Commun. Math. Phys. 127, 41–69 (1990)

Communications in Mathematical Physics

A Perturbed Mean Field Model of An Interacting Boson Gas and the Large Deviation Principle

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Abstract. This is a study of the equilibrium thermodynamics of a mean-field model of an interacting boson gas perturbed by a term quadratic in the occupation numbers of the free-gas energy-levels. We prove the existence of the pressure in the thermodynamic limit. We obtain also a variational formula for the pressure; this enables us to compare the effect of a smooth quadratic perturbation with that of a singular one (the Huang–Yang–Luttinger model). The proof uses a large deviation result for the occupation measure of the free boson gas which is of independent interest.

1. Introduction

The rigorous investigation of the thermodynamics of a system of bosons based on the full quantum mechanical hamiltonian and using a realistic pair- potential seems beyond the reach of present methods. Either one must use a very special pair-potential or one must truncate the hamiltonian. The first course was followed by Lieb and Liniger [1] who diagonalized the full hamiltonian for a boson gas with a δ-function pair-potential in one dimension using what is now known as the Bethe Ansatz. Using the results of [1], Yang and Yang [2] gave a formula for the pressure in this model; in a recent paper [3], we proved that if the Bethe Ansatz wave-functions form a complete set then the grand canonical pressure is given by the Yang–Yang formula. The proof in [3] uses probabilistic methods (Varadhan's Laplacian asymptotics [4] based on the large deviation principle).

The second course has been followed by many authors. One such approach is to use a hamiltonian which is a function of the free-gas occupation numbers; the reader is referred to the book of Thouless [5] for an introduction to these models. In this paper, we continue our investigation by probabilistic methods of such models: we study a smooth perturbation of a mean-field model of interacting bosons (for perturbations of mean-field models in other contexts, see Bricmont and Fontaine [6]).

In [7], we studied the equilibrium thermodynamics of the Huang–Yang–Luttinger model [12] of a boson gas with a hard-sphere repulsion using large deviation methods, contrasting its properties with those of the mean-field model. We proved the existence of condensate for values of the chemical potential above a critical value and verified a prediction of Thouless [5] that there is a jump in the density of condensate at the critical value. In the mean-field model, condensation occurs if and only if it occurs in the corresponding free-gas; in the HYL-model, there is always condensation provided the density is sufficiently high. For example, when the integrated density of states of the free-gas is given as a function of the energy $\lambda (\lambda \geqq 0)$ by $F(\lambda) = C_d \lambda^{d/2} (d > 0)$ there is no condensation in the mean-field model if $d \leqq 2$, while there is condensation in the HYL-model for all $d > 0$.

The difference in the conditions for the existence of condensate in the mean-field and HYL-models reflects the difference in origin of the phase-transition. The mean-field phase-transition is a consequence of the balance between entropy and kinetic energy; the HYL phase-transition results from the balance between entropy and interaction energy. The HYL-hamiltonian differs from the mean-field hamiltonian by the addition of a singular quadratic form

$$\frac{a}{2V}\left\{\left(\sum_{j=1}^{\infty} n(j)\right)^2 - \sum_{j=1}^{\infty} n(j)^2\right\} \tag{1.1}$$

in the occupation numbers $n(j), j = 1, 2, \ldots$, of the energy-levels of the single-particle hamiltonian of the free-gas. This quadratic form is a purely quantum mechanical term which reflects the boson statistics. It is smallest when all the particles are in the same energy level, and therefore tends to produce condensation in momentum space. It is of interest to know what is the effect of a less singular quadratic perturbation of the mean-field hamiltonian. In this paper we consider a model in which the mean-field hamiltonian is perturbed by addition of a quadratic form

$$\sum_{j,j'} \frac{1}{2V} v(\lambda(j), \lambda(j')) n(j) n(j'), \tag{1.2}$$

where $v(\cdot,\cdot)$ is a positive-definite bounded continuous function and $\lambda(j)$ is the j^{th} eigenvalue of the single-particle hamiltonian. We prove that if $v(\cdot,\cdot)$ is sufficiently smooth and the integrated density of states of the free-gas is given by $F(\lambda) = C_d \lambda^{d/2} (d > 0)$ then there is no condensation if $d \leqq 2$.

The methods employed in this paper are extensions of those used in [7]. To prove the existence of the pressure, we use large deviation methods; these yield a variational formula for the pressure which is investigated by means of the differential calculus. In both respects, the techniques of this paper are more sophisticated than those of [7] because the spaces involved are infinite-dimensional; nevertheless, the principles are the same and this paper should be read as a sequel to [7] which contains a review of results on the free-boson gas obtained in [8] and an introduction to large deviation methods.

In models in which the hamiltonian is diagonal in the occupation number operators, it is possible to consider the occupation numbers as random variables rather than as operators; we shall do this. The probability space on which we define our random variables is the countable set Ω of terminating sequences of non-negative integers: an element ω of Ω is a sequence $\{\omega(j)\in\mathbb{N}: j=1,2,\ldots\}$ satisfying $\sum_{j\geq 1}\omega(j)<\infty$. The basic random variables are the occupation numbers $\{\sigma_j: j=1,2,\ldots\}$; they are the evaluation maps $\sigma_j:\Omega\to\mathbb{N}$ defined by $\sigma_j(\omega)=\omega(j)$ for each ω in Ω. The total number of particles $N(\omega)$ in the configuration ω is defined by

$$N(\omega)=\sum_{j\geq 1}\sigma_j(\omega). \tag{1.3}$$

Let $\Lambda_1,\Lambda_2\ldots$ be a sequence of regions in $\mathbb{R}^d$ and denote the volume of Λ_l by V_l; we assume that $V_l\to\infty$ as $l\to\infty$. We associate with the region Λ_l the free-gas hamiltonian H_l given by

$$H_l(\omega)=\sum_{j\geq 1}\lambda_l(j)\sigma_j(\omega), \tag{1.4}$$

where $0=\lambda_l(1)\leqq\lambda_l(2)\leqq\cdots$, and the mean-field hamiltonian H_l^{MF} given by

$$H_l^{MF}(\omega)=H_l(\omega)+\frac{a}{2V_l}N(\omega)^2, \tag{1.5}$$

where a is a strictly positive real number.

The principal model considered in this paper is the perturbed mean-field model with hamiltonian $\tilde{H}_l$ given by

$$\tilde{H}_l(\omega)=H_l^{MF}(\omega)+\frac{1}{2V_l}\sum_{j,j'}v(\lambda_l(j),\lambda_l(j'))\sigma_j(\omega)\sigma_{j'}(\omega). \tag{1.6}$$

The free-gas pressure $p_l(\mu)$, $\mu<0$, is defined by

$$e^{\beta V_l p_l(\mu)}=\sum_{\omega\in\Omega}e^{\beta\{\mu N(\omega)-H_l(\omega)\}}; \tag{1.7}$$

it is given in terms of the $\lambda_l(j)$ by

$$p_l(\mu)=\int_{[0,\infty)}p(\mu|\lambda)dF_l(\lambda), \tag{1.8}$$

where F_l is the distribution function

$$F_l(\lambda)=(V_l)^{-1}\#\{j:\lambda_l(j)\leqq\lambda\} \tag{1.9}$$

and $p(\mu|\lambda)$ is the partial pressure given by

$$p(\mu|\lambda)=\beta^{-1}\ln(1-e^{\beta(\mu-\lambda)})^{-1}. \tag{1.10}$$

Since Ω is a countable set, we may specify a probability measure on Ω by giving its value at each point ω of Ω. The free-gas grand canonical measure is defined for $\mu<0$ by

$$\mathbb{P}_l^\mu[\omega]=e^{\beta\{\mu N(\omega)-H_l(\omega)-V_l p_l(\mu)\}}. \tag{1.11}$$

The pressure $\tilde{p}_l(\mu)$ in the perturbed mean-field model is given by

$$e^{\beta V_l\tilde{p}_l(\mu)}=\sum_{\omega\in\Omega}e^{\beta\{\mu N(\omega)-\tilde{H}_l(\omega)\}}; \tag{1.12}$$

this is defined for all real values of μ because of the stability provided by the $(a/2V_l)N(\omega)^2$ term in the hamiltonian. However, the free-gas pressure is defined only for $\mu<0$; to express (1.12) as an expectation with respect to the free-gas grand canonical measure, we employ a device used in [7]. Fix $\alpha<0$; we may re-write (1.12) as an expectation with respect to the grand canonical measure $\mathbb{P}_l^\alpha$ as follows: introduce the occupation measure L_l by defining for each Borel subset A of $[0,\infty)$ and ω in Ω

$$L_l[\omega;A]=\frac{1}{V_l}\sum_{j\geqq 1}\sigma_j(\omega)\delta_{\lambda_l(j)}[A], \tag{1.13}$$

where $\delta_x[\cdot]$ is the Dirac measure concentrated at x so that, for each ω in Ω, $A\to L_l[\omega;A]$ is a bounded positive measure; for each bounded positive measure m on $[0,\infty)$, define

$$\langle m,Vm\rangle=\int_{[0,\infty)\times[0,\infty)}\int v(\lambda,\lambda')m(d\lambda)m(d\lambda'), \tag{1.14}$$

and put

$$G^\mu[m]=\mu\|m\|-\tfrac{1}{2}\{a\|m\|^2+\langle m,Vm\rangle\}, \tag{1.15}$$

where

$$\|m\|=\int_{[0,\infty)}m(d\lambda); \tag{1.16}$$

then

$$e^{\beta V_l\tilde{p}_l(\mu)}=e^{\beta V_l p_l(\alpha)}\sum_{\omega\in\Omega}e^{\beta V_l G^{\mu-\alpha}[L_l[\omega;\cdot]]}\mathbb{P}_l^\alpha[\omega]. \tag{1.17}$$

The next step is to re-write (1.17) as an integral over $E=\mathscr{M}_+^b(\mathbb{R}_+)$, the space of bounded positive measures on $[0,\infty)$. We equip E with the narrow topology. Define $\langle m,f\rangle$ *by*

$$\langle m,f\rangle=\int_{[0,\infty)}f(\lambda)m(d\lambda) \tag{1.18}$$

for m in E and f in $\mathscr{C}^b(\mathbb{R}_+)$, the bounded continuous functions on $[0,\infty)$ equipped with the norm of uniform convergence; the narrow topology on E is the weakest topology for which the mappings $m\to\langle m,f\rangle$ are continuous for all f in $\mathscr{C}^b(\mathbb{R}_+)$. Let $\mathbb{K}_l$ be the probability measure induced on E by L_l:

$$\mathbb{K}_l^\alpha=\mathbb{P}_l^\alpha\circ L_l^{-1}, \tag{1.19}$$

we can re-write (1.17) as an integral with respect to $\mathbb{K}_l^\alpha$:

$$e^{\beta V_l\tilde{p}_l(\mu)}=e^{\beta V_l p_l(\alpha)}\int_E e^{\beta V_l G^{\mu-\alpha}[m]}\mathbb{K}_l^\alpha[dm], \tag{1.20}$$

so that

$$\tilde{p}_l(\mu)=p_l(\alpha)+\frac{1}{\beta V_l}\ln\int_E e^{\beta V_l G^{\mu-\alpha}[m]}\mathbb{K}_l^\alpha[dm]. \tag{1.21}$$

Conditions on the double sequence $\{\lambda_l(j)\}$ sufficient to ensure the existence of the limit $p(\alpha) = \lim_{t \to \infty} p(\alpha)$ were given in [8] and reviewed in [7]; for convenience, we restate them here. Define $\phi_l(\beta)$ for $0 < \beta < \infty$ by

$$\phi_l(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} dF_l(\lambda), \tag{1.22}$$

and introduce the conditions

(S1) $\phi(\beta) = \lim_{l \to \infty} \phi_l(\beta)$ *exists for all* β *in* $(0, \infty)$.

