Commun. Math. Phys. 129, 69–94 (1990)

Communications in Mathematical Physics

Orbifolds as Configuration Spaces of Systems with Gauge Symmetries

C. Emmrich and H. Römer
Fakultät für Physik der Universität Freiburg, Hermann-Herder-Straße 3, D-7800 Freiburg, FRG

Abstract. In systems like Yang–Mills or gravity theory, which have a symmetry of gauge type, neither phase space nor configuration space is a manifold but rather an orbifold with singular points corresponding to classical states of non-generically higher symmetry. The consequences of these symmetries for quantum theory are investigated. First, a certain orbifold configuration space is identified. Then, the Schrödinger equation on this orbifold is considered. As a typical case, the Schrödinger equation on (double) cones over Riemannian manifolds is discussed in detail as a problem of selfadjoint extensions. A marked tendency towards concentration of the wave function around the singular points in configuration space is observed, which generically even reflects itself in the existence of additional bound states and can be interpreted as a quantum mechanism of symmetry enhancement.

1. Introduction

Let a Lie group $\mathbf{G}$ act on a manifold M. Then it is natural to perform a "symmetry reduction" by identifying points which can be transformed into one another and, hence, going over to the space $M/\mathbf{G}$, the set of orbits in M under the action of $\mathbf{G}$. Now, $M/\mathbf{G}$ will, in general, not be a manifold, but rather an orbifold. Singular points in $M/\mathbf{G}$ will arise whenever a jump occurs in the conjugacy class $(\mathbf{H})$ of the isotropy groups $\mathbf{H} \subset \mathbf{G}$ of points in different orbits.

In this note, we shall investigate the impact of such orbifold singularities in configuration and phase space of mechanical systems to the associated quantum systems, thus expanding and elaborating on a programme announced in [1].

This introduction contains a description of the main ideas and results, all technical details will be deferred to following sections. Our framework will be Lagrangian mechanics (and field theory) with a (possibly infinite dimensional) manifold Q as configuration space and its cotangent bundle $P = T^*Q$ as phase space. Let now $\mathbf{G}$ act on Q. For every $\xi \in \mathcal{G}$, the Lie algebra of $\mathbf{G}$, there is a fundamental vector field ξ_Q on Q, defined by

$$(\xi_Q u)(m) = \frac{d}{dt} u(e^{t\xi} m)|_{t=0}$$

for $u \in \mathscr{C}^\infty(Q)$, an induced action on $P = T^*Q$ with fundamental vector field ξ_P and a map

$$\mathscr{J}_\xi : P = T^*Q \to \mathbb{R}$$

with

$$\xi_P f = \{\mathscr{J}_\xi, f\} \quad \text{for} \quad f \in \mathscr{C}^\infty(P)$$

and

$$\mathscr{J}_{\mathrm{Ad}_g \xi}(p) = \mathscr{J}_\xi(g^{-1} p).$$

In other words: there is an equivariant momentum map $\mathscr{J} : P \to \mathscr{G}^*$, defined by

$$\langle \mathscr{J}(p), \xi \rangle = \mathscr{J}_\xi(p).$$

In local coordinates $\mathscr{J}_\xi = \sum p_i dq^i(\xi_Q)$.

For an ordinary symmetry group like rotations in space, which transforms physical configurations into one another, the transition to the quotient $P/\mathbf{G}$ is not indispensible and may be avoided, if singularities in $P/\mathbf{G}$ turn out to be troublesome.

Gauge transformations in Yang–Mills theory and coordinate transformations in General Relativity theory are examples of different types of symmetries, which one could denote as redundancy symmetries: rather than transforming physical configurations they transform different redundant descriptions of the same physical state by changing redundant variables used to describe this state. In this case, it is only $Q/\mathbf{G}$ which should be identified with the set of physical states, and the transition to the quotient becomes a conceptual necessity.

Typically, for instance in the two principal examples mentioned above, there is freedom to fix redundant variables for each time t separately. As a result, the Lagrangian will be invariant under time dependent transformations $g(t) \in \mathbf{G}$. For an infinitesimal transformation $e^{\xi \cdot \varepsilon(t)}$ of this kind, Noether's theorem will give

$$0 = \frac{d}{dt} \sum p_i \xi_Q q^i \varepsilon(t) = \varepsilon(t) \frac{d}{dt} \mathscr{J}_\xi + \dot{\varepsilon} \mathscr{J}_\xi$$

and, hence

$$\mathscr{J}_\xi \equiv 0 = \sum p_i \xi_Q q^i \quad \text{for all} \quad \xi \in \mathscr{G}.$$

The system is constrained to the zero set of the momentum map. The singular points of the zero set $\mathscr{J}^{-1}(0)$ are points in P with non-generically large isotropy group $\mathbf{H} \subset \mathbf{G}$. Under quite general conditions [2] the nature of these singularities is quite simple and explicitly known: In the neighbourhood of a singularity, $\mathscr{J}^{-1}(0)$ is locally equivalent to a quadratic cone.

For Yang–Mills and gravity theory the constraint $\mathscr{J} \equiv 0$ is identical with Gauß's law and the ADM constraints respectively. The singular points are gauge potentials which are left fixed by some group $\mathbf{H} \neq \{e\}$ of gauge transformations or metrics

which admit some isometry. The quadratic conical nature of these singularities has been established also in this infinite dimensional situation.

The symplectic form ω is degenerate on the level set $\mathcal{J}^{-1}(0)$, which is, hence, not a good candidate for a symmetry reduced phase space. Rather one has to consider the quotient

$$\tilde{P}_0 = \mathcal{J}^{-1}(0)/\mathbf{G}.$$

More generally, for $\mu \in \mathcal{G}^*$ one defines

$$\tilde{P}_\mu = \mathcal{J}^{-1}(\mu)/\mathbf{G}_\mu,$$

where $\mathbf{G}_\mu$ is the isotropy group of $\mu \in \mathcal{G}^*$ with respect to the coadjoint representation. Unfortunately, $\tilde{P}_\mu$ fails to be a manifold whenever jumps occur in the orbit type.

As far as classical mechanics is concerned, according to a theorem of M. Otto [3], this difficulty can be circumvented under very general circumstances in the following way:

Let $P_{(H)} \subset P$ be the set of points in P with orbit type $(\mathbf{H})$, i.e. isotropy groups conjugate to $\mathbf{H} \subset \mathbf{G}$. $P_{(H)}$ is a (not necessarily symplectic) submanifold of P. Then take for arbitrary $\mu \in \mathcal{G}^*$: $P_{(H)\mu} = \mathcal{J}^{-1}(0) \cap P_{(H)}$. $P_{(H)\mu}$ is invariant under $\mathbf{G}_\mu$ and

$$\widetilde{P_{(H)\mu}} = (\mathcal{J}^{-1}(0) \cap P_{(H)})/\mathbf{G}_\mu$$

is a (nonsingular) symplectic manifold in a natural way. For a $\mathbf{G}$-invariant Hamiltonian, the classical trajectories will never leave $\mathcal{J}^{-1}(0) \cap P_{(H)}$. Hence, for classical mechanics, it will always suffice to take $\widetilde{P_{(H)\mu}}$ as a reduced phase space.

The situation is more involved for quantum mechanics for at least two reasons: First, quantum mechanics uses configuration space rather than phase space, or at least a polarization of phase space. Secondly, and more importantly, there is no justification for restricting oneself to a given orbit type $(\mathbf{H})$ because quantum transitions between classical configurations of different orbit types are always possible, as there is no reason to assume that the wave function(al) has support in only one orbit type.

Restriction to the zero set $\mathcal{J}^{-1}(0)$ and going over from $\mathcal{J}^{-1}(0)$ to the reduced phase space $\tilde{P}_0 = \mathcal{J}^{-}(0)/\mathbf{G}$ by a procedure of "gauge fixing" both amount to imposing constraints. Hence, we are facing a problem of quantization of a constrained system with the additional complication that the solution set of the constraints does not form a manifold.

Two different approaches are conceivable:

1. *Extrinsic Quantization*: One first quantizes an unconstrained system, which still contains redundant degrees of freedom. Then, one translates the constraint functions into operators and constructs a Hilbert space of physical states by imposing vanishing conditions, when (annihilation parts of) the constraint operators are applied. The Gupta–Bleuler and the BRST formalisms are two examples of extrinsic quantization.

2. *Intrinsic Quantization*: One first solves for the classical constraints by introducing appropriate coordinates and then only quantizes in terms of these coordinates.

The equivalence of these two alternative quantization schemes is not guaranteed. In particular, extrinsic quantization, at least on intermediate stages, allows for fluctuations into configurations which violate the constraint equations. The equivalence problem is even more serious [1], if the constraint set fails to be a manifold. An example of non-equivalence in quantum gravity, which is of particular relevance in the context of this work, is given in [4].

In any case, for a system with a redundancy symmetry the intrinsic approach seems to be conceptionally preferable. Unfortunately, it is in general hard if not impossible to solve for the constraints even locally.

One very promising and attractive method of intrinsic quantization has been announced by Śniatycki and Weinstein [5], who show, that, even in the singular case, one can define a classical Poisson algebra of functions on $\tilde{P}_0 = \mathcal{J}^{-1}(0)/\mathbf{G}$ by identifying functions on $\tilde{P}_0$ with **G**-invariant elements in $\mathscr{C}^\infty(P)/\langle \mathcal{J} \rangle$, where $\langle \mathcal{J} \rangle$ is the ideal generated by the momentum map $\mathcal{J}$. However, the difficult and crucial step of quantizing this Poisson algebra has, so far, only been performed in simple special cases.

The strategy of intrinsic quantization we are going to follow in this work is the formulation and solution of a Schrödinger equation on an orbifold.

For this approach, it is a crucial observation, that, at least for the cases of interest for us, it is possible to identify a substitute for a reduced configuration space. Indeed, if $P = T^*Q$ is a cotangent bundle and if **G** acts on Q with only one orbit type, then under suitable technical conditions (always met in physically interesting cases) there is a natural identification of the reduced phase space with a cotangent bundle:

$$\tilde{P}_0 = \mathcal{J}^{-1}(0)/\mathbf{G} \cong T^*(Q/\mathbf{G}).$$

In general, of course, there are several orbit types in Q, but there is a generic orbit type of lowest symmetry, which is dense and open in Q.

