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The Fixed Boundary Value Problems for the Equations of Ideal Magneto-Hydrodynamics with a Perfectly Conducting Wall Condition

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Abstract. The equations of ideal Magneto-Hydrodynamics are investigated concerning initial boundary value problems with a perfectly conducting wall condition. The local in time solution is proved to exist uniquely, provided that the normal component of the initial magnetic field vanishes everywhere or nowhere on the boundary.

1. Introduction

The equations of ideal Magneto-Hydrodynamics (ideal MHD) are the model which describes the macroscopic motion of an electrically conducting fluid. Here "ideal" means the model to be free from the effect of viscosity and electrical resistivity. In this paper we study initial boundary value problems for the ideal MHD with a perfectly (electrically) conducting wall condition.

Although there are several studies of these problems relevant to the plasma confinement from physical and engineering viewpoints (cf. [6]), any mathematical exploration into these problems, as far as we know, has not been found. Even the boundary conditions themselves, which are not only mathematically proper but also fully consistent with the physical situation, have not been well investigated. Therefore we first investigate and propose such adequate boundary conditions to a perfectly conducting wall. Then we show local in time existence of a unique classical solution to two special cases of these conditions. Now we state our problems more precisely. The problems we will treat are the equations of ideal MHD,

$$\varrho_p(\partial_t + (u \cdot \nabla))p + \varrho \nabla \cdot u = 0, \tag{1.1$_\mathrm{a}$}$$

$$\varrho(\partial_t + (u \cdot \nabla))u + \nabla p + \mu H \times (\nabla \times H) = 0, \tag{1.1$_\mathrm{b}$}$$

$$\partial_t H - \nabla \times (u \times H) = 0, \quad \text{in} \quad [0, T] \times \Omega, \tag{1.1$_\mathrm{c}$}$$

$$(\partial_t + (u \cdot \nabla))S = 0, \tag{1.1$_\mathrm{d}$}$$

$$\nabla \cdot H = 0, \tag{1.1$_\mathrm{e}$}$$

with initial conditions

$$ {}^t(p, u, H, S)|_{t=0} = {}^t(p_0, u_0, H_0, S_0) = U_0 \quad \text{in } \Omega, \tag{1.2} $$

and with "perfectly conducting wall boundary conditions," which we will formulate in Sect. 2,

$$ \begin{gathered} u \cdot n = 0, \quad H \cdot n = 0 \quad \text{on} \quad [0, T] \times \Gamma_0, \\ u = 0 \quad \text{on} \quad [0, T] \times \Gamma_1, \end{gathered} \tag{1.3} $$

where

$$ \Gamma_0 = \{x \in \Gamma \mid H_0(x) \cdot n(x) = 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma \mid H_0(x) \cdot n(x) \neq 0\}. $$

Here Ω is a bounded or unbounded domain in $\mathbb{R}^3$ with sufficiently smooth and compact boundary Γ, or a half space $\mathbb{R}^3_+ = \{x \mid x_1 > 0\}$ with the boundary $\partial\mathbb{R}^3_+ = \{x \mid x_1 = 0\}$; pressure $p = p(t, x)$ (scalar), entropy $S = S(t, x)$ (scaler), velocity $u = u(t, x) = (u^1, u^2, u^3)$, and magnetic field $H = (H^1, H^2, H^3)$ are unknown functions of time t and space variables $x = (x_1, x_2, x_3)$. For simplicity, we denote the unknown functions ${}^t(p, u, H, S)$ by U; density ϱ is determined by the equation of state $\varrho = f(p, S)$, where f is a given function so that $f > 0$ and $\partial f/\partial p$ $(= \varrho_p) > 0$ for $p > 0$ and each S; magnetic permeability μ is assumed to be a positive constant; we write $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ $(i = 1, 2, 3)$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and use the conventional notations in the vector analysis; $n = n(x) = (n_1, n_2, n_3)$ denotes the unit outward normal at $x \in \Gamma$.

We will show local in time existence theorems of these problems when Γ consists only of Γ_0 or Γ_1 in (1.3). In each case we can reduce (1.1)–(1.3) into initial boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary. The proof of the theorems is proceeded through the iteration scheme, and the main ingredient is to get a priori estimates of the linearized problems subordinate to the scheme.

We first note the structure of the problem in which Γ consists only of Γ_1 is very similar to that studied in [17]. So we can show the local in time existence of the solution by the same line of the proof as in [17] (see Remark 2.9).

We next note that the problem in which Γ consists only of Γ_0 seems peculiar one compared with initial boundary value problems appeared in other physical problems (a typical example is the problem for the compressible Euler equations, see [13]) by the following reasons:
(i) the condition on the boundary, $H \cdot n = 0$, seems to be an excess boundary condition when we solve the linearized problem as an initial boundary problem of symmetric hyperbolic systems (see, for example, Lemma 3.3).
(ii) it seems difficult to show that the solution has *full regularity*, i.e., that the solution has the same order of regularity in the direction normal to Γ as in the direction tangential to Γ.

By virtue of the fact we may regard the condition $H \cdot n = 0$ as the restriction on the initial data U_0 instead of on the lateral surface $[0, T] \times \Gamma$, we can overcome the first difficulty by adding lower order terms to the linearized equations subordinate to the usual iteration scheme [see (3.10) and Lemma 5.2]. We owe this idea to Taira Shirota.

Next, to evade the second difficulty we introduce a weighted Sobolev space with respect to space variables in which the order of the partial differentiation in the direction transversal to Γ is half of that in the direction tangential to Γ (as for the definition of function spaces, see Sect. 2). This space seems to be suitable to get a priori estimates of solutions for our linearized problems.

In preparation of this manuscript, we heard of the results by Chen Shuxing [3]. He develops the general theory for the initial boundary value problems for a quasilinear symmetric hyperbolic system with boundary characteristic of constant multiplicity. Although his approach is close to ours, the theorem of [3] is too restrictive to apply it to our problems directly. In addition, some further considerations are needed because of the first difficulty (i) cited above. (As for the linear problems, see [9, 11, 15].)

Finally we point out two open problems related to these initial boundary value problems. The first problem is the initial boundary value problem (1.1)–(1.3) in which both Γ_0 and Γ_1 are not empty. Any linearized problem of this problem requires the study of the initial boundary value problem with boundary characteristic of *nonconstant* multiplicity. The second problem is the question of whether the solution of the problem in which Γ consists only of Γ_0 has *full regularity* or not (cf. [7, 10, 14]).

This paper is the full and extended version of [16]. The plan of this paper is as follows. In Sect. 2 we present formulation of a perfectly conducting wall condition and then give the statements of the main theorems. In Sects. 3–5 we give the proof of the theorems for the case in which we are most interested that Γ consists only of Γ_0.

2. Formulation of Perfectly Conducting Wall Condition and Statements of Main Theorems

In this section we formulate the conditions on the lateral surface when the boundary wall consists of a perfectly conducting wall. Since the boundary is supposed to be rigid and fixed (independent of time variable), the same condition as in the fluid dynamics is imposed on velocity u:

$$u\cdot n=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{2.1}$$

Since we suppose the boundary wall is perfectly conducting, i.e. electric conductivity σ on the boundary wall is infinite, the tangential components of the electric field E must vanish:

$$E\times n=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{2.2}$$

Further, by Ohm's law $J=\sigma(E+u\times\mu H)$, we get formally

$$E=-u\times\mu H \quad \text{on} \quad [0,T]\times\Gamma.$$

Accordingly, by virtue of vector identity that $(A\times B)\times C=(A\cdot C)B-(B\cdot C)A$, we get from (2.2),

$$u(H\cdot n)=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{2.3}$$

Now we can summarize as perfectly conducting wall conditions

$$u\cdot n=0, \qquad u(H\cdot n)=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{2.4}$$

The second part of (2.4) forms nonlinear conditions. However, we can reduce them to linear ones by the following proposition.

Proposition 2.1. *Let u and H be*[1] *a classical solution of* (1.1) *with initial conditions* $(u, H)|_{t=0}=(u_0, H_0)$. *Then the following conditions are equivalent to* (2.4)*:*

$$\begin{aligned} u \cdot n &= 0 \quad \text{on} \quad [0, T] \times \Gamma_0, \\ u &= 0 \quad \text{on} \quad [0, T] \times \Gamma_1, \end{aligned} \tag{2.5}$$

where Γ_0, Γ_1 are the same as in (1.3).

Proof. Let (u, H) satisfy (2.4). Let S be any portion on Γ. Since the condition (2.4) yields $n \times (u \times H)=0$ on $[0, T] \times \Gamma$, we find from $(1.1)_c$ by integration by parts

$$\partial_t \int_S H \cdot n \, d\Gamma = 0 \quad \text{on } [0, T],$$

where $d\Gamma$ denotes surface element on Γ.

Accordingly, we get

$$H \cdot n = H_0 \cdot n \quad \text{on} \quad [0, T] \times \Gamma.$$

Thus (u, H) satisfies (2.5). Next we prove the reversion. Let u satisfy (2.5). If we show that

$$H \cdot n = 0 \quad \text{on} \quad [0, T] \times \Gamma_0, \tag{2.6}$$

we easily see that (u, H) satisfy (2.4). To prove (2.6), we first note the following vector identity: $-\nabla \times (u \times H) = u \cdot \nabla H - H \cdot \nabla u - u \nabla \cdot H$. Since $u \cdot n = 0$ on $[0, T] \times \Gamma$, we get by direct calculations

$$(u \cdot \nabla H) \cdot n = u \cdot \nabla(H \cdot n) - H \cdot (u \cdot \nabla n),$$

$$-(H \cdot \nabla u) \cdot n = -(H \cdot n)(n \cdot \nabla(u \cdot n)) + u \cdot (H \cdot \nabla n) \quad \text{on} \quad [0, T] \times \Gamma.$$

Accordingly, we obtain from $(1.1)_c$,

$$\partial_t(H \cdot n) + u \cdot \nabla(H \cdot n) + b(\nabla u, u, n) H \cdot n + c(u, H, \nabla n) = 0 \quad \text{on} \quad [0, T] \times \Gamma, \tag{2.7}$$

where $b(\nabla u, u, n) = \nabla \cdot u - n \cdot \nabla(u \cdot n)$ and $c(u, H, \nabla n) = -H \cdot (u \cdot \nabla n) + u \cdot (H \cdot \nabla n)$. Since $n = n(x)$ is expressed by $-\nabla \operatorname{dist}(x, \Gamma)$, we see that $c(u, H, \nabla n) = 0$. Thus by the standard method of characteristic curves, we get (2.6). This completes the proof.

Remark 2.2. The above proof shows that we can regard the condition

$$H \cdot n = 0 \quad \text{on} \quad [0, T] \times \Gamma_0$$

as the restriction only on the initial data.