(S2) $\phi(\beta)$ *is non-zero for at least one value of* β *in* $(0, \infty)$.

When (S1) holds, there exists a unique distribution function F, the integrated density of states, such that

$$\phi(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} dF(\lambda) \tag{1.23}$$

and $F_l(\lambda) \to F(\lambda)$ at least at the points of continuity of F. When in addition (S2) holds, the limit $p(\mu) = \lim_{l \to \infty} p_l(\mu)$ exists for $\mu < 0$ and $p(\mu)$ is given by

$$p(\mu) = \int_{[0,\infty)} p(\mu|\lambda) dF(\lambda). \tag{1.24}$$

In this paper, we find it necessary to introduce a further restriction on $\{\lambda_l(j)\}$:

(S3) *The measure dF determined by the integrated density of states F is absolutely continuous with respect to Lebesgue measure with a density which is strictly positive almost everywhere on* $[0, \infty)$.

(In the standard example, where the single particle hamiltonian is a constant multiple of the Laplacian in Λ_l with Dirichlet boundary conditions on $\partial\Lambda_l$, and $\{\Lambda_l : l = 1, 2, \ldots\}$ is a sequence of bounded convex open sets in $\mathbb{R}^d$ which eventually fills out the whole of $\mathbb{R}^d$, all three conditions hold and $F(\lambda) = c_d \lambda^{d/2}$.)

The formula (1.21) suggests the use of Laplace's method to complete the proof that $\tilde{p}(\mu) = \lim_{l \to \infty} \tilde{p}_l(\mu)$ exists. Varadhan's theorem [4] provides an efficient way of doing this; the conditions on the sequence $\{\mathbb{K}_l^\alpha : l = 1, 2, \ldots\}$ are stated abstractly:

Let E be a regular topological space and $\{V_l : l = 1, 2, \ldots\}$ a sequence of positive numbers diverging to $+\infty$. Let $\{\mathbb{K}_l : l = 1, 2, \ldots\}$ be a sequence of probability measures on the Borel subsets of E. We say that $\{\mathbb{K}_l\}$ obeys the large deviation principle with constants $\{V_l\}$ and rate-function $I : E \to [0, \infty]$ if the following conditions are satisfied:

(LD1) $I[\cdot]$ *is lower semi-continuous.*

(LD2) *For each* $b < \infty$, *the set* $\{m : I[m] \leqq b\}$ *is compact.*

(LD3) *For each closed set* C,

$$\limsup_{l \to \infty} \frac{1}{V_l} \ln \mathbb{K}_l[C] \leqq -I[C].$$

(LD4) *For each open set* G,

$$\liminf_{l \to \infty} \frac{1}{V_l} \ln \mathbb{K}_l[G] \geqq -I[G].$$

Here we have used the notational device of defining $I[A]$, where A is a subset of E, to be the infimum of the set $\{I[m]:m\in A\}$ if $A \neq \emptyset$ and $I[\emptyset] = +\infty$.

Varadhan's Theorem [4]. *Let $\{\mathbb{K}_l\}$ be a sequence of probability measures on the Borel subsets of a regular topological space E satisfying the large deviation principle with constants $\{V_l\}$ and rate-function $I[\cdot]$. Then, for any continuous function $G:E\to\mathbb{R}$ which is bounded above, we have*

$$\lim_{l\to\infty}\frac{1}{V_l}\ln\int_E e^{V_lG[m]}\mathbb{K}_l[dm] = \sup_E\{G[m]-I[m]\}. \tag{1.25}$$

In applying Varadhan's theorem to the proof of the existence of the limit $\tilde{p}(\mu)$, our first task is to find a candidate for the rate-function $I[\cdot]$. There is a standard trick which we employ. An inspection of (1.25) suggests that it might be the case that

$$\lim_{l\to\infty}\frac{1}{\beta V_l}\ln\int_E e^{\beta V_l\langle m,t\rangle}\mathbb{K}_l^\alpha[dm] = \sup_E\{\langle m,t\rangle - I^\alpha[m]\}; \tag{1.26}$$

now the left-hand side is finite, provided the function is suitably restricted, and can be evaluated explicitly. Put

$$C^\alpha[t] = \lim_{l\to\infty}\frac{1}{\beta V_l}\ln\int_E e^{\beta V_l\langle m,t\rangle}\mathbb{K}_l^\alpha[dm]; \tag{1.27}$$

then

$$C^\alpha[t] = \lim_{l\to\infty}\{p_l[\alpha+t]-p_l(\alpha)\} = p[\alpha+t]-p(\alpha),$$

where

$$p_l[\alpha+t] = \int_{[0,\infty)} p(\alpha+t(\lambda)\,|\,\lambda)dF_l(\lambda)$$

and

$$p[\alpha+t] = \int_{[0,\infty)} p(\alpha+t(\lambda)\,|\,\lambda)dF(\lambda), \tag{1.28}$$

provided

$$\inf_{\lambda\geqq 0}\{\lambda-\alpha-t(\lambda)\}>0. \tag{1.29}$$

That is, (1.27) suggests that

$$C^\alpha[t] = \sup_E\{\langle m,t\rangle - I^\alpha[m]\}; \tag{1.30}$$

this expresses $C^\alpha[t]$ as the Legendre transform of $I^\alpha[\cdot]$. Inverting the Legendre transform, we get the following candidate for the rate-function:

$$I^\alpha[m] = \sup_t\{\langle m,t\rangle - C^\alpha[t]\}, \tag{1.31}$$

where the supremum is taken over those t in $\mathscr{C}^b(\mathbb{R}_+)$ which satisfy (1.29), and

$$C^\alpha[t] = p[\alpha + t] - p(\alpha). \tag{1.32}$$

In Sect. 3 we verify that conditions (LD1) to (LD4) are satisfied whenever conditions (S1), (S2) and (S3) hold.

In Sect. 4 we verify that $G^\mu[\cdot]$ satisfies the hypotheses of Varadhan's theorem and conclude that the limit $\tilde{p}(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_E \{G^{\mu-\alpha}[m] - I^\alpha[m]\}.$$

In Sect. 5 we examine this variational expression for $\tilde{p}(\mu)$; the expression (1.31) for $I^\alpha[m]$ is not convenient for this purpose. A more useful as well as a more enlightening expression for $I^\alpha[m]$ is

$$I^\alpha[m] = p(\alpha) + f[m] - \alpha \| m \|, \tag{1.33}$$

where

$$f[m] = \int_{[0,\infty)} \lambda m(d\lambda) - \beta^{-1} \int_{[0,\infty)} s(\rho(\lambda)) dF(\lambda); \tag{1.34}$$

here $\rho(\cdot)$ is the density of the absolutely continuous part of m with respect to dF and

$$s(x) = \begin{cases} (1+x)\ln(1+x) - x\ln x, & x > 0, \\ 0, & x = 0. \end{cases} \tag{1.35}$$

Readers of the treatise of Landau and Lifschitz [9] will recognize $f[m]$ as the *non-equilibrium free energy of free bosons.* The proof of the equivalence of (1.33) and (1.31) is a major task; the key is the Approximation Theorem. Call a measure m a *Bose–Einstein measure* if it can be written as $m^t(d\lambda) = \rho^{\alpha+t}(\lambda) dF(\lambda)$, where

$$\rho^{\alpha+t}(\lambda) = (e^{\beta(\lambda - \alpha - t(\lambda))} - 1)^{-1}. \tag{1.36}$$

Approximation Theorem. *Let m be an element of E such that $I^\alpha[m]$, given by (1.31), is finite; then there exists a sequence $\{m_n : n = 1, 2 \ldots\}$ of Bose–Einstein measures such that*

1. $$\lim_{n\to\infty} m_n = m,$$

2. $$\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m].$$

The Approximation Theorem is not only necessary for the proof of the equivalence of (1.33) and (1.31), it is also crucial in establishing (LD4). For this reason, we place the proof of the equivalence of the two expressions for the rate-function in Sect. 2.

Using the expression (1.33) for $I^\alpha[m]$, we obtain in Sect. 4 the following expression for $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = -\inf_E \mathscr{E}^\mu[m], \tag{1.37}$$

where

$$\mathscr{E}^{\mu}[m]=f[m]+\tfrac{1}{2}\{a\|m\|^2+\langle m,Vm\rangle\}-\mu\|m\|. \tag{1.38}$$

In Sect. 5 we use a compactness argument to prove the existence of at least one minimizer of $\mathscr{E}^{\mu}[\cdot]$ and go on to show that a measure is a minimizer if and only if it satisfies the Euler–Lagrange equations associated with $\mathscr{E}^{\mu}[\cdot]$. This result enables us to prove that the singular part of a minimizing measure must be concentrated on the subset R_F of $[0,\infty)$ defined by

$$R_F=\{\lambda\in[0,\infty):\lambda'\mapsto(\lambda-\lambda')^{-1}\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}_+;dF)\},$$

provided $v(\cdot,\cdot)$ is sufficiently smooth.

Specifically, we prove that if v is continuously differentiable and $\partial v/\partial\lambda$ is bounded then $m_s(R_F^c)=0$ for m a minimizing measure. For example, if $F(\lambda)=C_d\lambda^{d/2}$, $d>0$, then $R_F=\emptyset$ for $d\leqq 2$ and $R_F=\{0\}$ for $d>2$.

In the mean-field model (v identically zero), we can prove more: in this case, the Euler–Lagrange equations become

$$\lambda-\mu+a\|m\|=0,\quad m_s-\text{a.e.},$$

$$\lambda-\mu+a\|m\|=\beta^{-1}\ln\frac{1+\rho(\lambda)}{\rho(\lambda)},\quad dF-\text{a.e.}.$$

They have a unique solution with

$$\rho(\lambda)=(e^{\beta(\lambda-\mu+a\|m\|)}-1)^{-1} \tag{1.39}$$

and $a\|m\|\geqq\mu$. It follows that

$$\rho(\lambda)\leqq(e^{\beta\lambda}-1)^{-1}$$

and

$$\|m_a\|=\int_{[0,\infty)}\rho(\lambda)dF(\lambda)\leqq\int_{[0,\infty)}(e^{\beta\lambda}-1)^{-1}dF(\lambda). \tag{1.40}$$

Now $\rho_c=\int_{[0,\infty)}(e^{\beta\lambda}-1)^{-1}dF(\lambda)$ is the free-gas critical density, so that (1.40) implies that the total mass of the absolutely continuous part of m cannot exceed the free-gas critical density: if $\mu>a\rho_c$, then $\|m_s\|>0$. (Note that if $F(\lambda)=C_d\lambda^{d/2}, d>0$, then $\rho_c=\infty$ for $d\leqq 2$ and $\rho_c<\infty$ for $d>2$.) In the case of the mean-field model, the location of the support of the singular measure is determined by the necessity of minimizing the internal energy of the free-gas since $\int_{[0,\infty)}\lambda m(d\lambda)$ is the only term in $\mathscr{E}^{\mu}[m]$ which is sensitive to the location of m_s; it follows that $m_s(d\lambda)=\|m_s\|\delta_0(d\lambda)$. Assume that m is a minimizer and $\|m_s\|>0$; then, by the first Euler–Lagrange equation,

$$\mu=a\|m\|=a\{\|m_s\|+\|m_a\|\}$$

and, by the second, $\rho(\lambda)=(e^{\beta\lambda}-1)^{-1}$ so that $\|m_a\|=\rho_c$; hence $\mu>a\rho_c$. Thus $\mu\leqq a\rho_c$ implies that $\|m_s\|=0$; in this case,

$$\rho(\lambda)=(e^{\beta(\lambda-\mu+a\|m\|)}-1)^{-1},$$

where $\|m\|$ is the positive root of the equation

$$x = \int_{[0,\infty)} (e^{\beta(\lambda-\mu+ax)} - 1)^{-1} dF(\lambda).$$

In summary, in the mean-field model (v identically zero) we have the following result:

there exists a unique minimizer m of $\mathscr{E}^{\mu}[\cdot]$ given by

$$m(d\lambda) = \begin{cases} (\mu/a - \rho_c)\delta_0(d\lambda) + p'(0|\lambda)dF(\lambda), & \mu > a\rho_c, \\ p'(\mu - a\|m\| |\lambda)dF(\lambda), & \mu \leqq a\rho_c, \end{cases}$$

so that bosons condense into a state of zero kinetic energy, in order to minimize the free-energy, once the particle number density exceeds a critical value ρ_c; below ρ_c, the free-energy is minimized by having almost all particles in a state with strictly positive energy.

