-P_{A(z)})$ has growth <1. But this is a well known fact (cf. e.g. [DD]) and can be seen as follows: One expresses $P_{A(z)}$ in terms of fundamental solutions $Y_j(\cdot,z)\in C^\infty(\mathbb{R},\mathrm{End}\,\mathbb{C}^N)$ by

$$
P(A) = \begin{vmatrix}
Y_1(1) & Y_2(1) & \cdots & Y_n(1) \\
\frac{d}{dx}Y_1(1) & \frac{d}{dx}Y_2(1) & \cdots & \frac{d}{dx}Y_n(1) \\
\vdots & \vdots & \vdots\vdots\vdots & \\
\frac{d^{n-1}}{dx^{n-1}}Y_1(1) & \frac{d^{n-1}}{dx^{n-1}}Y_2(1) & \cdots & \frac{d^{n-1}}{dx^{n-1}}Y_n(1)
\end{vmatrix},
$$

where $Y_j(x)$ solves $\sum_{k=0}^{n}A^k(x)(-1)^k\frac{d^k}{dx^k}Y_j + z\sum_{k=0}^{n-2}Q_k(x)(-1)^k\frac{d}{dx^k}Y_j = 0$ and satisfies

$\left(\frac{d^k}{dx^k} Y_j\right)(0) = \delta_{jk}\,\mathrm{Id}$. By converting the system of differential equations into one of integral equations one proves (cf. [DD]), by considering successive approximations, that there exist constants C_1 and C_2 such that

$$\left|\frac{d^k}{dx^k} Y_j(x,z)\right| \leqq C_1 \exp\{C_2|z|^{1/2}\}.$$

Proof of Theorem 1. The proof is presented for the case n odd only – for n even it is similar and in fact even a little bit simpler. So let us assume that n is odd. We have to show that $f_\theta(A) = \mathrm{Det}_\theta(A) - (i)^{(n+1)N} S_\theta(A) R(A) \det(\mathrm{Id} - P_A)$ vanishes identically on $\mathrm{EDO}^j_{n;\theta}$, the space of elliptic operators $A = \sum_{k=0}^{n} A_k D^k$ on $S^1 \times \mathbb{C}^N$, satisfying $\dim \Pi^+_\theta(A) = j$ and having θ as a principal angle. If A is not injective, then $\mathrm{Det}_\theta A = \det(\mathrm{Id} - P_A) = 0$ so we might assume, in addition, that A is $1-1$. Observe that due to Propositions 2.4 and 2.8 it suffices to consider the angle, say, $\theta = \pi/2$. For $0 \leqq j \leqq N$, choose a subspace E_j of $\mathbb{C}^N$ of dimension j and denote by Π_j the orthogonal projection onto E_j. Define the grading operator $\Gamma_j = \Pi_j - (\mathrm{Id} - \Pi_j)$. In the following we mostly drop the subscript $\theta = \pi/2$.

I Deformation 1. It suffices to prove that $f(A) = 0$ for A in EDO^j_n with A_n an involution. To see it write A in the form $A = A_n(D^n + H)$, where H is of order $\leqq n-1$. Define the smooth 1-parameter family in EDO^j_n of the form $A(t) = \alpha_t(D^n + H)$ $(0 \leqq t \leqq 1)$ with $\alpha_t(x)$ given by $\alpha_t(x) = tA_n(x) + (1-t)(\Pi^+_{\pi/2}(A) - \Pi^-_{\pi/2}(A))$. Then $\mathrm{spec}\,\alpha_t(x) \cap \mathbb{R} = \emptyset$ and $\frac{d}{dt} a_t(x)$ commutes with $\alpha_t(x)$ $(0 \leqq t \leqq 1, x \in S^1)$. Thus Corollary 2.9 implies that $\mathrm{Det}_{\pi/2}(A(1)) = \mathrm{Det}_{\pi/2}(A(0))$. Together with Proposition 2.4, $f(A(t))$ is independent of t.

