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The Pressure in the Huang–Yang–Luttinger Model of an Interacting Boson Gas

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Abstract. This completes our study of the equilibrium thermodynamics of the Huang–Yang–Luttinger model of a boson gas with a hard-sphere repulsion. In an earlier paper we obtained a lower bound on the pressure, but our proof of an upper bound held only for a truncated version of the model. In this paper we establish an upper bound on the pressure in the full model; the upper and lower bounds coincide and provide a variational formula for the pressure. The proof relies on recent second-level large deviation results for the occupation measure of the free boson gas.

1. Introduction

Huang, Yang and Luttinger [1] introduced a model of a boson gas with a hard-sphere repulsion which may be described thus: let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of regions in $\mathbb{R}^d$ with V_l, the volume of Λ_l, tending to infinity with l; with each region Λ_l, we associate the sequence $\varepsilon_l(1) \leqq \varepsilon_l(2) \leqq \cdots$ of ordered real numbers interpreting $\varepsilon_l(j)$ as the j^{th} eigenvalue of the single-particle Hamiltonian of the non-interacting system in the region Λ_l, so that the free-gas Hamiltonian H_l^0 is given by

$$H_l^0 = \sum_{j \geqq 1} \varepsilon_l(j) n_l(j), \tag{1.1}$$

where $n_l(j)$ is the occupation number of the j^{th} level; then the Huang–Yang–Luttinger model is described by the Hamiltonian

$$H_1^{\mathrm{HYL}} = H_l^0 + \frac{a}{2V_l}\left\{2N_l^2 - \sum_{j \geqq 1} n_l(j)^2\right\}, \tag{1.2}$$

where $N_l = \sum_{j \geqq 1} n_l(j)$ is the total number of particles and $a > 0$. The physics of this model was discussed by Huang, Yang and Luttinger [1] and by Thouless [2] and reviewed in our recent paper [3]; we do not repeat the discussion here, except to recall that in [1] the authors argued that the condensate, if any, would occupy the ground state and concluded that (1.2) could be replaced by

$$H_l^1 = H_l^0 + \frac{a}{2V_l}\{2N_l^2 - n_l(1)^2\}; \tag{1.3}$$

this replacement leads to the following formula for the pressure:

$$p^{\mathrm{HYL}}(\mu) = \sup_{x_0 \geqq x_1 \geqq 0} \left\{ \mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1) \right\}, \tag{1.4}$$

where $f(\rho)$ is the canonical free-energy density of the free boson gas at density ρ, given in terms of the free boson gas pressure $p(\mu)$ by

$$f(\rho) = \sup_{\alpha < 0} \{\alpha\rho - p(\alpha)\}. \tag{1.5}$$

In [3], we gave a rigorous proof that formula (1.4) remains valid for the Hamiltonian

$$H_l^{m_l} = H_l^0 + \frac{a}{2V_l}\left\{2N_l^2 - \sum_{j=1}^{m_l} n_l(j)^2\right\}, \tag{1.6}$$

where $\{m_l : l = 1, 2, \ldots\}$ is any sequence of positive integers satisfying

$$\lim_{l \to \infty} m_l / V_l = 0. \tag{1.7}$$

Condition (1.7) was imposed for technical reasons and has no physical significance. Because of it, nagging doubts remained. Could it be that the striking behaviour of the condensate (the Thouless effect, established rigorously in [3] on the basis of formula (1.4)) disappears when the tail in the sum $\sum_{j \geqq 1} n_l(j)^2$ is included in the interaction? This is not the case. In this paper, we prove that formula (1.4) holds good for the full HYL Hamiltonian (1.2).

The Hamiltonian H_l^1 is obviously an upper bound for the full HYL Hamiltonian H_l^{HYL} so that the result proved in [3] provides a lower bound for the pressure corresponding to H_l^{HYL}:

$$\liminf_{l \to \infty} p_l^{\mathrm{HYL}}(\mu) \geqq \sup_{x_0 \geqq x_1 \geqq 0} \left\{ \mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1) \right\}. \tag{1.8}$$

In this paper, we prove that

$$\limsup_{l \to \infty} p_l^{\mathrm{HYL}}(\mu) \leqq \sup_{x_0 \geqq x_1 \geqq 0} \left\{ \mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1) \right\}, \tag{1.9}$$

thus establishing the result

$$p^{\mathrm{HYL}}(\mu) = \lim_{l \to \infty} p_l^{\mathrm{HYL}}(\mu) = \sup_{x_0 \geqq x_1 \geqq 0} \left\{ \mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1) \right\}. \tag{1.10}$$

The consequences of this formula have been discussed fully in [3]. The proof is accomplished by expressing the interaction term

$$\frac{a}{2V_l}\left\{\sum_{j\geqq 1} n_l(j)^2 - 2N_l^2\right\} \tag{1.11}$$

as a functional of the occupation measure introduced in [4]. This enables us to write the partition function as an integral with respect to a probability measure on the space of positive bounded measures on $\mathbb{R}_+$. Next we apply Varadhan's theorem [5] concerning the Laplacian asymptotics of integrals with respect to probability measures on topological spaces. In this instance, the topological space is the space of positive bounded measures on $\mathbb{R}_+$ equipped with the narrow topology. The essential fact, proved in Sect. 3, is that the interaction term, regarded as a functional on the space of positive bounded measures, is upper semicontinuous in the narrow topology. Varadhan's theorem thus provides an upper bound to the pressure which leads to the bound (1.9).