Orbifold singularities in $Q/\mathbf{G}$ correspond to points of higher symmetry. In this situation, the above identification is true for a dense open subset $\hat{P}_0$, which leaves out singular points of $Q/\mathbf{G}$. Then it is possible to define a Hamiltonian operator $\hat{H}$ and a Schrödinger equation first on the regular points in $Q/\mathbf{G}$. (For the quantization of classical operators there may still be ordering problems.) The Hamilton operator and the Schrödinger equation on the orbifold $Q/\mathbf{G}$ are then constructed by selfadjoint extensions of $\hat{H}$. The behaviour of the wave function at the orbifold points of $Q/\mathbf{G}$ is of particular physical interest. It is the principal concern of this work.

Singular points in $Q/\mathbf{G}$ are linearization unstable. In [6] it was shown that for these points the linearized constraints have to be supplemented by additional quadratic constraint conditions, which suppress spurious solutions of the linearized constraint equations. The effect of these additional constraints is a suppression of transitions to configurations of lower symmetry. This leads to the conjecture, formulated in [1], that in the Schrödinger picture, the wave function(al) shows a particular concentration at the orbifold points. A very peculiar quantum mechanism for the enhancement of symmetric configurations would emerge this way. We test this conjecture by a detailed treatment of some finite dimensional models.

In the vicinity of an orbifold singularity one expects $Q/\mathbf{G}$ to have an iterated cone structure ("cones over cones"), the prototype being the set $\mathcal{G}/\mathbf{G}$ of adjoint orbits of a semisimple group $\mathbf{G}$, which is just given by a Weyl chamber.

This provides a motivation to consider Schrödinger equations on a cone or double cone $\mathcal{C}$ over a (compact Riemannian) manifold M of dimension n [1], see also [8]. The Schrödinger equation on $\mathcal{C}$ has the form

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_M\right)\psi + V\psi = E\psi,$$

where $n = \dim M$ and Δ_M is the Laplace operator on M with respect to its metric g_M. Rescaling g_M corresponds to changing the opening angle of the cone $\mathcal{C}$.

We consider two different potentials:

i) $V \equiv 0$: free motion
ii) $V = \frac{1}{2}\omega^2 r^2$: oscillator potential.

After separation of the radial coordinate r we have

$$h_l\phi = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}-\frac{l^2}{r^2}\right)\phi + V\phi = E\phi.$$

Here $l^2 \geqq 0$ is an eigenvalue of $-\Delta_M$.

The radial Hamiltonians are first defined on the domain $\mathcal{D} = \mathcal{C}_0^\infty(\mathbb{R}\backslash\{0\})$ of smooth functions with compact support in $\mathbb{R}\backslash\{0\}$. Clearly, h_l is symmetric on $\mathcal{D}$. The Schrödinger equation on the cone and its solutions are obtained by selfadjoint extensions of h_l.

The main results are the following:

- For $n \geqq 3$, h_l is essentially selfadjoint. The unique selfadjoint extension is obtained by closure.

- More complicated and also more interesting is the case $n \leqq 2$. It is relevant for our considerations, since we expect low dimensions to enter because of the orbifold structure ("cones over cones") of $Q/\mathbf{G}$. In this case, for $\nu(l) = \sqrt{((n-1)^2/4) + l^2} < 1$, the selfadjoint extension is non-unique and depends on several real parameters. Generically, a bound state occurs even for $V \equiv 0$.

- For both potentials and $n \leqq 2$, one can always achieve by a suitable choice of the selfadjoint extension arbitrarily low energy $E < 0$ and arbitrary l^2 with $\nu(l) < 1$ for the ground state.

- There is a clear tendency towards a concentration of the wave function around the symmetric configuration $r = 0$ for decreasing opening angle of $\mathcal{C}$ (decreasing l^2). This tendency is particularly conspicuous if an additional bound state occurs. This confirms our expectations formulated above.

Details are described in Sect. 3, which also contains a discussion of scattering solutions for $V \equiv 0$.

2. The Reduced Phase and Configuration Space

In the following we are going to study the structure of the reduced phase and configuration space in detail, restricting ourselves for technical reasons to the finite dimensional case. However, in the physically most important examples with infinite dimensional phase and configuration space—Yang–Mills theories and General Relativity—analogous results are known to hold [2, 7].

Let Q be a Riemannian manifold with metric $\mathbf{g}$ and Φ be a proper and isometric action of a Lie group $\mathbf{G}$ on Q. Here, proper means that if (x_n) is a convergent sequence in Q and (g_n) an arbitrary sequence in $\mathbf{G}$ for which $\Phi(g_n, x_n)$ converges in Q, then (g_n) has a convergent subsequence. This is a technical assumption which guarantees the existence of $\mathbf{G}$-slices, needed for the proofs of the following theorems about the structure of the reduced phase and configuration space (see definitions and remarks below). Especially, any action of a compact group is obviously proper.

Before studying the structure of the reduced configuration space $Q/\mathbf{G}$, the reduced phase space $\mathcal{J}^{-1}(0)/\mathbf{G}$ and the question in which sense $\mathcal{J}^{-1}(0)/\mathbf{G}$ may be considered a cotangent bundle over the reduced configuration space $Q/\mathbf{G}$, we first give some standard definitions in order to fix our notation, which essentially coincides with that of [9]:

On the cotangent bundle $\pi: T^*Q \to Q$ there is a natural 1-form θ, defined by

$$\theta(\alpha_q)(X_{\alpha_q}) := \alpha_q(T_{\alpha_q}\pi X_{\alpha_q}),$$

inducing a canonical symplectic form $\omega := -d\theta$. Any group action Φ on Q induces a group action Φ^* on T^*Q, the lift of Φ, defined by:

$$\Phi_g^* \alpha_q := (T_{g\cdot q}\Phi_{g^{-1}})^* \alpha_q.$$

This action leaves the canonical 1-form invariant and hence is a symplectic action. For this action, there is an Ad*-equivariant momentum mapping $\mathcal{J}$, i.e., a mapping

$$\mathcal{J}: T^*Q \to \mathcal{G}^*$$

with

$$d\mathcal{J}_\xi = i_{\xi_P}\omega \tag{1}$$

defined by:

$$\mathcal{J}_\xi(\alpha_q) = \theta(\alpha_q)(\xi_{T^*Q}(\alpha_q)) = \alpha_q(\xi_Q(q)), \tag{2}$$

where $\mathcal{J}_\xi(\alpha_q) := \mathcal{J}(\alpha_q)(\xi)$ for arbitrary $\xi \in \mathcal{G}$.

This momentum mapping is Ad*-equivariant, i.e., for arbitrary $g \in \mathbf{G}$ and $\alpha_q \in T^*Q$ the following equation holds:

$$\mathcal{J}(g\cdot\alpha_q) = \mathrm{Ad}^*_{g^{-1}}\mathcal{J}(\alpha_q). \tag{3}$$

(The concept of a momentum mapping is a generalization of the concept of ordinary momentum and angular momentum to arbitrary symmetry groups instead of translations or rotations. Here, (1) means that the momentum mapping generates—via the Poisson bracket—infinitesimal transformations, whereas (3) fixes the trans-

formation behaviour of the momentum mapping, generalizing the statements that ordinary momentum is a polar vector, angular momentum an axial vector.)

The isotropy group $\mathbf{G}_q$ of a point $q \in Q$ is defined as the set of all $g \in \mathbf{G}$ leaving q fixed. For an arbitrary subgroup $\mathbf{H} \subset \mathbf{G}$ we denote by $(\mathbf{H})$ the conjugacy class of $\mathbf{H}$, i.e., $(\mathbf{H}) := \{g\mathbf{H}g^{-1} | g \in \mathbf{G}\}$, by Q_H the submanifold of Q of symmetry type $\mathbf{H}$, the set of points with isotropy group $\mathbf{H}$, and by $Q_{(H)}$ the submanifold of orbit type $(\mathbf{H})$, the set of points with isotropy group in $(\mathbf{H})$.

With these definitions we may formulate the following theorem:

Theorem 1. *Let Q be a Riemannian manifold with metric $\mathbf{g}$, Φ be a proper and isometric action of a Lie group $\mathbf{G}$ on Q and J be the* Ad*-*equivariant momentum mapping* (2) *for the symplectic action Φ^* on T^*Q. If the isotropy groups $\mathbf{G}_q$ are conjugate for all $q \in Q$, the following statements hold:*

i) *$Q/\mathbf{G}$ is a Riemannian manifold with a metric $\tilde{\mathbf{g}}$ canonically induced by $\mathbf{g}$.*
ii) *All isotropy groups in $\mathscr{J}^{-1}(0)$ are conjugate and coincide for any $\alpha_q \in \mathscr{J}^{-1}(0)$ with the isotropy group $\mathbf{G}_q$ of the corresponding base point $q = \pi(\alpha_q)$ in Q.*
iii) *$\mathscr{J}^{-1}(0)$ is a subbundle of T^*Q invariant under the action of $\mathbf{G}$ with*

$$\pi(\mathscr{J}^{-1}(0)) = Q.$$

iv) *$\mathscr{J}^{-1}(0)/\mathbf{G}$ is a symplectic manifold symplectomorphic to $T^*(Q/\mathbf{G})$.*

The statements of this theorem are well known for free group actions [9, Theorem 4.3.3]. The generalization to conjugate isotropy groups is nontrivial, as we only require the isotropy groups in Q to be conjugate, not those in T^*Q. (If we required the isotropy groups in T^*Q to be conjugate, it could be easily shown that they even had to be equal and hence normal in $\mathbf{G}$. Hence, the statements of the theorem could be reduced to the free action of the group $\mathbf{G}/\mathbf{G}_q$, yielding only a trivial generalization.) Especially, as the rank of $d\mathscr{J}$ generally is not constant in any neighbourhood of $\mathscr{J}^{-1}(0)$ (as may be easily seen by the example $Q = \mathbb{R}^n \setminus \{0\}$, $\mathbf{G} = \mathrm{SO}(n)$ acting on $\mathbb{R}^n \setminus \{0\}$ in the natural way), the fact that $\mathscr{J}^{-1}(0)$ is a subbundle of T^*Q is nontrivial.