II Deformation 2. Is suffices to prove that $f(A) = 0$ for A in EDO^j_n with $A_n(x) = \Gamma_j$.

Let A be in EDO^j_n with $A_n(x)$ an involution (all x in S^1). By Proposition 2.1 there exists a smooth 1-parameter family $\beta_t(x)$ in $C^\infty(S^1; GL_N(\mathbb{C}))$ $(0 \leqq t \leqq 1)$ such that $\beta_0(x) = \mathrm{Id}$ and $\beta_1(x)^{-1} A_n(x) \beta_1(x) = \Gamma_j$. Thus the $\beta_1(x)^{-1} A \beta_1(x) = \Gamma_j D^n +$ lower order terms. The claim now follows as $f(A)$ is invariant under reparametrization.

III Deformation 3. It suffices to prove $f(A) = 0$ for A in EDO^j_n with $A_n(x) \equiv \Gamma_j$ and A_{n-1} upper triangular. To see it, consider $A = \sum_{k=0}^{n} A_k D^k$ in EDO^j_n with $A_n(x) \equiv \Gamma_j$. Introduce the family of operators $A(t) = \Gamma_j \cdot B(t)$, where $B(t) = D^n + \sum_{k=0}^{n-1} B_k(x,t) D^k$ is a smooth 1-parameter family in $C^\infty(S^1, \mathrm{EDO}^j_n)$ $(0 \leqq t \leqq 1)$ such that $A(0) = A$ and $A_{n-1}(x,1) = \Gamma_j B_{n-1}(x,1)$ is upper triangular (all x). Clearly $R(A(t)) = R(B(t))$ and $\det(\mathrm{Id} - P_{A(t)}) = \det(\mathrm{Id} - P_{B(t)})$. In order to study $\mathrm{Det}_{\pi/2} A(t)$ we first compute $\frac{d}{dt}(\log \mathrm{Det}_{\pi/2} A(t) - \log \mathrm{Det}_{\pi/2} B(t))$ and then integrate in t. Clearly $A(t)$ is in EDO^j_n. Then

$$\frac{d}{dt}\log \mathrm{Det}_{\pi/2} A(t) = \mathrm{Tr}\left(\frac{d}{dt}B(t)\right)B(t)^{-1}\Gamma_j^{-1}(\Gamma_j B(t))^{-s}\Gamma_j|_{s=0}$$
$$= \mathrm{Tr}\left(\frac{d}{dt}B(t)\right)B(t)^{-1}(B(t)\Gamma_j)^{-s}|_{s=0}.$$

Thus, by Proposition 2.10

$$\frac{d}{dt}\log \mathrm{Det}_{\pi/2} A(t) - \frac{d}{dt}\log \mathrm{Det}_{\pi/2} B(t) = \mathrm{Tr}\left(\frac{d}{dt}B(t)\right)B(t)^{-1}\{(B(t)\Gamma_j)^{-s} - B(t)^{-s}\}|_{s=0}$$
$$= -\frac{1}{2\pi}\sum_{\xi=\pm 1}\frac{1}{\xi}\int_{S^1} dx\, \mathrm{tr}\, q(x,\xi)\cdot h(x,\xi),$$

where

$$q(x,\xi) = \sigma_L\left(\frac{d}{dt}B(t)B(t)^{-1}\right) = \frac{d}{dt}(B_{n-1}(t))\frac{1}{\xi},$$

$$h(x,\xi) = \lim_{s\to 0}\frac{1}{2\pi i s}\int_{\Gamma_{\pi/2}} \lambda^{-s/n}\{(\lambda - \Gamma_j\xi^n)^{-1} - (\lambda - \xi^n)^{-1}\}\,d\lambda$$
$$= -\frac{i}{n}e^{i\pi/2}\int_0^\infty (r + i\Gamma_j\xi^n)^{-1} - (r+\xi^n)^{-1}\,dr.$$