2. The Main Result

In the HYL model, the Hamiltonian is diagonal in the occupation number operators; it follows that it is possible to regard the occupation numbers as random variables rather than as operators. We shall do this. The probability space on which we define our random variables is the countable set Ω of terminating sequences of non-negative integers: an element ω of Ω is a sequence $\{\omega(j)\in\mathbb{N}: j=1,2,\dots\}$ satisfying $\sum_{j\geqq 1}\omega(j)<\infty$. The basic random variables are the occupation numbers $\{\sigma_j: j=1,2,\dots\}$; they are the evaluation maps $\sigma_j:\Omega\to\mathbb{N}$ defined by $\sigma_j(\omega)=\omega(j)$ for each ω in Ω. The total number of particles $N(\omega)$ in the configuration ω is defined by

$$N(\omega)=\sum_{j\geqq 1}\sigma_j(\omega). \tag{2.1}$$

Motivated by the discussion in Sect. 1, we define, for each integer $l\geqq 1$, the free-gas Hamiltonian H_l^0 by

$$H_l^0(\omega)=\sum_{j\geqq 1}\lambda_l(j)\sigma_j(\omega), \tag{2.2}$$

where $0=\lambda_l(1)\leqq\lambda_l(2)\leqq\cdots$, and the HYL Hamiltonian H_l^{HYL} by

$$H_l^{\mathrm{HYL}}(\omega)=H_l^0(\omega)+\frac{a}{2V_l}\left\{2N(\omega)^2-\sum_{j\geqq 1}\sigma_j(\omega)^2\right\}, \tag{2.3}$$

where a is a strictly positive real number. Since Ω is a countable set, we may specify a probability measure on Ω by giving its value at each point of Ω. The free-gas grand canonical measure $\mathbb{P}_l^\mu[\cdot]$ is defined for all $\mu<0$ by

$$\mathbb{P}_l^\mu[\omega]=\exp(\beta\{\mu N(\omega)-H_l^0(\omega)-V_l p_l(\mu)\}), \tag{2.4}$$

provided $\sum_{j \geqq 1} e^{-\beta\lambda_l(j)}$ is finite for all $\beta > 0$. Here $p_l(\mu)$ is the free-gas pressure given in terms of the $\lambda_l(j)$ by

$$p_l(\mu) = (\beta V_l)^{-1} \sum_{j \geqq 1} \ln(1 - e^{\beta(\mu - \lambda_l(j))})^{-1}; \tag{2.5}$$

it satisfies

$$e^{\beta V_l p_l(\mu)} = \sum_{\omega \in \Omega} \exp(\beta\{\mu N(\omega) - H_l^0(\omega)\}). \tag{2.6}$$

The HYL pressure $p_l^{\mathrm{HYL}}(\mu)$ satisfies, for $\alpha < 0$,

$$\exp(\beta V_l p_l^{\mathrm{HYL}}(\mu)) = \sum_{\omega \in \Omega} \exp(\beta V_l g_l^{\mu-\alpha}(\omega)) \exp(\beta\{\alpha N(\omega) - H_l^0(\omega)\}), \tag{2.7}$$

where

$$g_l^{\mu-\alpha}(\omega) = (\mu - \alpha)\frac{N(\omega)}{V_l} + \frac{a}{2}\left\{ \sum_{j \geqq 1} \left(\frac{\sigma_j(\omega)}{V_l}\right)^2 - 2\left(\frac{N(\omega)}{V_l}\right)^2 \right\}; \tag{2.8}$$

using (2.4), we may re-write (2.7) as

$$\exp(\beta V_l p_l^{\mathrm{HYL}}(\mu)) = \exp(\beta V_l p_l(\alpha)) \sum_{\omega \in \Omega} \exp(\beta V_l g_l^{\mu-\alpha}(\omega)) \mathbb{P}_l^{\alpha}[\omega]. \tag{2.9}$$

The next step is to make use of the occupation measure L_l introduced in [4]. For each Borel subset A of $[0, \infty)$ and ω in Ω, put

$$L_l[\omega; A] = \frac{1}{V_l} \sum_{j \geqq 1} \sigma_j(\omega) \delta_{\lambda_l(j)}[A], \tag{2.10}$$

where δ_x is the Dirac measure concentrated at x so that, for each ω in Ω, the map $A \mapsto L_l[\omega; A]$ is a bounded positive measure. We denote by $\bar{E} = \mathscr{M}_+^b(\mathbb{R}_+)$ the space of bounded positive measures on $[0, \infty)$ equipped with the narrow topology (the weak topology induced by $\mathscr{C}^b(\mathbb{R}_+)$, the bounded continuous functions on $[0, \infty)$ equipped with the norm of uniform convergence); it is the weakest topology for which the mapping