We use the following notations for the function spaces. Let $H^m(\Omega)$ be a usual vector-valued Sobolev space of order m, with the associated inner product denoted by $(\,,)_m$ or $(\,,)_{m,\Omega}$ and norm denoted by $|\cdot|_m$ or $|\cdot|_{m,\Omega}$. Define the function space $H^m_*(\Omega)$ to be the set of functions $U(x)$ taking values in $\mathbb{R}^8$ and satisfying the following properties:

Let k be an integer such that $0 \leq k \leq m$ and let Λ_i $(i=1, \ldots, k)$ be an arbitrary vector field tangential to Γ, i.e. Λ_i belongs to $B^\infty(\bar{\Omega}; \mathbb{R}^3)$ and satisfies $\langle \Lambda_i(x), n(x) \rangle = 0$ for all $x \in \Gamma$ and $i = 1, \ldots, k$. Then $\Lambda_1 \ldots \Lambda_k U(x) \in H^{[(m-k)/2]}(\Omega)$, where [ ] denotes Gaussian bracket.

[1] We use here and hereafter the terms of "being a classical solution" to represent that each component of the solution belongs to $C^1([0, T] \times \bar{\Omega})$

In connection with $H_*^m(\Omega)$, we define the function space $X_T^m(\Omega)$ by

$$X_T^m(\Omega)=\bigcap_{k=0}^{m} L_\infty^k(0,T;H_*^{m-k}(\Omega)). \tag{2.8}$$

Here $L_\infty^k(0,T;H_*^{m-k}(\Omega))$ is the space of all functions $U=U(t,x)$ such that $\partial_t^i U$, $0\leqq i\leqq k$, are essentiallly bounded and strongly measurable functions on $[0,T]$ taking values in $H_*^{m-k}(\Omega)$. We introduce norm $\|\cdot\|_{m,T}$ associated to the space $X_T^m(\Omega)$ (cf. Lemma 1 in [2]). At first, cover $\bar\Omega$ by a finite family of open sets $\{\mathcal{O}_i\}_{i=0}^{l}$ such that $\mathcal{O}_i\cap\bar\Omega$, $i=1,\ldots,l$, are diffeomorphic to $\mathscr{B}_+=\{x_1\geqq 0\}\cap\{|x|<1\}$ with Γ mapping to $\{x_1=0\}$ and $\mathcal{O}_0\subset\subset\Omega$. Denote these diffeomorphisms from $\mathcal{O}_i\cap\bar\Omega$, $i=1,\ldots,\ell$, to $\mathscr{B}_+$ by γ_i. Choose a finite number of partition of unity $\{\varphi_i\}_{i=0}^{\ell}$ subordinate to this covering $\{\mathcal{O}_i\}_{i=0}^{\ell}$ such that $\sum_{i=0}^{\ell}\varphi_i^2=1$ in $\bar\Omega$. Now define the norm $\|\cdot\|_{m,T}$ by

$$\begin{aligned}
&\|u\|_{m,T}=\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|_m,\\
&\|u(t)\|_m^2=\sum_{k=0}^{m}\|\partial_t^k u(t)\|_{m-k,*}^2,\\
&\|u(t)\|_{m,*}^2=|\varphi_0 u(t)|_m^2+\sum_{i=1}^{\ell}|\varphi_i u(t)\circ\gamma_i^{-1}|_{m,*}^2,
\end{aligned} \tag{2.9}$$

where

$$|f|_{m,*}^2=\sum_{s=0}^{[m/2]}\sum_{|\alpha|\leqq m-2s}|\partial_*^\alpha\partial_1^s f|_{0,\mathbb{R}_+^3}^2$$

with $\partial_*^\alpha=(\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. The weight $\sigma(x_1)$ is a smooth and positive function such that $\sigma(x_1):=x_1$ for small enough x_1, $:=1$ for $x_1\geqq 1$ and

$$\sigma'(x_1)=(\partial/\partial x_1)\sigma(x_1)>0 \quad \text{for} \quad 0\leqq x_1<1.$$

We remark that the norms arising from different choice of $\mathcal{O}_i$, γ_i, φ_i are equivalent. So when $\Omega=\mathbb{R}_+^3$, we can define $\|\cdot\|_{m,*}$ by $\|\cdot\|_{m,*}=|\cdot|_{m,*}$. Throughout this paper c, C, and c_i, C_i $(i=0,1,\ldots)$ denote positive constants which may change from line to line.

Now we state our main theorems. For the problem (1.1)–(1.3) in which Γ consists only of Γ_0, the statement is

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ. Let $m\geqq 8$ be an integer. Suppose that the initial data $U_0\in H^m(\Omega)$ satisfies*

$$\nabla\cdot H_0=0, \quad p_0>0 \ \textit{in}\ \Omega, \quad H_0\cdot n=0 \ \textit{on}\ \Gamma, \tag{2.10}$$

and the compatibility conditions:

$$\mathring{\partial}_t^k u_0\cdot n=0, \quad \textit{for} \quad k=0,\ldots,m-1, \ \textit{on}\ \Gamma. \tag{2.11}$$

Then there exists a constant $T_0>0$ such that the initial boundary value problem (1.1)–(1.3) has a unique solution $U\in X_{T_0}^m(\Omega)$.

When Ω is unbounded, we get

Theorem 2.4. *Let Ω be an unbounded domain in $\mathbb{R}^3$ with sufficiently smooth and compact boundary Γ or a half space $\mathbb{R}_+^3$. Let $m\geqq 8$ be an integer. Suppose that*

$U_0 - {}^t(c,0) \in H^m(\Omega)$ for some constant $c>0$ and that U_0 satisfies the conditions given in Theorem 2.3. Then there exists a constant $T_1>0$ such that the problem (1.1)–(1.3) *has a unique solution U satisfying $U - {}^t(c,0) \in X^m_{T_1}(\Omega)$.*

Remark 2.5. (a) Since $H^m_*(\Omega) \subset H^{[m/2]}(\Omega)$ and $m \geqq 8$, these solutions U are classical.
(b) The term $\mathring{\partial}_t^k u_0$ in (2.11) is determined by less than or equal to k^{th} order derivatives of U_0 by using Eqs. (1.1) at $t=0$ successively. We can also determine the terms $\mathring{\partial}_t^k H_0$, and $\mathring{\partial}_t^k U_0$ analogously. [As for precise definitions, see (3.5).]
(c) The compatibility conditions associated with the boundary condition $H \cdot n = 0$:

$$\mathring{\partial}_t^k H_0 \cdot n = 0, \quad \text{for} \quad k=0,1,\ldots,m-1, \text{ on } \Gamma \tag{2.12}$$

are automatically satisfied by the condition $H_0 \cdot n = 0$ in (2.10) and $\mathring{\partial}_t^k u_0 \cdot n = 0$, for $k=0,\ldots,m-1$, in (2.11).

For the problem (1.1)–(1.3) in which Γ consists only of Γ_1, we can show the following theorems. This result was pointed out by Taira Shirota before us.

Theorem 2.6 (T. Shirota). *Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ. Let $m \geqq 3$ be an integer. Suppose that $U_0 \in H^m(\Omega)$ and that U_0 satisfies*

$$\nabla \cdot H_0 = 0, \quad p_0 > 0 \text{ in } \Omega, \quad H_0 \cdot n \neq 0 \text{ on } \Gamma, \tag{2.13}$$

and the compatibility conditions

$$\mathring{\partial}_t^k u_0 = 0, \quad \text{for} \quad k=0,1,\ldots,m-1, \text{ on } \Gamma. \tag{2.14}$$

Then there exists a constant $T_2>0$ such that the problem (1.1)–(1.3) *has a unique solution $U \in \bigcap_{j=0}^{m} C^j(0,T_2;\, H^{m-j}(\Omega))$.*

When Ω is unbounded, we get

Theorem 2.7. *Let Ω be an unbounded domain in $\mathbb{R}^3$ with sufficiently smooth and compact boundary Γ (respectively a half space $\mathbb{R}^3_+$). Let $m \geqq 3$ be an integer. Suppose that $U_0 - {}^t(x,0) \in H^m(\Omega)$ for some constant $c>0$ (respectively $U_0 - {}^t(c,0,0,0,c',0,0,0) \in H^m(\mathbb{R}^3_+)$ for some constants $c>0$, $c' \neq 0$) and that U_0 satisfies the conditions given in Theorem 2.6. Then there exists a constant $T_3>0$ such that the problem* (1.1)–(1.3) *has a unique solution*

$$U - {}^t(c,0) \in \bigcap_{j=0}^{m} C^j(0,T_3;H^{m-j}(\Omega))$$

$$\left(\textit{respectively } U - {}^t(c,0,0,0,c',0,0,0) \in \bigcap_{j=0}^{m} C^j(0,T_3;H^{m-j}(\mathbb{R}^3_+))\right).$$

Remark 2.8. The assumptions that $\nabla \cdot H_0 = 0$ in Ω and $H_0 \cdot n \neq 0$ on Γ in (2.13) imply that the boundary Γ consists of more than two connected components except when Ω is a half space.

Remark 2.9. We present a sketch of the proof of Theorem 2.6 and 2.7. We set $U = {}^t(p,u,H,S)$ and rewrite Eqs. $(1.1)_{\text{a-d}}$ into the symmetric form

$$A_0(U)\partial_t U + \sum_{i=1}^{3} A_i(U)\partial_i U = 0 \qquad (\text{cf. [17]}). \tag{2.15}$$

In order to solve the problem by iteration, we consider the linearization of (2.15) around an arbitrary function $U' = {}^t(p',u',H',S')$ near the initial data, satisfying

$u' \cdot n = 0$ and $H' \cdot n \neq 0$ on Γ. The linearized equations form a symmetric hyperbolic system with singular boundary matrix, $A_n(U') = \sum_{i=1}^{3} A_i(U')n_i$, which in fact has constant rank 6 on Γ. First, we find by direct calculation that the null space of boundary conditions, i.e.

$$\{V = {}^t(v_1, \ldots, v_8) \in \mathbb{R}^8 \mid v_2 = v_3 = v_4 = 0\},$$

is a maximally nonnegative subspace of the boundary matrix. Next, although the boundary is characteristic, we can estimate, by virtue of the special structure of the equation to $\operatorname{div} H$, $\partial_t |\operatorname{div} H|^2_{m-1,\Omega}$ in terms of $|U|^2_{m,\Omega}$ (see Eq. (5.3) in [17]). Further, we can also estimate $\partial_t |S|^2_{m,\Omega}$ in terms of $|U|^2_{m,\Omega}$ from the equation for S by standard energy estimates. Then by using the nonzero part of $A_n(U')$ and these estimates, we can estimate normal derivatives of U. All the rest procedure proceeds as in the proof in [17]. So we omit it.

