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Loop Groups and Yang-Mills Theory in Dimension Two

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Abstract. Given a connection ω in a G-bundle over S^2, then a process called radial trivialization from the poles gives a unique clutching function, i.e., an element γ of the loop group ΩG. Up to gauge equivalence, ω is completely determined by γ and a map $f: S^2 \to \mathfrak{g}$ into the Lie algebra. Moreover, the Yang-Mills functional of ω is the sum of the energy of γ and the square of a certain norm of f. In particular, the Yang-Mills functional has the same Morse theory as the energy functional on ΩG. There is a similar description of connections in a G-bundle over an arbitrary Riemann surface, but so far not of the Yang-Mills functional.

1. Introduction

The purpose of this paper is to point out that the Yang-Mills functional on S^2 with gauge group G essentially is the same as the energy functional on the loop group ΩG. More precisely, I prove that up to gauge equivalence, a connection in a G-bundle P over S^2 is completely described by a loop $\gamma \in \Omega G$ and a map $f: S^2 \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Furthermore, the Yang-Mills functional $\mathscr{YM}(\omega)$ is the sum $\pi\mathscr{E}(\gamma) + \langle f, f \rangle$, of the energy of γ and a certain inner product of f with itself (Theorem 2.6).

In [1] Atiyah and Bott mentioned that the Yang-Mills functional on S^2 and the energy functional on ΩG have the same Morse theory and in [2] Friedrich and Habermann prove it in all details, but it is of course a trivial consequence of the above description of the Yang-Mills functional.

One consequence is that the space of solutions to the Yang-Mills equations on S^2 is the same as the space of homomorphisms $S^1 \to G$. There are similar results in higher dimensions: the space of instantons on $\mathbb{R}^4$ is the same as the space of holomorphic maps $S^2 \to \Omega G$ and the space of monopoles on $\mathbb{R}^3$ is the same as the space of holomorphic maps $S^2 \to G/T$, where T is a maximal torus. In both cases there are suitable boundary conditions at infinity. But the proofs of these results are done in a very roundabout way and the connection between Yang-Mills theory and mapping spaces are not fully understood. The Yang-Mills equation on the sphere is the simplest example and here the present paper explains why solutions to equations in the "Yang-Mills family" can be identified with certain mapping spaces.

The radial trivialization works on any S^n and gives a map $\mathscr{C}(P)\to\Omega^{n-1}G$, so one could hope to explain the higher dimensional results by methods similar to the present, but so far this has not been possible.

It seems to be more likely that one can generalize the result to an arbitrary Riemann surface. In Sect. 4 it is shown that connections over any Riemann surface X can be described in a manner similar to the one for the sphere. The loop γ is replaced by an element of an infinite Grassmannian Gr^X which like ΩG has an energy functional. In fact both spaces are subsets of a larger Grassmannian Gr, and the functionals are the restriction of an energy functional defined on all of Gr, see [4].

Let $\mathscr{C}(P)$ be the space of connections on P. The definition of the map $\mathscr{C}(P)\to\Omega G$ uses a process which I call radial trivialization and which was told to me by G. Segal. It is the lift of the great circles through the poles to horizontal curves in P, and the loop is the difference on the equator between the lift from the south pole and the lift from the north pole. Friedrich and Habermann use the same construction, but the rest of their proof is different.

After the completion of this work I learned that Nahm and Uhlenbeck too have compared the Yang-Mills functional and the energy functional and have arrived at the same result and even use the same method, see [3].

2. The Yang-Mills Functional on the Sphere

Consider the two dimensional unit sphere

$$S^2=\{(x,y,z)\in\mathbb{R}^3\,|\,x^2+y^2+z^2=1\}.$$

Denote $(0,0,1)$ by ∞ (the North Pole), $(0,0,-1)$ by 0 (the South Pole), the complement of 0 by U_∞, the complement of ∞ by U_0, the part of S^2 where $z>0$ by D_∞ (the northern hemisphere) and the part of S^2 where $z<0$ by D_0 (the southern hemisphere). Finally let $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$ be spherical coordinates on S^2.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let P be a principal G-bundle over S^2 with a connection ω. Choose a base point p_∞ in the fiber over ∞.