$$m \mapsto \langle m, f \rangle = \int_{[0,\infty)} f(\lambda) m(d\lambda)$$

is continuous for every f in $\mathscr{C}^b(\mathbb{R}_+)$. The norm $\|m\|$ of an element m of $\bar{E}$ is given by $\|m\| = \int_{[0,\infty)} m(d\lambda)$; we note that the map $m \mapsto \|m\|$ is continuous. Our aim is to bound the interaction term $g_l^{\mu-\alpha}(\omega)$ by a functional of the occupation measure $L_l[\omega; \cdot]$. To this end, we introduce the map $S: \bar{E} \to \mathbb{R}_+$ as follows: Let P be an ordered subset of $[0, \infty)$, $0 = x_0 < x_1 < \cdots$, let $\Delta_n = [x_{n-1}, x_n]$ and let $\delta(P) = \inf_n (x_n - x_{n-1})$; if $\delta(P) > 0$ we say that P is a *partition* of $[0, \infty)$. Let $\mathscr{P}$ denote the set of all partitions of $[0, \infty)$; for each m in $\bar{E}$, define $S[m]$ by

$$S[m] = \inf_{P \in \mathscr{P}} S_P[m], \tag{2.11}$$

where

$$S_P[m] = \sum_{n \geqq 1} m[\Delta_n]^2. \tag{2.12}$$

(Using closed intervals in the definition of $S_P[m]$ has the disadvantage that it may introduce double counting of atoms in m; the disadvantage is outweighed by the advantage that, as we prove in Sect. 3, $m \mapsto S_P[m]$ is upper semicontinuous. The double counting disappears when we take the infimum over all partitions.) For a measure m whose support is a finite set $a_1 \leqq a_2 \leqq \cdots \leqq a_r$, so that $m = \sum_{n=1}^{r} \alpha_n \delta_{a_n}$, it is clear that $S[m] \geqq \sum_{n=1}^{r} \alpha_n^2$ with equality if and only if all the points a_n are distinct.

Define $\bar{G}^{\mu-\alpha}[m]$ for each m in $\bar{E}$ by

$$\bar{G}^{\mu-\alpha}[m] = (\mu - \alpha) \| m \| + \frac{a}{2}\{S[m] - 2 \| m \|^2\}. \tag{2.13}$$

Then

$$\bar{G}^{\mu-\alpha}[L_l[\omega_j; \cdot]] \geqq g_l^{\mu-\alpha}(\omega) \tag{2.14}$$

with equality for all ω if and only if all the $\lambda_l(j)$ are distinct. Thus we have

$$\exp(\beta V_l p_l^{\mathrm{HYL}}(\mu)) \leqq \exp(\beta V_l p_l(\alpha)) \sum_{\omega \in \Omega} \exp(\beta V_l \bar{G}^{\mu-\alpha}[L_l[\omega_j; \cdot]]) \mathbb{P}_l^{\alpha}[\omega]. \tag{2.15}$$

Let $\bar{\mathbb{K}}_l^{\alpha}$ be the probability measure induced on $\bar{E}$ by L_l:

$$\bar{\mathbb{K}}_l^{\alpha} = \mathbb{P}_l^{\alpha} \circ L_l^{-1}. \tag{2.16}$$

Then (2.15) may be written as

$$\exp(\beta V_l p_l^{\mathrm{HYL}}(\mu)) \leqq \exp(\beta V_l p_l(\alpha)) \int_{\bar{E}} \exp(\beta V_l \bar{G}^{\mu-\alpha}[m]) \bar{\mathbb{K}}_l^{\alpha}[dm], \tag{2.17}$$

so that

$$p_l^{\mathrm{HYL}}(\mu) \leqq p_l(\alpha) + \frac{1}{\beta V_l} \ln \int_{\bar{E}} \exp(\beta V_l \bar{G}^{\mu-\alpha}[m]) \bar{\mathbb{K}}_l^{\alpha}[dm]. \tag{2.18}$$

Conditions on the double sequence $\{\lambda_l(j)\}$ sufficient to ensure the existence of the limit $p(\alpha) = \lim_{l \to \infty} p_l(\alpha)$ were given in [6] and reviewed in [3]; for convenience, we restate them here. Define $\phi_l(\beta)$ for $0 < \beta < \infty$ by

$$\phi_l(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} dF_l(\lambda) \tag{2.19}$$

and introduce the conditions

(S1): $\phi(\beta) = \lim_{l \to \infty} \phi_l(\beta)$ *exist for all* β *in* $(0, \infty)$.

(S2): $\phi(\beta)$ *is non-zero for at least one value of* β *in* $(0, \infty)$.