In the case where v is not identically zero, our analysis is, at present, less complete; it may be summarized by saying that, when $F(\lambda) = C_d\lambda^{d/2} (d > 0)$, the mean-field result that there is no condensation for $d \leqq 2$ is stable with respect to a smooth quadratic perturbation. The implications of this result for the hamiltonians of many-body theory will be discussed elsewhere.

2. The Rate-Function

We have seen that a natural candidate for the rate-function $I^{\alpha}[\cdot]$ of the sequence $\mathbb{K}_l^{\alpha}: l = 1, 2, \ldots$ of Kac measures is the Legendre transform of the cumulant generating function $C^{\alpha}[\cdot]$. Our strategy is to define $I^{\alpha}[\cdot]$ in this way and then to prove that the Large Deviation Principle holds for $\mathbb{K}_l^{\alpha}$ with this choice of rate-function. Let $\mathscr{C}^{\alpha}$ be the class of functions defined, for $\alpha < 0$, by

$$\mathscr{C}^{\alpha} = \left\{ t \in C^b(\mathbb{R}_+): \inf_{\lambda \geqq 0} \{\lambda - \alpha - t(\lambda)\} > 0 \right\}. \tag{2.1}$$

For t in $\mathscr{C}^{\alpha}$, define $p[\alpha + t]$ by

$$p[\alpha + t] = \int_{[0,\infty)} p(\alpha + t(\lambda)|\lambda) dF(\lambda), \tag{2.2}$$

where

$$p(\alpha|\lambda) = \beta^{-1} \ln(1 - e^{\beta(\alpha - \lambda)})^{-1}, \tag{2.3}$$

and put

$$C^{\alpha}[t] = p[\alpha + t] - p(\alpha). \tag{2.4}$$

Define $I^{\alpha}[m]$, for m in E, by

$$I^{\alpha}[m] = \sup_{t \in \mathscr{C}^{\alpha}} \{\langle m, t\rangle - C^{\alpha}[t]\}. \tag{2.5}$$

Our first task is to find a more useful expression for $I^{\alpha}[m]$ than the defining relation (2.5); we accomplish this by a sequence of lemmas.

Lemma 2.1. *Let m be a measure in E; then, for all $\alpha < 0$, $I^\alpha[m]$ is finite if and only if $\int_{[0,\infty)} \lambda m(d\lambda)$ is finite.*

Proof. It is convenient to introduce the function $I^\alpha(\cdot,\cdot): E \times \mathscr{C}^b(\mathbb{R}_+) \to \mathbb{R}$, defined by

$$I^\alpha(m,t) = \langle m,t\rangle - C^\alpha[t]. \tag{2.6}$$

For t in $\mathscr{C}^\alpha$, $p[\alpha + t]$ is positive and $t(\lambda) < \lambda - \alpha$ so that

$$\begin{aligned} I^\alpha(m,t) &= \langle m,t\rangle - p[\alpha+t] + p(\alpha) \\ &< \langle m,t\rangle + p(\alpha) < \int_{[0,\infty)} \lambda m(d\lambda) - \alpha\|m\| + p(\alpha). \end{aligned}$$

Taking the supremum over t in $\mathscr{C}^\alpha$, we have

$$I^\alpha[m] \leqq \int_{[0,\infty)} \lambda m(d\lambda) - \alpha\|m\| + p(\alpha).$$

This proves that $I^\alpha[m]$ is finite whenever $\int_{[0,\infty)} \lambda m(d\lambda)$ is finite.

For the converse, we define a sequence $\{t_n : n = 1,2,\dots\}$ in $\mathscr{C}^\alpha$ by

$$t_n(\lambda) = \begin{cases} (\lambda-\alpha)/2, & \lambda \leqq n, \\ (n-\alpha)(n+1-\lambda)/2, & n \leqq \lambda \leqq n+1, \\ 0, & n+1 \leqq \lambda. \end{cases}$$

Then

$$I^\alpha[m] \geqq I^\alpha(m,t_n) \geqq \frac{1}{2}\int_{[0,n)} (\lambda-\alpha) m(d\lambda) - p[\alpha+t_n] + p(\alpha);$$

taking $n \to \infty$, we find that

$$I^\alpha[m] \geqq \frac{1}{2}\int_{[0,\infty)} \lambda m(d\lambda) - \frac{1}{2}\alpha\|m\| - \int_{[0,\infty)} p\left(\frac{1}{2}\alpha \middle| \frac{1}{2}\lambda\right) dF(\lambda) + p(\alpha);$$

We conclude that $\int_{[0,\infty)} \lambda m(d\lambda)$ is finite whenever $I^\alpha[m]$ is finite. □

Next, we split off that part of a measure m which is singular with respect to dF and deal with it separately. Let m be a measure in E; let $m = m_s + m_a$ be the Lebesgue decomposition of m with respect to the density-of-states measure dF into the singular part m_s and the absolutely continuous part m_a; let $\rho(\cdot)$ be the density of m_a so that $m_a(d\lambda) = \rho(\lambda) dF(\lambda)$.

Lemma 2.2. *For each m in E and each $\alpha < 0$, we have*

$$I^\alpha[m] = \int_{[0,\infty)} (\lambda-\alpha) m_s(d\lambda) + I^\alpha[m_a]. \tag{2.7}$$

Proof. For t in $\mathscr{C}^\alpha$, we have $t(\lambda) < \lambda - \alpha$ so that

$$I^\alpha(m,t) = \langle m_s,t\rangle + I^\alpha(m_a,t) \leqq \int_{[0,\infty)} (\lambda-\alpha) m_s(d\lambda) + I^\alpha(m_a,t);$$

hence

$$I^\alpha[m] \leqq \int_{[0,\infty)} (\lambda - \alpha) m_s(d\lambda) + I^\alpha[m_a].$$

In proving the reversed inequality, we may assume that $m_s \neq 0$. Given t in $\mathscr{C}^\alpha$, we construct a sequence $\{t_n : n = 1, 2, \ldots\}$ such that $I^\alpha(m, t_n)$ approximates $\int_{[0,\infty)} (\lambda - \alpha) m_s(d\lambda) + I^\alpha(m_a, t)$ for n large.

First define, for $0 < \delta < \inf_{\lambda \geqq 0} \{\lambda - \alpha - t(\lambda)\}$,

$$s_{n,\delta}(\lambda) = \min\{\lambda - \alpha - \delta, n\}.$$

Now let $B \subset [0, \infty)$ be a Borel subset such that $m_s(B^c) = 0$ and $\int_B dF(\lambda) = 0$, so that m_s is concentrated on B. Choose a compact subset $K \subset B$ such that $m_s(B \backslash K) < \delta$ and choose an open set $O \supset B$ so that

$$\int_O \rho(\lambda) dF(\lambda) < \delta \quad \text{and} \quad \int_O dF(\lambda) < \delta^2.$$

By Urysohn's Lemma, there exists a function τ in $\mathscr{C}^b_+(\mathbb{R}_+)$ such that $0 \leqq \tau(\lambda) \leqq 1$ for all $\lambda \geqq 0$, $\tau(\lambda) = 1$ for λ in K and $\tau(\lambda) = 0$ for λ in the complement of O. Define $t_{n,\delta}$ by

$$t_{n,\delta}(\lambda) = \tau(\lambda) s_{n,\delta}(\lambda) + (1 - \tau(\lambda)) t(\lambda).$$

Then $t_{n,\delta}$ is in $\mathscr{C}^\alpha$ and

$$\begin{aligned}
\langle m, t_{n,\delta} \rangle &= \langle m_s, s_{n,\delta} \rangle + \langle m_a, t \rangle + \int_{B \backslash K} \{1 - \tau(\lambda)\}\{t(\lambda) - s_{n,\delta}(\lambda)\} m_s(d\lambda) \\
&\quad + \int_{O \backslash B} \tau(\lambda)\{s_{n,\delta}(\lambda) - t(\lambda)\} m_a(d\lambda) \\
&\geqq \langle m_s, s_{n,\delta} \rangle + \langle m_a, t \rangle - 2\max\{\|t\|_\infty, n\} m_s(B \backslash K) \\
&\quad - 2\max\{\|t\|_\infty, n\} m_a(O) \\
&\geqq \langle m_s, s_{n,\delta} \rangle + \langle m_a, t \rangle - 4\max\{\|t\|_\infty, n\}\delta,
\end{aligned}$$

while $C^\alpha[t_{n,\delta}] \leqq \int_O p(\lambda - \delta | \lambda) dF(\lambda) + C^\alpha[t]$ (since $t_{n,\delta}(\lambda) + \alpha < \lambda - \delta$) and

$$\int_O p(\lambda - \delta | \lambda) dF(\lambda) = \int_O p(0 | \delta) dF(\lambda) \leqq \frac{1}{\beta^2 \delta} \int_O dF(\lambda) \leqq \frac{\delta}{\beta^2}.$$

Thus we have

$$I^\alpha(m, t_{n,\delta}) \geqq \langle m_s, s_{n,\delta} \rangle + I^\alpha(m_a, t) - 4\delta \cdot \max\{\|t\|_\infty, n\} - \beta^{-2}\delta.$$

Now put $t_n = t_{n,\delta}$ with $\delta = 1/n^2$, and $s_n = s_{n,\delta}$ with $\delta = 1/n^2$; then $\langle m_s, s_n \rangle \to \int_{[0,\infty)} (\lambda - \alpha) m_s(d\lambda)$ by Lebesgue's monotone convergence theorem, and it follows that

$$I^\alpha[m] \geqq \int_{[0,\infty)} (\lambda - \alpha) m_s(d\lambda) + I^\alpha(m_a, t);$$

since t was an arbitrary element of $\mathscr{C}^\alpha$, we have

$$I^\alpha[m] \geqq \int_{[0,\infty)} (\lambda-\alpha)m_s(d\lambda) + I^\alpha[m_a]. \qquad \square$$