By an elementary computation

$$\sum_{\xi=\pm 1}\frac{1}{\xi}\left(-\frac{i}{n}\right)\int_0^\infty (r + i\Gamma_j\xi^n)^{-1}\,dr = -\frac{1}{n}\int_0^\infty (-2i\Gamma_j)(r+i\Gamma_j)^{-1}(r - i\Gamma_j)^{-1}\,dr$$
$$= -\frac{2}{n}\Gamma_j\int_0^\infty (r^2 + \mathrm{Id})^{-1}\,dr = -\frac{\pi}{n}\Gamma_j.$$

Similarly

$$\sum_{\xi=\pm 1}\frac{1}{\xi}\left(-\frac{i}{n}\right)\int_0^\infty (r + i\xi^n)^{-1}\,dr = -\frac{\pi}{n}\mathrm{Id},$$

then in all

$$\frac{d}{dt}\log \mathrm{Det}_{\pi/2} A(t) - \frac{d}{dt}\log \mathrm{Det}_{\pi/2} B(t) = \frac{i}{2n}\int_{S^1} dx\, \mathrm{tr}\,(\Gamma_j - \mathrm{Id})\frac{d}{dt}B_{n-1}(t),$$

and thus

$$(\log \mathrm{Det}_{\pi/2} A(1) - \log \mathrm{Det}_{\pi/2} B(1)) - (\log \mathrm{Det}_{\pi/2} A(0) - \log \mathrm{Det}_{\pi/2} B(0))$$
$$= \frac{i}{2n}\int_{S^1} dx\, \mathrm{tr}\,(\Gamma_j - \mathrm{Id})B_{n-1}(1) - \frac{i}{2n}\int_{S^1} dx\, \mathrm{tr}\,(\Gamma_j - \mathrm{Id})B_{n-1}(0).$$

This last expression is, by an easy verification, equal to

$$(\log S_{\pi/2}A(1) - \log S_{\pi/2}B(1)) - (\log S_{\pi/2}A(0) - \log S_{\pi/2}B(0)).$$

IV. Deformation 4. It suffices to prove $f(A) = 0$ for an injective operator A of the form $A = \Gamma_j D^n + A_{n-1} D^{n-1} + a$, where A_{n-1} is upper triangular and $a \in \mathbb{C}$. This follows from Proposition 3.1.

V. By IV and Proposition 2. We may assume that $N = 1$.

VI. So let us consider A of the form $A = D^n + G(x)D^{n-1} + a$; by the analyticity of $f(A)$ in A, it suffices to check that $f(A) = 0$ for g in $C^\infty(S^1, \mathbb{R}^+)$.

Let us consider the operator $\alpha(x)^{-1} A \alpha(x)$, where $\alpha(x)$ is the complex number given by $\exp\left\{\frac{i}{n}\int_0^x (g(t) - \bar{g})dt\right\}$, where $\bar{g} = \int_0^1 g(t)dt > 0$. With respect to this new parametrization induced by α, the operator takes the form $D^n + i\bar{g}D^{n-1} +$ terms involving lower order derivatives. This operator can be written $\left(D + \frac{i}{n}\bar{g}\right)^n + H$, where H is an operator of order $\leqq n-2$. Applying Proposition 3.1 once more it suffices to consider the operator $\left(D + \frac{i}{n}\bar{g}\right)^n$.

VII. Verification that $f((D+a)^n) = 0$, where $a = -i\alpha$ is a constant with $\alpha > 0$. Define $A = D + a$, then $R(A^n), S_{\pi/2}(A^n), \det(\mathrm{Id} - P_{A^n})$ and $\det_{\pi/2} A^n$ are entire functions of a. The zeroes of $\mathrm{Det}_{\pi/2} A^n$ and of $R(A^n)S_{\pi/2}(A^n)\det(I - P_{A^n})$ are the same. Both functions are of growth < 1. Applying Hadamard's theorem (cf. [T]) it suffices to show that $\mathrm{Det}_{\pi/2} A^n$ and $(i)^{n+1}R(A^n)S_{\pi/2}(A^n)\det(I - P_{A^n})$ have the asymptotic behavior as $\alpha \to +\infty$.