When (S1) holds, there exists a unique distribution function F, the *integrated density of states*, such that

$$\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} dF(\lambda) \tag{2.20}$$

and $F_l(\lambda) \to F(\lambda)$ at least at the points of continuity of F. When in addition (S2) holds, the limit $p(\mu) = \lim_{l \to \infty} p_l(\mu)$ exists for $\mu < 0$, and $p(\mu)$ is given by

$$p(\mu) = \int_{[0,\infty)} p(\mu|\lambda) dF(\lambda), \tag{2.21}$$

where

$$p(\mu|\lambda) = \beta^{-1} \ln(1 - e^{\beta(\mu - \lambda)})^{-1}. \tag{2.22}$$

It is necessary at this point to introduce a further condition on $\{\lambda_l(j)\}$:

(S3): *The measure dF determined by the integrated density of states F is absolutely continuous with respect to Lebesgue measure with a density which is strictly positive almost everywhere on* $[0, \infty)$. (In the standard example, where the single-particle hamiltonian is a constant multiple of the Laplacian in Λ_l with Dirichlet boundary conditions on $\partial\Lambda_l$, $\varepsilon_l(1) < \varepsilon_l(2) \leqq \cdots$ are its eigenvalues, $\lambda_l(j) = \varepsilon_l(j) - \varepsilon_l(1)$, $j = 1, 2, \ldots$, and $\{\Lambda_l : l = 1, 2, \ldots\}$ is a sequence of bounded convex open sets in $\mathbb{R}^d$ which eventually fills out the whole of $\mathbb{R}^d$, all three conditions are satisfied and $F(\lambda) = C_d \lambda^{d/2}$, where d is the dimension of the ambient Euclidean space.)

The expression (2.18) suggests the use of Laplace's method to complete the proof of the bound for $\limsup_{l \to \infty} p_l^{\mathrm{HYL}}(\mu)$. Varadhan's theorem [5] provides an efficient way of doing this; we use the following version of it:

Varadhan's Theorem [5]. *Let* $\{\mathbb{K}_l : l = 1, 2, \ldots\}$ *be a sequence of probability measures on the Borel subsets of a regular topological space E satisfying the large deviation principle with rate-function* $I : E \to \mathbb{R}_+$ *and constants* $\{V_l\}$. *Suppose that* $G : E \to \mathbb{R}$ *is upper semicontinuous and bounded above, then*

$$\limsup_{l \to \infty} \frac{1}{V_l} \ln \int_E e^{V_l G[m]} \mathbb{K}_l[dm] \leqq \sup_E \{G[m] - I[m]\}. \tag{2.23}$$

Suppose that $G : E \to \mathbb{R}$ *is lower semicontinuous, then*

$$\liminf_{l \to \infty} \frac{1}{V_l} \ln \int_E e^{V_l G[m]} \mathbb{K}_l[dm] \geqq \sup_E \{G[m] - I[m]\}. \tag{2.24}$$

In [4] we proved a large deviation result for the measures $\{\bar{\mathbb{K}}_l^\alpha\}$ defined at

(2.16). It is best stated in terms of the rate-function

$$\bar{I}^{\alpha}[m] = p(\alpha) + f[m] - \alpha \| m \|, \tag{2.25}$$

where $p(\alpha)$ is the free-gas pressure and $f[m]$ is the free-energy functional on $\bar{E}$, by

$$f[m] = \int_{[0,\infty)} \lambda m(d\lambda) - \beta^{-1} \int_{[0,\infty)} \left(s \circ \frac{dm}{dF} \right)(\lambda) dF(\lambda); \tag{2.26}$$

here s is the boson entropy function defined by

$$s(x) = \begin{cases} (1+x)\ln(1+x) - x\ln x, & x > 0, \\ 0, & x = 0, \end{cases} \tag{2.27}$$

and dm/dF is the Radon–Nikodym derivative of m with respect to the measure dF so that the Lebesgue decomposition of m may be written

$$m(d\lambda) = m_s(d\lambda) + \frac{dm}{dF}(\lambda) dF(\lambda) \tag{2.28}$$

with m_s singular with respect to dF.

In the spirit of Laudau and Lifshitz [7], we may call $f[m]$ the non-equilibrium free-energy density of the free-boson gas; it is related to the equilibrium free-energy density $f(\rho)$, defined at (1.5), by the following result which we prove in Sect. 3:

Proposition 1. *Suppose that* (S1), (S2) *and* (*S*3) *hold; then*

$$f(x) = \inf_{\{m \text{ abs. cts.}, \|m\| = x\}} f[m]. \tag{2.29}$$

In [4], we proved the following theorem:

Theorem 1. *Let* $\{\lambda_l(j)\}$ *be a double sequence satisfying* (S1), (S2) *and* (S3) *and let* F *be the corresponding integrated density of states; then the sequence* $\{\bar{\mathbb{K}}_l^{\alpha} = \mathbb{P}_l^{\alpha} \circ L_l^{-1}: l = 1, 2, \ldots\}$ *of probability measures on* $\bar{E}$ *satisfies the large deviation principle with constants* $\{V_l\}$ *and rate-function*

$$\bar{I}^{\alpha}[m] = p(\alpha) + f[m] - \alpha \| m \|. \tag{2.30}$$

To apply (2.23), we must check that $m \mapsto \bar{G}^{\mu}[m]$ is upper semicontinuous and bounded above; to this end, we prove in Sect. 3 the following result:

Proposition 2. *The functional* $S: \bar{E} \to \mathbb{R}_+$ *has the following properties*:

1. *Let* $m = m_s + m_a$ *be the Lebesgue decomposition of an element* m *of* $\bar{E}$ *into singular part* m_s *and absolutely continuous part* m_a; *then*

$$S[m] = S[m_s].$$

2. *For each element* m *of* $\bar{E}$,

$$S[m] \leqq \| m \|^2.$$

3. *The map* $m \mapsto S[m]$ *is upper semicontinuous on* $\bar{E}$.