There is a special class of measures for which the supremum in (2.5) can be computed explicitly with ease: for $t \in \mathscr{C}^\alpha$, define the measure m^t by

$$m^t(d\lambda) = \rho^{\alpha+t}(\lambda)dF(\lambda),$$

where

$$\rho^{\alpha+t}(\lambda) = p'(\alpha+t(\lambda)|\lambda) = (e^{\beta(\lambda-\alpha-t(\lambda))}-1)^{-1}; \tag{2.8}$$

we call these measures *Bose–Einstein measures*. Using the notation of Lemma 2.2 we define the *free-energy functional*

$$f[m] = \int_{[0,\infty)} \lambda m(d\lambda) - \beta^{-1} \int_{[0,\infty)} s(\rho(\lambda))dF(\lambda), \tag{2.9}$$

where the *entropy function* $s\colon [0,\infty) \to [0,\infty)$ is defined by

$$s(x) = \begin{cases} (1+x)\ln(1+x) - x\ln x, & x>0, \\ 0, & x=0. \end{cases} \tag{2.10}$$

Lemma 2.3. *Let m be a Bose–Einstein measure; then the rate function $I^\alpha[m]$ is given by*

$$I^\alpha[m] = p(\alpha) + f[m] - \alpha\|m\|. \tag{2.11}$$

Proof. Let $m(d\lambda) = m^t(d\lambda) = \rho^{\alpha+t}(\lambda)dF(\lambda)$ with t in $\mathscr{C}^\alpha$; we show that the supremum in (2.5) is attained at $\tilde{t} = t$. Then (2.11) follows since, for any α,

$$s(\rho^\alpha(\lambda)) = \beta\rho^\alpha(\lambda)(\lambda-\alpha) + \beta p(\alpha|\lambda), \tag{2.12}$$

and therefore

$$\begin{aligned} f[m^t] - \alpha\|m^t\| &= \int_{[0,\infty)} \lambda\rho^{\alpha+t}(\lambda)dF(\lambda) - \int_{[0,\infty)} (\lambda-\alpha-t(\lambda))\rho^{\alpha+t}(\lambda)dF(\lambda) \\ &\quad - \int_{[0,\infty)} p(\alpha+t(\lambda)|\lambda)dF(\lambda) - \alpha\int_{[0,\infty)} \rho^{\alpha+t}(\lambda)dF(\lambda) \\ &= \langle m^t, t\rangle - p[\alpha+t]. \end{aligned}$$

Now let $\tilde{t}$ be arbitrary; we want to show that $I^\alpha(m,\tilde{t}) \leqq I^\alpha(m,t)$. To this end, we define $q_x(y,\lambda) = yp'(\alpha+x|\lambda) - p(\alpha+y|\lambda)$; the function $y \mapsto q_x(y,\lambda)$ is strictly concave and $y=x$ is a stationary point so that, for all y, $q_x(y,\lambda) \leqq q_x(x,\lambda)$. Putting $x=t(\lambda)$ and $y=\tilde{t}(\lambda)$ and integrating with respect to $dF(\lambda)$ we find

$$\langle \tilde{t}, m^t\rangle - p[\alpha+\tilde{t}] \leqq \langle t, m^t\rangle - p[\alpha+t]. \qquad \square$$

Next we approximate an arbitrary measure in E by a sequence of Bose–Einstein measures to prove that (2.11) holds for all m in E.

Theorem 1 (The Approximation Theorem). *Let m be an element of E such that $I^\alpha[m]$ is finite; then there exists a sequence $\{t_n : n=1,2,\ldots\}$ of elements of $\mathscr{C}^\alpha$ such* *that the corresponding sequence of Bose–Einstein measures $m_n = m^{t_n}$ satisfies:*

1. $$\lim_{n\to\infty} m_n = m \text{ in the narrow topology,}$$

2. $$\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m].$$

Proof. The proof proceeds by a sequence of reductions. First we show that it is sufficient to prove the theorem for measures with compact support. Given m in E, define m_n by $m_n(d\lambda) = 1_{[0,n]}(\lambda)m(d\lambda)$; then, for each t in $\mathscr{C}^b(\mathbb{R}_+)$, we have

$$\langle m_n, t\rangle = \langle m, 1_{[o,n]}t\rangle \to \langle m, t\rangle$$

by Lebesgue's dominated convergence theorem. Hence $m_n \to m$ in the narrow topology. Now I^α is lower semi-continuous (being the supremum of a family of continuous functions), so that $\liminf_{n\to\infty} I^\alpha[m_n] \geqq I^\alpha[m]$; putting $t_N = t \vee (-N)$, we have

$$I^\alpha(m_n, t) = I^\alpha(m_n, t_N) + \int_{\{\lambda: t(\lambda) < -N\}} (t(\lambda) + N)m_n(d\lambda) + p[\alpha + t_N] - p[\alpha + t]$$

$$\leqq I^\alpha(m_n, t_N) + p[\alpha + t_N] - p[\alpha + t].$$

Next we estimate the right-hand side of the inequality.

Lemma 2.4. *Given $\varepsilon > 0$, there exists $N_0 > 0$ such that, for all $N \geqq N_0$ and all t in $\mathscr{C}^\alpha$,*

$$p[\alpha + t_N] - p[\alpha + t] < \varepsilon,$$

where $t_N = t \vee (-N)$.

Proof. We have

$$P[\alpha + t_N] - p[\alpha + t] = \beta^{-1} \int_{\{\lambda: t(\lambda) < -N\}} \ln\left(\frac{1 - e^{\beta(\alpha + t(\lambda) - \lambda)}}{1 - e^{\beta(\alpha - N - \lambda)}}\right) dF(\lambda)$$

$$< -\beta^{-1} \int_{[0,\infty)} \ln(1 - e^{\beta(\alpha - N - \lambda)}) dF(\lambda) \to 0$$

as $N \to \infty$ by Lebesgue's monotone convergence theorem. □

Using Lemma 2.4, we have: given $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$I^\alpha(m_n, t) \leqq I^\alpha(m_n, t_{N_0}) + \varepsilon$$

$$= I^\alpha(m, t_{N_0}) - \int_{(n,\infty)} t_{N_0}(\lambda)m(d\lambda) + \varepsilon$$

$$\leqq I^\alpha(m, t_{N_0}) + N_0 m((n, \infty)) + \varepsilon.$$

Now choose n_0 such that, for $n > n_0$, we have $N_0 m((n, \infty)) < \varepsilon$. Thus

$$I^\alpha[m_n] = \sup_{t\in\mathscr{C}^\alpha} I^\alpha(m_n, t) \leqq \sup_{\{t\in\mathscr{C}^\alpha: t > -N_0\}} I^\alpha(m, t) + 2\varepsilon$$

$$\leqq \sup_{t\in\mathscr{C}^\alpha} I^\alpha(m, t) + 2\varepsilon = I^\alpha[m] + 2\varepsilon.$$

Hence

$$\limsup_{n\to\infty} I^\alpha[m_n] \leqq I^\alpha[m];$$

we conclude that

$$\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m].$$

Thus it is enough to prove the theorem for measures m with compact support, say $\operatorname{supp} m \subset [0, T]$.

Next, we decompose m into a singular part m_s and an absolutely continuous part m_a with respect to dF as in the statement of Lemma 2.2.

Let $\{m_s^{(n)}: n = 1, 2, \ldots\}$ be a sequence of absolutely continuous measures on $[0, T]$ converging to m_s, and put

$$m^{(n)} = m_s^{(n)} + m_a;$$

such a sequence exists by virtue of assumption (S3). Then

$$I^\alpha[m^{(n)}] = \sup_{t\in\mathscr{C}^\alpha} I^\alpha(m^{(n)}, t) = \sup_{t\in\mathscr{C}^\alpha}\{\langle m_s^{(n)}, t\rangle + I^\alpha(m_a, t)\}$$

$$\leqq \sup_{t\in\mathscr{C}^\alpha}\left\{\int_{[0,T]} (\lambda-\alpha) m_s^{(n)}(d\lambda) + I^\alpha(m_a, t)\right\} = \int_{[0,T]} (\lambda-\alpha) m_s^{(n)}(\mathrm{d}\lambda) + I^\alpha[m_a],$$

which tends to $I^\alpha[m]$ as n tends ∞ by Lemma 2.2. Hence

$$\limsup_{n\to\infty} I^\alpha[m^{(n)}] \leqq I^\alpha[m].$$

On the other hand, by the lower semi-continuity of I^α, we have

$$\liminf_{n\to\infty} I^\alpha[m^{(n)}] \geqq I^\alpha[m],$$

and we conclude that

$$\lim_{n\to\infty} I^\alpha[m^{(n)}] = I^\alpha[m].$$

It follows that it is enough to prove the theorem for absolutely continuous measures with compact support.

Assume, therefore, that $m(d\lambda) = \rho(\lambda) dF(\lambda)$ with ρ in $\mathscr{L}^1_+([0, T]; dF)$. Since $\mathscr{C}_c^+([0, T))$ is dense in $\mathscr{L}^1_+([0, T]; dF)$, there exists a sequence $\{\rho_n: n = 1, 2, \ldots\}$ of functions in $\mathscr{C}_c^+([0, T))$ such that

$$\int_0^T |\rho_n(\lambda) - \rho(\lambda)| dF(\lambda) < \frac{1}{n^2}.$$

Define the following subsets of $[0, \infty)$:

$$E_n^< = \{\lambda \in [0, T): \rho_n(\lambda) < \rho^{\alpha-n}(\lambda)\} \cup [T, \infty),$$
$$E_n^0 = \{\lambda \in [0, T): \rho^{\alpha-n}(\lambda) \leqq \rho_n(\lambda) \leqq n\},$$
$$E_n^> = \{\lambda \in [0, T): \rho_n(\lambda) > n\};$$

put

$$t_n(\lambda) = \begin{cases} -n, & \lambda \in E_n^<, \\ \lambda - \alpha - \beta^{-1} \ln\left(1 + \dfrac{1}{\rho_n(\lambda)}\right), & \lambda \in E_n^0, \\ \lambda - \alpha - \beta^{-1} \ln\left(1 + \dfrac{1}{n}\right), & \lambda \in E_n^>, \end{cases} \tag{2.13}$$

and

$$m_n(d\lambda) = \rho^{\alpha + t_n}(\lambda) dF(\lambda).$$

It follows that $\{t_n : n = 1, 2, \dots\}$ is a sequence of elements of $\mathscr{C}^\alpha$ and $\{m_n : n = 1, 2, \dots\}$ is a sequence of Bose–Einstein measures with

$$\rho^{\alpha + t_n}(\lambda) = \begin{cases} \rho^{\alpha - n}(\lambda), & \lambda \in E_n^<, \\ \rho_n(\lambda), & \lambda \in E_n^0, \\ n, & \lambda \in E_n^>. \end{cases}$$

Furthermore, for t in $\mathscr{C}^b(\mathbb{R}_+)$, we have

$$\begin{aligned} \langle t, m_n \rangle - \langle t, m \rangle = & \int_{E_n^<} t(\lambda)\{\rho^{\alpha - n}(\lambda) - \rho_n(\lambda)\} dF(\lambda) \\ & + \int_{E_n^< \cup E_n^0} t(\lambda)\{\rho_n(\lambda) - \rho(\lambda)\} dF(\lambda) \\ & + \int_{E_n^>} t(\lambda)\{n - \rho(\lambda)\} dF(\lambda). \end{aligned}$$