For the proof of the theorem we use the well known existence theorems for $\mathbf{G}$-slices:

We call a submanifold S_q of Q with $q \in S_q$ *$\mathbf{G}$-slice in q* iff the following conditions are satisfied:

1. $gS_q = S_q$ for all $g \in \mathbf{G}_q$.
2. From $g \cdot S_q \cap S_q \neq \varnothing$ follows $g \in \mathbf{G}_q$.
3. There is a local section $\kappa: V \subset \mathbf{G}/\mathbf{G}_q \to \mathbf{G}$ defined on a neighbourhood V of $e \cdot \mathbf{G}_q$ such that the mapping:

$$\tau: S_q \times V \to Q, \quad (x, u) \to \kappa(u) \cdot x$$

defines a local diffeomorphism.

It is well known [2] that for a proper group action by isometries there is a $\mathbf{G}$-slice in any point $q \in Q$, which can be chosen as

$$S_q := \{\exp(X_q) | X_q \in (T_q(\mathbf{G} \cdot q))^{\perp}, |X_q| < \varepsilon\} \tag{4}$$

for a sufficient small $\varepsilon > 0$. A **G**-slice of the form (4) is called an affine **G**-slice.

Now, with this technical device, the proof of the theorem is quite simple: From the definition of a **G**-slice in a point $q \in Q$ obviously follows that the isotropy group of any point $\tilde{q} \in S_q$ is a subgroup of $\mathbf{G}_q$. Hence, by the assumed existence of only one orbit type in Q, the isotropy groups in S_q are identical. Therefore, by the defining properties of a **G**-slice, the intersection of any orbit in Q with S_q contains at most one point and for any $q \in Q$, we may identify $Q/\mathbf{G}$ in a suitable neighbourhood of $[q] = \mathbf{G} \cdot q$ with the slice S_q. By choosing the slices sufficiently small, we may thus define a chart for any slice, inducing by the above identification a chart in $Q/\mathbf{G}$. As the mapping

$$\rho: \mathbf{G} \cdot S_q = \{\mathbf{g} \cdot q \mid g \in \mathbf{G},\ q \in S_Q\} \to S_q$$

mapping any $q \in \mathbf{G} \cdot S_q$ to the intersection of the orbit [q] with S_q is smooth by the defining property 3. of **G**-slices, the transition functions of the charts defined in this way are easily seen to be smooth.

Introducing the orthogonal projections

$$\sigma_q: T_q Q \to (T_q(\mathbf{G} \cdot q))^{\perp}$$

and locally identifying $Q/\mathbf{G}$ with the affine slice S_q and hence $T_{[q]}(Q/\mathbf{G})$ with $T_q S_q = (T_q(\mathbf{G} \cdot q))^{\perp}$, the metric $\tilde{\mathbf{g}}$ may be defined by:

$$\tilde{\mathbf{g}}([q])(\tilde{X}_{[q]}, \tilde{X}_{[q]}) = g(q)(\sigma_q(X_q), \sigma_q(X_q)) \tag{5}$$

for any $q \in Q$, $\tilde{X}_{[q]} \in T_{[q]}(Q/\mathbf{G})$ and $X_q \in T_q Q$ with $\sigma(X_q) \cong \tilde{X}_{[q]}$. As we required the group to operate isometrically, $\tilde{\mathbf{g}}$ is well defined and we have shown i).

In order to prove statement ii) we obviously only have to show that the isotropy group $\mathbf{G}_{\alpha_q}$ for any $\alpha_q \in \mathscr{J}^{-1}(0)$ coincides with $\mathbf{G}_q$. As necessarily $\mathbf{G}_{\alpha_q} \subset \mathbf{G}_q$, we have to show, that $g \cdot \alpha_q = \alpha_q$ for any $g \in \mathbf{G}_q$, or equivalently, that

$$\alpha_q(T_q \Phi_{g^{-1}} X_q) = \alpha_q(X_q) \tag{6}$$

for any $g \in \mathbf{G}_q$ and $X_q \in T_q Q$. As any point in S_q is left invariant by g, any tangent vector to S_q in q is left invariant as well. Furthermore, any tangent vector to the orbit $\mathbf{G} \cdot q$ in q is obviously mapped by $T_q \Phi_{g^{-1}}$ to another tangent vector to the orbit. Now, by the definition (2) of the momentum mapping, any tangent vector to the orbit is annihilated by α_q. As we may decompose any $X_q \in T_q Q$ into the sum of a vector tangential to the orbit and a vector tangential to S_q, (6) follows.

The invariance of $\mathscr{J}^{-1}(0)$ under **G** is obvious by the Ad*-equivariance of $\mathscr{J}$ and the statement $\pi(\mathscr{J}^{-1}(0)) = Q$ is trivial, as 0_q, mapping any $X_q \in T_q Q$ to zero, is contained in $\mathscr{J}^{-1}(0)$ for any $q \in Q$. Furthermore, as $\mathscr{J}^{-1}(0) \cap T_q Q$ is by the definition of the momentum mapping (2) just the annihilator of the tangent space to the orbit $\mathbf{G} \cdot q$, it is a subspace of $T_q^* Q$ for any q, and, as the dimension of the orbits is constant by the assumed conjugacy of all isotropy groups, its dimension is independent of the point q.

Now, the rest of statement iii) and statement iv) may be easily proved by the use of affine **G**-slices:

For iii), only the smoothness of $\mathscr{T}^{-1}(0)$ remains to be shown. For any **G**-slice S_q in Q, the local diffeomorphism τ induces a local diffeomorphism

$$\phi: T^*U \subset T^*Q \to T^*S_q \times T^*(\mathbf{G}/\mathbf{G}_q),$$

$$\alpha_q \to (T_{\tau^{-1}(q)}\tau)^*\alpha_q.$$

By this local diffeomorphism $\mathscr{J}^{-1}(0) \cap T^*U$ is mapped onto the set $T^*S_q \times \{0_{[g]} | [g] \in (\mathbf{G}/\mathbf{G}_q)\}$. Hence, $\mathscr{J}^{-1}(0)$ obviously is a subbundle of T^*Q, and by i) and ii), $\mathscr{J}^{-1}(0)/\mathbf{G}$ is a manifold with smooth projection

$$\chi: \mathscr{J}^{-1}(0) \to \mathscr{J}^{-1}(0)/\mathbf{G},$$

locally diffeomorphic to T^*S_q. On $\mathscr{J}^{-1}(0)/\mathbf{G}$ there is a symplectic form $\tilde{\omega}$, uniquely defined by $\chi^*\tilde{\omega} = i^*\omega$ with the canonical imbedding $i: \mathscr{J}^{-1}(0) \to T^*Q$.

Now, denoting by P the canonical projection $P: Q \to Q/\mathbf{G}$, we may define a mapping

$$\psi: T^*(Q/\mathbf{G}) \to \mathscr{T}^{-1}(0)/\mathbf{G},$$

$$\alpha_{[q]} \to \mathbf{G} \cdot ((T_qP)^*\alpha_{[q]}).$$

It may be easily checked that the mapping ψ is a well defined bijection. Now, using a $\mathbf{G}$-slice S_q in Q, the local diffeomorphism from $Q/\mathbf{G}$ to S_q induces a local symplectomorphism from $T^*(Q/\mathbf{G})$ to T^*S_q. On the other hand, the local diffeomorphism ϕ from T^*Q to $T^*S_q \times T^*(\mathbf{G}/\mathbf{G}_q)$ induces a symplectomorphism from $\mathscr{J}^{-1}(0)/\mathbf{G}$ to T^*S_q. Now, under these local identifications of $T^*(Q/\mathbf{G})$ and $\mathscr{J}^{-1}(0)/\mathbf{G}$ with T^*S_q, the mapping ψ corresponds to the identity on T^*S_q. Hence, the smoothness of ψ is obvious and ψ is a symplectomorphism.

So far, we assumed that there is only one orbit type in Q, ensuring the smoothness of the reduced configuration and phase space. If we drop this assumption, $Q/\mathbf{G}$ is no longer a manifold and the concept of a cotangent bundle over $Q/\mathbf{G}$ is meaningless. However, we can make several general statements about the structure of the orbits in Q which ensure that we only have to cut out a subset of measure zero in order to get a manifold with conjugate isotropy groups.

First, the requirement of a proper action guarantees that the topology of $Q/\mathbf{G}$ is not too bad:

Theorem 2. *Let Q be a manifold and $\mathbf{G}$ a Lie group acting properly on Q. Then $Q/\mathbf{G}$ is Hausdorff.*

By [9, Proposition 4.1.19], it is sufficient to show, that the set

$$R := \{(q, g \cdot q) | q \in Q, g \in \mathbf{G}\}$$

is a closed subset of $Q \times Q$. Now, if $(q_n, g_n q_n)$ is a convergent sequence in $Q \times Q$, then (q_n) and $(g_n q_n)$ are convergent sequences in Q. Defining $q := \lim_{n \to \infty} q_n$, there is—by the definition of a proper action—a convergent subsequence (g_{i_k}) of (g_n). With $g := \lim_{k \to \infty} g_{i_k}$ follows: $\lim_{n \to \infty} g_n q_n = \lim_{k \to \infty} g_{i_k} q_{i_k} = gq$ and, hence:

$$\lim_{n \to \infty} (q_n, g_n q_n) = (q, gq) \in R.$$

Thus, R is closed.