Notice that it follows from (1) and (2) that

$$S[m] \leqq \|m_s\|^2. \tag{2.31}$$

Since $m \mapsto \|m\|$ is continuous in the narrow topology and a is strictly positive, it now follows that $m \mapsto \bar{G}^\mu[m]$ is upper semicontinuous and bounded above for all values of μ.

Applying (2.23) to (2.18), we have

$$\begin{aligned}\limsup_{l\to\infty} p_l^{\mathrm{HYL}}(\mu) &\leqq p(\alpha) + \sup_{\bar{E}} \{\bar{G}^{\mu-\alpha}[m] - \bar{I}[m]\} \\ &\leqq p(\alpha) + \sup_{\bar{E}} \{(\mu-\alpha)\|m\| - \frac{a}{2}(2\|m\|^2 - \|m_s\|^2) - \bar{I}^\alpha[m]\} \\ &= \sup_{\bar{E}} \{\mu\|m\| - \frac{a}{2}(2\|m\|^2 - \|m_s\|^2) - f[m]\}. \end{aligned} \tag{2.32}$$

We see from (2.26) that, using the Lebesgue decomposition $m = m_a + m_s$, we can write $f[m]$ as

$$f[m] = \int_{[0,\infty)} \lambda m_s(d\lambda) + f[m_a]. \tag{2.33}$$

Thus (2.32) becomes

$$\begin{aligned}\limsup_{l\to\infty} p_l^{\mathrm{HYL}}(\mu) &\leqq \sup_{\bar{E}} \left\{\mu\|m\| - \frac{a}{2}(2\|m\|^2 - \|m_s\|^2) - \int_{[0,\infty)} \lambda m_s(d\lambda) - f[m_a]\right\} \\ &= \sup_{\{m\in\bar{E}:\operatorname{supp} m_s = \{0\}\}} \left\{\mu\|m\| - \frac{a}{2}(2\|m\|^2 - \|m_s\|^2) - f[m_a]\right\} \\ &= \sup_{x_0 \geqq x_1 \geqq 0} \left\{\mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - \inf_{\{\|m\| = x_0 - x_1\}} f[m]\right\}. \end{aligned} \tag{2.34}$$

By Proposition 1, this yields (1.9); putting it together with the lower bound (1.8), proved in [3], we establish

Theorem 2. *Under the hypotheses of Theorem 1, the limit*

$$p^{\mathrm{HYL}}(\mu) = \lim_{l\to\infty} p_l^{\mathrm{HYL}}(\mu)$$

exists and is given by

$$p^{\mathrm{HYL}}(\mu) = \sup_{x_0 \geqq x_1 \geqq 0} \left\{\mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1)\right\}.$$

Perhaps it is worth pointing out that the final result is purely probabilistic in character; there is no topology in sight. Factoring the function $g_l^\mu : \Omega \to \mathbb{R}$ through the topology space $\bar{E}$ exploits the convergence of $\{\mathbb{K}_l^\alpha\}$ to the degenerate distribution. In this connection, we describe briefly a second proof of the lower bound (1.8) based on a result announced in [8]: Let $\underline{E}$ be the positive cone $l_+^1 = \left\{ x_j \geqq 0 \colon \sum_{j \geqq 0} x_j < \infty \right\}$ of the Banach space $l^1 = \left\{ x_j \in \mathbb{R} \colon \sum_{j \geqq 0} |x_j| < \infty \right\}$ equipped with the weak*—topology; let $X_l \colon \Omega \to \underline{E}$ be the map $x_0 = X_l(0; \omega) = (1/V_l) \sum_{j \geqq 1} \sigma_j(\omega)$, $x_n = X_l(n; \omega) = \sigma_n(\omega)/V_l$, $n \geqq 1$, and let $\underline{G}^{\mu-\alpha} \colon \underline{E} \to \mathbb{R}$ be the functional

$$\underline{G}^{\mu-\alpha}[x] = (\mu - \alpha)x_0 + \frac{a}{2}\left\{ \sum_{j \geqq 1} x_j^2 - 2x_0^2 \right\}. \tag{2.35}$$

Then

$$g_l^{\mu-\alpha}(\omega) = \underline{G}^{\mu-\alpha}[X_l(\omega)], \tag{2.36}$$

so that

$$p_l^{\mathrm{HYL}}(\mu) = p_l(\alpha) + \frac{1}{\beta V_l} \ln \int_E \exp(\beta V_l \underline{G}^{\mu-\alpha}[x]) \underline{\mathbb{K}}_l^\alpha[dx], \tag{2.37}$$

where

$$\underline{\mathbb{K}}_l^\alpha = \mathbb{P}_l^\alpha \circ X_l^{-1}. \tag{2.38}$$

