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Deformations and Renormalisations of W_∞

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Abstract. Deformations of the infinite N limit of the Zamolodchikov W_N algebra are discussed. A recent one, due to Pope, Romans and Shen with non-zero central extensions for every conformal spin is shown to be formally renormalisable to one representable in Moyal bracket form. Another deformation is discovered which, like the algebra of Pope et al. possesses automatic closure, but has non-zero central extension only in the Virasoro subalgebra.

In recent months there has been a great deal of interest in infinite-dimensional algebras which represent area preserving diffeomorphisms of various two-dimensional manifolds [1–3]. These algebras all possess a Poisson structure, and it is a current topic of great activity to extend these considerations to deformations of this Poisson structure. For example, the well-known Moyal bracket deformation was resurrected in the context of the algebra describing area preserving maps on a torus [2] as a means of relating this to the algebra of $SU(N)$ as $N \to \infty$. The idea was to replaced the structure constants of the algebra

$$[L_{j,m}, L_{k,n}] = (mk - nj)L_{j+k,m+n} + (aj + bm)\delta_{j+k,0}\delta_{m+n,0} \tag{1a}$$

by

$$[K_{j,m}, K_{k,n}] = \frac{i}{\lambda}\sin\lambda(mk - nj)K_{j+k,m+n} + (aj + bm)\delta_{j+k,0}\delta_{m+n,0}\,. \tag{1b}$$

The structure function which arises here is just a special case of the Moyal bracket [4] (see also [1]), acting on functions f, g of x, y, which is given by

$$\sin(\lambda\{f, g\}) = \sum_{p=0}^{\infty}(-1)^p \frac{\lambda^{2p+1}}{(2p+1)!}\sum_{k=0}^{2p+1}(-1)^k\binom{2p+1}{k} \times (\partial_x^k\partial_y^{2p+1-k}f)(\partial_x^{2p+1-k}\partial_y^k g)\,. \tag{2}$$

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The Moyal bracket algebra is given by [2]

$$[K_f, K_g] = iK_{(\sin\lambda\{f,g\})}, \tag{3a}$$

where

$$K_j = \tfrac{1}{2} f(x - i\lambda\partial_y, y + i\lambda\partial_x). \tag{3b}$$

The above construction of K_f is to be interpreted as the formal Taylor expansion of $\frac{1}{2}f(x - i\lambda\partial_y, y + i\lambda\partial_x)$ in powers of the partial derivatives, which are assumed to act on functions on the right; i.e.

$$K_f = \frac{1}{2}\sum_{j=0}\sum_{k=0}\lambda^{j+k}\frac{(-i)^j i^k}{j!k!}(\partial_x^j\partial_y^k f)\partial_x^k\partial_y^j. \tag{3c}$$

The substitution $f(x,y) = \lambda\exp i(mx + jy)$, $g(x,y) = \lambda\exp i(nx + ky)$ yields Eq. (1); the substitution $f(x,y) = x^{m+1}y^{j+1}$, $g(x,y) = x^{n+1}y^{k+1}$ gives the algebra of Bender and Dunne [5] and we shall provide further examples later. In a similar manner the Poisson bracket algebra

$$[L_{f(x,y)}, L_{g(x,y)}] = L_{\{f,g\}}, \tag{4a}$$

where

$$L_f = (\partial_x f)\partial_y - (\partial_y f)\partial_x \tag{4b}$$

and

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_x g)(\partial_y f), \tag{4c}$$

describes area preserving maps on the torus T^2 for the first of these substitutions, on the plane R^2 for the second and on the cylinder $C^1 \times R^1$ with the substitution $f(x,y) = \exp(mx)y^{j-1}$, $g(x,y) = \exp(nx)y^{k-1}$ giving the algebra:

$$[w_m^{(j)}, w_n^{(k)}] = ((k-1)m - (j-1)n)w_{m+n}^{(j+k-2)}, \tag{5}$$

which Bakas [6] found as the $N \to \infty$ limit of the Zamolodchikov W_N algebras [7]. In general W_N which describes the Virasoro algebra coupled to a set of generators of conformal spins $\leqq N$ is not an infinite Lie algebra, except for $N = 0$ and $N = \infty$.