3. Study of a Linearized Problem

We first rewrite Eqs. (1.1). We may assume $\mu = 1$ without loss of generality; otherwise it suffices to introduce new variables $\mu^{1/2} H$ instead of H. Then (1.1) can be converted into the following symmetric system:

$$\varrho_p(\partial_t + u \cdot \nabla)p + \varrho \nabla \cdot u = 0, \tag{3.1$_1$}$$

$$\varrho(\partial_t + u \cdot \nabla)u + \nabla p + H \times (\nabla \times H) = 0, \tag{3.1$_2$}$$

$$(\partial_t + u \cdot \nabla)H - H \cdot \nabla u + H \nabla \cdot u = 0, \tag{3.1$_3$}$$

$$(\partial_t + u \cdot \nabla)S = 0. \tag{3.1$_4$}$$

This equivalence of (3.1) and (1.1), under the initial and boundary conditions (1.2) and (1.3), can be seen by noting the fact that if the solution of $(3.1)_3$ satisfies $\nabla \cdot H = 0$ in Ω at $t = 0$, then $\nabla \cdot H = 0$ in Ω is true for all $t > 0$. Next, we introduce new unknown functions $V = {}^t(q - c, u, H, S)$ in place of U (when Ω is a bounded domain we omit a positive constant c in the above V and hereafter we do not mention this remark), where $q = p + 1/2|H|^2$ is the summation of (fluid dynamical) pressure and magnetic pressure, and rewrite Eqs. (3.1) in the form

$$\begin{pmatrix} \alpha & 0 & -\alpha H & 0 \\ 0 & \varrho I_3 & 0 & 0 \\ -\alpha\, {}^t H & 0 & I_3 + \alpha H \otimes H & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \partial_t V$$

$$+ \begin{pmatrix} \alpha(u \cdot \nabla) & \nabla & -\alpha H(u \cdot \nabla) & 0 \\ {}^t\nabla & \varrho(u \cdot \nabla) I_3 & -(H \cdot \nabla) I_3 & 0 \\ -\alpha\, {}^t H(u \cdot \nabla) & -(H \cdot \nabla) I_3 & (I_3 + \alpha H \otimes H) u \cdot \nabla & 0 \\ 0 & 0 & 0 & u \cdot \nabla \end{pmatrix} V$$

$$\equiv A_0(V)\partial_t V + \sum_{i=1}^{3} A_i(V)\partial_i V = 0. \tag{3.2}$$

Here we set $\alpha=\varrho_q/\varrho$, where $\varrho=f(q-1/2|H|^2,S)$ and $\varrho_q=\partial f/\partial q$, and

$$H\otimes H=(H^iH^j|i\rightarrow 1,2,3,j\downarrow 1,2,3).$$

Note that $\varrho>0$, $\varrho_q>0$ for $q-1/2|H|^2>0$ and each S.

Then the initial boundary value problem we consider is that for Eqs. (3.2) with initial conditions

$$V|_{t=0}={}^t(q_0-c,u_0,H_0,S_0)\equiv V_0 \quad \text{in } \Omega, \tag{3.3}$$

and with boundary conditions

$$u\cdot n=0, \qquad H\cdot n=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{3.4}$$

For setting an invariant set for iteration scheme we define "k-Cauchy data $\mathring{\partial}_t^k V_0$" for the Cauchy problem (3.2) and (3.3) as follows: Set $\mathring{\partial}_t^0 V_0=V_0$ and determine $\mathring{\partial}_t^k V_0$ successively by

$$\mathring{\partial}_t^k V_0=-\left\{\sum_{i=0}^{k-1}\sum_{j=0}^{3}\binom{k-1}{i}G_{i,j}(\mathring{\partial}_t^0 V_0,\dots,\mathring{\partial}_t^i V_0)\partial_j\mathring{\partial}_t^{k-i-1}V_0\right\}, \quad \text{for} \quad k=1,2,\dots,$$

where

$$G_{i,j}(\mathring{\partial}_t^0 V_0,\dots,\mathring{\partial}_t^i V_0)=\partial_t^i(A_0^{-1}A_j)(V)||(V,\dots,\partial_t^i V)=(\mathring{\partial}_t^0 V_0,\dots,\mathring{\partial}_t^i V_0).$$

We also define k-Cauchy data $\mathring{\partial}_t^k u_0$ and $\mathring{\partial}_t^k H_0$ as counterparts of (3.5).

Now we set an invariant subset for iteration scheme. Let κ, M_{m-1}, and M_m be positive constants and let $X_T^m(\Omega;\kappa,M_{m-1},M_m)$ be a set of functions $V'={}^t(q'-c,u',H',S')$ satisfying the following conditions:

$$\begin{gathered} V'\in X_T^m(\Omega), \qquad \|V'\|_{m-1,T}\leqq M_{m-1}, \qquad \|V'\|_{m,T}\leqq M_m, \\ q'-(1/2)|H'|^2\geqq\kappa \quad \text{in} \quad [0,T]\times\Omega, \\ u'\cdot n=H'\cdot n=0 \quad \text{on} \quad [0,T]\times\Gamma, \\ \partial_t^k V'(0)=\mathring{\partial}_t^k V_0 \quad \text{in } \Omega, \text{ for } 0\leqq k\leqq m-1. \end{gathered} \tag{3.6}$$

According to (2.10), the initial data V_0 satisfy that

$$\begin{gathered} \nabla\cdot H_0=0, \qquad q_0-(1/2)|H_0|^2>0 \quad \text{in } \Omega, \\ H_0\cdot n=0 \quad \text{on } \Gamma. \end{gathered} \tag{3.7}$$

We further suppose that V_0 satisfy the additional conditions

$$V_0\in H^{m+2}(\Omega), \qquad \mathring{\partial}_t^k u_0\cdot n=0, \quad \text{for} \quad k=0,\dots,m, \quad \text{on } \Gamma. \tag{3.8}$$

By Remark 2.5(c), we know that the compatibility conditions

$$\mathring{\partial}_t^k H_0\cdot n=0, \quad \text{for} \quad k=0,\dots,m, \quad \text{on } \Gamma \tag{3.9}$$

associated to the boundary condition $H\cdot n=0$ are automatically satisfied.

Let V' be a given function belonging to $X_T^m(\Omega;\kappa,M_{m-1},M_m)$. Then the linearized problem we study is

$$A_0(V')\partial_t V+\sum_{j=1}^{3}A_j(V')\partial_j V+B(V',V)=0 \quad \text{in} \quad [0,T]\times\Omega, \tag{3.10}$$

$$V|_{t=0}=V_0 \quad \text{in } \Omega, \tag{3.11}$$

$$u\cdot n=0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{3.12}$$

Here $B(V', V)$ is an 8 dimensional vector valued two form of V' and V defined by

$$B(V', V) = {}^t(0, 0, 0, 0, l(V', V), 0), \tag{3.13}$$

where $l(V', V) = \mathfrak{n}\{(u' \cdot \nabla \mathfrak{n}) \cdot H - (H' \cdot \nabla \mathfrak{n}) \cdot u\}$ with $\mathfrak{n} \in \mathscr{B}^\infty(\bar{\Omega})$ such that $\mathfrak{n}|_\Gamma = n$. We remark that $B(V', V) = -B(V, V')$. Hence it is easy to know that k-Cauchy data for (3.10) and (3.11) take the same value as $\mathring{\partial}_t^k V_0$ defined by (3.5).

For this linearized problem we can get the following a priori estimate in accordance with $H_*^m(\Omega)$.

Proposition 3.1 (A priori estimate). *Let $m \geqq 6$ be an integer and let*

$$V' \in X_T^m(\Omega; \kappa, M_{m-1}, M_m).$$

Then a solution $V \in \bigcap_{j=0}^{m+1} C^j(0, T; H^{m+1-j}(\Omega))$ of the problem (3.10) *and* (3.12) *satisfies*

$$\|V(t)\|_m \leqq C(M_{m-1}, \kappa)\, \|V(0)\|_m \exp(C(M_m)t) \quad \text{for} \quad 0 \leqq t \leqq T. \tag{3.14}$$

Here and hereafter $C(A, B, \ldots)$ denotes a positive constant depending smoothly on $A, B, \ldots,$ and m, Ω.

We postpone the proof of Proposition 3.1 till Sect. 4. On the basis of this a priori estimate, we get the main result in this section.

Theorem 3.2. *Let $m \geqq 8$ be an integer. Let $V' \in X_T^m(\Omega; \kappa, M_{m-1}, M_m)$. Then the problem* (3.10)–(3.12) *has a unique solution $V \in X_T^m(\Omega)$ with the estimate*

$$\|V\|_{m,T} \leqq C(M_{m-1}, \kappa) \sum_{k=0}^{m} \|\mathring{\partial}_t^k V_0\|_{m-k} \exp(C(M_m)t)$$
$$\text{for} \quad 0 \leqq t \leqq T. \tag{3.15}$$

To proceed the proof of this theorem, we need the following two lemmas. We first show

Lemma 3.3. *Let $V' = {}^t(q' - c, u', H', S')$ be a smooth given function satisfying the conditions $u' \cdot n = H' \cdot n = 0$ on $[0, T] \times \Gamma$. Then the null space of the boundary condition* (3.12) *is maximally nonnegative subspace of the boundary matrix $A_n(V')$. Here the boundary matrix $A_n(V')$ stands for $\sum_{j=1}^{3} A_j(V') n_j(x)$.*

Proof. Observe that the boundary matrix $A_n(V')$ takes the form

$$A_n(V') = \begin{pmatrix} \alpha' u' \cdot n & n & -\alpha H' u' \cdot n & 0 \\ {}^t n & \alpha' u' \cdot n I_3 & -H' \cdot n I_3 & 0 \\ -\alpha'' {}^t H' u' \cdot n & -H' \cdot n I_3 & (I_3 + \alpha' H' \otimes H') u' \cdot n & 0 \\ 0 & 0 & 0 & u' \cdot n \end{pmatrix}.$$

Since $u' \cdot n = H' \cdot n = 0$ on Γ, we can easily check the boundary space

$$\left\{ {}^t(v_1, \ldots, v_8) \in \mathbb{R}^8 \,\middle|\, \sum_{j=1}^{3} v_{j+1} n_j = 0 \right\}$$

is nonnegative subspace of $A_n(V')$. Further, the maximality is followed by the fact that eigenvalues of $A_n(V')$ consists of ± 1 (simple) and 0 (with multiplicity six).