In [8], we stated the following

Theorem 3. *Let $\{\lambda_l(j)\}$ be a double sequence satisfying* (S1), (S2) *and* (S3) *and let F be the corresponding integrated density of states; then the sequence $\{\underline{\mathbb{K}}_l^\alpha = \mathbb{P}_l^\alpha \circ X_l^{-1} \colon l = 1, 2, \dots\}$ of probability measures on $\underline{E}$ satisfies the large deviation principle with constants $\{V_l\}$ and rate-function*

$$I^\alpha[x] = \begin{cases} p(\alpha) + f\left(x_0 - \sum_{j \geqq 1} x_j \right) - \alpha x_0, & x_0 \geqq \sum_{j \geqq 1} x_j, \\ \infty, & \textit{otherwise}. \end{cases} \tag{2.39}$$

To apply (2.24), we must check that $x \mapsto \underline{G}^\mu[x]$ is lower semicontinuous; this follows from the fact that $x \mapsto \sum_{j=1}^{n} x_j^2$ is continuous and

$$\sum_{j \geqq 1} x_j^2 = \sup_n \sum_{j=1}^{n} x_j^2. \tag{2.40}$$

It then follows that the lower bound (1.8) holds:

$$\begin{aligned} \liminf_{l \to \infty} p_l^{\mathrm{HYL}}(\mu) &\geqq p(\alpha) + \sup_{x \in \underline{E}} \{ \underline{G}^{\mu-\alpha}[x] - I^\alpha[x] \} \\ &= \sup_{\left\{ x \in \underline{E} \colon x_0 \geqq \sum_{j \geqq 1} x_j \right\}} \left\{ \mu x_0 - \frac{a}{2}\left(2x_0^2 - \sum_{j \geqq 1} x_j^2 \right) - f\left(x_0 - \sum_{j \geqq 1} x_j \right) \right\} \end{aligned}$$

$$\geqq \sup_{\{(x_0,x_1):x_0 \geqq x_1 \geqq 0\}} \left\{ \mu x_0 - \frac{a}{2}(2x_0^2 - x_1^2) - f(x_0 - x_1) \right\}.$$

The strategy of the proof is summarized by the following diagram:

$$\begin{tikzcd}
 & \Omega \arrow[dl, "X_l"'] \arrow[dr, "L_l"] & \\
\underline{E} \arrow[dr, "\underline{G}^\mu"'] & & \bar{E} \arrow[dl, "\bar{G}^\mu"] \\
 & \mathbb{R} &
\end{tikzcd}$$

To prove the upper bound, the interaction g_l^μ is factored as $\bar{G}^\mu \circ L_l$ through the space $\bar{E}$ while, to prove the lower bound, it is factored as $\underline{G}^\mu \circ L_l$ through the space $\underline{E}$. These are technical devices introduced to deal with the estimates. Loosely speaking, the phenomena reflected in the bounds occur on different "scales"; these are picked up, not by the scaling which is the same in both cases, but by the different topologies.

3. The Promised Proofs

In this section, we provide the proofs, deferred from Sect. 2, of the Propositions.

Proof of Proposition 1. We are seeking to minimize $f[m_a]$, where $m_a(d\lambda) = \rho(\lambda)dF(\lambda)$ with ρ in $L_+^1(\mathbb{R}_+; dF)$. Defined ρ_μ for $\mu \leqq 0$ by

$$\rho_\mu(\lambda) = (e^{\beta(\lambda - \mu)} - 1)^{-1} \tag{3.1}$$

and ρ_c by

$$\rho_c = \begin{cases} \int\limits_{[0,\infty)} \rho_0(\lambda)dF(\lambda), & \text{whenever } \rho_0 \in L_+^1(\mathbb{R}_+; dF), \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

We first prove that, for $x \leqq \rho_c$, the infimum is attained by $\rho_{\mu(x)}$, where $\mu(x)$ is defined as the unique real root of

$$\int\limits_{[0,\infty)} \rho_\mu(\lambda)dF(\lambda) = x. \tag{3.3}$$

By abuse of notation, we write $f[\rho]$ for $f[m]$ whenever $m[d\lambda] = \rho(\lambda)dF(\lambda)$. Since $\rho_\mu(x)$ satisfies the constraint, we have

$$\inf_{\{\rho \in L_+^1(dF): \|\rho\|_1 = x\}} f[\rho] \leqq f[\rho_{\mu(x)}]. \tag{3.4}$$

To prove the reverse inequality, we write

$$f[\rho] = \int\limits_{[0,\infty)} (\varphi_\lambda \circ \rho)(\lambda)dF(\lambda), \tag{3.5}$$

where

$$\varphi_\lambda(\sigma) = \lambda\sigma - \beta^{-1}s(\sigma), \tag{3.6}$$

and exploit the convexity of $\sigma \mapsto \varphi_\lambda(\sigma)$ from which follows the inequality

$$\varphi_\lambda(\sigma_2) - \varphi_\lambda(\sigma_1) \geqq \varphi'_\lambda(\sigma_1)(\sigma_2 - \sigma_1). \tag{3.7}$$

But

$$\varphi'_\lambda(\rho_\mu(\lambda)) = \mu, \tag{3.8}$$

so that (3.7) yields

$$f[\rho] - f[\rho_\mu] \geqq \mu\left\{ \int_{[0,\infty)} \rho(\lambda)dF(\lambda) - \int_{[0,\infty)} \rho_\mu(\lambda)dF(\lambda)\right\}. \tag{3.9}$$