Using the fact, that $\mathbf{G}$ acts by proper isometries, which ensures the existence of $\mathbf{G}$-slices, we may make some general statements about the orbit structure of Q:

Theorem 3. *Let Q be a connected Riemannian manifold and G be a Lie group properly acting on Q by isometries. Let $m(x)$ be the number of components of the isotropy group $\mathbf{G}_x$ of the point $x \in Q$. For $u, v \in \mathbb{N}$ define:*

$$Q_{uv} := \{x \in Q \mid \dim(\mathbf{G} \cdot x) = u, m(x) = v\},$$

$$Q_u := \bigcup_{v \in \mathbb{N}} Q_{uv},$$

$$r := \max\{\dim(\mathbf{G} \cdot x) \mid x \in Q\},$$

$$s := \min\{m(x) \mid x \in Q, \dim(\mathbf{G} \cdot x) = r\}.$$

Then the following statements hold:

i) *For any point in Q there is a neighbourhood with only a finite number of orbit types. Especially, if Q is compact, there is only a finite number of orbit types in Q.*

ii) *For any $t < r$ the union of all orbits with dimension at most t has topological dimension at most $\dim Q - r + t - 1$.*

iii) *For any $u, v \in \mathbb{N}$ there is only one orbit type in any component of Q_{uv}.*

iv) *Q_r is connected.*

v) *Q_{rs} is an open dense subset of Q.*

vi) *$Q_{rs}/\mathbf{G}$ is connected. If $\mathbf{G}$ is connected, then Q_{rs} is connected.*

The statements of this theorem are well known for the action of compact groups [10–13]. In the more general context of proper actions by isometries, the proofs may be done completely analogous to those for the compact case, using the existence of $\mathbf{G}$-slices. Alternatively, most statements may be reduced to those for compact groups by the observation that for a proper action the isotropy group of any point is compact. Indeed, the definition for a proper action given above is equivalent to the requirement that for any compact subset $K \subset Q \times Q$ the set $\hat{\Phi}^{-1}(K)$ is compact, where $\hat{\Phi}$ denotes the mapping $\hat{\Phi}: \mathbf{G} \times Q \to Q \times Q, (g, q) \to (q, gq)$. As $\hat{\Phi}^{-1}(\{(q, q)\}) = \mathbf{G}_q \times q$, $\mathbf{G}_q$ obviously is compact.

Now, by the definition of a $\mathbf{G}$-slice, Q is locally diffeomorphic to $S_q \times \mathbf{G}/\mathbf{G}_q$, and most of the statements immediately follow from those for the action of the compact group $\mathbf{G}_q$ on S_q.

The only statements whose proofs are nontrivial are those concerning the connectedness of $Q_r, Q_{rs}/\mathbf{G}$ and Q_{rs}.

However, iv) is a consequence of ii) as the complement of Q_r may at most have topological dimension $\dim Q - 2$, and it is well known that a subset of a metric space with topological dimension D cannot separate the metric space unless it has topological dimension at least $D - 1$ (see [14]).

v) and vi) are most easily shown in complete analogy to the proofs for compact groups in [12], using the existence of $\mathbf{G}$-slices.

Now, iii), v) and vi) together imply the existence of one generic orbit type. Hence, by Theorem 1, $Q_{rs}/\mathbf{G}$ is a Riemannian manifold in a natural way, and there is an open dense subset $C \subset \mathcal{J}^{-1}(0)/\mathbf{G}$, such that $T^*(Q_{rs}/\mathbf{G})$ is symplectomorphic to C. Hence, we may consider a Schrödinger equation on the "reduced configuration space" $Q/\mathbf{G}$ by considering it as a differential equation on $Q_{rs}/\mathbf{G}$ and

taking the singular points into account by requiring suitable boundary conditions which may be determined by considering selfadjoint extensions of a Hamiltonian defined on the smooth functions with compact support in $Q_{rs}/\mathbf{G}$.

3. Schrödinger Equation on Cones

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold with metric g_M. We define the double cone $\mathcal{C}$ over $\mathcal{M}$ as $\mathbb{R} \times \mathcal{M}$ with the identification of all points of the form $(0, m)$ for arbitrary $m \in \mathcal{M}$, where the identified points form the tip of the cone. By cutting out the tip the punctured double cone $\mathcal{C}^*$ can be considered a Riemannian manifold with metric g given in coordinates $(r, x_1, \ldots, x_n)$ by:

$$ds^2 = dr^2 + r^2(g_M(\mathbf{x}))_{ik} dx^i \otimes dx^k. \tag{7}$$

Thus we may define a Schrödinger equation on the cone

$$(-\tfrac{1}{2}\Delta + V(r))\psi = E\psi \tag{8}$$

for an arbitrary real valued, smooth potential $V(r)$. Using (7) in the standard formula for the Laplacian yields:

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_M + 2(E - V)\right\}\psi = 0, \tag{9}$$

where Δ_M is the Laplacian on $\mathcal{M}$.

By the assumed compactness of $\mathcal{M}$ the spectrum of $-\Delta_M$ is purely discrete and nonnegative. For an arbitrary eigenvalue l^2 of $-\Delta_M$ with corresponding eigenfunction $\rho_l(\mathbf{x})$ the ansatz

$$\psi(r, \mathbf{x}) = \phi(r) \cdot \rho_l(\mathbf{x})$$

leads to the ordinary differential equation

$$\left\{\frac{d^2}{dr^2} + \frac{n}{r}\frac{d}{dr} - \frac{l^2}{r^2} + 2(E - V)\right\}\phi(r) = 0. \tag{10}$$

In the following we restrict to the cases of a harmonic potential $V(r) = 1/2\omega^2 r^2$ and of an identically vanishing potential. In both cases the Schrödinger equation may be explicitly solved. With the definition

$$\nu(l) = \sqrt{\frac{(n-1)^2}{4} + l^2} \tag{11}$$

two independent solution for $V(r) \equiv 0$ on the upper half of the double cone ($r > 0$) are:

$$\phi_r(r) = E^{1/4} r^{(1-n)/2} J_{\nu(l)}(\sqrt{2E}r), \tag{12}$$

$$\phi_s(r) = \begin{cases} E^{1/4} r^{(1-n)/2} J_{-\nu(l)}(\sqrt{2E}r), & \text{if } \nu(l) \notin \mathbb{N} \\ E^{1/4} r^{(1-n)/2} Y_{\nu(l)}(\sqrt{2E}r), & \text{if } \nu(l) \in \mathbb{N}, \end{cases} \tag{13}$$

where J_ν and Y_ν are the Bessel functions of the first and the second kind.

For the harmonic potential two independent solutions on the upper half cone are:

$$\phi_r(r) = e^{-(\omega/2)r^2} r^{((1-n)/2)+\nu(l)} M\left(\frac{1}{2}\left(1+\nu(l)-\frac{E}{\omega}\right), 1+\nu(l), \omega r^2\right), \tag{14}$$

$$\phi_s(r) = \begin{cases} e^{-(\omega/2)r^2} r^{((1-n)/2)+\nu(l)} U\left(\frac{1}{2}\left(1+\nu(l)-\frac{E}{\omega}\right), 1+\nu(l), \omega r^2\right), \\ \qquad\qquad \text{if} \quad -\frac{1}{2}\left(1+\nu(l)-\frac{E}{\omega}\right) \notin \mathbb{N} \\ e^{(\omega/2)r^2} r^{((1-n)/2)+\nu(l)} U\left(\frac{1}{2}\left(1+\nu(l)+\frac{E}{\omega}\right), 1+\nu(l), -\omega r^2\right), \\ \qquad\qquad \text{if} \quad -\frac{1}{2}\left(1+\nu(l)-\frac{E}{\omega}\right) \in \mathbb{N} \end{cases} \tag{15}$$

with the confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$.

Similarly we may define in both cases two independent solutions on the lower half cone by replacing r by $|r|$. Because of the singularity at 0 there is no a priori restriction on the solutions of the differential equation on both half cones. Thus, for fixed $\rho_l(r)$, the general solution of (9) depends on four arbitrary constants.

For both potentials the asymptotic behaviour of the solutions with $\nu(l) \neq 0$ is

$$\phi_r(r) \propto r^{((1-n)/2)+\nu(l)}(1+O(r^2))$$

$$\phi_s(r) \propto r^{((1-n)/2)-\nu(l)}(1+O(r^2)) + R(r) r^{((1-n)/2+\nu(l)}$$

with an at most logarithmically divergent function $R(r)$. For $\nu(l)=0$, i.e., for $n=1$ and $l=0$, the regular solution ϕ_r is analytical and ϕ_s is logarithmically divergent.

Already at this point we notice a clear tendency towards concentration of the wave function around $r=0$ as compared with the non-singular case, which becomes more marked for a smaller opening angle of the cone. This is clear if admixture of the singular solution occurs. But even for the regular solution one observes increasing dominance of $l=0$ over $l>0$ for decreasing opening angle.

The singular solution is locally square integrable at 0 iff $1-2\nu(l) > -1$, i.e. iff

$$\nu(l) < 1 \Leftrightarrow \frac{(n-1)^2}{4} + l^2 < 1. \tag{16}$$

Thus, the singular solution is excluded for $n \geqq 3$. For $n \in \{1,2\}$ however, the singular solution cannot be excluded for those values of l, for which (16) holds. As we can multiply the metric g_M with an arbitrary positive constant—corresponding to different opening angles of the cone—there may be an arbitrary number of such eigenvalues.

It is easy to see that in this case the admissible solutions of the Schrödinger equation form an overcomplete set. Especially, by combining the regular and the singular solution for $V(r) \equiv 0$, it is possible to form solutions which describe pure incoming waves which are completely absorbed in the tip and thus violate unitarity.

Furthermore, for the values of l fulfilling (16), the singular solution (15) of the Schrödinger equation with harmonic potential is square integrable for all positive values of the energy E. Consequently a further restriction of the set of admissible solutions is necessary.

An analogous problem arises for the usual Schrödinger equation in $\mathbb{R}^2$ or $\mathbb{R}^3$, written in polar coordinates. In this case the problem has its origin in the coordinate singularity which does not correspond to a real singularity of the configuration space. By using Cartesian coordinates one may see that the admissible solutions must be smooth even at the origin. So the singular solutions have to be abandoned. Contrary to this, in our problem there is a real singularity which cannot be removed by a change of the coordinate system: The double cone is not even a topological manifold. Therefore we cannot argue that the singular solution must be omitted.

In the functional analytical terminology of the following section the essential difference between $\mathbb{R}^2$ or $\mathbb{R}^3$ and the cone is that, in the first case, we may choose a larger domain for the Hamiltonian: In $\mathbb{R}^2$ or $\mathbb{R}^3$ differentiability at the origin is well defined. Hence, the domain of the Hamiltonian should at least contain the smooth functions with compact support on $\mathbb{R}^2$ or $\mathbb{R}^3$. But the Hamiltonian is already essentially selfadjoint on $\mathscr{C}_0^\infty(\mathbb{R}^2)$, respectively $\mathscr{C}_0^\infty(\mathbb{R}^3)$. Thus the problem with the polar coordinates arises only by the restriction to smooth functions with support outside the origin, necessary because of the coordinate singularity at the origin. On the cone however, a restriction of the domain is enforced by the singularity and there is no longer a natural choice of a domain on which the Hamiltonian is essentially selfadjoint.