Next we show

Lemma 3.4. *Let $V' \in X_T^m(\Omega; \kappa, M_{m-1}, M_m)$. Let $\delta, \varepsilon, \lambda, \varepsilon'$ be parameters such that $\delta > \varepsilon > \lambda > \varepsilon' > 0$. Then there exists a function $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ such that*

$$V_{\delta,\lambda}^{\varepsilon,\varepsilon'} \in \bigcap_{r=0}^{m+1} C^r(0, T; H^{m+1-r}(\Omega))$$

and

$$\left\| \begin{array}{l} \text{when } \varepsilon' \to 0,\ \lambda \to 0,\ \varepsilon \to 0,\ \delta \to 0 \text{ in this order,} \\ \quad \partial_t^k V_{\delta,\lambda}^{\varepsilon,\varepsilon'} \to \partial_t^k V' \text{ in } H_*^{m-k}(\Omega), \text{ for } 0 \leqq k \leqq m, \text{ for a.e. } t \in [0, T]. \end{array} \right.$$

Further, there exist positive constants $C_0 = C_0(|V_0|_{m+2})$ and c_0, independent of the parameters, such that

$$V_{\delta,\lambda}^{\varepsilon,\varepsilon'} \in X_T^m(\Omega; \kappa/2, M_{m-1} + C_0, M_m + C_0) \quad \textit{for} \quad 0 < \varepsilon' < \lambda < \varepsilon < \delta < c_0 .$$

Proof. We construct the function $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ by four steps as follows. (i) We first extend Cauchy data $(V_0, \ldots, \hat{\partial}_t^{m-1} V_0)$ to $[0, T] \times \Omega$. Since $V_0 \in H^{m+2}(\Omega)$ by assumption (3.8), we see that $\hat{\partial}_t^k V_0 \in H^{m+2-k}(\Omega)$ for $0 \leqq k \leqq m-1$. Hence, by virtue of Theorem 2.5.7 of [5], we can get an extension $\hat{V} = {}^t(\hat{q} - c, \hat{u}, \hat{H}, \hat{S})$ of the data $\hat{\partial}_t^k V_0$, for $0 \leqq k \leqq m-1$, such that $\hat{V} \in H^{m+2}([0, T] \times \Omega)$ $\left(\text{i.e. } \hat{V} \in \bigcap_{r=0}^{m+1} C^r(0, T; H^{m+1-r}(\Omega))\right)$ and $\partial_t^k \hat{V}(0) := \hat{\partial}_t^k V_0$ for $0 \leqq k \leqq m-1$, $:= 0$ for $k = m$ and $m+1$, in Ω, with the following estimate

$$|\hat{V}|_{m+2, [0, T] \times \Omega} \leqq C \sum_{k=0}^{m-1} |\hat{\partial}_t^k V_0|_{m+2-k, \Omega} . \tag{3.16}$$

Next we define an operator $(1 - \Delta)_d^{-1} f$, where f is a given function on Γ, by a bounded solution of the Dirichlet boundary value problem

$$(1 - \Delta) u = 0 \quad \text{in } \Omega ,$$
$$u = f \quad \text{on } \Gamma .$$

Let us put

$$\langle \hat{V} \rangle = \hat{V} - (1 - \Delta)_d^{-1} g(\hat{V}) , \tag{3.17}$$

where the function $g(\hat{V})$ is defined by

$$(0, (\gamma_\Gamma \hat{u} \cdot n) n, (\gamma_\Gamma \hat{H} \cdot n) n, 0)$$

and γ_Γ is the trace operator on Γ.

Obviously, this $\langle \hat{V} \rangle = {}^t(q - c, u, H, S)$ satisfies that $u \cdot n = H \cdot n = 0$ on $[0, T] \times \Gamma$. Further, by (3.8) and (3.9) we see that $\partial_t^k \langle \hat{V} \rangle (0) = \partial_t^k \hat{V}(0)$, for $0 \leqq k \leqq m-1$, in Ω. From (3.16) we also see that there exists a positive constant $C_1 = C_1(|V_0|_{m+2})$ such that $\| \langle \hat{V} \rangle \|_{m, T} \leqq C_1$.

(ii) For $\delta > 0$, let us define the function V_δ by

$$V_\delta = \langle \hat{V} \rangle - T_\delta \circ (\langle \hat{V} \rangle - V') , \tag{3.18}$$

where $T_\delta \circ f(t, x) = f(t - \delta, x)$ for $t \geqq \delta$, $= 0$ for $\delta > t \geqq 0$. Then there exists a constant $c_1 > 0$ such that V_δ belongs to

$$X_T^m(\Omega; 3\kappa/4, M_{m-1} + C_2, M_m + C_2) \quad \text{for} \quad \delta < c_1 ,$$

where $C_2=2C_1+1$. Further, we can see that when $\delta\to 0$

$$V_\delta \to V' \quad \text{in} \quad X_T^m(\Omega). \tag{3.19}$$

(iii) We take the mollifier of the second part of (3.18) with respect to time variable and spatial variable tangential to the boundary. Fix δ such that $c_1>\delta>0$. For $\varepsilon<\delta$, let us define the function V_δ^ε by

$$V_\delta^\varepsilon=\langle\hat{V}\rangle+J_{\varepsilon(t,x')}(V_\delta-\langle\hat{V}\rangle).$$

To make the meaning of $J_{\varepsilon(t,x')}$ clear, we carry over localization and flattening of the boundary by using the covering near the boundary $\{\mathcal{O}_i\}_{i=1}^{\ell}$, the diffeomorphisms γ_i, and the partition of unity φ_i appeared in the definition of (2.9). By this procedure, we may suppose that $V_\delta-\langle\hat{V}\rangle$ has support in $\mathscr{B}_+$ for each $t\in[0,T]$. Then the meaning of $J_{\varepsilon(t,x')}$ is a (Friedrichs') mollifier with respect to t and (x_2,x_3). As for the function $V_\delta-\langle\hat{V}\rangle$ localized on a covering far from the boundary, we take the mollifier with respect to t and x. Of course the integrand $V_\delta-\langle\hat{V}\rangle$ is naturally extended to $(-\delta,T+\delta)\times\Omega$. It is easily seen that there exists a constant $c_2(<c_1)$ such that V_δ^ε belongs to

$$X_T^m(\Omega;2\kappa/3,M_{m-1}+2C_2,M_m+2C_2) \quad \text{for} \quad \varepsilon<c_2.$$

Further, we see that when $\varepsilon\to 0$

$$\partial_t^k V_\delta^\varepsilon\to\partial_t^k V_\delta \text{ in } H_*^{m-k}(\Omega), \text{ for } 0\leqq k\leqq m, \text{ for a.e. } t\in[0,T]. \tag{3.20}$$

(iv) Lastly, we take the mollifier of the second part of V_δ^ε with respect to spatial variables normal to the boundary. Fix δ and ε such that $c_2>\delta>\varepsilon>0$. Let us define $\tilde{V}_\delta^\varepsilon$ by

$$\tilde{V}_\delta^\varepsilon=J_{\varepsilon(t,x')}(V_\delta-\langle\hat{V}\rangle).$$

Then we apply the same localization and flattening of the boundary as in the last step and suppose that the function $\tilde{V}_\delta^\varepsilon$ localized near the boundary has support in $\mathscr{B}_+$. For $\lambda>0$ and fixed $t\in[0,T]$, let us define the function $\tilde{V}_{\delta,\lambda}^\varepsilon$ by

$$\tilde{V}_{\delta,\lambda}^\varepsilon(t,x)=\tau_\lambda\circ\tilde{V}_\delta^\varepsilon(t,x)\equiv\tilde{V}_\delta^\varepsilon(t,x_1+\lambda,x').$$

Since $\partial_{x'}^{\alpha'}\tilde{V}_\delta^\varepsilon\in C^\infty(0,T;H_*^m(\mathbb{R}_{+,\lambda}^3))$ for each $\alpha'=(\alpha_2,\alpha_3)\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}$, where

$$\mathbb{R}_{+,\lambda}^3=\{x\,|\,x_1>-\lambda\}, \qquad \partial_{x'}^{\alpha'}=\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, \quad \text{and} \quad \dot{\mathbb{N}}=\mathbb{N}\cup\{0\},$$

we can see that

$$\partial_{x'}^{\alpha'}\tilde{V}_{\delta,\lambda}^\varepsilon(t,\cdot)\in C^\infty(0,T;H^m(\mathbb{R}_+^3)).$$

Next let us take a (Friedrichs') mollifier of $\tilde{V}_{\delta,\lambda}^\varepsilon$ with respect to x_1. For ε' such that $0<\varepsilon'<\lambda$, let us define $\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ by

$$\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}=J_{\varepsilon'(\overset{*}{x}_1)}\tilde{V}_{\delta,\lambda}^\varepsilon.$$

Clearly, the function $\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ belongs to $C^\infty([0,T]\times\mathbb{R}_+^3)$ and when $\varepsilon'\to 0$

$$\partial_{x'}^{\alpha'}\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}\to\partial_{x'}^{\alpha'}\tilde{V}_{\delta,\lambda}^\varepsilon \quad \text{in} \quad C^l(0,T;H^m(\mathbb{R}_+^3)),$$
$$\text{for each } l\in\dot{\mathbb{N}},\ \alpha'\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}. \tag{3.21}$$

Further, we will prove that when $\lambda\to 0$

$$\partial_{x'}^{\alpha'}\tilde{V}_{\delta,\lambda}^\varepsilon\to\partial_{x'}^{\alpha'}\tilde{V}_\delta^\varepsilon \quad \text{in} \quad C^l(0,T;H_*^m(\mathbb{R}_+^3)),$$
$$\text{for each } l\in\dot{\mathbb{N}},\ \alpha'\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}. \tag{3.22}$$

Let us fix α_1, α', s, and l such that $\alpha_1+2s\leqq m$, $l\in\dot{\mathbb{N}}$, $\alpha'\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}$ and denote

$$\partial_1^{\alpha_1+s}\partial_{x'}^{\alpha'}\partial_t^l\tilde{V}_\delta^\varepsilon \quad \text{by } \tilde{V}.$$

Then the same argument as on pp. 43–46 in [8] shows that when $\lambda\to0$

$$\sigma(x_1)^{\alpha_1}\tau_\lambda\circ\tilde{V}\to\sigma(x_1)^{\alpha_1}\tilde{V} \quad \text{in} \quad C^0(0,T;L^2(\mathbb{R}_+^3)). \tag{3.23}$$

Since $\partial_*^\alpha\partial_1^s(\partial_t^l V_\delta^\varepsilon)=\sigma(x_1)^{\alpha_1}\tilde{V}+\text{l.o.t.}$, where $\alpha=(\alpha_1,\alpha')$, we can easily see (3.22) from (3.23).