Thus

$$\begin{aligned} |\langle t, m_n \rangle - \langle t, m \rangle| \leqq & \|t\|_\infty \left\{ \int_{E_n^<} \rho^{\alpha - n}(\lambda) dF(\lambda) \right. \\ & \left. + \int_{[0,\infty)} |\rho_n(\lambda) - \rho(\lambda)| dF(\lambda) + \int_{\{\lambda : \rho(\lambda) > n\}} \rho(\lambda) dF(\lambda) \right\} \\ \leqq & \|t\|_\infty \left\{ \int_{[0,\infty)} \rho^{\alpha - n}(\lambda) dF(\lambda) + \frac{1}{n^2} + \int_{\{\lambda : \rho(\lambda) > n\}} \rho(\lambda) dF(\lambda) \right\} \end{aligned}$$

$\to 0$ as $n \to \infty$, so that $m_n \to m$ in the narrow topology. Again, we have

$$\begin{aligned} \langle t_n, m_n \rangle - \langle t_n, m \rangle = & \int_{E_n^<} t_n(\lambda)\{\rho^{\alpha - n}(\lambda) - \rho_n(\lambda)\} dF(\lambda) \\ & + \int_{E_n^< \cup E_n^0} t_n(\lambda)\{\rho_n(\lambda) - \rho(\lambda)\} dF(\lambda) + \int_{E_n^>} t_n(\lambda)\{n - \rho(\lambda)\} dF(\lambda). \end{aligned}$$

Thus

$$\begin{aligned} |\langle t_n, m_n \rangle - \langle t_n, m \rangle| \leqq & n \int_{[0,\infty)} \rho^{\alpha - n}(\lambda) dF(\lambda) \\ & + \{n \vee (T + |\alpha|)\} \int_{[0,\infty)} |\rho_n(\lambda) - \rho(\lambda)| dF(\lambda) \\ & + \int_{\{\lambda \in [0,T] : \rho(\lambda) > n\}} (\lambda - \alpha) \rho(\lambda) dF(\lambda) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

But m_n is a Bose–Einstein measure, so that, by Lemma 2.3,

$$I^\alpha[m_n] = \langle t_n, m_n\rangle - p[\alpha + t_n] + p(\alpha) = \langle t_n, m_n\rangle - \langle t_n, m\rangle + I^\alpha(m, t_n)$$
$$\leqq \langle t_n, m_n\rangle - \langle t_n, m\rangle + I^\alpha[m];$$

hence

$$\limsup_{n\to\infty} I^\alpha[m_n] \leqq I^\alpha[m].$$

But $m_n \to m$ in the narrow topology and $m \mapsto I^\alpha[m]$ is lower semi-continuous, so that $\liminf_{n\to\infty} I^\alpha[m_n] \geqq I^\alpha[m]$. We conclude that

$$\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m]. \quad \Box$$

We now have the corollary:

Theorem 2. *Let m be such that $I^\alpha[m]$ is finite; then*

$$I^\alpha[m] = p(\alpha) + f[m] - \alpha\|m\|, \tag{2.14}$$

where $f[m]$ is given by (2.9).

Proof. It follows from Lemma 2.2 that it is enough to prove the correctness of this expression for $I^\alpha[m]$ in the case where m is absolutely continuous. Moreover, we have seen in the proof of Theorem 1 that, for the truncated measures $m_n(d\lambda) = 1_{[0,n]}(\lambda)m(d\lambda)$ we have $\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m]$. On the other hand, it is evident that the sequence

$$\left\{\int_{[0,\infty)} (\lambda - \alpha)m_n(d\lambda) - \beta^{-1}\int_{[0,\infty)} s(\rho_n(\lambda))dF(\lambda) : n = 1, 2, \ldots\right\} \tag{2.15}$$

also converges; we conclude that it is sufficient to consider absolutely continuous measures $m(d\lambda) = \rho(\lambda)dF(\lambda)$ with ρ in $\mathscr{L}^1_+([0, T]; dF)$ for some $T < \infty$. We choose ρ_n as in the proof of Theorem 1 and define t_n by (2.13). We have seen that, with $m_n(d\lambda) = \rho^{\alpha + t_n}(\lambda)dF(\lambda)$, $\lim_{n\to\infty} I^\alpha[m_n] = I^\alpha[m]$; it remains to show that, with this choice of m_n, the sequence, (2.15) converges to the expression on the right-hand side of (2.14). Now

$$\int_{[0,\infty)} (\lambda - \alpha)m_n(d\lambda) = \int_{E_n^<} (\lambda - \alpha)\rho^{\alpha - n}(\lambda)dF(\lambda)$$
$$+ \int_{E_n^0} (\lambda - \alpha)\rho_n(\lambda)dF(\lambda) + n\int_{E_n^>} (\lambda - \alpha)dF(\lambda).$$

The first term on the right-hand side is bounded by $\int_{[0,\infty)} (\lambda - \alpha)\rho^{\alpha - n}(\lambda)dF(\lambda)$, which converges to zero. We split the second term into

$$\int_{E_n^0} (\lambda - \alpha)\rho(\lambda)dF(\lambda) + \int_{E_n^0} (\lambda - \alpha)\{\rho_n(\lambda) - \rho(\lambda)\}dF(\lambda).$$

Here

$$\left| \int_{E_n^0} (\lambda-\alpha)\{\rho_n(\lambda)-\rho(\lambda)\}dF(\lambda) \right| \leqq (T+|\alpha|) \int_{[0,\infty)} |\rho_n(\lambda)-\rho(\lambda)|dF(\lambda) \to 0;$$

as to $\int_{E_n^0} (\lambda-\alpha)\rho(\lambda)dF(\lambda)$, we remark that we may assume that $\rho_n(\lambda)\to\rho(\lambda)$ for almost all λ with respect to dF which implies that $1_{E_n^0}\to 1_{\mathbb{R}_+}$ a.e. and hence, using Lebesgue's dominated convergence theorem, we have

$$\int_{E_n^0} (\lambda-\alpha)\rho(\lambda)dF(\lambda) \to \int_{[0,\infty)} (\lambda-\alpha)m(d\lambda).$$

Finally, we have

$$n \int_{E_n^>} (\lambda-\alpha)dF(\lambda) \leqq \int_{\{\lambda:\rho_n(\lambda)>n\}} (\lambda-\alpha)\rho_n(\lambda)dF(\lambda)\to 0$$

since $\int_{[0,\infty)} (\lambda-\alpha)\rho_n(\lambda)dF(\lambda)$ is uniformly bounded. We conclude that

$$\lim_{n\to\infty} \int_{[0,\infty)} (\lambda-\alpha)m_n(d\lambda) = \int_{[0,\infty)} (\lambda-\alpha)m(d\lambda).$$

For the convergence of the entropy term, we use the following bound on $s(x)$:

Lemma 2.5. *For all x_1,x_2 in $[0,\infty)$, $|s(x_2)-s(x_1)|\leqq 4|x_2-x_1|^{1/2}$.*

Proof. We have

$$\begin{aligned}|s(x_2)-s(x_1)| &= \left|\int_{x_1}^{x_2} \ln\left(1+\frac{1}{x}\right)dx\right| = 2\left|\int_{x_1}^{x_2} \ln\left(1+\frac{1}{x}\right)^{1/2}dx\right| \\ &\leqq 2\left|\int_{x_1}^{x_2} \ln\left(1+\frac{1}{x^{1/2}}\right)dx\right| \leqq 2\left|\int_{x_1}^{x_2} \frac{1}{x^{1/2}}dx\right| = 4|x_2^{1/2}-x_1^{1/2}| \\ &\leqq 4|x_2-x_1|^{1/2}. \end{aligned}$$ □

Now put $\rho^{(n)}(\lambda)=\rho^{\alpha+t_n}(\lambda)$; then

$$\begin{aligned}\int_{[0,\infty)} |s(\rho^{(n)}(\lambda))-s(\rho(\lambda))|dF(\lambda) &\leqq 4\int_{[0,T]} |\rho^{(n)}(\lambda)-\rho(\lambda)|^{1/2}dF(\lambda) \\ &\leqq 4F(T)^{1/2}\|\rho^{(n)}-\rho\|_1^{1/2}\to 0.\end{aligned}$$

This concludes the proof of Theorem 2. □

3. The Large Deviation Theorem

Before proving that the sequence of Kac measures introduced in Sect. 1 satisfies the large deviation principle with the rate-function studied in Sect. 2, we need some results about certain subsets of E. For each positive integer N, define the function t_N in $\mathscr{C}^\alpha$ by

$$t_N(\lambda)=\begin{cases}0, & \lambda\leqq N,\\ N(\lambda-N)/2, & N\leqq\lambda\leqq N+1,\\ N/2, & \lambda\geqq N+1.\end{cases} \tag{3.1}$$

We note that

$$(\sup t_{N^1})(\lambda) = 0, \qquad \lambda \leqq N,$$
$$\Big(\sup_{N^1 \geqq N} t_{N^1}\Big)(\lambda) < \lambda/2, \quad \lambda > N. \tag{3.2}$$

Lemma 3.1. *For all $\alpha < 0$, we have*

(1) *for all N,*

$$C^\alpha[t_N] \leqq C^\alpha[\sup_{N^1 \geqq N} t_{N^1}]; \tag{3.3}$$

(2) *there exists $A > 0$ such that, for all N,*

$$C^\alpha[\sup_{N^1 \geqq N} t_{N^1}] \leqq A e^{-\beta N/4}. \tag{3.4}$$

Proof. (1) follows from the monotonicity of $t \mapsto C^\alpha[t]$ which is evident from the definition (1.27). To prove (2) we use (1.28):

$$C^\alpha[t] = p[\alpha + t] - p(\alpha) = \int_{[0,\infty)} \{p(\alpha + t(\lambda)|\lambda) - p(\alpha|\lambda)\} dF(\lambda). \tag{3.5}$$

Putting $t = \sup_{N^1 \geqq N} t_{N^1}$ and using (3.2), we have

$$C^\alpha[\sup_{N^1 \geqq N} t_{N^1}] \leqq \int_{[N,\infty)} \left\{p\left(\alpha + \frac{\lambda}{2}\middle|\lambda\right) - p(\alpha|\lambda)\right\} dF(\lambda);$$

using the convexity of $\alpha \mapsto p(\alpha|\lambda)$, we have

$$p\left(\alpha + \frac{\lambda}{2}\middle|\lambda\right) - p(\alpha|\lambda) \leqq \frac{\lambda}{2} p'\left(\alpha + \frac{\lambda}{2}\middle|\lambda\right),$$

so that

$$\begin{aligned} C^\alpha\left[\sup_{N^1 \geqq N} t_{N^1}\right] &\leqq \int_{[N,\infty)} \frac{\lambda/2}{e^{\beta(\alpha + \lambda/2)} - 1} dF(\lambda) \\ &\leqq e^{-\beta N/4} \int_{[N,\infty)} \frac{(\lambda/2)e^{\beta\lambda/4}}{e^{\beta\lambda/2} - 1} dF(\lambda) \\ &\leqq e^{-\beta N/4} \int_{[0,\infty)} \frac{(\lambda/4)}{\sinh \beta\lambda/4} dF(\lambda); \end{aligned}$$

it follows that (3.4) holds with

$$A = \int_{[0,\infty)} \frac{\lambda/4}{\sinh \beta\lambda/4} dF(\lambda). \qquad \square$$

For M a positive integer, define the set K_M by

$$K_M = \bigcap_{N \geqq 1} \{m \in E : \langle m, t_N \rangle \leqq M\};$$

for L a positive integer, define B_L by

$$B_L = \{m \in E: \|m\| \leqq L\},$$

and put $K_M^L = K_M \cap B_L$. We are going to prove that K_M^L is compact, using

Prokhorov's Criterion. *Let H be a subset of E; if H is uniformly bounded and uniformly tight then the closure of H is compact in the narrow topology.* (This is Theorem 1 of No. 5.5 of [10].)