In this situation one might try to find other physical arguments which restrict the set of solutions of the Schrödinger equation. One might for example look for another physical observable and require the eigenfunctions of the Hamiltonian to be elements of its domain. The only operators which—at first sight—could seem to be suitable to this end are the "radial momentum operator"

$$p_r := \frac{1}{2}\left\{ -i\frac{\partial}{\partial r} + \left(-i\frac{\partial}{\partial r} \right)^\dagger \right\} = -i\left(\frac{\partial}{\partial r} + \frac{n}{2r} \right),$$

and its powers. However, there are a lot of problems with this operator which render the above idea impossible: First, for the single cone, the radial momentum operator has deficiency indices $(1,0)$ on the domain of the smooth functions with compact support on the punctured single cone. Therefore, there are no selfadjoint extensions, and the radial momentum operator does not correspond to any physical observable. Next, even for the double cone, where the deficiency indices of p_r are $(1,1)$ and selfadjoint extensions exist, neither the regular nor the singular nor any linear combination of the regular and the singular solution of the Schrödinger equation for $n = 1$ and $l \neq 0$ is contained in the domain of any selfadjoint extension of p_r. Last, in the case of $n = 2$ and $l = 0$, the regular solution is an element of the domain of any, and the singular solution is in the domain of some selfadjoint extensions of p_r. Therefore, even in that case, an unambiguous discrimination of the solutions is not possible in this way.

The next simple operator, the square of the radial momentum operator, is even less suitable for our purposes: For any $l \neq 0$ which fulfills (16) none of the solutions

of (10) is an element of the domain of any selfadjoint extension of p_r^2, whereas on the eigenspace of $-\Delta_M$ to the eigenvalue $l=0$ the Hamiltonian and p_r^2 are—apart from a constant factor and the potential term—the same operators and thus endowed with the same problems.

Thus, it is not possible to find any simple physical argument which allows a sufficient restriction of the set of the admissible solutions of the Schrödinger equation. By the requirement of probability conservation, which is equivalent to the selfadjointness of the Hamiltonian, we are thus led to the study of all physically reasonable selfadjoint extensions of the symmetric Hamiltonian

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_M\right) + V(r) \tag{17}$$

defined on a suitable domain.

3.1 Selfadjoint Extensions and Boundary Conditions. Apparently, any reasonable choice of the domain of the Hamiltonian (17) should at least contain the smooth functions with compact support on the punctured cone. On the other hand, it is not possible to choose as the domain of H the set of all smooth functions on $\mathscr{C}^*$, because H is no longer symmetric on this domain. Thus—apart from the more technical aspect of forming the closure of this operator—the natural choice for the domain H, on which it is symmetric, but not necessarily selfadjoint, is:

$$\mathscr{D} = \mathscr{C}_0^\infty(\mathscr{C}^*). \tag{18}$$

By the well known regularity theorems for the Laplacian any weak solution of the equation $H\psi = \lambda\psi (\lambda \in \mathbb{C})$ is a strong solution and even a smooth function. Hence, all elements of the deficiency subspaces $\mathscr{H}_\pm = \ker(H^\dagger \mp i)$ are smooth functions on $\mathscr{C}^*$ which satisfy the differential equation (9) with E replaced by $\pm i$. From the asymptotic behaviour of these functions at the tip and at infinity follows that the deficiency subspaces $\mathscr{H}_\pm$ for the double cone with $V(r) \equiv 0$ are spanned by the functions:

$$\phi_\pm(r, \mathbf{x}) = (\alpha\theta(r) + \beta\theta(-r))|r|^{(1-n)/2} H^{(1)}_{\nu(l)}(\sqrt{2}e^{\pm i\pi/4}|r|)\rho_l(\mathbf{x}) \tag{19}$$

with arbitrary $\alpha, \beta \in \mathbb{C}$ and $\nu(l) < 1$, where $H_\nu^{(l)}$ denotes the Hankel function. Similarly, for $V(r) = \frac{1}{2}\omega^2 r^2$, the deficiency subspaces are spanned by:

$$\phi_\pm(r, \mathbf{x}) = (\alpha\theta(r) + \beta\theta(-r))e^{-(\omega/2)r^2}|r|^{((1-n)/2)+\nu(l)} \cdot U\left(\frac{1}{2}\left(1 + \nu(l) \mp \frac{i}{\omega}\right), 1 + \nu(l), \omega r^2\right)\rho_l(\mathbf{x}). \tag{20}$$

The deficiency subspaces for the single cone may be obtained by simply restricting those functions to the upper half cone. Hence, with $\mathscr{E}_l$ being the eigenspace of Δ_M to the eigenvalue $-l^2$, the deficiency indices of the Hamiltonian on the double cone are:

$$\eta_+(H) = n_-(H) = \sum_{\nu(l)<1} (2 \dim \mathscr{E}_l), \tag{21}$$

whereas they take only half that value on the single cone. By the standard theorems

on selfadjoint extensions the equality of the deficiency indices guarantees the existence of selfadjoint extensions of H, which are parametrized by the unitary mappings from $\mathscr{H}_+$ onto $\mathscr{H}_-$ and thus depend on $(n_+(H))^2$ real parameters.

With the identification

$$\mathscr{H} = L^2(\mathscr{C}^*, d\mu) = \bigoplus_l (L^2(\mathbb{R}, |r|^n dr) \otimes \mathscr{E}_l),$$

we may write the Hamiltonian on the double cone as:

$$H = \bigoplus_l (h_l \otimes \mathrm{id}) \tag{22}$$

with

$$h_l = -\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{n}{r}\frac{d}{dr} - \frac{l^2}{r^2}\right) + V(r). \tag{23}$$

Furthermore, the domain of the Hamiltonian may be restricted to

$$\mathscr{D} = \bigoplus_l (\mathscr{C}_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathscr{E}_l) \tag{24}$$

where only finite linear combinations are allowed, because the closures of the operators with the domains (18) and (24) are obviously the same.

In order to simplify concrete calculations, we restrict in the following to the special class of those selfadjoint extensions, which may be obtained by forming selfadjoint extensions of the operators h_l. Apart from computational convenience, this restriction may be motivated by the fact that we want to maintain the commutativity of the Hamiltonian and the observable $-\Delta_M$, which at least restricts the set of all admissible extensions to those which may be obtained by extending the operators $h_l \otimes \mathrm{id}$ in $L^2(\mathbb{R}, |r|^n dr) \otimes \mathscr{E}_l$. Furthermore, if the "$\mathscr{M}$-part" $\rho_l(\mathbf{x})$ is not allowed to change the tunneling through the tip, the stronger restriction follows. Especially, in the case of $\mathscr{M}$ being a sphere S^n on which the rotation group $\mathrm{SO}(n)$ operates in the natural way, we may identify $-\Delta_M$ with the square of the angular momentum operator. In this case, the restriction of the set of all selfadjoint extensions to the set of those which are obtained by extending the operators h_l is just equivalent to the requirement of commutativity of the Hamiltonian and the angular momentum operator, i.e., to rotational invariance. Hence, more general selfadjoint extensions would correspond to a symmetry breaking due to a rotational noninvariance strictly localized in the tip. However, even if we dropped this restriction on the selfadjoint extensions the essential results of the following computations would stay the same. The only real difference would be, that the eigenfunctions of the extensions of the Hamiltonian would no longer be eigenfunctions of $-\Delta_M$ and that, in the case of free motion on the cone, there were solutions corresponding to a tunneling through the tip, where the "angular momentum" changes in tunneling.

Thus, we are finally led to the study of all selfadjoint extensions of the operators h_l with domain $\mathscr{D}(h_l) = \mathscr{C}_0^\infty(\mathbb{R}\backslash\{0\})$. Obviously, the deficiency subspaces for those operators are spanned by the "radial part" of the functions (19), respectively (20). Hence, the operators h_l with $\nu(l) \geqq 1$ are essentially selfadjoint, whereas the

deficiency indices of the operators h_l with $v(l) < 1$ are $(2,2)$. By choosing

$$\phi^1_\pm(r) = \theta(r)|r|^{(1-n)/2} H^{(1)}_{v(l)}(\sqrt{2}e^{\pm i\pi/4}|r|),$$
$$\phi^2_\pm(r) = \theta(-r)|r|^{(1-n)/2} H^{(1)}_{v(l)}(\sqrt{2}e^{\pm i\pi/4}|r|)$$

for $V(r) \equiv 0$, or

$$\phi^1_\pm(r) = \theta(r)e^{-(\omega/2)r^2}|r|^{((1-n)/2)+v(l)} U\left(\frac{1}{2}\left(1 + v(l) \mp \frac{i}{\omega}\right), 1 + v(l), \omega r^2\right),$$
$$\phi^2_\pm(r) = \theta(-r)e^{-(\omega/2)r^2}|r|^{((1-n)/2)+v(l)} U\left(\frac{1}{2}\left(1 + v(l) \mp \frac{i}{\omega}\right), 1 + v(l), \omega r^2\right)$$

for $V(r) = \frac{1}{2}\omega^2 r^2$, respectively, as an orthogonal base of the deficiency subspaces, the selfadjoint extensions of any operator h_l with $v(l) < 1$ are in one-to-one correspondence to the unitary 2×2-matrices and are given by:

$$\mathscr{D}(h_l^U) = \mathscr{D}(\overline{h_l}) \oplus \mathbb{C}(\phi^1_+ + U_{11}\phi^1_- + U_{12}\phi^2_-) \oplus \mathbb{C}(\phi^2_+ + U_{21}\phi^1_- + U_{22}\phi^2_-),$$
$$h_l^U f = \left[-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{n}{r}\frac{d}{dr} - \frac{l^2}{r^2}\right) + V(r)\right] f \tag{25}$$

for any $U \in U(2)$. Here, the domain $\mathscr{D}(\overline{h_l})$ of the closure of h_l is explicitly given by:

$$\mathscr{D}(\overline{h_l}) = \begin{cases} \{f \in \mathscr{D}(h_l^\dagger) \mid \lim_{r\to 0} \phi(r) = \lim_{r\to 0} (r^{(n-1)/2}\phi'(r)) = 0\} & \text{if} \quad 0 \leqq v(l) < 1 \\ \mathscr{D}(h_l^\dagger) & \text{if} \quad v(l) \geqq 1. \end{cases} \tag{26}$$

with

$$\mathscr{D}(h_l^\dagger) = \Big\{ f \in \mathscr{C}^1(\mathbb{R}\backslash\{0\}) \mid f' \text{ is absolutely continuous on } \mathbb{R}\backslash\{0\},$$
$$\left[-\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{n}{r}\frac{d}{dr} - \frac{l^2}{r^2}\right) + V(r)\right] f \in L^2(\mathbb{R}, |r|^n dr) \Big\}.$$