Finally, we compensate the boundary value of $\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}$. Let us define the function $\langle\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}\rangle$ by

$$\langle\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}\rangle=\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}-(1-\Delta)_d^{-1}g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}), \tag{3.24}$$

where g is the same function as in (3.17). Note that $\partial_t^l\partial_{x'}^{\alpha'}g(\tilde{V}_\delta^\varepsilon)=0$, for each $l\in\dot{\mathbb{N}}$, $\alpha'\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}$, on $[0,T]\times\Gamma$. Hence, by (3.21) and (3.22), it is easily shown that when $\varepsilon'\to0$ and $\lambda\to0$ in this order,

$$\begin{aligned}&\partial_{x'}^{\alpha'}(1-\Delta)_d^{-1}g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'})\to0 \quad \text{in} \quad C^l(0,T;H^{[m/2]}(\mathbb{R}_+^3)),\\ &\text{for each } l\in\dot{\mathbb{N}},\ \alpha'\in\dot{\mathbb{N}}\times\dot{\mathbb{N}}.\end{aligned} \tag{3.25}$$

Further, we can show that when $\varepsilon'\to0$ and $\lambda\to0$ in this order

$$\begin{aligned}&\sigma(x_1)^s\partial_1^s(1-\Delta)_d^{-1}g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'})\to0 \quad \text{in} \quad C^l(0,T;H^{[m/2]}(\mathbb{R}_+^3)),\\ &\text{for each } l\in\dot{\mathbb{N}},\ s\in\dot{\mathbb{N}}.\end{aligned} \tag{3.26}$$

This is shown by the following manner. First, denote the function $(1-\Delta)_d^{-1}g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'})$ by G and fix $t\in[0,T]$. By the definition of $(1-\Delta)_d^{-1}$ we see that

$$\begin{aligned}&(1-\Delta)G=0 \quad \text{in } \mathbb{R}_+^3,\\ &G=g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}) \quad \text{on } \partial\mathbb{R}_+^3.\end{aligned}$$

Operating $\sigma(x_1)^s\partial_1^{s-1}$ to the above equations in $\mathbb{R}_+^3$ and taking inner product in $L^2(\mathbb{R}_+^3)$ with $\sigma(x_1)^s\partial_1^{s-1}G$, we find that by integration by parts

$$\begin{aligned}&|\sigma(x_1)^s\partial_1^{s-1}G|_0^2+|\sigma(x_1)^s\partial_1^sG|_0^2+|\sigma(x_1)^s\partial_1^{s-1}\nabla'G|_0^2\\ &\quad=-2s(\sigma(x_1)^{2s-1}\sigma'(x_1)\partial_1^sG,\partial_1^{s-1}G)_0,\end{aligned}$$

where $\nabla'=(\partial_2,\partial_3)$. Accordingly, by Young's inequality we get

$$|\sigma(x_1)^s\partial_1^sG|_0^2\leqq C|\sigma(x_1)^{s-1}\partial_1^{s-1}G|_0^2.$$

By using this estimate repeatedly we get

$$|\sigma(x_1)^s\partial_1^sG|_0^2\leqq C|G|_0^2.$$

Hence, we can see (3.26) easily by virtue of the standard estimates to $(1-\Delta)_d^{-1}$. So combining (3.26) with (3.25), we find that when $\varepsilon'\to0$ and $\delta\to0$ in this order

$$(1-\Delta)_d^{-1}g(\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'})\to0 \quad \text{in } C^l(0,T;H_*^m(\Omega)), \text{ for each } l\in\dot{\mathbb{N}}. \tag{3.27}$$

Now let us define $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ by

$$V_{\delta,\lambda}^{\varepsilon,\varepsilon'}=\langle\hat{V}\rangle+\langle\tilde{V}_{\delta,\lambda}^{\varepsilon,\varepsilon'}\rangle.$$

By (3.21), (3.22), and (3.27), we can show that when $\varepsilon'\to0$, $\lambda\to0$ in this order,

$$V_{\delta,\lambda}^{\varepsilon,\varepsilon'}\to V_\delta^\varepsilon \quad \text{in } C^l(0,T;H_*^m(\Omega)), \text{ for each } l\in\dot{\mathbb{N}}. \tag{3.28}$$

Further, we find easily that there exists a constant $c_0(<c_2)$ such that, for $c_0>\delta>\varepsilon>\lambda>\varepsilon'$, $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ belongs to

$$X_T^m(\Omega;\kappa/2, M_{m-1}+3C_2, M_m+3C_2)$$

and to $\bigcap_{j=0}^{m+1} C^j(0,T;H^{m+1-j}(\Omega))$. Hence, putting $C_0=3C_2$, we can show from (3.19), (3.20), and (3.28) that this $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ is a desired function. We complete the proof of Lemma 3.4.

We now give

Proof of Theorem 3.2. By virtue of Proposition 3.2, we first regularize V', the coefficients of the differential operator in (3.10), by $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$. Let us fix δ, ε, λ, and ε'. Denote the differential operator with the regularized coefficients $V_{\delta,\lambda}^{\varepsilon,\varepsilon'}$ by $\tilde{L}$:

$$\tilde{L}=A_0(V_{\delta,\lambda}^{\varepsilon,\varepsilon'})\partial_t+\sum_{j=1}^{3} A_j(V_{\delta,\lambda}^{\varepsilon,\varepsilon'})\partial_j+B(V_{\delta,\lambda}^{\varepsilon,\varepsilon'}),$$

where $B(V_{\delta,\lambda}^{\varepsilon,\varepsilon'})$ is the 8×8 matrix such that $B(V_{\delta,\lambda}^{\varepsilon,\varepsilon'})V=B(V_{\delta,\lambda}^{\varepsilon,\varepsilon'}, V)$. Next we apply *noncharacteristic regularization* to L. Let $\mathfrak{n}$ be one defined in (3.13). Let us define for a parameter $1\gg\gamma>0$,

$$\tilde{L}^\gamma=\tilde{L}+\gamma(\mathfrak{n}\cdot\nabla).$$

We find that for $\tilde{L}^\gamma$ with γ small, Γ is noncharacteristic and the boundary condition (3.12) is still maximality nonnegative. Now we consider the initial boundary value problems $\tilde{L}^\gamma V^\gamma=\gamma(\mathfrak{n}\cdot\nabla)\hat{V}$ in $[0,T]\times\Omega$ with (3.11) and (3.12). Here $\hat{V}$ is the function constructed in part (i) of the proof of Lemma 3.4. We remark that k-Cauchy data for these problems also satisfy the compatibility conditions of second part of (3.8) and (3.9). Hence, by virtue of the existence and regularity theorem for linear symmetric hyperbolic systems with noncharacteristic boundary (we refer to Theorem A.1 in [13]), we obtain a unique solution $V^\gamma\in\bigcap_{j=0}^{m+1} C^j(0,T;H^{m+1-j}(\Omega))$ for these problems. Further, it is easy to see that when $\gamma\to 0$, V^γ converges to a function $\tilde{V}$ in $C^0(0,T;L^2(\Omega))$, which depends on the parameters δ, ε, λ, and ε'. In addition, by retracing the derivation of the a priori estimate (3.14) of Proposition 3.1, we find the following estimate holds:

$$\begin{aligned}\|V^\gamma(t)\|_m\leqq &\Big(C_1(\|V_{\delta,\lambda}^{\varepsilon,\varepsilon'}\|_{m-1,T},\kappa)\sum_{k=0}^{m}\|\partial_t^k V_0\|_{m-k}\\ &+\gamma(1+T)C_2(\|V_{\delta,\lambda}^{\varepsilon,\varepsilon'}\|_{m,T})\,\|(\mathfrak{n}\cdot\nabla)\hat{V}\|_{m,T}\Big)\\ &\times\exp\int_0^t C_3(\|V_{\delta,\lambda}^{\varepsilon,\varepsilon'}(\tau)\|_m)d\tau,\quad\text{for}\quad 0\leqq t\leqq T.\end{aligned}\tag{3.29}$$

Hence, referring to the inequality $\|(\mathfrak{n}\cdot\nabla)\hat{V}\|_{m,T}\leqq c|\hat{V}|_{m+2,[0,T]\times\Omega}$, we know by (3.16) that the norm $\|V^\gamma\|_{m,T}$ is bounded uniformly to γ. Accordingly, by the standard argument (see, [11, 13]), we see that $\tilde{V}$ belongs to $X_T^m(\Omega)$ and is a solution of the problem such that $\tilde{L}\tilde{V}=0$ in $[0,T]\times\Omega$ with (3.11) and (3.12). We also find that the solution $\tilde{V}$ satisfies the estimate of (3.29) in which V^γ is replaced by $\tilde{V}$ and the term

$$\gamma(1+T)C_2(\|V_{\delta,\lambda}^{\varepsilon,\varepsilon'}\|_{m,T})\,\|(\mathfrak{n}\cdot\nabla)\hat{V}\|_{m,T}$$

is omitted. Next, by virtue of Proposition 3.4, we find by (3.29) that $\|\tilde{V}\|_{m,T}$ is bounded uniformly to δ, ε, λ, and ε', and we further find that $\tilde{V}$ converges to a function V in $C^0(0, T; L^2(\Omega))$ when $\varepsilon' \to 0$, $\lambda \to 0$, $\varepsilon \to 0$, and $\delta \to 0$ in this order. Hence, it is easy to show that this V is a desired solution, which belongs to $X_T^m(\Omega)$, of the problem (3.10), (3.11), and (3.12). By Lebesgue's dominated convergence theorem, we also see that the solution V satisfies the estimate (3.15). Uniqueness readily follows from L^2 estimate of (3.10) with (3.12). We complete the proof of Theorem 3.2.

4. A Priori Estimates

First we recall some inequalities of Sobolev and Hardy type.

Lemma 4.1. (i) *Let Ω be an arbitrary domain in $\mathbb{R}^3$ having the cone property. Then*

$$\begin{aligned} &\text{(a)} \quad |f|_{C^\sigma(\bar{\Omega})} \leqq C|f|_{2,\Omega}, \quad 0 \leqq \sigma < 1/2, \quad \text{for } f \in H^2(\Omega), \\ &\text{(b)} \quad |f|_{L^4(\Omega)} \leqq C|f|_{1,\Omega}, \quad \text{for } f \in H^1(\Omega) \end{aligned} \tag{4.1}$$

hold.
(ii) *For $f \in H^1(\mathbb{R}^1_+)$ such that $f(0)=0$, the inequality*

$$\int_0^\infty |f(x)/x|^2 dx \leqq 4 \int_0^\infty |f'(x)|^2 dx \tag{4.2}$$

holds.

Proof. For example, see [1, 4].

As a direct consequence of Lemma 4.1 (i) we get the following estimation of a product of functions.