By lifting the curves $\theta\mapsto(\theta,\varphi)$ for $\varphi\in[0,2\pi]$ to horizontal curves in P starting at p_∞, we get a section σ_∞ in P over U_∞, and by lifting the curves $\theta\mapsto(\pi-\theta,\varphi)$ to horizontal curves in P starting at some point p_0 in the fiber over 0, we get a section σ_0 in P over U_0. We say that σ_0 and σ_∞ are obtained by *radial trivialization* from the poles. Henceforth j denotes either 0 or ∞.

Lemma 2.1. *The sections σ_0 and σ_∞ in P are smooth.*

Proof. Use the connection on P to combine the metric on S^2 and the biinvariant metric on G to give a metric on P. Obviously a horizontal lift of a geodesic in S^2 gives a geodesic in P. Hence $\sigma_j=\exp_P\circ\pi_*^{-1}\circ\exp_{S^2}^{-1}$, where π_* is the projection from the horizontal space in P to TS^2 and $\exp_M$ is the exponential map from TM to M. $\square$

The transition function $\gamma: U_0 \cap U_\infty \to G$ is defined by

$$\sigma_0 = \sigma_\infty \gamma . \tag{2.2}$$

As the curve $\theta \mapsto \sigma_j(\theta, \varphi)$ is horizontal, $\theta \mapsto \gamma(\theta, \varphi)$ is constant for all φ, hence γ can be regarded as a smooth map $\gamma: S^1 \to G$, i.e. as an element of LG.

As p_0 is arbitrary, σ_0, and hence γ is only determined up to multiplication by an element of G, but we have a well-defined element of $LG/G = \Omega G$. Alternatively we can put $p_0 = \lim_{\theta \to \pi} \sigma_\infty(\theta, 0)$.

We put

$$\omega_j = \sigma_j^*(\omega) \in \Omega^1(E_{\mathfrak{g}}|_{U_j}) \cong C^\infty(\wedge^1(U_j, \mathfrak{g})), \tag{2.3}$$

where $E_{\mathfrak{g}} = P \times_G \mathfrak{g}$.

Lemma 2.4. *The loop γ and the forms ω_0, ω_∞ are gauge invariant.*

Proof. Let g be a based gauge transformation, i.e. an automorphism $g: P \to P$ with $g(p_\infty) = p_\infty$. Then $\tilde{\sigma}_j = g^{-1} \circ \sigma_j$ is the section induced by the gauge transformed connection $\tilde{\omega} = g^*(\omega)$. We can write $\tilde{\sigma}_j = \sigma_j g_j$, then $g_0 = \gamma^{-1} g_\infty \gamma$,

$$\tilde{\sigma}_0 = \sigma_0 g_0 = (\sigma_\infty \gamma)(\gamma^{-1} g_\infty \gamma) = \sigma_\infty g_\infty \gamma = \tilde{\sigma}_\infty \gamma$$

and

$$\tilde{\omega}_j = \tilde{\sigma}_j^*(\tilde{\omega}) = (g^{-1} \circ \sigma_j)^* (g^*(\omega)) = \sigma_j^* g^{-1*} g^*(\omega) = \sigma_j^*(\omega) = \omega_j . \quad \square$$

The curve $\theta \mapsto \sigma_j(\theta, \varphi)$ is horizontal, hence $\omega_0(0) = \omega_\infty(\infty) = 0$ and $\omega_j = f_j d\varphi$. As $f_0 = \gamma^{-1} f_\infty \gamma + \gamma^{-1} \gamma'$, we can put

$$f = f_0 - \cos^2\left(\frac{\theta}{2}\right) \gamma^{-1} \gamma' = \gamma^{-1}\left(f_\infty + \sin^2\left(\frac{\theta}{2}\right) \gamma' \gamma^{-1}\right) \gamma , \tag{2.5}$$

and obtain a well-defined map $f: S^2 \to \mathfrak{g}$.