In the course of this article, as if often the case in working with Jacobi identifies for infinite algebras, and redefinitions of generators, it is not easy to find an analytic proof of solutions claimed on the basis of algebraic computer calculations performed to high order (here using REDUCE), so it is more accurate to describe our "solutions" as conjectures in many cases.

Pope, Romans, and Shen [8–10] have recently produced a very interesting series of papers in which they construct a deformation of the w_∞ algebra. We shall examine in detail the structure of their algebra and produce an alternative solution, which has a nontrivial central term only in the Virasoro subalgebra, while theirs has one for all conformal spins. A natural question concerns the uniqueness of such deformations; we consider redefinitions of the generators which effect renormalisations of these algebras. We follow their convention of referring to the algebra (5) as w_∞, and to the deformation of Pope et al. as W_∞. They construct a deformation analogous to the Moyal (sine bracket) deformation (2) of the Poisson algebra (4c). They require an algebra of the following form, with nontrivial central terms of the form

$$[V_m^j, V_n^k] = \sum_{p=0}^{s=\left[\frac{j+k}{2}\right]} q^{2p} g_{2p}^{jk}(m,n) V_{m+n}^{j+k-2p} + q^{2j} c_j(m)\delta^{j,k}\delta_{m+n,0}. \tag{6}$$

It is important to stress that in the above $j, k \geq 0$ and the terms on the right-hand side of (5) terminate with either V^1_{m+n} or V^0_{m+n} according to whether $(j+k)$ is odd or even. They discover that the structure constants $g^{jk}_{2p}(m,n)$ factorize as follows (with a slight change of normalisation as compared to [8]);

$$g^{jk}_{2p}(m,n) = \phi^{jk}_{2p} N^{jk}_{2p}(m,n) \tag{7}$$

with

$$\phi^{jk}_{2p} = \sum_{r=0}^{p} \prod_{l=1}^{r} \frac{(2l-3)(2l+1)(2p-2l+3)(p-l+1)}{l(2j-2l+3)(2k-2l+3)(2j+2k-4r+2l+3)} \tag{8a}$$

or

$$\phi^{jk}_{2p} = {}_4F_3\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2}, -p-\frac{1}{2}, -p \\ -j-\frac{1}{2}, -k-\frac{1}{2}, j+k-2p+\frac{5}{2} \end{matrix}; 1\right]. \tag{8b}$$

They find the central terms to be simple monomials in m; that is, of the form

$$c_j(m) = m^{2j+3} \frac{2^{2j} j!(j+2)!}{(2j+1)!!(2j+3)!!} c, \tag{9}$$

and determine

$$N^{jk}_{2p}(m,n) = \frac{1}{(2p+1)!} \sum_{x=0}^{2p+1} (-1)^x \binom{2p+1}{x} m^{2p+1-x} n^x \frac{(2j-2p+1+x)!}{(2j-2p+1)!} \times \frac{(2k+2-x)!}{(2k-2p+1)!}. \tag{10}$$

It is remarkable that the properties of these functions are sufficient to guarantee the automatic termination of the right-hand side of (5), without any particular assumptions about the representation of V^j_m. A similar formal solution holds with $\phi^{jk}_{2p} = 1$, $N^{j,k}_{2p}(m,n)$ unchanged.

Let us call the case just described: Case (1).

We find also two further solutions, with $\phi^{jk}_{2p} = 1$ in both cases, no automatic termination and no central charge. Our reason for quoting them at all is that we shall require them to illustrate renormalisation isomorphisms of the algebra (5) and demonstrate a non-trivial alternative solution.

They are as follows:

– Case (2);

$$N^{jk}_{2p}(m,n) = \frac{1}{(2p+1)!} \sum_{x=0}^{2p+1} (-1)^x \binom{2p+1}{x} m^{2p+1-x} n^x \times \frac{(j+1)!}{(j+1-x)!} \frac{(k+1)!}{(k-2p+x)!} \tag{11}$$

– and Case (3);

$$N^{jk}_{2p}(m,n) = \frac{1}{(2p+1)!} \sum_{x=0}^{2p+1} (-1)^x \binom{2p+1}{x} m^{2p+1-x} n^x \times \frac{(j-2p+x)(j-2p-2+2x)!!}{(j-2p)!!} \times \frac{(k-x+1)(k+2p-2x)!!}{(k-2p)!!}. \tag{12}$$