However, there are certain restrictions on the physically reasonable extensions: First, in the case of the double cone, the physically acceptable extensions should respect the strict symmetry between both half cones. Furthermore, invariance under time reversal should hold. Either requirement reduces the allowed unitary matrices to those, which are of the form

$$U = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in U(2).$$

Here, by the unitary of U:

$$a = -\cos\psi e^{i\theta}, \quad b = -i\sin\psi e^{i\theta}$$

for some $\theta, \psi \in \mathbb{R}$. It is possible to characterize the selfadjoint extensions by generalized boundary conditions. To this end, we define for $f \in \mathscr{D}(h_l^U)$:

$$S_{\pm}(f) := \lim_{r \to 0^{\pm}} |r|^{(n-1)/2 + \nu(l)} f(r),$$
$$R_{\pm}(f) := \lim_{r \to 0^{\pm}} |r|^{(n-1)/2 - \nu(l)} (f(r) - S_{\pm}(f)|r|^{(1-n)/2 - \nu(l)})$$

if $0 < \nu(l) < 1$, and

$$S_{\pm}(f) := \lim_{r \to 0^{\pm}} (\log|r|)^{-1} f(r),$$
$$R_{\pm}(f) := \lim_{r \to 0^{\pm}} (f(r) - S_{\pm}(f) \log|r|),$$

if $\nu(l) = 0$, respectively.

By (25), (26) and the asymptotic behaviour of the deficiency functions at the tip, $R_{\pm}(f)$ and $S_{\pm}(f)$ are well defined for any $f \in \mathscr{D}(h_l^U)$. Using (25) it is easy to check that any solution $\phi(r)$ of the differential Eq. (9) is contained in the domain of h_l^U if and only if there are constants $\lambda, \mu \in \mathbb{C}$ with:

$$\begin{aligned} R_{\pm}(\phi) &= \lambda\{R_{\pm}(\phi_+^1) + aR_{\pm}(\phi_-^1) + bR_{\pm}(\phi_-^2)\} \\ &\quad + \mu\{R_{\pm}(\phi_+^2) + bR_{\pm}(\phi_-^1) + aR_{\pm}(\phi_-^2)\}, \\ S_{\pm}(\phi) &= \lambda\{S_{\pm}(\phi_+^1) + aS_{\pm}(\phi_-^1) + bS_{\pm}(\phi_-^2)\} \\ &\quad + \mu\{S_{\pm}(\phi_+^2) + bS_{\pm}(\phi_-^1) + aS_{\pm}(\phi_-^2)\}. \end{aligned}$$

Using the obvious identities

$$\begin{aligned} &R_+(\phi_+^1) = R_-(\phi_+^2) = R_+(\phi_+^1)^* = R_-(\phi_-^2)^*, \\ &S_+(\phi_+^1) = S_-(\phi_+^2) = S_+(\phi_-^1)^* = S_-(\phi_-^2)^*, \\ &R_+(\phi_+^2) = R_-(\phi_+^1) = R_+(\phi_-^2) = R_-(\phi_-^1) = 0, \\ &S_+(\phi_+^2) = S_-(\phi_+^1) = S_+(\phi_-^2) = S_-(\phi_-^1) = 0 \end{aligned} \tag{27}$$

and the definitions:

$$\delta := \mathfrak{Re}\left(\frac{R_+(\phi_+^1)}{S_+(\phi_+^1)}\right), \quad \varepsilon := \mathfrak{Im}\left(\frac{R_+(\phi_+^1)}{S_+(\phi_+^1)}\right) \tag{28}$$

the constants λ, μ may be eliminated, yielding for $b^2 \neq (1+a)^2$, i.e., $\theta \neq \pm\psi$, the boundary conditions:

$$\begin{aligned} R_+(\phi) &= C_1 S_+(\phi) + C_2 S_-(\phi) \\ R_-(\phi) &= C_2 S_+(\phi) + C_1 S_-(\phi)' \end{aligned} \tag{29}$$

with

$$C_1 = \delta + \frac{\varepsilon \sin\theta}{\cos\theta - \cos\psi}, \quad C_2 = \frac{-\varepsilon \sin\psi}{\cos\theta - \cos\psi}.$$

For $b^2 = (1+a)^2$ the boundary conditions are:

$$\begin{aligned} S_+(\phi) &= \sigma S_-(\phi), \\ R_-(\phi) &= -\sigma R_+(\phi) + 2\sigma(\delta - \varepsilon \cot\theta) S_+(\phi), \end{aligned} \tag{30}$$

if $b = \sigma(1 + a) \neq 0$ with $\sigma \in \{\pm 1\}$, i.e., if $\theta = \sigma\psi$, and

$$S_+(\phi) = S_-(\phi) = 0, \tag{31}$$

if $b = 0$, $a = -1$, respectively.

Completely analogous considerations may be made for the single cone. Here, any of the operators h_l with $\nu(l) < 1$ has deficiency indices $(1, 1)$. The selfadjoint extensions are parametrized by $U(1)$, i.e., by one angle θ. The boundary conditions may simply be obtained from those for the double cone by setting $\psi = 0$ and restricting to $r \geqq 0$:

$$R_+(\phi) = C_1 S_+(\phi), \tag{32}$$

with

$$C_1 = \delta + \frac{\varepsilon \sin\theta}{\cos\theta - 1} \tag{33}$$

if $\theta \neq 0$,
and

$$S_+(\phi) = S_-(\phi) = 0, \tag{34}$$

if $\theta = 0$.

3.2 Spectra and Eigenfunctions for $V(r) = \frac{1}{2}\omega^2 r^2$. By our restriction of the admissible selfadjoint extensions, the spectrum of the Hamiltonian is just the union of the spectra of the operators h_l. For $\nu(l) \geqq 1$ the singular solution (15) never is square integrable, whereas the regular solution (14) is square integrable iff:

$$E = (2j + 1 + \nu(l))\omega \tag{35}$$

with $j \in \mathbb{N}$, which is essentially the same relation as for the usual harmonic oscillator in $\mathbb{R}^n$. The eigenfunctions of the Hamiltonian to these eigenvalues are

$$(\alpha\theta(r) + \beta\theta(-r))\phi_r(|r|)\rho_l(\mathbf{x})$$

with arbitrary $\alpha, \beta \in \mathbb{R}$. Obviously, for these values of l, the half cones decouple and eigenfunctions exist which are completely localized on one half cone. Hence, for $n \geqq 3$ the spectrum of the Hamiltonian is just the union of the equidistant eigenvalues (35) with $l \in \mathrm{spec}(-\Delta_M)$, and both half cones completely decouple.

For $\nu(l) < 1$ the spectrum of h_l is generally more complicated and depends on the selfadjoint extension chosen. As we required the selfadjoint extensions to respect to strict symmetry between both half cones any eigenfunction of the Hamiltonian (17) is either symmetric, antisymmetric or may be decomposed into a symmetric and an antisymmetric eigenfunction. Hence, without loss of generality, we may assume

$$R_+(\phi) = \pm R_-(\phi) \quad S_+(\phi) = \pm S_-(\phi),$$

and the boundary conditions (29) for $\theta \neq \pm\psi$ may be simplified to:

$$R_+(\phi) = \hat{C}_\pm S_+(\phi), \tag{36}$$

where $\hat{C}_\pm = C_1 \pm C_2$, and the plus sign has to be chosen for the symmetric, the

minus sign for the antisymmetric eigenfunctions. In the other cases the boundary conditions for the symmetric solutions are:

$$\begin{aligned} &R_+(\phi) = (\delta - \varepsilon \cot\theta) S_+(\phi) \quad \text{if} \quad \theta = \psi \neq 0, \\ &S_+(\phi) = 0 \qquad\qquad\qquad\quad \text{if} \quad \theta = -\psi, \end{aligned} \tag{37}$$

and similarly for the antisymmetric eigenfunctions:

$$\begin{aligned} &S_+(\phi) = 0 \qquad\qquad\qquad\quad \text{if} \quad \theta = \psi, \\ &R_+(\phi) = (\delta - \varepsilon \cot\theta) S_+(\phi) \quad \text{if} \quad \theta = -\psi \neq 0. \end{aligned} \tag{38}$$

If we allow the value ∞ for $\hat{C}_\pm$, where (36) with $\hat{C}_\pm = \infty$ has to be interpreted as $S_+(\phi) = 0$, (37) and (38) may be considered as special cases of (36), and the selfadjoint extensions of any h_l with $\nu(l) < 1$ are via the boundary conditions (36) is one-to-one correspondence with the pairs $(\hat{C}_+, \hat{C}_-) \in (\mathbb{R} \cup \{\infty\})^2$.

By the asymptotic behaviour of the confluent hypergeometric functions at ∞ the radial part for $r > 0$ of any eigenfunction with $-\frac{1}{2}(1 + \nu(l) - E/\omega) \notin \mathbb{N}$ must be of the form (15), whereas, in the case $-\frac{1}{2}(1 + \nu(l) - E/\omega) \in \mathbb{N}$ it must be of the form (14). These functions are easily seen to fulfill the boundary conditions (36) for $\nu(l) \neq 0$ iff

$$\frac{\Gamma\left(\frac{1}{2}\left(1 + \nu(l) - \frac{E}{\omega}\right)\right)}{\Gamma\left(\frac{1}{2}\left(1 - \nu(l) - \frac{E}{\omega}\right)\right)} = -\hat{C}_\pm \omega^{-\nu(l)} \frac{\Gamma(1 + \nu(l))}{\Gamma(1 - \nu(l))}, \tag{39}$$

whereas the condition for $\nu(l) = 0$, i.e., for $n = 1$ and $l = 0$, reads:

$$\frac{1}{2}\left[\psi\left(\frac{1}{2}\left(1 - \frac{E}{\omega}\right)\right) + 2\gamma + \log\omega\right] = \hat{C}_\pm. \tag{40}$$

Hence, the spectrum of h_l consists of all E fulfilling (39) or (40) for $\hat{C}_+$ or $\hat{C}_-$, depending on the symmetry type of the eigenfunction and on $\nu(l)$.