As ω_0 has zero $d\theta$ component, $[\omega_0, \omega_0] = 0$, so the curvature R^ω has on U_0 the local expression

$$R_0^\omega = d\omega_0 = \frac{\partial f_0}{\partial \theta} d\theta \wedge d\varphi$$

and

$$\frac{\partial f_0}{\partial \theta} = \frac{\partial f}{\partial \theta} - \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \gamma^{-1} \gamma' = \frac{\partial f}{\partial \theta} - \frac{1}{2} \sin(\theta) \gamma^{-1} \gamma' .$$

The Yang-Mills functional of the connection ω is given by

$$\mathcal{YM}(\omega) = \frac{1}{2} \int_{S^2} \|R^\omega\|^2 = \frac{1}{2} \int_{S^2} \langle *R^\omega, *R^\omega \rangle d\Omega ,$$

where $*$ is the Hodge star operator, $d\Omega = \sin(\theta) d\theta \wedge d\varphi$ is the standard volume form on S^2 and the bracket is the usual bi-invariant inner product on the Lie algebra $\mathfrak{g}$.

We have $*(d\theta\wedge d\varphi)=\dfrac{1}{\sin(\theta)}$, so

$$\begin{aligned}\mathcal{YM}(\omega)&=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left\|\frac{\partial f_0}{\partial\theta}\frac{1}{\sin(\theta)}\right\|^2\sin(\theta)d\theta d\varphi\\&=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left\langle\frac{\partial f}{\partial\theta}-\frac{1}{2}\sin(\theta)\gamma^{-1}\gamma',\frac{\partial f}{\partial\theta}-\frac{1}{2}\sin(\theta)\gamma^{-1}\gamma'\right\rangle\frac{1}{\sin(\theta)}d\theta d\varphi\\&=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left\|\frac{1}{\sin(\theta)}\frac{\partial f}{\partial\theta}\right\|^2\sin(\theta)d\theta d\varphi\\&\quad-\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left\langle\frac{\partial f}{\partial\theta},\gamma^{-1}\gamma'\right\rangle d\theta d\varphi\\&\quad+\frac{1}{8}\int_0^{2\pi}\int_0^{\pi}\|\gamma^{-1}\gamma'\|^2\sin(\theta)d\theta d\varphi\,.\end{aligned}$$

We have

$$\int_0^{2\pi}\int_0^{\pi}\left\langle\frac{\partial f}{\partial\theta},\gamma^{-1}\gamma'\right\rangle d\theta d\varphi=\int_0^{2\pi}[\langle f,\gamma^{-1}\gamma'\rangle]_0^{\pi}d\varphi=0$$

and

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\|\gamma^{-1}\gamma'\|^2\sin(\theta)d\theta d\varphi&=\int_0^{\pi}\sin(\theta)d\theta\int_0^{2\pi}\|\gamma^{-1}\gamma'\|^2d\varphi\\&=2\int_0^{2\pi}\|\gamma^{-1}\gamma'\|^2d\varphi\\&=8\pi\mathcal{E}(\gamma)\,,\end{aligned}$$

where $\mathcal{E}(\gamma)$ is the *energy* of the loop γ. Finally

$$\langle f_1,f_2\rangle=\frac{1}{2}\int_{S^2}\left\langle\frac{1}{\sin(\theta)}\frac{\partial f_1}{\sin(\theta)},\frac{1}{\sin(\theta)}\frac{\partial f_2}{\partial\theta}\right\rangle d\Omega$$

is an inner product on the vector space $\{f\in C^\infty(S^2,\mathfrak{g})\,|\,f(0)=f(\infty)=0\}$. If we let $\mathcal{H}$ be the completion, then we have proved

Theorem 2.6. *Let $\mathcal{C}$ be the space of connections, $\mathcal{G}_0$ the group of based gauge transformations and $\mathcal{B}=\mathcal{C}/\mathcal{G}_0$ the orbit space. Then the map $\omega\mapsto(\gamma,f)$ is an injection*

$$\mathcal{B}\hookrightarrow\Omega G\times\mathcal{H}\,,$$

such that

$$\mathcal{YM}(\omega)=\pi\mathcal{E}(\gamma)+\langle f,f\rangle\,.$$

The image of $\mathcal{B}$ is not a product of ΩG and a subspace of $\mathcal{H}$, but consists of pairs (γ,f) such that f_0 and f_∞ defined by (2.5) are smooth on U_0 and U_∞ respectively. So f is smooth outside $\{0,\infty\}$, but will in general have singularities at 0 and ∞. None the less all of the fibres of $\mathcal{B}\to\Omega G$ are dense in $\mathcal{H}$ and this gives us