Lemma 4.2. *Under the same conditions as of Lemma 4.1* (i), *we get*

$$\begin{aligned} &|fg|_{0,\Omega} \leqq C|f|_{2,\Omega}|g|_{0,\Omega}, \quad \text{for } f \in H^2(\Omega), \ g \in L^2(\Omega), \\ &|fg|_{0,\Omega} \leqq C|f|_{1,\Omega}|g|_{1,\Omega}, \quad \text{for } f \in H^1(\Omega), \ g \in H^1(\Omega). \end{aligned} \tag{4.3}$$

Let Ω be a bounded or unbounded domain in $\mathbb{R}^3$. We first note the invariance of the principal part of Eqs. (3.10) and the boundary condition (3.12) for $0(3)$ (the orthogonal group of order 3). Then, by applying localization and flattening of the boundary of the problems (3.10)–(3.12) localized near the boundary, we can reduce them to the problems for a half space $\mathbb{R}^3_+$. Although in the process of localization we must add lower order terms of V and V' to (3.10), they give no essential change in deriving a priori estimate (3.14). So we neglect them.

Hence, we begin to treat the problem (3.10)–(3.12) for $\Omega = \mathbb{R}^3_+ = \{x \mid x_1 > 0\}$ and suppose that V has support in $\mathscr{B}_+$. Note that $B(V', V) = 0$ in this case. For convenience, we write

$$A_i(V) = \begin{pmatrix} P_i(V) & Q_i(V) \\ {}^tQ_i(V) & R_i(V) \end{pmatrix}, \quad \text{for} \quad i = 0, 1, \ldots, 3, \tag{4.4}$$

where $P_i(V)$, $Q_i(V)$, and $R_i(V)$ are 2×2, 2×6, and 6×6 submatrices, respectively. We write also $v = {}^t(q-c, u^1)$, $w = {}^t(u^2, u^3, H^1, H^2, H^3, S)$. Hence $V = {}^t(v, w)$. Notice that

$$P_1(V)|_{x_1=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_1(V)|_{x_1=0} = 0, \quad R_1(V)|_{x_1=0} = 0. \tag{4.5}$$

if $u^1|_{x_1=0}=H^1|_{x_1=0}=0$. We find that the boundary matrix has constant rank 2 on Γ.

Now we begin to obtain the estimate of tangential derivatives of the solution V of (3.10)–(3.12).

Lemma 4.3. *We have*

$$\|V(t)\|_{m,\tan}\leqq C(\kappa,M_{m-1})\,\|V(0)\|_{m,\tan}+\int_0^t (C(\|V'(\tau)\|_m)\times(|[v(\tau)]|_m+\|V(\tau)\|_m)d\tau,\quad \text{for}\quad 0\leqq t\leqq T, \tag{4.6}$$

$$\|V(t)\|_{m-1}\leqq C(\kappa,M_{m-1})\,\|V(0)\|_{m-1}+\int_0^t C(\|V'(\tau)\|_{m-1})\times(|[v(\tau)]|_m+\|V(\tau)\|_m)d\tau,\quad \text{for}\quad 0\leqq t\leqq T, \tag{4.7}$$

where $\|V(t)\|^2_{m,\tan}=\sum_{|\alpha|\leqq m}|\partial^\alpha_{\tan}V(t)|^2_0$ *with*

$$\partial^\alpha_{\tan}=\partial_t^{\alpha_0}(\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_1}\partial_3^{\alpha_3}\quad \text{for}\quad \alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$$

and $|[v(\tau)]|^2_m=\sum_{k=1}^{[m/2]}\sum_{|\alpha|\leqq m-2k+1}\partial^\alpha_{\tan}\partial_1^k v(\tau)|^2_0.$

Proof. For α such that $|\alpha|\leqq m$, differentiate Eqs. (3.10) by $\partial^\alpha_{\tan}$ and take inner product with $\partial^\alpha_{\tan}V$ in $L^2(\mathbb{R}^3_+)$. Set $V^\alpha=\partial^\alpha_{\tan}V$. Then integration by parts gives

$$\partial_t(A_0(V')V^\alpha,V^\alpha)_0=(\operatorname{div}\vec{A}(V')V^\alpha,V^\alpha)_0-\int_{\partial\mathbb{R}^3_+}{}^tV^\alpha A_1(V')V^\alpha d\Gamma-(G^\alpha,V^\alpha)_0, \tag{4.8}$$

where

$$\operatorname{div}\vec{A}(V')=\partial_tA_0(V')+\sum_{j=1}^3\partial_jA_j(V')$$

and

$$G^\alpha=[\partial^\alpha_{\tan},A_0(V')]\partial_tV+\sum_{j=1}^3[\partial^\alpha_{\tan},A_j(V')]\partial_jV+A_1(V')[\partial^\alpha_{\tan},\partial_1]V.$$

Since the tangential derivatives $\partial^\alpha_{\tan}V$ of the solution (3.10)–(3.12) satisfy the same boundary condition as in (3.12), we can get by (4.1)(a),

$$|V^\alpha(t)|^2_{0,A_0(t)}\leqq|V^\alpha(0)|^2_{0,A_0(0)}+\int_0^t(C(|V'(\tau)|_3)|V^\alpha(\tau)|^2_0+|G^\alpha(\tau)|^2_0)d\tau,\quad \text{for}\quad 0\leqq t\leqq T, \tag{4.9}$$

where $|V^\alpha(t)|^2_{0,A_0(t)}=(A_0(V'(t))V^\alpha(t),\ V^\alpha(t))_0$.

Since $q'-(1/2)|H'|^2>\kappa$ and (4.1)(a), we find that there exist a positive constant $c(\kappa,M_{m-1})$ such that $c(\kappa,M_{m-1})^{-1}\leqq A_0(V')\leqq c(\kappa,M_{m-1})$. Hence, if the following estimate is shown

$$|G^\alpha(\tau)|_0\leqq C(\|V'(\tau)\|_m)\Big(\sum_{|\beta|=m-1}|\partial^\beta_{\tan}\partial_1v(\tau)|_0+\|V(\tau)\|_m\Big), \tag{4.10}$$

we obtain the desired estimate (4.6) by plugging (4.10) in (4.9) and summing them over all α with $|\alpha|\leqq m$. In deriving (4.10), the crucial terms to be estimated are

$[\partial_{\tan}^{\alpha}, A_1(V')]\partial_1 V$, for $|\alpha| = m$, which contains the terms such as $\partial_{\tan}^{l} Q_1(V')\partial_{\tan}^{\alpha-l}\partial_1 w$ and $\partial_{\tan}^{l} R_1(V')\partial_{\tan}^{\alpha-l}\partial_1 w$ with $|l| = 1$. Recall $Q_1(V') = 0$, $R_1(V') = 0$ on $[0, T] \times \partial\mathbb{R}_+^3$. We deal with these terms by regarding $\partial_{\tan}^{l} Q_1(V')\partial_{\tan}^{\alpha-l}\partial_1$ and $\partial_{\tan}^{l} R_1(V')\partial_{\tan}^{\alpha-l}\partial_1$ as vector fields tangential to $\partial\mathbb{R}_+^3$. (Refer to Rauch [11] on this technique.) For instance, we have

$$\partial_{\tan}^{l} Q_1(V')\partial_{\tan}^{\alpha-l}\partial_1 = H(V')x_1\partial_{\tan}^{\alpha-l}\partial_1,$$

where

$$H(V') = \int_0^1 \partial_1\partial_{\tan}^{l} Q_1(V')|_{(t,\theta x_1, x')} d\theta .$$

Accordingly, we obtain by (4.1)(a)

$$\begin{aligned} |\partial_{\tan}^{l} Q_1(V')\partial_{\tan}^{\alpha-l}\partial_1 w|_0 &\leqq |H(V')|_{L^\infty(\Omega)}|x_1\partial_{\tan}^{\alpha-l}\partial_1 w|_0 \\ &\leqq C|H(V')|_2|\sigma(x_1)\partial_{\tan}^{\alpha-l}\partial_1 w|_0 \\ &\leqq C|H(V')|_2\|V\|_{m,\tan} . \end{aligned} \tag{4.11}$$

Further, we can get by a change of variable and (4.2)

$$\begin{aligned} |H(V')|_0^2 &= \int_0^\infty \int_{\mathbb{R}^2} \left(\int_0^1 \partial_1\partial_{\tan}^{l} Q_1(V')|_{(t,\theta x_1, x')} d\theta \right)^2 dx' dx_1 \\ &= \int_{\mathbb{R}^2} \left(\int_0^\infty \left(\frac{1}{x_1} \int_0^{x_1} \partial_1\partial_{\tan}^{l} Q_1(V')|_{(t,\theta,x')} d\theta \right)^2 dx_1 \right) dx' \\ &\leqq 4|\partial_1\partial_{\tan}^{l} Q_1(V')|_0^2 . \end{aligned}$$

Hence, we get higher order estimates similarly

$$|H(V')|_2 \leqq C|\partial_1\partial_{\tan}^{l} Q_1(V')|_2 \leqq C(|V'|_4) \leqq C(\|V'\|_m), \tag{4.12}$$

by recalling that $H^{[m/2]}(\Omega) \supset H_*^m(\Omega)$ and $m \geqq 8$. This is the reason why we take $m \geqq 8$. Accordingly, by (4.11) and (4.12) we can evaluate the crucial terms, and by applying (4.3) to the other terms repeatedly we can get the estimate (4.10). The estimate (4.7) is shown by a direct calculation. Now the proof of Lemma 4.3 is completed.

Next we shall obtain the estimates of normal derivatives of v. To this aim, we first note by (4.4) that $P_1(V')$ is invertible on $[0, T] \times \{x | x_1 = 0\}$. Hence, we can take a small constant δ_0 such that there exists a positive constant $c = c(\delta_0)$ satisfying

$$|\det P_1(V')| \geqq c(\delta_0) \quad \text{for} \quad (t, x) \in [0, T] \times \{x | 0 \leqq x_1 \leqq \delta_0\} . \tag{4.13}$$

By (4.1)(a) and $m \geqq 8$, we see that this δ_0 depends only on M_{m-1}. Next, define a smooth cut off function $\chi(x)$ supported in $\{x | 0 \leqq x_1 \leqq \delta_0\}$ and equal to one in $\{x | 0 \leqq x_1 \leqq \delta_0/2\}$, which satisfies the estimate

$$\sup_{x \in [0,\delta_0]} |(\partial/\partial x)^\alpha \chi(x)| \leqq C(\delta_0),$$

with $|\alpha| \leqq m$, for a constant $C(\delta_0) > 0$. By using this cut off function $\chi(x)$, we divide v into the form

$$v = \chi v + (1 - \chi)v = v_I + v_{II} .$$

For this v_I and v_{II}, we get the following estimates.

Lemma 4.4. *We have*

$$
\begin{aligned}
|[v_I(t)]|_m \leqq C(\delta_0, \|V'(t)\|_{m-1})(\|w(t)\|_m + \|V(t)\|_{m-1} \\
+ \|V(t)\|_{m,\tan} + \|v_{II}(t)\|_m), \quad \textit{for} \quad 0 \leqq t \leqq T,
\end{aligned} \tag{4.14}
$$

$$
\|v_{II}(t)\|_m \leqq \hat{C}(\kappa, \delta_0)\, \|V(0)\|_m + \int_0^t C(\|V'(\tau)\|_m)\, \|V(\tau)\|_m d\tau, \quad \textit{for} \quad 0 \leqq t \leqq T, \tag{4.15}
$$

where $\delta_0 = \delta_0(M_{m-1})$ *is the positive constant defined in* (4.13).