First observe that $R(A^n) = R(A)^n = e^{(ia/2)n} = e^{+(\alpha/2)n}$, $\Gamma_{\pi/2}(A^n) = -1$ and $S_{\pi/2}(A^n) = S_{\pi/2}(A) = -e^{-i(a/2)} = -e^{-\alpha/2}$. Moreover $\det(\mathrm{Id} - P_{A^n}) = (\det(\mathrm{Id} - P_A))^n = (1 - e^{ia})^n = (1 - e^{-\alpha})^n$ as $(D + a)e^{-iax} = 0$.

Thus

$$
\begin{aligned}
&(i)^{n+1}R(A^n)S_{\pi/2}(A^n)\det(\mathrm{Id} - P_{A^n}) \\
&= (-1)^{(n-1)/2} e^{(\alpha/2)(n-1)}(1 - e^{-\alpha})^n \quad \text{and} \quad \log(i)^{n+1}R(A^n)S_{\pi/2}(A^n)\det(\mathrm{Id} - P_{A^n}) \\
&= \frac{i\pi(n-1)}{2} + \frac{\alpha(n-1)}{2} + n\log(1 - e^{-\alpha}) = \frac{n-1}{2}\alpha + i\pi\frac{n-1}{2} + 0\left(\frac{1}{|\alpha|}\right).
\end{aligned}
$$

It remains to compute the asymptotics of $\mathrm{Det}_{\pi/2}((D - i\alpha)^n)$. Following [Fr] we obtain $(\alpha > 0) \log \mathrm{Det}_{\pi/2}((D - i\alpha)^n) = p_{-1}|\alpha| + q_0 \log|\alpha| + p_0 + \left(\frac{1}{|\alpha|}\right)$, where p_{-1}, q_0 and p_0 are given by local formulas. One verifies that $p_{-1} = \frac{n-1}{2}$, $p_0 = i\pi\frac{n-1}{2}$ and $q_0 = 0$.

4. Proof of Theorem 2.

In this section we establish a number of results concerning the elliptic finite difference operators described in the introduction and prove Theorem 2.

Proposition 4.1. *If $\tilde{A}$ is an elliptic periodic finite difference operator of type (m, n) and period T and $\alpha: \mathbb{Z} \to GL_N(\mathbb{C})$ with $\alpha(k + T) = \alpha(k)$, then*

(1) $\det(\mathrm{Id}-P_{\alpha A})=\det(\mathrm{Id}-P_A)$,

(2) $\det(\alpha A)=\left(\prod_{k=0}^{T-1}\det\alpha(k)\right)\det A$.

Proof. (1) follows from the fact that α establishes an isomorphism between $\operatorname{Null}\tilde{A}$ and $\operatorname{Null}(\widetilde{\alpha A})$ which intertwines the translation by T on $\operatorname{Null}\tilde{A}$ and $\operatorname{Null}(\widetilde{\alpha A})$.

(2) follows from the multiplicative property of the determinant of finite matrices.

Proposition 4.2. *Let $\tilde{A}$ be an elliptic finite difference operator of type (m,n) of period T with coefficients $A_{-m}(k),\dots,A_n(k)$. Consider the 1-parameter family $\tilde{A}(t)$ of elliptic finite difference operators of type (m,n) with coefficients $A_{-m}(k), tA_j(k)$ $(-m+1\leqq j\leqq n-j)$ and $A_n(k)$. Then for all t,*

$$\det A(t)\det(\mathrm{Id}-P_{A(0)})=\det A(0)\det(\mathrm{Id}-P_{A(t)}).$$

Proof. It suffices to check the result for the case $\det A(0)\neq 0$, since, in general, $\det A=0$ iff $\det(\mathrm{Id}-P_A)=0$. The conclusion of Proposition 4.2 holds certainly for $t=0$ and thus for any t, as $\det A(t)\det(\mathrm{Id}-P_{A(0)})$ and $\det A(0)\det(\mathrm{Id}-P_{A(t)})$ are polynomials in t with the same zeroes.