Corollary 2.7. *The map $\mathcal{B}\to\Omega G$ gives a one to one correspondence between the critical points for the Yang-Mills functional on S^2 and the critical points for the* *energy functional on ΩG. Moreover, at corresponding critical points the Morse-indices are equal.*

We could replace (2.5) with

$$\tilde{\omega}=\omega_0-h(\theta)\gamma^{-1}\gamma' d\varphi=\gamma^{-1}(\omega_\infty+h(\theta)\gamma'\gamma^{-1})\gamma, \tag{2.8}$$

where $h(\theta)$ is a smooth function, which is 0 in a neighbourhood of π and 1 in a neighbourhood of 0. Then $\tilde{\omega}$ is a smooth one form on S^2 with values in $\mathfrak{g}$ and with $\tilde{\omega}\left(\frac{\partial}{\partial\theta}\right)=0$ at all points. Conversely, given $\gamma\in\Omega G$ and such a one form $\tilde{\omega}$, then (2.8) gives a pair ω_0, ω_∞, which in turn defines a connection in the bundle defined by γ. I.e., we have

Theorem 2.9. *The map $\mathscr{B}\to\Omega G$ is a trivial vector bundle.*

But we do not have the nice formula for the Yang-Mills functional, because if we try to use (2.8) in the calculation, then the mixed term $\int\langle d\tilde{\omega}, h'\gamma^{-1}\gamma'\rangle$ does not necessarily vanish.

3. Holomorphic Trivialization

If $G_{\mathbb{C}}$ is the complexification of G and $LG_{\mathbb{C}}$ is the space of free smooth loops in $G_{\mathbb{C}}$, then $\Omega G=LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$, where $L^+G_{\mathbb{C}}$ is the space of loops which are the boundary values of holomorphic maps $D\to G_{\mathbb{C}}$ and D is the unit disk in $\mathbb{C}$, see [4]. We can use this description of ΩG to get a different map $\mathscr{C}\to\Omega G$.

Using stereographic projection, we identify S^2 with $\mathbb{C}\cup\{\infty\}$. Then U_0 corresponds to $\mathbb{C}$, D_0 corresponds to the unit disk, and the meaning of 0 and ∞ are unchanged.

The connection ω on P induces a holomorphic structure on the complexified bundle $P_{\mathbb{C}}=P\times_G G_{\mathbb{C}}$, see [1], and the base point p_∞ in P is also a base point in $P_{\mathbb{C}}$. We still let σ_∞ be obtained by radial trivialization from the North Pole, but now σ_0 shall be any holomorphic section over $\bar{D}_0$ (i.e., σ_0 is smooth on $\bar{D}_0$ and holomorphic on D_0). As before $\gamma\in LG_{\mathbb{C}}$ is defined by $\sigma_0=\sigma_\infty\gamma$, and clearly σ_0, and hence γ is well-defined up to multiplication with an element of $L^+G_{\mathbb{C}}$, i.e. we have a well-defined element $[\gamma]\in LG_{\mathbb{C}}/L^+G_{\mathbb{C}}=\Omega G$.

Let $\hat{\sigma}_0$ denote the section over D_0 and $\hat{\gamma}$ the loop in G obtained by radial trivialization. We want to find a holomorphic section of $P_{\mathbb{C}}$ over D_0. Such a section has the form $\hat{\sigma}_0 g$, where $g:\bar{D}_0\to G_{\mathbb{C}}$, and then $\gamma=\hat{\gamma}g$.