While Cases 2 and 3 do not implement automatic termination, there is a subalgebra of Case 3 consisting of the generators indexed by j even which contrives to do it with $\phi^{jk}_{2p}=1$ and with a central charge

$$\delta_{i,0}\delta_{j,0}\delta_{m+n,0}(cm^3+c'm). \tag{13}$$

The coefficients $N^{j,k}_{2p}(m,n)$ can be expressed in terms of Moyal brackets for Case 1 by the substitutions

$$\begin{gathered} f(x,y)=i\lambda\exp\left(\frac{mx}{y}\right)y^{2(j+1)}, \quad g(x,y)=i\lambda\exp\left(\frac{nx}{y}\right)y^{2(k+1)} \\ q^2=-\lambda^2 \end{gathered} \tag{14}$$

and for Case 2 by the substitutions

$$\begin{gathered} f(x,y)=i\lambda\exp(mx)y^{j+1}, \quad g(x,y)=i\lambda\exp(nx)y^{k+1} \\ q^2=-\lambda^2. \end{gathered} \tag{15}$$

The differential operator realisation for the V^j_m constructed from this $f(x,y)$ via (3) implement the termination in this case. Since there is no central term here, it is possible that all representations are equivalent to this one in Case 2. The Moyal bracket form for case 3 is given by the substitution:

$$\begin{gathered} f(x,y)=i\lambda\exp(mxy^{j/2}), F_0\left(\frac{j}{4}+\frac{3}{2}, \left(\frac{m}{y}\right)^2\right) \\ g(x,y)=i\lambda\exp(nxy^{k/2}), F_0\left(\frac{j}{4}+\frac{3}{2}, \left(\frac{n}{y}\right)^2\right). \end{gathered}$$

possible that all representations are equivalent to this one in Case 2. We have not succeeded in expressing Case 3 in Moyal form.

The subalgebra of Case 3 discussed above is so important, we rewrite it by renaming the variables $U^j_m=V^{2j}_m/2$, $\varrho=2q$, $j\geqq 0$ and call it U_∞,

$$[U^j_m, U^k_n]=\sum_{p=0}^{j+k}\varrho^{2p}u^{jk}_p(m,n)U^{j+k-p}_{m+n}+\delta^{j,0}\delta^{k,0}\delta_{m+n,0}(cm^3+c'm), \tag{16}$$

where

$$\begin{aligned} u^{j,k}_p(m,n)=&\frac{1}{(2p+1)!}\sum_{x=0}^{2p+1}(-1)^x\binom{2p+1}{x}m^{2p+1-x}n^x \\ &\times\frac{(2j-2p+x)(j-p-1+x)!}{(j-p)!} \\ &\times\frac{(2k-x+1)(k+p-x)!}{(k-p)!}. \end{aligned} \tag{17}$$

The contribution $c'm$ to the central term may be renormalised away by redefinition fo U^0_0. The coefficients in the above expression possess the remarkable property that they vanish when $j\geqq 0$ and $k\geqq 0$ for all $p\geqq\max(j,k)$, hence making the upper index $j+k-p$ in (16) always positive. This is the property which causes automatic termination of the right-hand side of (12) for any given value of $j,k\geqq 0$ and makes (16) an acceptable alternative deformation of the Bakas algebra w_∞ to one constructed by Pope et al. [9, 10]. The algebra represented by (16) is obviously a *subalgebra* of algebra of Case 3 (12).

We thus have two possible non-trivial deformations of w_∞; one with a non-trivial central term appearing in all conformal spins, the other only in the Virasoro subalgebra $j=0$. This remarkable situation calls for further analysis, and we should like to elucidate several features of the m,n independent factors ϕ^{jk}_{2p} which appear, and analyse to what extent they may be transformed away in Case 1.