Let $M_{N_1,N_2}(\mathbb{C})$ denote the space of (N_1+N_2) square matrices α of the form $\begin{pmatrix}\alpha' & \beta\\ 0 & \alpha''\end{pmatrix}$, where α' is a N_1 square matrix and α'' is a N_2 square matrix. The following result is obvious:

Proposition 4.3. *If $\tilde{A}y(k)=\sum_{j=-m}^{n}A_j(k)y(k+j)$ is an elliptic difference operator of period T with $A_j(k)=\begin{pmatrix}A'_j(k) & B_j(k)\\ 0 & A''_j(k)\end{pmatrix}$ in M_{N_1,N_2} then*

(1) $\det A=\det(A')\det(A'')$,

(2) $\det(\mathrm{Id}-P_A)=\det(\mathrm{Id}-P_{A'})\det(\mathrm{Id}-P_{A''})$,

where $\tilde{A}'$ and $\tilde{A}''$ are elliptic finite difference operators with coefficients $A'_{-m},\dots,A'_n$ and $A''_{-m},\dots,A''_n$ respectively.

Proposition 4.4. *Let $\tilde{A}$ and $\tilde{B}$ be elliptic difference operators of type (m_1,n_1) and (m_2,n_2) respectively, both of period T acting on functions $y:\mathbb{Z}\to V$. Then $(\widetilde{B\cdot A})$ is an elliptic difference operator of type (m_1+m_2,n_1+n_2) such that*

$$\det(\mathrm{Id}-P_{B\cdot A})=\det(\mathrm{Id}-P_A)\det(\mathrm{Id}-P_B).$$

Proof. Observe that $\dim\operatorname{Null}(\tilde{A})=(m_1+n_1)N$. One then verifies that the following sequence is short exact:

$$0\to\operatorname{Null}(\tilde{A})\hookrightarrow\operatorname{Null}(\widetilde{B\cdot A})\xrightarrow{A}\operatorname{Null}(\tilde{B})\longrightarrow 0$$

and that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Null}(\tilde{A}) & \hookrightarrow & \operatorname{Null}(\widetilde{B\cdot A}) & \longrightarrow & \operatorname{Null}(\tilde{B}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{Id}-P_A} & & \downarrow{\scriptstyle \mathrm{Id}-P_{B\cdot A}} & & \downarrow{\scriptstyle \mathrm{Id}-P_B} & & \\
0 & \longrightarrow & \operatorname{Null}(\tilde{A}) & \hookrightarrow & \operatorname{Null}(\widetilde{B\cdot A}) & \xrightarrow{\tilde{A}} & \operatorname{Null}(\tilde{B}) & \longrightarrow & 0
\end{array}$$

This proves Proposition 4.4.

Let us now consider scalar elliptic difference operators of type (0, 1), i.e. $n=N=1$ and $m=0$. For these operators one verifies easily the following:

Lemma 4.5. $\det A=(-1)^{T-1}\left(\prod_{k=1}^{T} A_1(k)\right)\det(\mathrm{Id}-P_A)$.

Proof. If we view the space of periodic functions $y:\mathbb{Z}\to V$ as identified to the space of functions $f:\{1,\dots,T\}\to\mathbb{Z}$, then A is given by the following matrix

$$\left|\begin{array}{ccccc} 1 & A_1(1) & 0 & \text{---} & 0 \\ 0 & 1 & A_1(2) & & | \\ | & | & 1 & & 0 \\ 0 & & | & & A_1(T-1) \\ A_1(T) & 0 & 0 & \text{---} & 1 \end{array}\right|$$