In the trivialization defined by the section $\hat{\sigma}_0$, the covariant derivative induced by ω is $\nabla=d+f_0d\theta$. We have $z=\cot\left(\frac{\varphi}{2}\right)e^{i\theta}$, so

$$dz=d\left(\cot\left(\frac{\varphi}{2}\right)\right)e^{i\theta}+\cot\left(\frac{\varphi}{2}\right)ie^{i\theta}d\theta,$$

$$d\theta=\frac{i}{2\cot\left(\frac{\varphi}{2}\right)}(e^{i\theta}d\bar{z}-e^{-i\theta}dz)=\frac{i}{2|z|^2}(z\,d\bar{z}-\bar{z}dz),$$

and the operator $\bar{\partial}_\omega : C^\infty(S^2) \to C^\infty(\wedge^1 S^2)$ is on U_0 given by

$$\bar{\partial}_\omega = \left(\frac{\partial}{\partial \bar{z}} + \frac{iz}{2|z|^2} f_0\right) d\bar{z}.$$

If $g : \bar{D}_0 \to G_{\mathbb{C}}$, then $\hat{\sigma}_0 g : \bar{D}_0 \to P_{\mathbb{C}}$ is holomorphic if and only if

$$g^{-1} \frac{\partial g}{\partial \bar{z}} + \frac{iz}{2|z|^2} g^{-1} f_0 g = 0. \tag{3.1}$$

In general we have $[\gamma] \neq [\hat{\gamma}]$ because equality holds if and only if $g|_{S^1} \in L^+ G_{\mathbb{C}}$ and this need not be the case. Choose for example $\gamma = 1$ and $f(z) = 4i|z|^2 \bar{z} A$ on D_0 with $A \in \mathfrak{g}$, then (3.1) becomes

$$g^{-1} \frac{\partial g}{\partial \bar{z}} - 2z\bar{z} g^{-1} A g = 0.$$

A solution is $g(z) = \exp(z\bar{z}^2 A)$, but $g(e^{i\theta}) = \exp(e^{-i\theta} A)$ so $g|_{S^1} \notin L^+ G_{\mathbb{C}}$.

But, if ω is a solution to the Yang-Mills equations, i.e., a critical point for the Yang-Mills functional, then $f = 0$ and $\gamma = \exp(\theta A)$ for some $A \in \mathfrak{g}$, hence

$$f_0 = f + \cos^2\left(\frac{\varphi}{2}\right) \gamma^{-1} \gamma' = \frac{|z|^2}{1 + |z|^2} A.$$

So (3.1) becomes

$$g^{-1} \frac{\partial g}{\partial \bar{z}} + \frac{i}{2} \frac{z}{1 + |z|^2} g^{-1} A g = 0,$$

and $g(z) = \exp\left(-\frac{i}{2} \log(1 + |z|^2) A\right)$ is a solution. As $g(e^{i\theta}) = \exp\left(-\frac{i \log 2}{2} A\right)$ is constant, $g \in L^+ G_{\mathbb{C}}$ and $[\gamma] = [\hat{\gamma} g] = [\hat{\gamma}]$.

4. The General Case

The process described in Sect. 3 makes sense on any closed Riemann surface X. Choose a point x_∞ on X and a local parameter around x_∞. We shall write the local parameter as z^{-1}, thus z is a holomorphic map from a neighbourhood of x_∞ to a neighbourhood of ∞ in the Riemann sphere. We may assume that $z(x_\infty) = \infty$, and that z is an isomorphism between a neighbourhood U_∞ of x_∞ and the region $|z| > \frac{1}{2}$ on the Riemann sphere. The standard circle S^1 can then be identified with the circle $|z| = 1$ around x_∞ on X. We denote the part of X, where $|z| > 1$ by X_∞, and the complement of the region where $|z| \geqq 1$ by X_0. Thus

$$\bar{X}_0 \cap \bar{X}_\infty = S^1.$$

The space of smooth loops $\gamma : S^1 \to G_{\mathbb{C}}$, which are the boundary values of holomorphic maps $X_0 \to G_{\mathbb{C}}$, is denoted $L_X^+ G_{\mathbb{C}}$. Both $\Omega G = LG_{\mathbb{C}}/L^+ G_{\mathbb{C}}$ and the quotient $\mathrm{Gr}^X = LG_{\mathbb{C}}/L_X^+G_{\mathbb{C}}$ are subvarieties of an infinite dimensional Grassmannian Gr, see [4].