Lemma 3.2. *The set K_M is uniformly tight.*

Proof. Let m be an element of K_M; then $\langle m, t_N \rangle \leqq M$ for all $N \geqq 1$, so that

$$\frac{N}{2} \int_{N+1}^{\infty} m(d\lambda) \leqq \langle m, t_N \rangle \leqq M \quad \text{for all} \quad N \geqq 1.$$

Hence, given $\varepsilon > 0$, there exists N_ε such that

$$\int_{N_\varepsilon}^{\infty} m(d\lambda) < \varepsilon \quad \text{for all} \quad m \text{ in } K_M. \quad \square$$

Lemma 3.3. *The set K_M^L is compact in the narrow topology.*

Proof. Since $K_M^L \subset B_L$, the set K_M^L is uniformly bounded; since $K_M^L \subset K_M$, the set K_M^L is uniformly tight. It follows from Prokhorov's Criterion that the closure of K_M^L is compact; but $m \mapsto \langle m, t \rangle$ is continuous so that K_M^L is closed. $\square$

Lemma 3.4.

$$\lim_{L \to \infty} \limsup_{l \to \infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha [B_L^c] = -\infty.$$

Proof. Notice that $L_l[\cdot; [0, \infty)]$ is the particle number density, denoted in [8] by X_l. It follows that

$$\mathbb{K}_l^\alpha [B_L^c] = \mathbb{P}_l^\alpha [X_l \in (L, \infty)]$$

which, by Theorem A1 of [8], satisfies the large deviation principle with rate-function

$$I_0^\alpha(\rho) = p(\alpha) + f(\rho) - \alpha\rho,$$

where

$$f(\rho) = \sup_{\alpha < 0} \{\alpha\rho - p(\alpha)\} \geqq -p(0). \tag{3.6}$$

It follows that, for $L > p'(\alpha)$, we have

$$\limsup_{l \to \infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha [B_L^c] \leqq p(0) - p(\alpha) + \alpha L,$$

and the right-hand side decreases to $-\infty$ as $L \to \infty$ since $\alpha < 0$. $\square$

Lemma 3.5.

$$\lim_{M \to \infty} \limsup_{l \to \infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha [K_M^c] = -\infty.$$

Proof. We have

$$\mathbb{E}_l^\alpha\left[\exp\left\{\beta V_l \sup_N \langle t_N, \cdot\rangle\right\}\right] \leqq \mathbb{E}_l^\alpha\left[\exp\left\{\beta V_l\left\langle \sup_N t_N, \cdot\right\rangle\right\}\right]$$
$$= \exp\left\{\beta V_l C_l^\alpha[\sup_N t_N]\right\}.$$

Hence

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{E}_l^\alpha\left[\exp\left\{\beta V_l \sup_N \langle t_N, \cdot\rangle\right\}\right] \leqq C^\alpha[\sup_N t_N] \leqq A$$

by Lemma 3.1.
But, by Markov's inequality,

$$\mathbb{K}_l^\alpha[K_M^c] \leqq e^{\beta V_l M}\mathbb{E}_l^\alpha\left[\exp\left\{\beta V_l \sup_N \langle t_N, \cdot\rangle\right\}\right]$$

so that

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K_M^c] \leqq -M + A$$

and

$$\lim_{M\to\infty} \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K_M^c] = -\infty. \quad \square$$

We are now ready to prove the following result:

Theorem 3. *Suppose that* (S1), (S2) *and* (S3) *hold; then, for each* $\alpha < 0$, *the sequence* $\{\mathbb{K}_l^\alpha = \mathbb{P}_l^\alpha \circ L_l^{-1} : l = 1, 2, \ldots\}$ *of Kac measures satisfies the large deviation principle with constants* $\{V_l\}$ *and rate-function*

$$I^\alpha[m] = \sup_{t\in\mathscr{C}^\alpha}\{\langle m, t\rangle - C^\alpha[t]\}.$$

Proof. (LD1) holds because the supremum of a family of continuous functions is lower semicontinuous. To prove that (LD2) holds, we note first that it follows from (LD1) that the level set $K_b = \{m \in E : I^\alpha[m] \leqq b\}$ is closed. Next we prove that there exists L such that $K_b \subset B_L$: for t in $\mathscr{C}^\alpha$ and m in K_b, we have

$$b \geqq I^\alpha[m] \geqq \langle m, t\rangle - C^\alpha[t]; \tag{3.7}$$

putting $t(\lambda) = -\alpha/2$, we have

$$b \geqq -\frac{\alpha}{2}\|m\| - p\left(\frac{\alpha}{2}\right) + p(\alpha).$$

It follows that m lies in B_L provided $L > \{b + p(\alpha/2) - p(\alpha)\}/(-\alpha/2) > 0$. Finally, putting $t = t_N$ in (3.7), we have

$$b \geqq \langle m, t_N\rangle - C^\alpha[t_N]$$

so that, using Lemma 3.1, we have

$$\langle m, t_N\rangle \leqq b + A \quad \text{for all} \quad N \geqq 1. \tag{3.8}$$

It follows that $K_b \subset K_M$ for all $M \geqq A+b$. We have proved that, for suitable choices of L and M, the level set K_b is contained in $K_M^L = K_M \cap B_L$ and K_b is closed. It follows from Lemma 3.3 that K_b is compact in E in the narrow topology so that (LD2) holds.

To prove that (LD3) holds, we first establish the upper bound in the case of compact sets.

Following Ellis [11], we make use of Gärtner's Lemma:

Gärtner's Lemma. *Given t in $\mathscr{C}^\alpha$ and γ in $\mathbb{R}$, define $H_+(t,\gamma)$ by*

$$H_+(t,\gamma) = \{m \in E : \langle m,t\rangle - C^\alpha[t] \geqq \gamma\}.$$

Let K be a compact subset of E; if $\gamma < I^\alpha[K]$ then there exists a finite set $t_1,\ldots,t_r$ of non-zero elements of $\mathscr{C}^\alpha$ such that

$$K \subset \bigcup_{j=1}^{r} H_+(t_j,\gamma).$$

Let K be a compact subset of E; for each $\gamma < I^\alpha[K]$, by Gärtner's Lemma and Markov's Inequality, we have

$$\begin{aligned}
\mathbb{K}_l^\alpha[K] &\leqq \sum_{j=1}^{r} \mathbb{K}_l^\alpha[H_+(t_j;\gamma)] \\
&\leqq \sum_{j=1}^{r} e^{\beta V_l\{C^\alpha[t_j]+\gamma\}} \int_E e^{\beta V_l\langle m,t_j\rangle} \mathbb{K}_l^\alpha[dm] \\
&= e^{-\beta V_l \gamma} \sum_{j=1}^{r} e^{\beta V_l\{C_l^\alpha[t_j]-C^\alpha[t_j]\}},
\end{aligned}$$

so that

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K] \leqq -\gamma;$$

since this holds for all γ less than $I^\alpha[K]$, we conclude that

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K] \leqq -I^\alpha[K].$$

To establish the upper bound for an arbitrary closed set C, we note that $C \subset C_M^L \cup K_M^c \cup B_L^c$, where $C_M^L = C \cap K_M^L$.
It follows that

$$\mathbb{K}_l^\alpha[C] \leqq \mathbb{K}_l^\alpha[C_M^L] + \mathbb{K}_l^\alpha[K_M^c] + \mathbb{K}_l^\alpha[B_L^c]$$

from which we deduce that

$$\begin{aligned}
\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[C] \leqq \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[C_M^L] \vee \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K_M^c] \\
\vee \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[B_L^c]
\end{aligned}$$

$$\leqq (-I^\alpha[C_M^L]) \vee \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[K_M^c]$$

$$\vee \limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[B_L^c],$$

since C_M^L is compact (by Lemma 3.3); using Lemma 3.4 and Lemma 3.5, we have

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \mathbb{K}_l^\alpha[C] \leqq -I^\alpha[C_M^L] \leqq -I^\alpha[C],$$

since $C_M^L \subset C$. Thus (LD3) holds.

To prove (LD4), let G be an open subset of E; for each $\varepsilon > 0$ there exists m in G such that $I^\alpha[m] < I^\alpha[G] + \varepsilon$. By Theorem 1, there exists t in $\mathscr{C}^\alpha$ such that the corresponding Bose-Einstein measure m_t, defined by $m_t(d\lambda) = \rho^{\alpha+t}(\lambda) dF(\lambda)$, is in G and $I^\alpha[m_t] < I^\alpha[m] + \varepsilon$ so that

$$I^\alpha[m_t] < I^\alpha[G] + 2\varepsilon. \tag{3.9}$$

Now let $G_\varepsilon = G \cap \{m \in E : |\langle m, t\rangle - \langle m_t, t\rangle| < \varepsilon\}$ and define the shifted Kac distribution $\tilde{\mathbb{K}}_l^\alpha$ by $\tilde{\mathbb{K}}_l^\alpha[dm] = e^{\beta V_l\{\langle t, m\rangle - C_l^\alpha[t]\}} \mathbb{K}_l^\alpha[dm]$.

Lemma 3.6. *For l sufficiently large,*

$$\tilde{\mathbb{K}}_l^\alpha[G_\varepsilon] > \tfrac{1}{2}.$$

Proof. Since G_ε is open and m_t is in G_ε, there exist $t_1, \ldots, t_r$ in $\mathscr{C}^b(\mathbb{R}_+)$ and $\delta > 0$ such that G_ε contains the neighbourhood N_δ of m_t defined by

$$N_\delta = \bigcap_{j=1}^{r} \{m \in E : |\langle t_j, m - m_t\rangle| < \delta\}.$$

Define the sequence $\{\mathbb{Q}_l^\alpha : l = 1, 2, \ldots\}$ of probability measures on $\mathbb{R}^r$ by

$$\int_{\mathbb{R}^r} e^{-(s_1 x_1 + \cdots + s_r x_r)} \mathbb{Q}_l^\alpha(dx_1, \ldots, dx_r)$$

$$= \int_E e^{-\langle s_1 t_1 + \cdots + s_r t_r, m - m_t\rangle} \tilde{\mathbb{K}}_l^\alpha[dm]$$

$$= e^{\langle s_1 t_1 + \cdots + s_r t_r, m_t\rangle} \cdot \exp\left[\beta V_l\left\{C_l^\alpha\left[t - \frac{s_1 t_1 + \cdots + s_r t_r}{\beta V_l}\right] - C_l^\alpha[t]\right\}\right];$$

the righthand side converges to 1 as $l \to \infty$ so that the sequence $\{\mathbb{Q}_l : l = 1, 2, \ldots\}$ converges to the Dirac measure δ_0. It follows that $\tilde{\mathbb{K}}_l^\alpha[G_\varepsilon] > \frac{1}{2}$ for l sufficiently large since $\tilde{\mathbb{K}}_l^\alpha[G_\varepsilon] \geqq \tilde{\mathbb{K}}_l^\alpha[N_\delta] = \mathbb{Q}_l^\alpha[[-\delta, \delta]^r] \to 1$ as $l \to \infty$. □