For $x \leqq \rho_c$, we choose $\mu = \mu(x)$ to be the unique real root of

$$\int_{[0,\infty)} \rho_\mu(\lambda)dF(\lambda) = x. \tag{3.10}$$

It follows that

$$\inf_{\{\rho\in L^1_+(dF):\|\rho\|_1 = x\}} f[\rho] \geqq f[\rho_{\mu(x)}]; \tag{3.12}$$

combining this with (3.4), we have

$$\inf_{\{\rho\in L^1_+(dF):\|\rho\|_1 = x\}} f[\rho] = f[\rho_{\mu(x)}], \quad x \leqq \rho_c. \tag{3.13}$$

In the case $x > \rho_c$, the infimum is not attained; we construct a minimizing sequence. Put, for $n = 1, 2, \ldots,$

$$\rho^{(n)}(\lambda) = \rho_0(\lambda) + h^{(n)}(\lambda), \tag{3.14}$$

where

$$h^{(n)}(\lambda) = \begin{cases} \dfrac{(x-\rho_c)}{F((\beta n)^{-1})}, & 0 \leqq \lambda \leqq (\beta n)^{-1}, \\ 0, & \lambda > (\beta n)^{-1}, \end{cases} \tag{3.15}$$

so that

$$\int_{[0,\infty)} \rho^{(n)}(\lambda)dF(\lambda) = x. \tag{3.16}$$

Now (3.9) with $\mu = 0$ yields

$$f[\rho^{(n)}] \geqq f[\rho_0]. \tag{3.17}$$

Again the convexity of $\sigma \mapsto \varphi_\lambda(\sigma)$ yields the inequality

$$\varphi_\lambda(\sigma_2) - \varphi_\lambda(\sigma_1) \leqq \varphi'_\lambda(\sigma_2)(\sigma_2 - \sigma_1); \tag{3.18}$$

but

$$\varphi_\lambda'(\sigma_2) = \lambda - \beta^{-1} \ln\left(1 + \frac{1}{\sigma_2}\right) \leqq \lambda, \tag{3.19}$$

so that

$$f[\rho^{(n)}] \leqq f[\rho_0] + \int_{[0,\infty)} \lambda h^{(n)}(\lambda) dF(\lambda), \tag{3.20}$$

since $h^{(n)}(\lambda) \geqq 0$ for all λ in $[0, \infty)$. Combining (3.17) and (3.20) and using (3.15), we have

$$f[\rho_0] \leqq f[\rho^{(n)}] \leqq f[\rho_0] + (x - \rho_c).(\beta n)^{-1}. \tag{3.21}$$

Thus, for $x > \rho_c$, we have proved that

$$\inf_{\{\rho \in L^1_+[dF] \cdot \|\rho\|_1 = x\}} f[\rho] = f[\rho_0]. \tag{3.22}$$

A straightforward calculation establishes that $f[\rho_{\mu(x)}] = f(x)$, $x \leqq \rho_c$, and $f[\rho_0] = f(\rho_c) = f(x)$, $x > \rho_c$. □

Proof of Proposition 2(1). Let m be an arbitrary element of $\bar{E}$; given $\varepsilon > 0$, choose L such that $m[[L, \infty)] < \varepsilon$ and let A_m be the set of atoms of m. Choose a sequence $\{\tilde{P}_k : k = 1, 2, \ldots\}$ of partitions such that

$$0 \leqq S[m_s] \leqq S_{\tilde{P}_k}[m_s] \leqq S[m_s] + \frac{1}{k},$$

so that

$$\lim_{k \to \infty} S_{\tilde{P}_k}[m_s] = S[m_s].$$

For each k, construct a new partition P_k by adding points $x_1, x_2, \ldots$ to $\tilde{P}_k$ according to the following scheme: choose x_1 in $(1/2k, 1/k) \backslash (A_m \cup \tilde{P}_k)$, then choose x_2 in $(x_1 + (1/2k), x_1 + (1/k)) \backslash (A_m \cup \tilde{P}_k)$, and so on. This construction is possible because the set $A_m \cup \tilde{P}_k$ is countable so that, for each open interval I, the set $I \backslash (A_m \cup \tilde{P}_k)$ is non-empty. Denote the n^{th} interval of $P_k : 0 = x_0^{(k)} < x_1^{(k)} < \cdots$ by Δ_n^k. The partition P_k has the following properties:

$$0 \leqq S[m_s] \leqq S_{P_k}[m_s] \leqq S_{\tilde{P}_k}[m_s] \leqq S[m_s] + \frac{1}{k} \tag{1}$$

so that

$$\lim_{k \to \infty} S_{P_k}[m_s] = S[m_s];$$

$$d(P_k) \leqq \frac{1}{k}. \tag{2}$$

Now we have

$$\begin{aligned} 0 &\leqq S[m_s] \leqq S[m] \leqq \liminf_{k\to\infty} S_{P_k}[m] \\ &= \liminf_{k\to\infty} \sum_{n\geqq 1} \{m_s[\Delta_n^k] + m_a[\Delta_n^k]\}^2 \\ &\leqq \lim_{k\to\infty} S_{P_k}[m_s] \\ &\quad + 5\,\|m\| \limsup_{k\to\infty}\left(\sup_n m_a[\Delta_n^k]\right). \end{aligned}$$

But

$$\sup_n m_a[\Delta_n^k] \leqq \varepsilon + \sup_n m_a[\Delta_n^k \cap [0, L+1]],$$

so it is enough to prove that

$$\limsup_{k\to\infty}\left(\sup_n m_a[\Delta_n^k \cap [0, L+1]]\right) = 0$$

in order to conclude that $S[m] = S[m_s]$ for all m in $\bar{E}$.