Let P be a principal G-bundle over X with a connection ω and choose a base point p_∞ in the fiber over x_∞. Let $\theta \in [0, 2\pi]$ and $r \in (\frac{1}{2}, \infty]$ be polar coordinates on U_∞ and lift the curves $t \mapsto \left(\frac{1}{t}, \theta\right)$, with $t \in [0, 1]$ and $\theta \in [0, 2\pi]$, to horizontal curves in P starting at p_∞. As for S^2, we get a smooth section σ_∞ in P over $\bar{X}_\infty$.

Let σ_0 be a holomorphic section in the complexified bundle $P_{\mathbb{C}}$ over $\bar{X}_0$. The transition function $\gamma: S^1 \to G_{\mathbb{C}}$ is defined by

$$\sigma_0|_{S^1} = \sigma_\infty|_{S^1}\gamma. \tag{4.1}$$

As the section σ_0 is only determined up to multiplication by a holomorphic map $\bar{X}_0 \to G_{\mathbb{C}}$, the loop γ is only determined up to multiplication by an element of $L_X^+G_{\mathbb{C}}$, but we have a well-defined element $[\gamma] \in LG_{\mathbb{C}}/L_X^+G_{\mathbb{C}} = \mathrm{Gr}^X$. We put

$$\omega_j = \sigma_j^*(\omega).$$

Not any triple $(\gamma, \omega_0, \omega_\infty)$ can be obtained this way. At least we must have $\omega_0 \in C^\infty(\wedge^{1,0}(\bar{X}_0, \mathfrak{g}_{\mathbb{C}}))$ and $\omega_\infty = f_\infty d\theta$ with $f_\infty : \bar{X} \to \mathfrak{g}$. Given such a triple let us try to construct a G-bundle with a connection.

Using the connection ω_0 on $\bar{X}_0 \times G_{\mathbb{C}}$, we extend γ by radial trivialization from S^1 and obtain a smooth map $\tilde{\gamma}: \bar{X}_0 \cap U_\infty \to G_{\mathbb{C}}$, which satisfies

$$\tilde{\gamma}^{-1}\frac{\partial \tilde{\gamma}}{\partial r} = \omega_0\left(\frac{\partial}{\partial r}\right) \quad \text{and} \quad \tilde{\gamma}|_{S^1} = \gamma. \tag{4.2}$$

We can now use $\tilde{\gamma}$ as the transition function in a smooth $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$. Next we must extend ω_∞ to a one form $\tilde{\omega}_\infty = \tilde{f}_\infty d\theta$ with $\tilde{f}_\infty : U_\infty \to \mathfrak{g}$, such that

$$\omega_0 = \tilde{\gamma}^{-1}\tilde{\omega}_\infty\tilde{\gamma} + \tilde{\gamma}^{-1}d\tilde{\gamma}.$$

In view of (4.2) this is equivalent to

$$\omega_0\left(\frac{\partial}{\partial \theta}\right) = \tilde{\gamma}^{-1}\tilde{\omega}_\infty\left(\frac{\partial}{\partial \theta}\right)\tilde{\gamma} + \tilde{\gamma}^{-1}\frac{\partial \tilde{\gamma}}{\partial \theta},$$

or

$$\tilde{f}_\infty = \tilde{\omega}_\infty\left(\frac{\partial}{\partial \theta}\right) = \tilde{\gamma}\omega_0\left(\frac{\partial}{\partial \theta}\right)\tilde{\gamma}^{-1} + \frac{\partial \tilde{\gamma}}{\partial \theta}\tilde{\gamma}^{-1}.$$

We can get such an extension if and only if on S^1 we have that

$$f_\infty = \omega_\infty\left(\frac{\partial}{\partial \theta}\right) = \tilde{\gamma}\omega_0\left(\frac{\partial}{\partial \theta}\right)\tilde{\gamma}^{-1} + \frac{\partial \tilde{\gamma}}{\partial \theta}\tilde{\gamma}^{-1} \text{ to all orders.} \tag{4.3}$$

If (4.3) is satisfied, then we get a connection on $P_{\mathbb{C}}$. Finally, as ω_∞ has values in $\mathfrak{g}$ we have a G-structure over $\bar{X}_\infty$. And over $\bar{X}_0$ the complex structure together with the connection give a G-structure. So all in all we get a G-bundle with a connection.