Returning to the proof of the theorem, we have

$$\mathbb{K}_l^\alpha[G] \geqq \mathbb{K}_l^\alpha[G_\varepsilon] = e^{\beta V_l C_l^\alpha[t]} \int_{G_\varepsilon} e^{-\beta V_l \langle m, t\rangle} \tilde{\mathbb{K}}_l^\alpha[dm]$$

$$\geqq e^{\beta V_l\{C_l^\alpha[t] - \langle m_t, t\rangle - \varepsilon\}} \tilde{\mathbb{K}}_l^\alpha[G_\varepsilon].$$

It follows that

$$\liminf_{l\to\infty}\frac{1}{\beta V_l}\ln\tilde{\mathbb{K}}_l^\alpha[G]\geqq C^\alpha[t]-\langle m_t,t\rangle-\varepsilon$$
$$=-I^\alpha[m_t]-\varepsilon$$
$$\geqq -I^\alpha[G]-3\varepsilon,$$

by (3.9); but ε was an arbitrary positive number so that (LD4) holds. □

4. The Existence of the Pressure

We saw in Sect. 1 that $\tilde{p}_l(\mu)$, the grand canonical pressure in volume V_l, can be expressed in terms of an integral with respect to the Kac measure:

$$\tilde{p}_l(\mu)=p_l(\alpha)+\frac{1}{\beta V_l}\ln\int_E e^{\beta V_l G^{\mu-\alpha}[m]}\mathbb{K}_l^\alpha[dm], \tag{4.1}$$

where $p_l(\alpha)$ is the free-gas pressure and

$$G^\mu[m]=\mu\|m\|-\tfrac{1}{2}\{a\|m\|^2+\langle m,Vm\rangle\}$$

with $a>0$ and $(Vm)(\lambda)=\int v(\lambda,\lambda')m(d\lambda')$. It is assumed that $v(\cdot,\cdot):\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is bounded, continuous and positive-definite. Having proved in Sect. 3 that the Kac measures satisfy the large deviation principle, the next step is to use Varadhan's Theorem to prove the existence of the limit $\tilde{p}(\mu)=\lim_{l\to\infty}\tilde{p}_l(\mu)$ and to derive a variational principle for $\tilde{p}(\mu)$. To carry this through, it remains to check that the functional G^μ satisfies the hypotheses of Varadhan's Theorem. Since $v(\cdot,\cdot)$ is positive-definite, $\langle m,Vm\rangle$ is non-negative; it follows that $G^\mu[\cdot]$ is bounded above for all real values of μ since a is strictly positive. Moreover, $m\mapsto\|m\|$ is continuous in the narrow topology so that $m\mapsto\mu\|m\|-\frac{1}{2}a\|m\|^2$ is continuous; the continuity of $m\mapsto\langle m,Vm\rangle$ is a little more troublesome.

Lemma 4.1. *The functional $m\mapsto\langle m,Vm\rangle$ is continuous in the narrow topology on E.*

Proof. Since $\mathbb{R}_+$ is a Polish space, the space $E=\mathcal{M}_+^b(\mathbb{R}_+)$ is a Polish space in the narrow topology (Proposition 10 of No. 5.4 of [10]) so in investigating continuity it is enough to consider sequences. Let m be an element of E and let $\{m_n:n=1,2,\ldots\}$ be a sequence converging to m in E; for each $s\geqq 0$,

$$\int_{[0,\infty)}e^{-s\lambda}m_n(d\lambda)\to\int_{[0,\infty)}e^{-s\lambda}m(d\lambda).$$

It follows that, for $s_1\geqq 0$, $s_2\geqq 0$, we have

$$\int_{[0,\infty)\times[0,\infty)}e^{-(s_1\lambda_1+s_2\lambda_2)}(m_n\otimes m_n)(d\lambda_1,d\lambda_2)$$
$$=\int_{[0,\infty)}e^{-s_1\lambda_1}m_n(d\lambda_1)\int_{[0,\infty)}e^{-s_2\lambda_2}m_n(d\lambda_2)$$
$$\to\int_{[0,\infty)}e^{-s_1\lambda_1}m(d\lambda_1)\int_{[0,\infty)}e^{-s_2\lambda_2}m(d\lambda_2)$$
$$=\int_{[0,\infty)\times[0,\infty)}e^{-(s_1\lambda_1+s_2\lambda_2)}(m\otimes m)(d\lambda_1,d\lambda_2). \tag{4.2}$$

Let $X_0 = \{\lambda_1, \lambda_2 \mapsto e^{-(s_1\lambda_1 + s_2\lambda_2)}: s_1 > 0, s_2 > 0\}$; then X_0 is the set consisting of those real characters on $\mathbb{R}_+^2$ which tend to zero at infinity. Moreover, X_0 is full and the constant function $\lambda_1, \lambda_2 \mapsto 1$ is in the closure of X_0 in the topology of compact convergence; since $s_1, s_2 \mapsto \int_{[0,\infty)} e^{-(s_1\lambda_1 + s_2\lambda_2)}(m \otimes m)(d\lambda_1, d\lambda_2)$ is continuous at $(0,0)$, it follows by Theorem 3 of no. 5.7 of [10] that the sequence $\{m_n \otimes m_n : n = 1, 2, \ldots\}$ converges to $m \otimes m$ in the narrow topology on $\mathcal{M}_+^b(\mathbb{R}_+^2)$. But $\lambda_1, \lambda_2 \mapsto v(\lambda_1, \lambda_2)$ is a bounded continuous function on $\mathbb{R}_+^2$, so that by (4.2)

$$\langle m_n, Vm_n \rangle = \langle m_n \otimes m_n, v \rangle \to \langle m \otimes m, v \rangle = \langle m, Vm \rangle$$

as $n \to \infty$, establishing the continuity of $m \mapsto \langle m, Vm \rangle$ at m. □

We are now ready to prove the main theorem of this paper.

Theorem 4. *Suppose that* (S1), (S2) *and* (S3) *hold; suppose also that the potential* $v(\cdot,\cdot)$ *is a bounded continuous positive-definite function on* $\mathbb{R}_+^2$; *then the pressure* $\tilde{p}(\mu) = \lim_{l\to\infty} \tilde{p}_l(\mu)$ *exists for the perturbed mean-field model determined by* $v(\cdot,\cdot)$ *and is given by*

$$\tilde{p}(\mu) = -\inf_E \mathscr{E}^\mu[m],$$

where

$$\mathscr{E}^\mu[m] = f[m] + \tfrac{1}{2}\{a\|m\|^2 + \langle m, Vm \rangle\} - \mu\|m\|,$$

and $f[m]$ *is the free energy functional for free bosons given by* (2.9).

Proof. Since $m \mapsto G^\mu[m]$ is continuous and bounded above under the hypothesis on $v(\cdot,\cdot)$ and since, by Theorem 3, $\{\mathbb{K}_l^\alpha : l = 1, 2, \ldots\}$ satisfies the large-deviation principle with rate-function

$$I^\alpha[m] = p(\alpha) + \sup_{\mathscr{C}^\alpha}\{\langle m, t \rangle - p[\alpha + t]\},$$

it follows by Varadhan's theorem applied to (4.1) that $\tilde{p}(\mu) = \lim_{l\to\infty} \tilde{p}_l(\mu)$ exists and

$$\begin{aligned}\tilde{p}(\mu) &= p(\alpha) + \sup_E\{G^{\mu-\alpha}[m] - I^\alpha[m]\}\\ &= \sup_E\Big\{(\mu - \alpha)\|m\| - \tfrac{1}{2}(a\|m\|^2 + \langle m, Vm \rangle)\\ &\quad - \int_{[0,\infty)} (\lambda - \alpha)m(d\lambda) + \beta^{-1}\int_{[0,\infty)} s(\rho(\lambda))dF(\lambda)\Big\}\\ &= -\inf_E \mathscr{E}^\mu[m],\end{aligned}$$

where we have used the expression for $I^\alpha[m]$ given by Theorem 2. □

Remark. Using the Euler–Lagrange equations derived in Sect. 5, we may obtain the following alternative expression for $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \tfrac{1}{2}a\|m\|^2 + \tfrac{1}{2}\langle m, Vm \rangle + p[\mu - t^m], \tag{4.3}$$

where m satisfies the Euler–Lagrange equations (5.3) and

$$t^m(\lambda) = a\|m\| + (Vm)(\lambda). \tag{4.4}$$

5. The Minimization Problem

Recall that the functional $\mathscr{E}^\mu[\cdot]$ may be written as

$$\begin{aligned}\mathscr{E}^\mu[m] &= I^\alpha[m] - G^{\mu-\alpha}[m] - p(\alpha)\\ &= \tfrac{1}{2}\{a\|m\|^2 + \langle m, Vm\rangle\} + I^\alpha[m] + (\alpha-\mu)\|m\| - p(\alpha),\end{aligned}$$

where $(Vm)(\lambda) = \int_{[0,\infty)} v(\lambda,\lambda')m(d\lambda')$ and $v(\cdot,\cdot)$ is a positive-definite bounded continuous function on $\mathbb{R}_+ \times \mathbb{R}_+$, and a is strictly positive.

Lemma 5.1. *Let $e = \inf_{m\in E} \mathscr{E}^\mu[m]$; then there exists m^* in E such that $\mathscr{E}^\mu[m^*] = e$.*

Proof. Since $\mathscr{E}^\mu[0] = 0$, we have $e \leqq 0$. It follows that there exists a sequence $\{m_n\}$ in E such that $e \leqq \mathscr{E}^\mu[m_n] \leqq 0$ and $\lim_{n\to\infty} \mathscr{E}^\mu[m_n] = e$. Since $\langle m, Vm\rangle$ is non-negative, we have,

$$\begin{aligned}I^\alpha[m_n] &\leqq \mathscr{E}^\mu[m_n] - \frac{1}{2a}\{a\|m_n\| + (\alpha-\mu)\}^2 + \frac{1}{2a}(\alpha-\mu)^2 + p(\alpha)\\ &\leqq \frac{1}{2a}(\alpha-\mu)^2 + p(\alpha)\end{aligned} \tag{5.1}$$

which implies that the sequence $\{m_n\}$ lies inside a level set of I^α which, by (LD2), is compact. Hence $\{m_n : n = 1,2,\ldots\}$ contains a convergent subsequence $\{m_{n_k} : k = 1,2,\ldots\}$. Let $m^* = \lim_{k\to\infty} m_{n_k}$; since, by (LD1), the function $m \mapsto I^\alpha[m]$ is lower semi-continuous, we have

$$\liminf_{k\to\infty} I^\alpha[m_{n_k}] \geqq I^\alpha[m^*]. \tag{5.2}$$

Furthermore $m \mapsto G^\mu[m]$ is continuous, so that by (5.2) and the definition of e we have

$$\mathscr{E}^\mu[m^*] \geqq e = \lim_{k\to\infty} \mathscr{E}^\mu[m_{n_k}] = \liminf_{k\to\infty} \{I^\alpha[m_{n_k}] - G^{\mu-\alpha}[m_{n_k}]\} - p(\alpha) \geqq \mathscr{E}^\mu[m^*]. \quad \square$$

Recall that

$$s(x) = \begin{cases}(1+x)\ln(1+x) - x\ln x, & x > 0,\\ 0, & x = 0,\end{cases}$$

so that $s'(x) = \ln(1 + 1/x)$ for $x > 0$; we define $s'(x) = \infty$ for $x = 0$.