Suppose that

$$\limsup_{k\to\infty}\left(\sup_n m_a[\Delta_n^k \cap [0, L+1]]\right) = \alpha > 0;$$

then, for k sufficiently large,

$$\sup_n m_a[\Delta_n^k \cap [0, L+1]] > \frac{\alpha}{2},$$

so that there exists n_k such $\Delta_{n_k}^k \subset [0, L+1]$ and $m_a[\Delta_{n_k}^k] > \alpha/2$. Let x_k be the mid-point of $\Delta_{n_k}^k$; the sequence $\{x_k : k = 1, 2, \ldots\} \subset [0, L+1]$ has at least one limit point, x say. Without risk of confusion, we denote by $\{x_k\}$ a subsequence converging to x. Given $\varepsilon > 0$, choose r such that x_r is in $[x - \varepsilon/2,\ x + \varepsilon/2]$ and $d(P_r) < \varepsilon/2$. Then $\Delta_{n_r}^r \subset [x-\varepsilon, x+\varepsilon]$ so that $m_a[[x-\varepsilon, x+\varepsilon]] > \alpha/2 > 0$. But ε is an arbitrary positive number, hence m_a is not absolutely continuous; contradiction. □

Proof of Proposition 2(2). Given $\varepsilon > 0$, choose a point of continuity L of m such that $m[[L, \infty)] < \varepsilon$. Let P be the partition $0 = x_0 < x_1 = L$; then

$$\begin{aligned} S_P[m] &= m[[0, L]]^2 + m[[L, \infty)]^2 \\ &= \|m\|^2 - 2\,\|m\|\, m[[L, \infty)] + 2m[[L, \infty)]^2 \\ &\leqq \|m\|^2 + 2\varepsilon^2. \end{aligned}$$

Hence

$$S[m] \leqq S_P[m] \leqq \|m\|^2 + 2\varepsilon^2.$$

Since ε is an arbitrary positive number, we have

$$S[m] \leqq \|m\|^2. \quad \square$$

Proof of Proposition 2(3). First notice that, for each closed bounded interval $\Delta = [a, b]$, the map $m \mapsto m[\Delta]$ is upper semicontinuous. (This is a consequence of the fact that the infimum of a family of continuous functions is upper semicontinuous since, by Lebesgue's Dominated Convergence Theorem,

$$m[\Delta] = \lim_{n\to\infty} \langle m, t_n \rangle = \inf_n \langle m, t_n \rangle$$

for a decreasing sequence $\{t_n\}$ of continuous "tent-functions" majorizing the indicator function 1_Δ of the interval Δ.) Now let P be a partition of $[0, \infty)$; for each N, the map

$$m \mapsto \sum_{j=1}^{N} m[\Delta_j]^2$$

is upper semicontinuous. If the partition is finite we have proved that $m \mapsto S_P[m]$ is upper semicontinuous; suppose, therefore, that the partition is not finite so that $x_n \to \infty$.

Let $\{m_r : r = 1, 2, \ldots\}$ be a sequence in $\bar{E}$ converging to m. Given $\varepsilon > 0$, choose a point of continuity A of m such that $m[[A, \infty)] < \varepsilon$. Then for k large enough, $k \geqq k_0$ say, $m_k[[A, \infty)] < 2\varepsilon$ and thus, for $x_N > A$, we have

$$\begin{aligned}\sum_{n \geqq N+1} m_k[\Delta_n]^2 &\leqq \Big(\sum_{n \geqq N+1} m_k[\Delta_n] \Big)^2 \\ &\leqq 4 m_k[[x_N, \infty)]^2 \leqq 16\varepsilon^2.\end{aligned}$$

Thus, for $k \geqq k_0$, we have

$$S_P[m_k] \leqq \sum_{n \leqq N} m_k[\Delta_n]^2 + 16\varepsilon^2,$$

so that

$$\begin{aligned}\limsup_{k\to\infty} S_P[m_k] &\leqq \limsup_{k\to\infty} \sum_{n \leqq N} m_k[\Delta_n]^2 + 16\varepsilon^2 \\ &\leqq \sum_{n \leqq N} m[\Delta_n]^2 + 16\varepsilon^2 \\ &\leqq S_P[m] + 16\varepsilon^2.\end{aligned}$$

It follows that $m \mapsto S_P[m]$ is upper semicontinuous, since ε is an arbitrary positive number; hence $m \mapsto S[m] = \inf_{\mathscr{P}} S_P[m]$ is upper semicontinuous. $\square$

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