The subject of unitary equivalence of infinite Lie algebras is a difficult but important question. Consider the basic Moyal bracket (1b). This is an illustrative example as it is possible to carry out all the calculations analytically in this case, and exhibit a solution with extra m, n independent factors, and demonstrate that these factors may be formally transformed away. Were this not so, then, in consequence of the connection of the Moyal bracket with quantum mechanics [4, 11], an extension of the latter theory would become viable. The validity of the following more general algebra is easy to demonstrate:

$$[K'_{j,m}, K'_{k,n}] = \frac{i}{\lambda}\sin\lambda(mk-nj)\exp\frac{q(mk-nj)^2\lambda^2}{jk(j+k)}K'_{j+k,m+n}. \tag{18}$$

Here q is an arbitrary parameter. The reason this works is that the factor

$$\frac{(mk-nj)^2}{jk(j+k)} \equiv \frac{m^2}{j}+\frac{n^2}{k}-\frac{(m+n)^2}{j+k}. \tag{19}$$

Hence the extra factor can be formally normalised away, by redefinition of $K'_{j,m} = \exp\frac{q\lambda^2m^2}{j}K_{j,m}$. We caution that this is formal as the renormalisation factor contains an essential singularity at $j=0$, an admissible value of the index j, unless we restrict to the positive subalgebra $j, k>0$. On the other hand, by restricting the indices to $i>0$, $j>0$, writing the sine in exponentials, and recognising the form of the generating function for Hermite polynomials, it is possible to expand the above expression and identify the term in λ^{2p+1},

$$\frac{(mk-nj)^{2p+1}(-1)^p}{(2p+1)!}\left(\sqrt{\frac{q}{jk(j+k)}}\right)^{2p+1}H_{2p+1}\left(\frac{1}{2}\sqrt{\frac{jk(j+k)}{q}}\right), \tag{20}$$

as the product of the Moyal (sine) bracket function $\frac{(mk-nj)^{2p+1}(-1)^{2p+1}}{(2p+1)!}$ multiplied by a factor which depends on j and k only and which plays the role of ϕ in this case. The factor itself is the product of a coefficient by a Hermite polynomial, i.e. a hypergeometric function [see (24) below]. It can be seen that upon multiplication the square roots disappear in these ϕ's.

One might wonder whether something similar could happen in Case 1. Let us consider the effect of a redefinition of the V^j_m of the form

$$V'^j_m = \sum_{r=0} \chi(r,j,m)V^{j-2r}_m \tag{21}$$

and ask whether a renormalisation of this type is possible which will mimic the effect of the factors ϕ^{jk}_{2p}. We are here assuming that the operators V^j_m satisfy the algebra of Case 1;

$$[V^j_m, V^k_n] = \sum_{s=0} N^{jk}_{2s}(m,n)V^{j+k-2s}_{m+n}, \tag{22}$$

where the structure constants are given by Eq. (10). We find

$$\begin{aligned}\chi(r,j,m) &= -\frac{s(s+1)m^{2r}f^r}{j+\frac{1}{2}}\,{}_3F_2\left[\begin{matrix}1-s, 2+s, -r+1\\ -j+\frac{1}{2}, 2\end{matrix};1\right]\\ &= \lim_{\alpha\to 0}\frac{\alpha}{r}\,{}_3F_2\left[\begin{matrix}-s, 1+s, -r\\ -j-\frac{1}{2}, \alpha\end{matrix};1\right],\end{aligned} \tag{23}$$

where s, f are parameters of the solution. Then the generators $V_m'^j$ satisfy the algebra (6) with ϕ given by

$$\phi_{2p}^{jk} = {}_4F_3\left[\begin{matrix} -s, s+1, -p-\frac{1}{2}, -p \\ -j-\frac{1}{2}, -k-\frac{1}{2}, j+k-2p+\frac{5}{2} \end{matrix}; 1\right]. \tag{24}$$

The central terms in the renormalised algebra are of the form

$$\frac{q^{2j} c m^{2j+3}}{(2j+3)!} \delta^{j,k} \delta_{m+n,0}. \tag{25}$$

The renormalised algebra is that of Pope et al. when $s = \frac{1}{2}$. In this case the purpose of the ϕ's as explained in [8–10] is to provide additional zeros to those furnished by the N's to cause the algebra (6) to terminate for given i, j without additional assumptions, or special choice of representation, as for example in the representations by differential operators. In a subsequent article [13], Pope et al. find a second solution with the property of termination, which they call $W_{1+\infty}$, corresponding to the above when $s = 0$. (Note that we define s differently.) This solution has slightly different central terms from (9). Recently Bakas and Kiritsis [14] have claimed a realisation of W_∞ in terms of a complex free bosonic field which reproduces the central terms (9) with $c = 2$.