Lemma 5.2. *Let m be a minimizer of $\mathscr{E}^\mu$ and let $m(d\lambda) = m_s(d\lambda) + \rho(\lambda)dF(\lambda)$ be the Lebesgue decomposition of m with respect to dF; then $\rho(\lambda) > 0$ a.e. with respect to dF.*

Proof. We write the functional $\mathscr{E}^\mu[m]$ as

$$\mathscr{E}^\mu[m] = \int_{[0,\infty)} (\lambda - \mu)m(d\lambda) + \tfrac{1}{2}\{a\|m\|^2 + \langle m, Vm\rangle\} - \beta^{-1} \int_{[0,\infty)} s(\rho(\lambda))dF(\lambda).$$

Let $m(t; d\lambda) = t\tilde{m}(d\lambda) + (1-t)m(d\lambda)$, $0 \leqq t \leqq 1$, with $\tilde{m}(d\lambda) = m_s(d\lambda) + 1_{A^c}(\lambda)\rho(\lambda)dF(\lambda) + 1_A(\lambda)e^{-\lambda}dF(\lambda)$, where $A = \{\lambda : \rho(\lambda) = 0\}$, and let

$$m(t; d\lambda) = m_s(t; d\lambda) + \rho(t; \lambda)dF(\lambda)$$

be its Lebesgue decomposition. Introducing $f(t) = \mathscr{E}^\mu[m(t)]$, $0 \leqq t \leqq 1$, we have, for $0 < t \leqq 1$,

$$\begin{aligned} f'(t) = & \int_{[0,\infty)} (\lambda - \mu)\{\tilde{m}(d\lambda) - m(d\lambda)\} + a\|m(t)\|\{\|\tilde{m}\| - \|m\|\} \\ & + \langle \tilde{m} - m, Vm(t)\rangle - \beta^{-1} \int_{[0,\infty)} s'(\rho(t))\{\tilde{\rho}(\lambda) - \rho(\lambda)\}dF(\lambda) \\ = & \int_A (\lambda - \mu)e^{-\lambda}dF(\lambda) + a\|m(t)\| \int_A e^{-\lambda}dF(\lambda) \\ & + \int_A e^{-\lambda}(Vm)(t,\lambda)dF(\lambda) - \beta^{-1}\int_A s'(te^{-\lambda})e^{-\lambda}dF(\lambda). \end{aligned}$$

Now suppose that $\int_A dF(\lambda) > 0$; then $\lim_{t\downarrow 0} f'(t) = -\infty$ since $\lim_{x\downarrow 0} s'(x) = +\infty$. Since $t \mapsto f(t)$ is convex, $f'_+(0) = \lim_{t\downarrow 0} f'(t)$; hence $\lim_{t\downarrow 0} \frac{1}{t}\{\mathscr{E}^\mu[t\tilde{m} + (1-t)m] - \mathscr{E}^\mu[m]\} = -\infty$, so that, for t sufficiently small,

$$\mathscr{E}^\mu[t\tilde{m} + (1-t)m] < \mathscr{E}^\mu[m],$$

contradicting the assumption that m is a minimizer and proving that $\int_A dF(\lambda) = 0$. □

Our next task is to introduce the Euler–Lagrange equations of the variational problem. We define the function $L^\mu(m; \cdot)$ by

$$L^\mu(m; \lambda) = \lambda + a\|m\| + (Vm)(\lambda) - \mu.$$

By the assumptions on $v(\cdot,\cdot)$, the function $\lambda \mapsto L^\mu(m; \lambda)$ is continuous for each m in E. We define the Euler–Lagrange equations as follows:

$$L^\mu(m; \lambda) = 0, \quad m_s - \text{a.e.}, \tag{5.3a}$$

$$L^\mu(m; \lambda) = \beta^{-1}s'(\rho(\lambda)), \quad dF - \text{a.e.}, \tag{5.3b}$$

where m_s is the singular part of m and ρ is the density of its absolutely continuous part in the Lebesgue decomposition of m with respect to dF. We remark that it follows from (5.3b) that

$$L^\mu(m; \lambda) \geqq 0, \quad dF - \text{a.e.},$$

so that the set $\mathscr{N}$ on which

$$L^\mu(m; \lambda) < 0 \tag{5.4}$$

has dF-measure zero. But it follows from (S3) that the Lebesgue measure of $\mathscr{N}$ is also zero which implies that $\mathscr{N}^c$ is dense in $[0, \infty)$. Since $\lambda \mapsto L^\mu(m; \lambda)$ is continuous, we conclude that

$$L^\mu(m; \lambda) \geqq 0, \quad \lambda \in [0, \infty). \tag{5.5}$$

Theorem 5. *A measure m in E is a minimizer of $\mathscr{E}^\mu$ if and only if m satisfies the Euler-Lagrange equations (5.3).*

Proof. Suppose first that m satisfies the Euler–Lagrange equations (5.3); as in the proof of Lemma 5.2, let $f:[0,1]\to\mathbb{R}$ be the convex function defined by

$$f(t)=\mathscr{E}^\mu[t\tilde{m}+(1-t)m]$$

for some $\tilde{m}$ in E; then

$$f'_+(0)=(\mathscr{E}^\mu)'(m,\tilde{m}-m), \tag{5.6}$$

where

$$(\mathscr{E}^\mu)'(m,\hat{m})=\int_{[0,\infty)}\{L^\mu(m;\lambda)-\beta^{-1}s'(\rho(\lambda))\}\hat{\rho}(\lambda)dF(\lambda)+\int_{[0,\infty)}L^\mu(m;\lambda)\hat{m}_s(d\lambda). \tag{5.7}$$

Using (5.3), we have

$$f'_+(0)=\int_{[0,\infty)}L^\mu(m;\lambda)\tilde{m}_s(d\lambda)$$

so that, by (5.5), $f'_+(0)$ is non-negative; but $t\mapsto f(t)$ is convex which implies that

$$f(1)-f(0)\geqq f'_+(0)\geqq 0.$$

We have proved that

$$\mathscr{E}^\mu[\tilde{m}]-\mathscr{E}^\mu[m]\geqq 0$$

for an arbitrary element $\tilde{m}$ of E so that m is a minimizer of $\mathscr{E}^\mu$.

To prove the converse, assume that m is a minimizer of $\mathscr{E}^\mu$; putting

$$f(t)=\mathscr{E}^\mu[t\tilde{m}+(1-t)m]$$

for an arbitrary element $\tilde{m}$ of E, we have $f(t)\geqq f(0)$ for all t in $[0,1]$ so that $f'_+(0)\geqq 0$ and hence

$$(\mathscr{E}^\mu)'(m,\tilde{m}-m)\geqq 0.$$

Now let $\tilde{m}$ be defined by

$$\tilde{m}(d\lambda)=(1+\alpha g(\lambda))m(d\lambda),$$

where g is in $\mathscr{L}^\infty(\mathbb{R}_+,m)$ for α sufficiently small, $\tilde{m}$ is an element of E; then

$$(\mathscr{E}^\mu)'(m,\tilde{m}-m)=\alpha(\mathscr{E}^\mu)'(m,gm)\geqq 0.$$

Since α can be either positive or negative, we conclude that $\mathscr{E}'(gm,m)=0$; explicitly,

$$\int_{[0,\infty)}\{L^\mu(m;\lambda)-\beta^{-1}s'(\rho(\lambda))\}g(\lambda)\rho(\lambda)dF(\lambda)+\int_{[0,\infty)}L^\mu(m;\lambda)g(\lambda)m_s(d\lambda)=0. \tag{5.8}$$

Now g is an arbitrary element of $\mathscr{L}^\infty(\mathbb{R}_+;m)$ and ρ is strictly positive dF – a.e.; we conclude that (5.3a) and (5.3b) hold. □

Lemma 5.3. *Let m and $\tilde{m}$ be minimizers of $\mathscr{E}^\mu$; then their absolutely continuous parts coincide and $\|m\|=\|\tilde{m}\|$.*

Proof. The function $x \mapsto s(x)$ is strictly concave and the function $m \mapsto \mathscr{E}^{\mu}[m]$ is convex; this proves the first statement. The second statement follows from the strict convexity of the function $\|m\| \mapsto \|m\|^2$. □

Let R_F denote the subset of $[0, \infty)$ on which the function $\lambda' \mapsto (\lambda - \lambda')^{-1}$ is locally dF-integrable:

$$R_F = \{\lambda \in [0, \infty) : \lambda' \mapsto (\lambda - \lambda')^{-1} \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}_+; dF)\}.$$

Lemma 5.4. *Let v be such that $\lambda \mapsto v(\lambda, \lambda')$ is continuously differentiable for such λ' in $[0, \infty)$ and $\lambda, \lambda' \mapsto (\partial/\partial\lambda)v(\lambda, \lambda')$ is bounded; then the singular part m_s of a minimizing measure m is concentrated on the set $R_F : m_s(R_F^c) = 0$.*

Proof. Let $S = \{\lambda \in [0, \infty) : L^{\mu}(m; \lambda) = 0\}$; it follows from Theorem 5 that $m_s(S^c) = 0$, and that

$$L^{\mu}(m; \lambda) = \beta^{-1} s'(\rho(\lambda)), \quad dF - \text{a.e.},$$

so that

$$\rho(\lambda) = \{e^{\beta L^{\mu}(m;\lambda)} - 1\}^{-1};$$

hence

$$\|m\| \geqq \int_{[0,\infty)} \frac{dF(\lambda)}{e^{\beta L^{\mu}(m;\lambda)} - 1}. \tag{5.9}$$

Now fix λ_0 in S; then $\mu - a\|m\| = \lambda_0 + (Vm)(\lambda_0)$. It follows that, for all λ in $[0, \infty)$, we have $L^{\mu}(m; \lambda) = \lambda - \lambda_0 + (Vm)(\lambda) - (Vm)(\lambda_0)$ since $\lambda \mapsto (Vm)(\lambda)$ is continuously differentiable, we have

$$\lambda - \lambda_0 + (Vm)(\lambda) - (Vm)(\lambda_0) = (\lambda - \lambda_0)(1 + (Vm)'(\lambda_1))$$

for some λ_1 in (λ_0, λ). It follows from the assumption that $\lambda, \lambda' \mapsto (\partial/\partial\lambda)v(\lambda, \lambda')$ is bounded that

$$L^{\mu}(m; \lambda) \leqq K|\lambda - \lambda_0| \tag{5.10}$$

with $0 < K < \infty$; hence, using (5.9), we have

$$\|m\| \geqq \int_{[0,\infty)} \frac{dF(\lambda)}{e^{\beta K|\lambda - \lambda_0|} - 1}$$

so that $\lambda \mapsto (\lambda - \lambda_0)^{-1}$ is locally dF-integrable. We have proved that $S \subset R_F$; it follows that

$$0 \leqq m_s(R_F^c) \leqq m_s(S^c) = 0. \quad \square$$

Theorem 6. *Suppose that v satisfies the smoothness conditions of Lemma 5.4 and that $F(\lambda) = C_d \lambda^{d/2}$, $d > 0$; then a minimizer m of $\mathscr{E}^{\mu}$ has the following properties*:

1. *if $d \leqq 2$ then $m_s = 0$,*
2. *if $m_s \neq 0$ then m_s is concentrated at $\lambda = 0$,*
3. *m is the unique minimizer of $\mathscr{E}^{\mu}$.*

Proof. If $F(\lambda) = C_d \lambda^{d/2}$, $d > 0$, then, by Lemma 5.4, $R_F \subset \{0\}$, so that (2) holds; when $d \leqq 2$, $R_F = \varnothing$, so that (1) holds. Let m and $\tilde{m}$ be two minimizers; it follows from (2) and Lemma 5.4 that $m = \tilde{m}$.

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Communicated by J. Fröhlich

Received January 23, 1989; in revised form July 5, 1989