In Fig. 1, the eigenvalues for a fixed symmetry type and $\nu(l) = 0.5$ are sketched as a function of

$$\tilde{C}_\pm := -\hat{C}_\pm \omega^{-\nu(l)} \frac{\Gamma(1 + \nu(l))}{\Gamma(1 - \nu(l))}.$$

Obviously, for $\hat{C}_\pm \neq \infty$, the eigenvalues for fixed l and fixed symmetry type are no longer equidistant but are of the form:

$$E_{jl} = (2j + 1 - \nu(l))\omega + \varepsilon_{jl} \tag{41}$$

with

$$2(\nu(l) - 1)\omega < \varepsilon_{jl} < 2\nu(l)\omega \quad \text{and} \quad \lim_{j \to \infty} \varepsilon_{jl} = 0.$$

Thus, even asymptotically, the eigenvalues for the generic extensions are different from those for $\hat{C}_\pm = \infty$, which are of the form (35).

Completely analogous results hold for the single cone: The two parameters $\hat{C}_\pm$

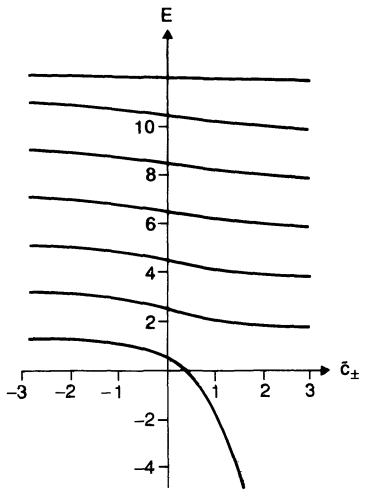

Fig 1. The eigenvalues of the symmetric or antisymmetric eigenfunctions of the operator h_l with oscillator potential for $v(l) = 0.5$ and $\omega = 1$ as a function of $\tilde{C}_\pm$

just have to be replaced by the one parameter C_1 which, according to (33), is a function of one angle θ. The resulting eigenvalues as a function of this angle θ are depicted in Fig. 2 for two values of $v(l)$. Here, the points $\theta = \pm\pi$ must be identified, as the selfadjoint extensions are parametrized by $U(1)$ and not by $\mathbb{R}$.

The most striking features of these spectra are:
- the existence of an eigenvalue with an arbitrary low negative energy, depending on the selfadjoint extension
- the shift of the eigenvalues when θ changes from 0 to $-\pi \sim \pi$ and back from π to 0
- the vanishing of the lowest eigenvalue, which tends to $-\infty$ as $\theta \to 0^+$.

Especially, as the constants $C_{1,2}$ may be chosen independently for any h_l with $v(l) < 1$, the ground state of the Hamiltonian H may be an eigenfunction of $-\Delta_M$ to any eigenvalue l with $v(l) < 1$ and arbitrarily low energy, depending on the selfadjoint extension. Especially, the ground state may be highly degenerate.

It may be shown by simple estimates that the eigenfunction corresponding to the lowest eigenvalue of any operator h_l becomes more and more concentrated at the tip as θ decreases to zero and the energy decreases to $-\infty$. If one tries to give a physical interpretation of the different selfadjoint extensions in terms of a kind of highly singular potential strictly localized in the tip, this behaviour might be interpreted as a tighter and tighter binding of the lowest eigenstate by a more and more attractive singular potential. In the limit $\theta = 0$ the formerly lowest eigenstate becomes completely concentrated in the tip with $E = -\infty$, but completely decouples from the other states. Hence, the next eigenstate becomes the real lowest eigenstate.

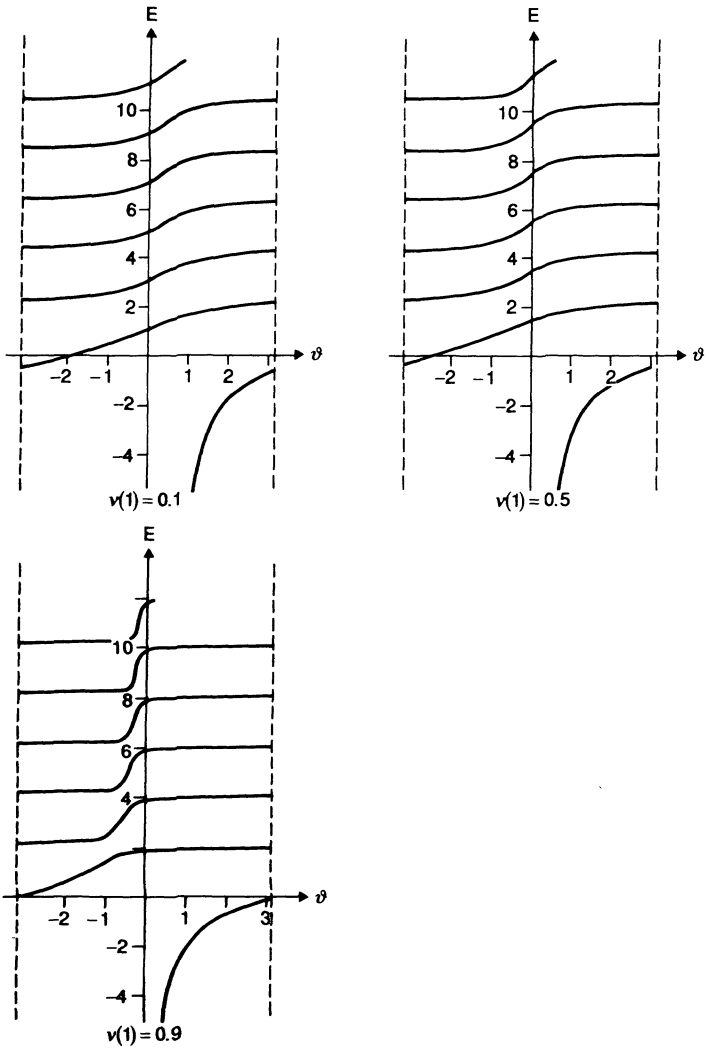

Fig. 2. The eigenvalues of h_l with oscillator potential on the single cone for different values of $\nu(l)$ and $\omega = 1$ as a function of the angle θ

3.3 Spectrum and Tunneling Probabilities for $V(r) \equiv 0$

3.3.1 Spectra and Eigenvalues. Before considering the scattering at the tip in detail, we first discuss the spectra of the operators h_l and hence of the Hamiltonian for the free motion on the double cone. As will be explicitly seen in the next section, there are generalized eigenfunctions of h_l for any $E \geqq 0$. Hence, the spectrum of any h_l contains $\mathbb{R}_0^+$. On the other hand, it is easy to see that any $\lambda \in \text{spec}(h_l) \setminus \mathbb{R}_0^+$ necessarily is an eigenvalue of h_l and we have to examine the question, whether there are any negative eigenvalues of h_l for $\nu(l) < 1$.

By the asymptotic behaviour of the Bessel functions at infinity, the restriction to $\mathbb{R}^+$ of any eigenfunction of h_l with $E<0$ must be of the form:

$$\phi(r) \propto \begin{cases} e^{-i\pi\nu(l)}\phi_r(r) - \phi_s(r), & \text{if } 0<\nu(l)<1 \\ \phi_r(r) + i\phi_s(r), & \text{if } \nu(l)=0 \end{cases} \tag{42}$$

with ϕ_r, ϕ_s given by (12), (13). From the asymptotic behaviour of the Bessel function at 0 follows for the function (42):

$$\frac{R_+(\phi)}{S_+(\phi)} = \begin{cases} -\xi|E|^{\nu(l)} & \text{if } 0<\nu(l)<1 \\ \frac{1}{2}\log E - \frac{1}{2}\log 2 + \gamma, & \text{if } \nu(l)=0 \end{cases}$$

with $\xi := 2^{-\nu(l)}\Gamma(1-\nu(l))/\Gamma(1+\nu(l))$.

As (42) is the restriction of a symmetric or antisymmetric eigenfunction of h_l to $\mathbb{R}^+$ iff it satisfies the boundary condition (36), there is at most one symmetric and one antisymmetric eigenfunction for any h_l with $\nu(l)<1$. For the different extensions, the eigenfunctions and eigenvalues are explicitly given by:

1. $\nu(l)\neq 0$:
(a) $b^2 \neq (1+a)^2$:
- one symmetric eigenfunction with energy

$$E = -\xi^{-1}|C_+|^{1/\nu(l)} \text{ exists iff } C_+ < 0$$

- one antisymmetric eigenfunction with energy

$$E = -\xi^{-1}|C_-|^{1/\nu(l)} \text{ exists iff } C_- < 0$$

(b) $b=(1+a)\neq 0$:
- one symmetric eigenfunction with energy

$$E = -\xi^{-1}|\delta - \varepsilon\cot\theta|^{1/\nu(l)} \text{ exists iff } \delta - \varepsilon\cot\theta < 0$$

(c) $b=-(1+a)\neq 0$:
- one antisymmetric eigenfunction with energy

$$E = -\xi^{-1}|\delta - \varepsilon\cot\theta|^{1/\nu(l)} \text{ exists iff } \delta - \varepsilon\cot\theta < 0$$

(d) $b=1+a=0$:
- no eigenfunctions exist

2. $\nu(l)=0$:
(a) $b^2 \neq (1+a)^2$:
- one symmetric eigenfunction with energy

$$E = -2e^{-2(C_+ + \gamma)}$$

- one antisymmetric eigenfunction with energy

$$E = -2e^{-2(C_- + \gamma)}$$

(b) $b=(1+a)\neq 0$:

- one symmetric eigenfunction with energy

$$E = -2e^{2(\delta - \varepsilon\cot\theta - \gamma)}$$

(c) $b = -(1+a) \neq 0$:
- one antisymmetric eigenfunction with energy

$$E = -2e^{2(\delta - \varepsilon \cot\theta - \gamma)}$$

(d) $b = 1 + a = 0$:
- no eigenfunction

Especially, for $n = 1, h_0$ and hence the Hamiltonian H has for any generic extension at least one negative eigenvalue, which may be arbitrarily low. Again by simple estimates, it may be shown that—as in the case of a harmonic potential—the eigenfunctions of any h_l and hence of the Hamiltonian become more and more concentrated near the tip as the energy tends to $-\infty$ by a continuous change of the selfadjoint extension.