whose determinant is $1+(-1)^{T-1}\prod_{j=1}^{T} A_1(j)$. Clearly

$$P_a=(-1)^T\prod_{j=1}^{T}\frac{1}{A_1(j)}.$$

Proof of Theorem 2. Theorem 2 is proved by a degeneration argument as follows: We have to prove that $f(A)=\det A-(-1)^{nN(T-1)}R_d(A)\det(\mathrm{Id}-P_A)$ vanishes. Clearly $f(A)$ is an analytic function of the coefficients of A. In view of Propositions 4.1 and 4.2 it suffices to check that $f(A)=0$ for operators $\tilde{A}$ of type $(0,n)$ with $A_j(k)=0$ for $1\leqq j\leqq n-1$ (all k). Conjugating $\tilde{A}$ by a suitable linear isomorphism $Q(k)$ with $Q(k+T)=Q(k)$ $(k\in\mathbb{Z})$ one can suppose that the matrices $a_0(k)$ $(k\in\mathbb{Z})$ are upper triangular. In view of Proposition 4.4 and 4.1 it then suffices to prove that $f(A)$ vanishes in the case where $n=N=1$ and $m=0$, which is treated in Lemma 4.5.

Theorem 2 is now applied to approximate the ζ-regularized determinant of an elliptic differential operator $A=\sum_{k=0}^{n} A_k(x)d_x^k\left(d_x=\frac{d}{dx}\right)$ by the determinant of suitably chosen finite difference operators. As usual A_k is in $C^\infty(S^1,\mathrm{End}\,V)$ and V is a complex vector space of dimension N. The T^{th} finite difference approximation a_T of A $(T\geqq n+1)$ is defined as follows:

$$\tilde{a}_T y(k)=\sum_{j=0}^{n} a_{T,j}(k)y(k+j),$$

where for each k in $\mathbb{Z}$, $a_{T,j}(k):V\to V$ are given by $a_{T,n}(k)=A_n(\exp(i2\pi(k-1)/T))$ and for $1\leqq l\leqq n$ and $h=\dfrac{2\pi}{N}$

$$a_{T,n-l}(k)=\sum_{j=0}^{l}(-1)^j\binom{n-j}{l}A_{n-j}(e^{i(2\pi(k-1))/T})h^j.$$

It is easy to see that $h^{-n}a_T$ is obtained from A by replacing $A_j(x)$ by $A_j(e^{i(2\pi(k-1))/T})$ and $d_x^j y(x)$ by $\Delta^j y(k)$, where $\Delta^j=\Delta\circ\Delta^{j-1}$ and $(\Delta y)(k)=\dfrac{y(k+1)-y(k)}{h}$. It is not hard to see that $a_{T,j}(k+T)=a_{T,j}(k)$. Moreover if h is sufficiently small, i.e. T

sufficiently large, then $a_{T,n}(k)$ and $a_{T,0}(k)$ are both invertible for all k, provided that the differential operator is elliptic (i.e. $A_n(x)$ is invertible). Hence for T sufficiently large $\tilde{a}_T$ is an elliptic finite difference operator of type $(0,n)$. Denote by $Y_{t,r}: \mathbb{Z} \to \operatorname{End}(V)$ $(0 \leqq r \leqq n-1)$ a system of matrices of fundamental solutions of $\tilde{a}_T$, i.e.

(i) $\sum_{j=0}^{n} a_{T,j}(k) Y_{T,r}(k+j) = 0$,

(ii) $Y_{T,r}(k) = 0 \quad \text{for} \quad n-r < k \leqq n$

$Y_{T,r}(k) = h^r \operatorname{Id}_V \quad \text{for} \quad 1 \leqq k \leqq n-r$.

Introduce the $nN \times nN$ matrix Y_T

$$Y_T = \begin{vmatrix} Y_{T,1}(T+1) & Y_{T,2}(T+1) & \cdots & Y_{T,n}(T+1) \\ \Delta Y_{T,1}(T+1) & \Delta Y_{T,2}(T+1) & & \Delta Y_{T,n}(T+1) \\ \vdots & \vdots & & \vdots \\ \Delta^{n-1} Y_{T,1}(T+1) & \Delta^{n-1} Y_{T,2}(T+1) & \cdots & \Delta^{n-1} Y_{T,n}(T+1) \end{vmatrix}.$$

Let Y be the canonical system of fundamental solutions for A, i.e. $Y = (Y_1, \ldots, Y_n)$ where $Y_k \in C^\infty(\mathbb{R}, \operatorname{End} V)$, $d_x^j Y_k(0) = \delta_{kj} \operatorname{Id}$, $A Y_k = 0$.