Proof. By the definition of $\chi(x)$, we can solve Eq. (3.10) with respect to $\partial_1 v$ and deduce the expression of $\partial_1 v$ such that

$$
\partial_1 v_I = -P_1^{-1}(V')F(V', \partial_{\tan} V, \partial_1 v_{II}), \tag{4.16}
$$

where

$$
\begin{aligned}
F(V', \partial_{\tan} V, \partial_1 v_{II}) = P_0(V')\partial_t v + Q_0(V')\partial_t w + \sum_{j=2}^{3} (P_j(V')\partial_j v + Q_j(V')\partial_j w) \\
+ P_1(V')\partial_1 v_{II} + Q_1(V')\partial_1 w\,.
\end{aligned}
$$

Note that we can regard $Q_1(V')\partial_1 w$ as a tangential derivative of w, so the right-hand side of (4.16) contains only first order tangential derivatives of V and $\partial_1 v_{II}$. Let us fix $t \in [0, T]$ and fix α and k such that $|\alpha| \leqq m - 2k + 1$ and $1 \leqq k \leqq [m/2]$. Then by using the expression (4.16) of $\partial_1 v_I$ repeatedly, and by applying (4.3), we obtain

$$
\begin{aligned}
&|\partial_{\tan}^{\alpha} \partial_1^{k-1}(P^{-1}(V')F(V', \partial_{\tan} V, \partial_1 v_{II})|_0 \\
&\leqq C(\delta_0, \|V'\|_{m-1})(\|w\|_m + \|V\|_{m-1} + \|V\|_{m,\tan} + \|v_{II}\|_m)\,.
\end{aligned} \tag{4.17}
$$

Hence, we get (4.14) by these estimates and (4.16).

As for v_{II}, we observe that

$$
A_0(V')\partial_t((1-\chi)V) + \sum_{j=1}^{3} A_j(V')\partial_j((1-\chi)V) + \sum_{j=1}^{3} A_j(V')(\partial_j \chi)V = 0\,.
$$

Then, by applying the same argument as in the Cauchy Problem to these equations with respect to $(1-\chi)V$, we readily obtain (4.15). We complete the proof of this lemma.

Lastly we have to show the estimates of normal derivatives of w.

Lemma 4.5. *We have*

$$
\begin{aligned}
\left(\sum_{k=1}^{[m/2]} \sum_{|\alpha| \leqq m-2k} |\partial_{\tan}^{\alpha} \partial_1^k w(t)|_0^2\right)^{1/2} \leqq C(\kappa, M_{m-1}) \left(\sum_{k=1}^{[m/2]} \sum_{|\alpha| \leqq m-2k} |\partial_{\tan}^{\alpha} \partial_1^k w(0)|_0^2\right)^{1/2} \\
+ \int_0^t C(\|V'(\tau)\|_m)(|[v(\tau)]|_m + \|V(\tau)\|_m) d\tau, \quad \textit{for} \quad 0 \leqq t \leqq T.
\end{aligned} \tag{4.18}
$$

Proof. We observe that w satisfies

$$
R_0(V')\partial_t w + \sum_{j=1}^{3} R_j(V')\partial_j w = -\left({}^t Q_0(V')\partial_t v + \sum_{j=1}^{3} {}^t Q_j(V')\partial_j v\right). \tag{4.19}
$$

For α and k such that $|\alpha| \leqq m - 2k$ and $1 \leqq k \leqq [m/2]$, differentiate the equations (4.19) by $\partial_{\tan}^{\alpha} \partial_1^k$ and take inner product with $\partial_{\tan}^{\alpha} \partial_1^k w$ in $L^2(\mathbb{R}_+^3)$. Set $w^{\alpha,k} = \partial_{\tan}^{\alpha} \partial_1^k w$.

Then in view of $R_1(V')=0$ on $[0,T]\times\partial\mathbb{R}^3_+$, we get by integration by parts

$$|w^{\alpha,k}(t)|^2_{0,R_0(t)}\leqq|w^{\alpha,k}(0)|_{0,R_0(0)}+\int_0^t C(|V'(\tau)|_3)\,|w^{\alpha,k}(\tau)|^2_0$$
$$+|G^{\alpha,k}(\tau)|^2_0)d\tau \quad \text{for} \quad 0\leqq t\leqq T, \tag{4.20}$$

where

$$|w^{\alpha,k}(t)|^2_{0,R_0(t)}=(R_0(V'(t))w^{\alpha,k}(t),w^{\alpha,k}(t))_0,$$

and

$$G^{\alpha,k}(t)=[\partial^\alpha_{\tan}\partial^k_1,R_0(V'(t))]\partial_t w(t)+\sum_{j=1}^3[\partial^\alpha_{\tan}\partial^k_1,R_j(V'(t))]\partial_j w(t)$$
$$+\partial^\alpha_{\tan}\partial^k_1({}^tQ_0(V'(t))\partial_t v(t)+\sum_{j=1}^3{}^tQ_j(V'(t))\partial_j v(t))$$
$$+R_1(V'(t))\,[\partial^\alpha_{\tan}\partial^k_1,\partial_1]w(t).$$

Since $R_1(V')=0$ and ${}^tQ_1(V')=0$ on $[0,T]\times\partial\mathbb{R}^3_+$, we can apply similar estimates of (4.11) and (4.12) to the counterpart of $|G^{\alpha,k}(t)|_0$. Hence, by the same argument as in deriving (4.10), we get

$$|G^{\alpha,k}(t)|_0\leqq C(\|V'(t)\|_m)(|[v(t)]|_m+\|V(t)\|_m).$$

Accordingly, since $R_0(V')$ is positive definite, we can obtain the estimate (4.18). We complete the proof of Lemma 4.5.

As for the problem localized on the region far from the boundary, we can get

$$\|\varphi_0V(t)\|_m\leqq\|\varphi_0V(0)\|_m+\int_0^t C(\|V'(\tau)\|_m)\,\|V(\tau)\|_m d\tau, \quad \text{for} \quad 0\leqq t\leqq T. \tag{4.21}$$

Hence, by combining (4.6), (4.7), (4.14), (4.15), and (4.18) for each localized problems, with (4.21), and by applying Gronwall's inequality, we obtain the estimate (3.14).

5. Proof of Theorem 2.3 and 2.4

We first show

Lemma 5.1. *Let $m\geqq 8$ be an integer. Let $\bar\kappa$ and $\bar M$ be positive constants such that $q_0-(1/2)|H_0|^2>\bar\kappa$ in Ω, and $|V_0|_m\leqq\bar M$. Then there exists a function $V^0=V^0(t,x)={}^t(q^0-c,u^0,H^0,S^0)$ and positive constants $\bar T=\bar T(\bar\kappa,\bar M)$, $\bar M_{m-1}=\bar M_{m-1}(\bar\kappa,\bar M)$, $\bar M_m=\bar M_m(\bar\kappa,\bar M)$ such that V^0 belongs to $X^m_{\bar T}(\Omega;\bar\kappa/2,\bar M_{m-1},\bar M_m)$.*

Proof. We construct the function V^0 as the solution of the following problem:

$$A_0(V_0)\partial_tV^0+\sum_{j=1}^3A_j(V_0)\partial_jV^0+B(V_0,V^0)=G \quad \text{in} \quad [0,T]\times\Omega,$$
$$V^0|_{t=0}=V_0 \quad \text{in } \Omega, \tag{5.1}$$
$$u^0\cdot n=0 \quad \text{on} \quad [0,T]\times\Gamma.$$

Here $G=G(t,x)$ is the function which belongs to $H^m([0,T]\times\Omega)$ and satisfies the following conditions:

$$N_H\cdot G(t,x)=0 \quad \text{on} \quad [0,T]\times\Gamma, \tag{5.2}$$

$$\begin{aligned}\partial_t^k G(0,x)= & -[\partial_t^k, A_0(V^0)]\partial_t V^0|_{t=0}-\sum_{j=1}^3[\partial_t^k, A_j(V^0)]\partial_j V^0|_{t=0}\\ & -[\partial_t^k, B_j(V^0)]V^0|_{t=0}, \quad \text{for} \quad 0\leqq k\leqq m-1, \text{ in } \Omega,\end{aligned} \tag{5.3}$$

$$|G|_{m,[0,T]\times\Omega}\leqq C(|V_0|_m). \tag{5.4}$$

Here $N_H={}^t(0,0,n,0)$ and $\partial_t^k V^0|_{t=0}=\mathring{\partial}_t^k V_0$, $k=0,\dots,m-1$, where $\mathring{\partial}_t^k V_0$ is defined by (3.5).

Once we get such a function $G(t,x)$, by virtue of the version of Theorem 3.2 we obtain the solution $V^0\in X_T^m(\Omega)$ of the problem (5.1). Since $V^0|_{t=0}=V_0$, it follows, by inductions, from the equalities (5.3) that this V^0 satisfies that

$$\begin{aligned}\partial_t^k V^0|_{t=0} &\equiv -A_0(V_0)^{-1}\left(\sum_{j=1}^3 A_j(V_0)\partial_j\partial_t^{k-1}V^0+B(V_0)\partial_t^{k-1}V^0-\partial_t^{k-1}G\right)\Bigg|_{t=0}\\ &=\mathring{\partial}_t^k V_0, \quad \text{for} \quad 0\leqq k\leqq m-1, \quad \text{in } \Omega.\end{aligned} \tag{5.5}$$

Next take inner product of N_H and the equations of (5.1) on $[0,T]\times\Gamma$ and retrace the same calculations as in deriving Eq. (2.7). Since $u^0\cdot n=0$ on $[0,T]\times\Gamma$, $H_0\cdot n=0$ on Γ and (5.2), we obtain

$$\partial_t(H^0\cdot n)+u_0\cdot\nabla(H^0\cdot n)=0 \quad \text{on} \quad [0,T]\times\Gamma.$$

We note that the lower order terms of V_0 and V^0 corresponding to $c(u,H,\nabla n)$ of (2.7) are canceled by the term $N_H\cdot B(V_0,V^0)$. It is easily seen that this equation with the assumption $H_0\cdot n=0$ on Γ yields

$$H^0\cdot n=0, \quad \text{on} \quad [0,T]\times\Gamma. \tag{5.6}$$

On the other hand, the estimate of V^0 such as (3.14) and (5.4) show that we can take positive constants $\bar T=\bar T(\bar\kappa,\bar M)$, $\bar M_{m-1}=\bar M_{m-1}(\bar\kappa,\bar M)$, $\bar M_m=\bar M_m(\bar\kappa,\bar M)$ such that

$$\begin{gathered} q^0-1/2|H^0|^2>\bar\kappa/2 \quad \text{in } \Omega,\\ \|V^0\|_{m-1,\bar T}\leqq\bar M_{m-1}, \qquad \|V^0\|_{m,\bar T}\leqq\bar M_m.\end{gathered} \tag{5.7}$$

By (5.5), (5.6), and (5.7), we see that this function V^0 is a desired one.