3.3.2 Scattering at the Tip. In the case of a vanishing potential on the double cone we are not primarily interested in the symmetric and antisymmetric eigenfunctions but rather in scattering solutions of the form:

$$\begin{aligned}\psi(r, \mathbf{x}) = [&\theta(r)\{(\phi_s(r) - e^{i\pi\nu(l)}\phi_r(r)) + \mu(e^{-i\pi\nu(l)}\phi_r(r) - \phi_s(r))\} \\ &+ \theta(-r)\{e^{-i\pi\nu(l)}\phi_r(|r|) - \phi_s(|r|)\}]\rho_l(\mathbf{x}).\end{aligned} \tag{43}$$

Obviously, a function of this form has the asymptotic behaviour

$$\begin{aligned}\psi(r, \mathbf{x}) = [&\theta(r)\{e^{(i\pi/2)(\nu(l)-1/2)}e^{-i\sqrt{2E}r} - \mu e^{-(i\pi/2)(\nu(l)-1/2)}e^{i\sqrt{2E}r}\} \\ &- \rho\theta(-r)e^{-(i\pi/2)(\nu(l)-1/2)}e^{i\sqrt{2E}|r|}]|r|^{-n/2}\left(1 + O\left(\frac{1}{r}\right)\right)\rho_l(\mathbf{x})\end{aligned}$$

as $|r| \to \infty$, thus describing a wave incoming on the upper half cone, which is partially reflected at the tip and partially tunnels through the tip. The reflecting and tunneling probabilities are just $|\mu|^2$ and $|\rho|^2$, respectively.

For $\nu(l) \geqq 1$, the solution must be regular at the tip. Hence, $\rho = 0$, $\mu = 1$ and no tunneling through the tip is possible.

For $\nu(l) < 1$ however, it may be shown, either by a limit process or by the use of generalized eigenfunctions in a Gelfand triple, that ψ is an admissible scattering solution iff it satisfies the boundary conditions at the tip. This condition uniquely determines the values of the coefficients μ and ρ, yielding, with the abbreviation $\xi := 2^{-\nu(l)}\Gamma(1-\nu(l))/\Gamma(1+\nu(l))$, for the different cases:

1. $\nu(l) \neq 0$:
(a) $b^2 \neq (1+a)^2$:

$$\mu = \frac{|C_1 + e^{i\pi\nu(l)}\xi E^{\nu(l)}|^2 - C_2^2}{(C_1 + e^{-i\pi\nu(l)}\xi E^{\nu(l)})^2 - C_2^2},$$

$$\rho = \frac{-2iC_2\xi E^{\nu(l)}\sin(\pi\nu(l))}{(C_1 + e^{-i\pi\nu(l)}\xi E^{\nu(l)})^2 - C_2^2}.$$

(b) $b = \sigma(1+a)$:

$$\mu = \frac{\xi\cos(\pi\nu(l))E^{\nu(l)} + \delta - \varepsilon\cot\theta}{e^{-i\pi\nu(l)}\xi E^{\nu(l)} + \delta - \varepsilon\cot\theta},$$

$$\rho = \frac{i\sigma\xi E^{v(l)} \sin(\pi v(l))}{e^{-i\pi v(l)}\xi E^{v(l)} + \delta - \varepsilon \cot\theta}.$$

(c) $b = 1 + a = 0$:

$$\mu \equiv 1,$$
$$\rho = 0.$$

2. $v(l) = 0$:
(a) $b^2 \neq (1+a)^2$:

$$\mu = \frac{-\frac{\pi^2}{4} - (u(E) - C_1)^2 + C_2^2}{\left(\frac{\pi}{2} + iu(E) - iC_1\right)^2 + C_2^2},$$

$$\rho = \frac{-i\pi C_2}{\left(\frac{\pi}{2} + iu(E) - iC_1\right)^2 + C_2^2},$$

with $u(E) = \gamma + \frac{1}{2}\log E - \frac{1}{2}\log 2$.
(b) $b = \sigma(1+a)$:

$$\mu = i\frac{u(E) - \delta + \varepsilon\cos\theta}{\frac{\pi}{2} + i(u(E) - \delta + \varepsilon\cot\theta)},$$

$$\rho = -\frac{\sigma\pi}{\pi + 2i(u(E) - \delta + \varepsilon\cot\theta)}.$$

(c) $b = 1 + a = 0$:

$$\mu \equiv 1,$$
$$\rho \equiv 0.$$

The resulting tunneling probabilities $p(E) = |\rho|^2$ as a function of the energy E are depicted in Fig. 3 for some values of $v(l)$ and $C_{1,2}$. Depending on the selfadjoint extensions, i.e., the two angle θ, ψ, the following qualitative features of the tunneling probabilities are possible:

1. $v(l) \neq 0$:

(a) $b^2 \neq (1+a)^2$:

- $p(0) = 0$, $\lim_{E\to\infty} p(E) = 0$, one maximum with value 1
- $p(0) = 0$, $\lim_{E\to\infty} p(E) = 0$, two maxima with value 1 and one minimum
- $p(0) = 0$, $\lim_{E\to\infty} p(E) = 0$, one maximum with value $\leqq 1$
- $p(0) > 0$, $\lim_{E\to\infty} p(E) = 0$, p monotonously decreasing
- $p(0) > 0$, $\lim_{E\to\infty} p(E) = 0$, one maximum with value 1

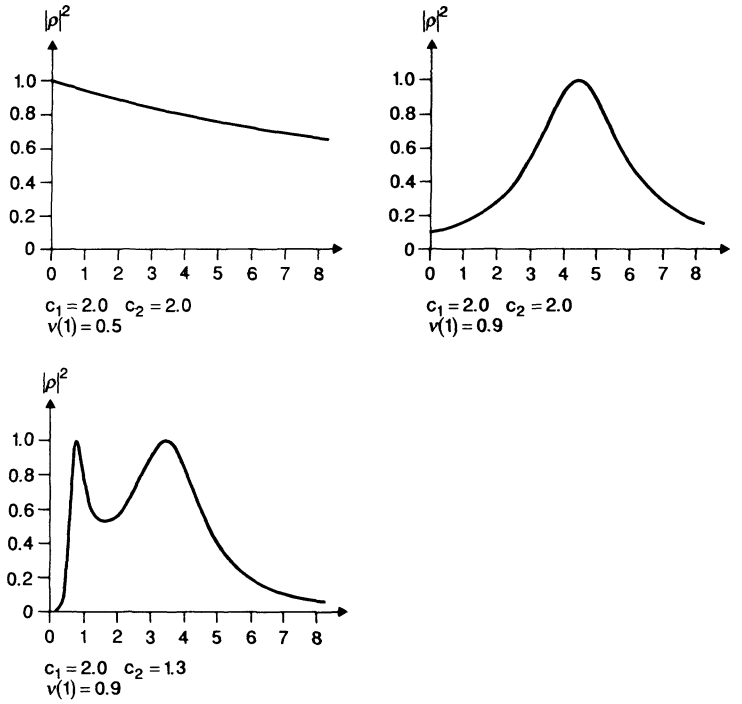

Fig. 3. Tunneling probabilities for the free motion on the double cone for different values of $c_{1,2} := \xi^{-1} C_{1,2}$ and $\nu(l)$

(b) $b = \sigma(1 + a) \neq 0$:

- $p(0) = 0$, $\lim_{E \to \infty} p(E) > 0$, p monotonously increasing
- $p(0) = 0$, $\lim_{E \to \infty} p(E) > 0$, one maximum with value 1
- $p = \text{const.} < 1$.

(c) $b = 1 + a = 0$:

- $p \equiv 0$

2. $\nu(l) = 0$:
Here, $p(0) = 0$, $\lim_{E \to \infty} p(E) = 0$ holds for any extension

(a) $b^2 \neq (1 + a)^2$:

- one maximum with value < 1
- two maxima with value 1

(b) $b = \sigma(1 + a) \neq 0$:

- one maximum with value 1

(c) $b = 1 + a = 0$:

- $p \equiv 0$.

Acknowledgements. We should like to thank M. Bordemann, Th. Filk and M. Forger for many valuable discussions.

References

1. de Barros Cobra, B.: Doctoral Thesis, Freiburg 1986; de Barros Cobra, B., Römer, H.: Lett. Math. Phys. **13**, 189 (1987)
2. Arms, J. M., Marsden, J. E., Moncrief, V.: Commun. Math. Phys. **78**, 455 (1981)
3. Otto, M.: J. Geom. Phys. **4**, 101 (1987)
4. Ashtekar, A., Horowitz, G. T.: Phys. Rev. **D26**, 3342 (1982); Gotay, M. J.: Class quantum gravity **3**, 487 (1986)
5. Śniatycki, J., Weinstein, A.: Lett. Math. Phys. **7**, 155 (1983)
6. Arms, J.: J. Math. Phys. **20**, 443 (1979)
7. Arms, J.: J. Math. Phys. **90**, 361 (1981)
8. In a completely different framework scattering on a cone has been considered by de Sousa Gerbert, P., Jackiw, R.: MIT preprint, CTP#1594. Selfadjoint extensions of the oscillator Hamiltonian on $\mathbb{R}^3$ minus a line is discussed in: Doebner, H. D., Elmers, H. J., Heidenreich, W. F.: J. Math. Phys. **30**, 1053 (1989), whereas Exner, P., Šeba, P.: Lett. Math. Phys. **12**, 193 (1986) treat free movement on two planes with one point identified.
9. Abraham, R., Marsden, J. E.: Foundations of mechanics. Second ed., Updated 1985 Printing, Reading MA: The Benjamin/Cummings 1978
10. Montgomery, D.: Compact groups of transformations. In: Differential analysis, papers presented at the Bombay Colloquium 1964. Oxford: Oxford University Press
11. Mostow, G. D.: Ann. Math. **65**, 432 (1957); Mostow, G. D.: Ann. Math. **65**, 513 (1957)
12. Montgomery, D., Samelson, H., Yang, C. T.: Ann. Math. **64**, 131 (1955)
13. Montgomery, D., Samelson, H., Zippin, L.: Ann. Math. **63**, 1 (1955)
14. Hurewicz, W., Wallman, H.: Dimension theory. Princeton, NJ: Princeton University Press 1948

Communicated by A. Jaffe

Received July 21, 1989