The following result is well known:

Proposition 4.6 (*Approximation*).

(1) $Y(2\pi) = \lim_{T \to \infty} Y_T$.

(2) $\det(\operatorname{Id} - P_A) = \lim_{T \to \infty} \det(\operatorname{Id} - P_{a_T})$.

As a consequence one obtains in combination with Theorems 1 and 2

Corollary 4.7. *Let A be an elliptic differential operator, of the form* $A = \sum_{k=0}^{n} A_k d_x^k$,

(1) *If A is of even order* (i.e. *n even*) *then* $\det A = (i)^{Nn} R(A) \det(I - P_A)$ *is given by*

$$(i)^{Nn} R(A) \lim_{T \to \infty} \left\{ (-1)^{(T-1)Nn} \left(\prod_{k=1}^{T} \frac{1}{\det A_n\left(\exp \frac{2\pi i(k-1)}{T}\right)} \right) \det a_T \right\}.$$

(2) *If A is odd order* (i.e. *n odd*) *and θ is a principal angle, then* $\operatorname{Det}_\theta A$ *is given by*

$$(i)^{(n+1)N} S_\theta(A) R(A) \lim_{T \to \infty} \left\{ (-1)^{(T-1)Nn} \left(\prod_{k=1}^{T} \frac{1}{\det A_n\left(\exp \frac{2\pi i(k-1)}{T}\right)} \right) \det a_T \right\}.$$

5. Applications

In this section we give various applications of Theorem 1.

Corollary 5.1.

(1) *For n even,* $\operatorname{Det}_\theta(A)$ *can be extended to be defined on all of* EDO_n *instead of* $\operatorname{EDO}_{n;\theta}$ *by the formula*

$$\operatorname{Det} A = (i)^{nN} R(A) \det(I - P_A).$$

Moreover $\mathrm{Det}(A \cdot B) = \mathrm{Det}\, A\, \mathrm{Det}\, B$ *in the case where both* A *and* B *are of even order and* $\mathrm{Det}\, A$ *is homogeneous of degree* 0.

(2) *If* $A \in \mathrm{EDO}_{n;\theta_1}, B \in \mathrm{EDO}_m$ *and* $A \cdot B$ *(or respectively* $B \cdot A$*) is in* $\mathrm{EDO}_{n+m;\theta}$, *where* m *is even and* n *is odd, then*

$$\mathrm{Det}_\theta(A \cdot B) = \frac{S_\theta(A \cdot B)}{S_{\theta_1}(A)} \mathrm{Det}_{\theta_1} A\, \mathrm{Det}\, B,$$

and respectively

$$\mathrm{Det}_\theta(B \cdot A) = \frac{S_\theta(B \cdot A)}{S_{\theta_1}(A)} (-1)^{\deg A} \mathrm{Det}_{\theta_1} A\, \mathrm{Det}\, B,$$

where $\deg A$ *is the degree of* A *as introduced in Sect. 2.*

(3) *If* $A \in \mathrm{EDO}_{n;\theta_1}, B \in \mathrm{EDO}_{m,\theta_2}$ *where* n *and* m *are odd, then*

$$\mathrm{Det}(A \cdot B) = (-1)^N \frac{(-1)^{\deg B}}{S_{\theta_1}(A) S_{\theta_2}(B)} \mathrm{Det}_{\theta_1}(A)\, \mathrm{Det}_{\theta_2}(B).$$

Corollary 5.1 can be viewed as formulae for the so-called multiplicative anomaly, as introduced by Wodzicki (cf. [W] and [Fr]).

Corollary 5.2. *On the space of self-adjoint elliptic differential operators of even or odd order with* θ *as a principal angle, the absolute value for the* ζ*-regularized determinant is independent of the choice of* θ, *is multiplicative and homogeneous of degree* 0.