To complete the proof, we have only to construct $G\in H^m([0,T]\times\Omega)$ satisfying (5.2), (5.3), and (5.4). Since the right-hand side of (5.3) belongs to $H^{m-k}(\Omega)$, we can get a function

$$\tilde G(t,x)\in H^{m+1/2}([0,T]\times\Omega)$$

satisfying

$$\begin{aligned}&\partial_t^k\tilde G(0,x)=\text{the right-hand side of (5.3)},\\ &\qquad\qquad \text{for} \quad 0\leqq k\leqq m-1, \quad \text{in } \Omega,\end{aligned} \tag{5.8}$$

$$|\tilde G|_{m+1/2,[0,T]\times\Omega}\leqq C(|V_0|_m) \quad \text{(cf. Theorem 2.5.7 of [5])}. \tag{5.9}$$

Here we note that, from the assumptions (3.8) and (3.9),

$$N_H\cdot\gamma_\Gamma(\partial_t^k\tilde G(0,x))=0, \quad \text{for} \quad 0\leqq k\leqq m-1, \quad \text{on } \Gamma. \tag{5.10}$$

Next we take a trace of $\tilde{G}(t,x)$ on $[0,T]\times\Gamma$, $\gamma_{[0,T]\times\Gamma}(\tilde{G})$, which belongs to $H^m([0,T]\times\Gamma)$ and satsifies the estimate

$$|\gamma_{[0,T]\times\Gamma}(\tilde{G})|_{m,[0,T]\times\Gamma}\leqq c|\tilde{G}|_{m+1/2,[0,T]\times\Omega}. \tag{5.11}$$

Then we can easily construct an extension of $N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G})$ to $[0,T]\times\Omega$, $S(N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G}))$, which is independent of t and satisfies the following:

$$\begin{gathered} S(N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G}))\in H^m([0,T]\times\Omega), \\ |S(N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G}))|_{m,[0,T]\times\Omega}\leqq c|N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G})|_{m,[0,T]\times\Gamma}. \end{gathered} \tag{5.12}$$

Note that by (5.10) we obtain

$$\partial_t^k S(N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G}))=0, \quad \text{for} \quad 0\leqq k\leqq m-1, \quad \text{on} \quad \{t|t=0\}\times\Gamma. \tag{5.13}$$

Now let us define a function $G(t,x)$ by

$$G=\tilde{G}-\mathbb{N}_H S(N_H\cdot\gamma_{[0,T]\times\Gamma}(\tilde{G})),$$

where $\mathbb{N}_H={}^t(0,0,\mathfrak{n},0)$. By (5.8), (5.9), (5.11), (5.12), and (5.13), we find that this G is a desired function. So the proof is completed.

We next show

Lemma 5.2. *Let $m\geqq 8$ be an integer. Let $\bar{\kappa}$ and $\bar{M}$ be the same constants as defined in Lemma 5.1. Then there exist positive constants $\tilde{T}(\leqq\bar{T})$, $\tilde{M}_{m-1}(\geqq\bar{M}_{m-1})$, and $\tilde{M}_m(\geqq\bar{M}_m)$ depending only on $\bar{\kappa}$ and $\bar{M}$ such that, if V' belongs to $X^m_{\tilde{T}}(\Omega;\bar{\kappa}/3,\tilde{M}_{m-1},\tilde{M}_m)$, then the problem* (3.10), (3.11), *and* (3.12) *with additional conditions* (3.8) *has a unique solution $V\in X^m_{\tilde{T}}(\Omega;\bar{\kappa}/3,\tilde{M}_{m-1},\tilde{M}_m)$. Further, this V satisfies the estimate* (3.14) *in which T,κ,M_{m-1},M_m are replaced by $\tilde{T},\bar{\kappa},\tilde{M}_{m-1},\tilde{M}_m$, respectively.*

Proof. Let V' be a given function belonging to $X^m_{\tilde{T}}(\Omega;\bar{\kappa}/3,\tilde{M}_{m-1},\tilde{M}_m)$ for some positive constants $\tilde{M}_{m-1}$, $\tilde{M}_m$, and $\tilde{T}$. Existence of the solution, V, of the problem (3.10)–(3.12) with the conditions (3.8), which satisfies the estimate corresponding to (3.14), is proven in Theorem 3.2. Hence, combined with mean value theorem, we see that this V satisfies

$$\|V(t)\|_m\leqq C_1(\bar{M},\bar{\kappa},\tilde{M}_{m-1})\exp(C_2(\tilde{M}_m)t), \quad \text{for} \quad 0\leqq t\leqq\tilde{T}.$$

and

$$\|V(t)\|_{m-1}\leqq C_3(\bar{M})+t\|V\|_{m,\tilde{T}}, \quad \text{for} \quad 0\leqq t\leqq\tilde{T}.$$

Here, we choose $\tilde{M}_{m-1}$ so large that $(1+C_3(\bar{M}))\vee\bar{M}_{m-1}\leqq\tilde{M}_{m-1}$. Next, we choose $\tilde{M}_m$ so large that $2C_1(\bar{M},\bar{\kappa},\tilde{M}_{m-1})\vee\bar{M}_m\leqq\tilde{M}_m$. Lastly, we choose $\tilde{T}$ so small that $\exp(C_2(\tilde{M}_m))\tilde{T}\leqq 2$, $\tilde{T}\tilde{M}_m\leqq 1$, $\tilde{T}\leqq\bar{T}$, and $q-(1/2)|H|^2\geqq\bar{\kappa}/3$ in $[0,\tilde{T}]\times\Omega$; the last condition is ensured by the inequality

$$\sup_{(t,x)\in[0,\tilde{T}]\times\Omega}|V-V_0|\leqq\tilde{T}\|V\|_{m,T}.$$

Further, by the same argument as in the proof of Lemma 5.1 we see that $H \cdot n = 0$ on $[0, \tilde{T}] \times \Gamma$. Now we can easily see that if we take the function V' in $X_{\tilde{T}}^m(\Omega; \bar{\kappa}/3, \tilde{M}_{m-1}, \tilde{M}_m)$ then the solution V of (3.10)–(3.12) belongs to the same space again. So we complete the proof.

To remove the additional conditions (3.8) from initial data we show

Lemma 5.3. *Let V_0 be a function such that $V_0 \in H^m(\Omega)$ and $\partial_t^k u_0 \cdot n = 0$, for $k = 0, 1, \ldots, m-1$, on Γ. Then there exists a sequence, $\{V_0^j\}_{j=0}^{\infty}$, such that*

$$\begin{gathered} V_0^j \in H^{m+2}(\Omega), \\ \partial_t^k u_0^j \cdot n = 0, \quad \text{for} \quad k = 0, 1, \ldots, m, \quad \text{on } \Gamma, \\ V_0^j \to V_0 \quad \text{in} \quad H^m(\Omega) \quad \text{as} \quad j \to \infty. \end{gathered} \tag{5.14}$$

Here "$\partial_t^k u_0^j$" is defined by the same manner as in "$\partial_t^k u_0$" but using V_0^j instead of V_0.

Proof. Recall $N_H = (0, n, 0, 0)$. If we get the relations

$$\operatorname{Range} N_H (A_n(V_0))^k = \operatorname{Range} N, \quad \text{on } \Gamma, \text{ for } k = 1, 2, \ldots, m,$$

by the same arguments as on pp. 52–53 in [13] we can translate the proof of Lemma 3.3 in [12] to our case and construct the desired sequence. Since $A_n(V_0)$ is symmetric, we find that $\operatorname{Range}(A_n(V_0)) = \operatorname{Range}\,(A_n(V_0))^k$ for $k = 1, \ldots, m$. So it is sufficient to show that

$$\operatorname{Range} N_H (A_n(V_0)) = \operatorname{Range} N_H \quad \text{on } \Gamma. \tag{5.15}$$

Since $\operatorname{Ker} N_H$ is maximally nonnegative subspace of $A_n(V_0)$, we see that $\operatorname{Ker} A_n(V_0) \subset \operatorname{Ker} N_H$, so that by the symmetry of $A_n(V_0)$, $(\operatorname{Ker} N_H)^\perp \subset \operatorname{Range} A_n(V_0)$, which implies (5.15).

Now we give the proof of Theorems 2.3 and 2.4. Let us define the iteration scheme associated with the linearized problem (3.10)–(3.12) by

$$\begin{gathered} A_0(V^{i-1}) \partial_t V^i + \sum_{j=1}^{3} A_j(V^{i-1}) \partial_j V^i + B(V^{i-1}, V^i) = 0 \quad \text{in} \quad [0, T] \times \Omega, \\ V^i|_{t=0} = V_0 \quad \text{in } \Omega, \\ u^i \cdot n = 0 \quad \text{on} \quad [0, T] \times \Gamma, \quad \text{for} \quad i = 1, 2, \ldots. \end{gathered}$$

Here V^0 is the function constructed in Lemma 5.1. By virtue of Lemma 5.2, we find that the space $X_{\tilde{T}}^m(\Omega; \bar{\kappa}/3, \tilde{M}_{m-1}, \tilde{M}_m)$ is an invariant subset of this iteration scheme. Further, it is easy to see that, when $j \to \infty$, V^j converges to a function V in $C^0(0, \tilde{T}; L^2(\Omega))$ by taking $\tilde{T}$ smaller if it is needed. Hence, combined with the uniform boundedness of $\| V^j \|_{m, \tilde{T}}$, we can show by interpolation argument that a subsequence of $\{V^j\}$ converges to the function V in

$$C^0(0, \tilde{T}; H^{[m/2]-\varepsilon}(\Omega)) \cap C^1(0, \tilde{T}; H^{[m/2]-1-\varepsilon}(\Omega)), \quad \text{for} \quad 0 < \varepsilon \ll 1.$$

Further we know that this V belongs to $X_{\tilde{T}}^m(\Omega)$ and satisfies the estimate corresponding to (3.14). So, by recalling $m \geqq 8$, we find that this is a solution of (3.1), (3.3), and (3.4) with additional conditions (3.8). However, by virtue of the estimate of V corresponding to (3.14) and Lemma 5.3, we can remove the condition (3.8) from the initial data. Uniqueness of the solution of the problem (3.1), (3.3), and (3.4) is a direct consequence of L^2 estimate such as (5.20) in [17]. Now the proof is completed.

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