Let us rephrase what we have done. It is easily seen that the transformation (21) is represented by an infinite upper triangular matrix with diagonal element $\chi(0, j, m) = 1$. This matrix is obviously invertible. Hence the algebras represented by V_m^j and $V_m'^j$ are formally equivalent *when the indices run on all values of j, negative as well as positive.*

If the indices j in $V_m'^j$ are restricted to positive values only, this defines a subalgebra of the $V_m'^j$ for all values of j. Hence, formally at least, the algebra found by Pope et al. with the ϕ's is a *subalgebra of the algebra of Case 1 without the ϕ's.* These formal renormalisation effects or formally invertible equivalences are, we feel, important steps in the understanding of the relations between these infinite algebras.

Hence this formally resembles what we have obtained for Case 3. A terminating algebra is a subalgebra of a non-terminating one.

We now turn to the question of the associative product implied by these algebras, which Pope et al., in recognition of their affiliation, designate by the "Lone Star Product." It is given by

$$V_m^j * V_n^k = \sum_{s=-1} q^s g_s^{jk}(m, n) V_{m+n}^{j+k-s}, \tag{26}$$

where the coefficients $g_s^{jk}(m, n)$ ($g_{-1}^{jk} = 1$) are simply obtained by analytic continuation of the coefficients defined in Cases 1, 2, and 3. The associativity requirement is simply

$$(V_m^j * V_n^k) * V_p^l = V_m^j * (V_n^k * V_p^l). \tag{27}$$

The antisymmetric part of this gives the Lie algebra W_∞; the symmetric part gives its supersymmetric extension. The corresponding associative product for the Moyal bracket has long been recognised [11, 12], and leads to the introduction of the exponential bracket, whose imaginary part is the sine, or Moyal bracket, as the more fundamental construct. Our algebra U_∞ does not apparently admit an associative product, because it is not possible to restrict the indices in the associative product for Case 3 to even values alone.

References

1. Fairlie, D.B., Fletcher, P., Zachos, C.K.: Phys. Lett. B **218**, 203 (1989)
2. Fairlie, D.B., Zachos, C.K.: Phys. Lett. B **224**, 101 (1989)
3. Fairlie, D.B., Fletcher, P., Zachos, C.K.: Infinite dimensional algebras and a trigonometric basis for the classical Lie algebras. ANL HEP PR 89-64, to appear in J. Math. Phys.
4. Moyal, J.: Proc. Camb. Phil. Soc. **45**, 99, (1949); Baker, G.: Phys. Rev. **109**, 2198 (1958); Fairlier, D.B.: Proc. Camb. Phil. Soc. **60**, 581 (1964)
5. Bender, C.M., Dunne, G.: Phys. Rev. D **40**, 3504 (1989)
6. Bakas, I.: The Structure of the W_∞ algebra. Commun. Math. Phys. (to appear)
7. Zamolodchikov, A.B.: Teo. Mat. Fiz. **65**, 347 (1985); Fateev, V.A., Lykyanov, S.: Int. J. Mod. Phys. A **3**, 507 (1988)
8. Pope, C.N., Shen, X.: Phys. Lett. B **236**, 21 (1990)
9. Pope, C.N., Romans, L.J., Shen, X.: Phys. Lett. B **236**, 173 (1990)
10. Pope, C.N., Romans, L.J., Shen, X.: W_∞ and the Racah-Wigner algebra. Preprint, CTP TAMU-72/89, USC-89/HEP 040
11. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Ann. Phys. N.Y. **111**, 61, 111 (1978)
12. Estrada, R., Garcia-Bondia, J.M., Varilly, J.C.: J. Math. Phys. **30**, 2789 (1989)
13. Pope, C.N., Romans, L.J., Shen, X.: A new higher-spin algebra and the Lone-Star product. Preprint, CTP TAMU-17/90, USC-90/HEP 01
14. Bakas, I., Kiritsis, E.: Bosonic realisation of a universal W-algebra and Z_∞ parafermions. Preprint LBL-28714, UCB-PTH-90/8, UMD-PP 90-160

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