Corollary 5.3. *Let* A *be in* $\mathrm{EDO}_{n;\theta}$ *with* n *odd. Assume* A *is injective (thus* $\mathrm{Det} A^2 \neq 0$*). Then*

(1) $S_\theta(A)^2 = \dfrac{(\mathrm{Det}_\theta A)^2}{\mathrm{Det}\, A^2} (-1)^N (-1)^{\deg A}$. *Thus* $S_\theta(A)^2$ *is, up to a* sign, *a spectral invariant.*

(2) *If the principal symbol of* A *is scalar valued and positive,* i.e. *of the form* $a_n(x)\,\mathrm{Id}$, *with* $a_n(x) > 0$, *then* $S_\theta(A)$ *is a spectral invariant.*

Proof. (1) follows from Corollary 5.1 (3) with $A = B, \theta = \theta_1 = \theta_2$.

(2) follows with (1) and the fact that $\Gamma_\theta(A) = \pm \mathrm{Id}$ from the well known result (cf. [N] e.g.) that the eigenvalues of $\frac{i}{n} \int_{S^1} A_n^{-1}(x) A_{n-1}(x) dx$ appear in the second term of the asymptotics of the eigenvalues of A. □

For a self-adjoint injective elliptic differential operator $A: \Gamma(E) \to \Gamma(E)$ on a hermitian complex vector bundle E, Atiyah, Patodi and Singer (cf. [APS]) have considered a variant of the ζ-function, called the η-function, $\eta_A(s) = \sum_{k \in \mathbb{Z}} \mathrm{sgn}\, \lambda_k (\lambda_k)^{-s}$. It has been noted by Shubin [Sh] that

$$\eta_A(s) = \frac{e^{i\pi s}}{e^{i\pi s} - 1} \zeta_{A,-\pi/2}(-s) + \frac{e^{-i\pi s}}{e^{-i\pi s} - 1} \zeta_{A,-3\pi/2}(-s).$$

It follows that η is a meromorphic function with at most simple poles. It was proved by Atiyah, Patodi and Singer that for compact manifolds of odd dimension $\eta_A(s)$ is regular at $s=0$. In the case where $M \cong S^1$ this also follows from Sect. 2 where it was proved that $\zeta_{A,-\pi/2}(0) = \zeta_{A,-3\pi/2}(0) = 0$. The η-invariant is defined as

$\eta(A) = [\eta_A(0)(\bmod 2\mathbb{Z})]$. A simple computation shows that $\eta(A)$ is given mod $2\mathbb{Z}$, by $\frac{i}{\pi}(\zeta'_{A,-\pi/2}(0) - \zeta'_{A,-3\pi/2}(0))$.

Corollary 5.4. *Let $E \xrightarrow{p} S^1$ be a vector bundle of dimension N. If $A : \Gamma(E) \to \Gamma(E)$ is a self-adjoint elliptic differential operator with respect to some hermitian structure on E then $\eta(A) = 0$ if the order is even and $\eta(A) = \left[-N + \frac{2i}{\pi}\log S_{-\pi/2}(A)^2\right](\bmod 2\mathbb{Z})$ if the order of A is odd.*

Proof. If the order of $A, n,$ is even then the conclusion follows by Theorem 1.

If n is odd, then by Theorem 1,

$$\frac{\mathrm{Det}_{-\pi/2} A}{\mathrm{Det}_{-3\pi/2} A} = \frac{S_{-\pi/2} A}{S_{-3\pi/2} A}.$$

Observe that $S_\theta(A) = (-1)^N (S_{\theta-\pi}(A))^{-1}$. One obtains that $\frac{\mathrm{Det}_{-\pi/2} A}{\mathrm{Det}_{-3\pi/2} A} = (-1)^N \cdot (S_{-\pi/2}(A))^2$. Then $\eta(A)$ is given, mod $2\mathbb{Z}$, and with respect to an admissible parametrization by

$$-N - \frac{1}{\pi n} \int_{S^1} \mathrm{tr}(\Gamma_{-\pi/2}(x) A_{n-1}(x) A_n(x)^{-1}) dx.$$

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