Furthermore, given (γ, ω_0) such that the right-hand side of (4.3) lies in $\mathfrak{g}$ at S^1, then by [5] we can find ω_∞ satisfying (4.3), and ω_∞ can be chosen such that it depends continuously on (γ, ω_0). Let

$$\Omega^1_{d\theta}(\bar{X}_\infty, \mathfrak{g}) = \left\{ \alpha \in \Omega^1(\bar{X}_\infty, \mathfrak{g}) \,\middle|\, \alpha\left(\frac{\partial}{\partial r}\right) = 0 \right\},$$

$$\mathscr{K} = \{(\gamma, \omega_0, \omega_\infty) \in LG_{\mathbb{C}} \times \Omega^{1,0}(\bar{X}_0, \mathfrak{g}_{\mathbb{C}}) \times \Omega^1_{d\theta}(\bar{X}_\infty, \mathfrak{g}) \mid (4.3) \text{ is satisfied}\}$$

and

$$\mathscr{L} = \left\{ (\gamma, \omega_0) \in LG_{\mathbb{C}} \times \Omega^{1,0}(\bar{X}_0, \mathfrak{g}_{\mathbb{C}}) \,\middle|\, \tilde{\gamma}\omega_0\left(\frac{\partial}{\partial \theta}\right)\tilde{\gamma}^{-1} + \frac{\partial \tilde{\gamma}}{\partial \theta}\tilde{\gamma}^{-1} \in \mathfrak{g} \text{ on } S^1 \right\}.$$

We let $\mathscr{C}$ denote the space of connections, $\mathscr{G}_0$ the group of based gauge transformations and $\mathscr{B}$ the orbit space $\mathscr{C}/\mathscr{G}_0$.

The group $L^+_X G_{\mathbb{C}}$ acts on $\mathscr{L}$ (and $\mathscr{K}$) by $(\gamma, \omega_0)g = (\gamma g, g^{-1}\omega_0 g + g^{-1}dg)$, and we have well defined maps

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\cong} & \mathscr{K}/L^+_X G_{\mathbb{C}} \\ & & \downarrow \\ & & \mathscr{L}/L^+_X G_{\mathbb{C}} \\ & & \downarrow \\ & & \mathrm{Gr}^X . \end{array}$$

The map $\mathscr{K}/L^+_X G_{\mathbb{C}} \to \mathscr{L}/L^+_X G_{\mathbb{C}}$ is an affine bundle, and the vector space of translations is in each fibre

$$\{\omega_\infty \in \Omega^1_{d\theta}(\bar{X}_\infty, \mathfrak{g}) \mid \omega_\infty \text{ vanishes to all orders on } S^1\}.$$

Likewise $\mathscr{L}/L^+_X G_{\mathbb{C}} \to \mathrm{Gr}^X$ is an affine bundle with fibre

$$\{\omega_0 \in \Omega^{1,0}(\bar{X}_0, \mathfrak{g}_{\mathbb{C}}) \mid \omega_0 \in \gamma^{-1}\mathfrak{g}\gamma - \gamma^{-1}\gamma' \text{ on } S^1\}.$$

Hence we have

(4.4) Theorem. *The map $\mathscr{B} \to \mathrm{Gr}^X$ is a homotopy equivalence.*

It is already known that $\mathrm{Map}(X, BG)$ is homotopy equivalent to both $\mathscr{B}$, see [1], and Gr^X, see [4]. We have now shown the third homotopy equivalence directly.

It would be nice if one could get a result like Theorem 2.6 for a general Riemann surface. In Sect. 3 we saw that the map we get by radial trivialization is different from the map given by holomorphic trivialization, so perhaps Theorem 2.6 is too much to ask for. But the two maps agree on the moduli space of instantons, so we could look for a function $\mathscr{F} : \mathrm{Gr}^X \to \mathbb{R}$, such that the moduli space of instantons is mapped bijectively to the critical points of $\mathscr{F}$.

The energy-function on ΩG is the restriction of a function defined on all of Gr, see [4], so the restriction to Gr^X could be a candidate for $\mathscr{F}$.

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