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Inverse Spectral Problem for the Schrödinger Equation with Periodic Vector Potential

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Abstract. For the Schrödinger operator with periodic magnetic (vector) and electric (scalar) potentials a new system of spectral invariants is found. These invariants are enough to prove the rigidity of isospectral deformations in the class of generic even and real analytic magnetic and electric potentials.

1. Introduction

Let L be a lattice in $\mathbf{R}^2$ with a basis d_1, d_2, i.e. any $d \in L$ can be represented in the form

$$d = md_1 + nd_2, \quad m, n \in \mathbf{Z}.$$

Denote by L' the dual lattice, i.e. $L' = \{\delta = m\delta_1 + n\delta_2\}$, where $\delta_k \cdot d_k = 1$, $k = 1, 2$, $\delta_i \cdot d_k = 0$ for $i \neq k$, $\delta \cdot d$ is the scalar product in $\mathbf{R}^2$. Let $A_k(x_1, x_2), k = 1, 2, V(x_1, x_2)$ be real-valued C^∞ functions periodic with respect to the lattice L. Consider the Schrödinger equation describing the election in an electromagnetic field (see, for example [1])

$$\left(i\frac{\partial}{\partial x_1} + A_1(x)\right)^2 \psi + \left(i\frac{\partial}{\partial x_2} + A_2(x)\right)^2 \psi + V(x)\psi(x) = \lambda\psi(x), \tag{1.1}$$

where $\vec{A}(x) = (A_1(x), A_2(x))$ is the vector potential and $V(x)$ is the scalar (electric) potential. Without loss of generality we shall assume that

$$\operatorname{div} \vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} = 0. \tag{1.2}$$

Since $\vec{A}(x)$ is periodic we have that the magnetic field

$$B(x) = \operatorname{curl} \vec{A} = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \tag{1.3}$$

is also periodic and moreover

$$\iint_{\mathbf{R}^2/L} B(x_1, x_2)\, dx_1 dx_2 = 0. \tag{1.4}$$

Using the Fourier series expansion it is easy to check that having periodic magnetic field $B(x)$ satisfying (1.4) one can find a unique periodic $\vec{A}(x)$ such that (1.2) and (1.3) hold and

$$\iint_{\mathbf{R}^2/L} A_k(x)dx = 0, \quad k = 1,2. \tag{1.5}$$

Denote by $\mathrm{Spec}_0 H$ the periodic spectrum of the operator $H = \sum_{k=1}^{2} (i(\partial/\partial x_k) + A_k)^2 + V(x)$, i.e.

$$H\psi_n = \lambda_n \psi_n, \quad n = 1,2,\ldots, \tag{1.6}$$

where

$$\psi_n(x+d) = \psi_n(x), \quad \forall d \in L. \tag{1.7}$$

Also, denote by $\mathrm{Spec}_k H, k \in \mathbf{R}^2/L'$, the Floquet spectrum of H, i.e.

$$H\varphi_n(x) = \lambda_n(k)\varphi_n, \quad n = 1,2,\ldots, \tag{1.8}$$

where

$$\varphi_n(x+d) = e^{2\pi i k\cdot d}\varphi_n(x), \quad \forall d \in L. \tag{1.9}$$

We shall study the inverse spectral problem of recovering $B(x)$ and $V(x)$ (or $\vec{A}(x)$ and $V(x)$) from $\mathrm{Spec}_0 H$ or $\mathrm{Spec}_k H$, $\forall k \in \mathbf{R}^2/L'$. The case when $\vec{A}(x) = 0$ was considered in [3, 4]. We shall use some results and tools from [3, 4]. Note that the case of the vector potential is quite different and requires new ideas and new techniques. Repeating the proof of Theorem 6.2 in [3] we get

Theorem 1.1. *Assume that $\vec{A}(x)$ and $V(x)$ are real analytic and the lattice L has the following property:*

$$|d| = |d'| \textit{ implies } d' = \pm d \quad \textit{for any} \quad d, d' \in L. \tag{1.10}$$

Then $\mathrm{Spec}_0 H$ *determines* $\mathrm{Spec}_k H$ *for all* $k \in \mathbf{R}^2/L'$.

As in [3] denote by S the set of all "directions" in L', i.e. for any $\delta \in L'$ there is $\delta_0 \in S$ such that $\delta = m\delta_0$ for some integer m and $m\delta_0 \notin S$ for $m \neq 1$. Periodic functions $A_k(x), k = 1,2$, have the following decomposition:

$$A_k(x) = \sum_{\delta \in S} A_{k\delta}\left(\frac{\delta \cdot x}{|\delta|}\right), \quad k = 1,2, \tag{1.11}$$

where

$$A_{k\delta}(s) = \sum_{n=-\infty}^{\infty} a_{k\delta n} e^{2\pi i k|\delta|s}, \tag{1.12}$$

$$a_{k\delta n} = |\delta| \int_0^{|\delta|^{-1}} A_{k\delta}(s) e^{-2\pi i k|\delta|s} ds = \frac{1}{|T^2|} \iint_{T^2} A_k(x) e^{-2\pi i n(\delta\cdot x)} dx, \tag{1.12'}$$

$T^2 = \mathbf{R}^2/L$, $|T^2|$ is the area of T^2. Analogous decomposition holds for $V(x)$. Take arbitrary $\delta_0 \in S$. There is $d_0 \in L$ such that $\delta_0 \cdot d_0 = 0$ and $nd_0, n \in \mathbf{Z}$, span the subspace of L orthogonal to δ_0. Note that there exists a basis $(d_0, d^{(0)})$ in L that includes d_0 and $d^{(0)}$ must be such that $d^{(0)} \cdot \delta_0 = 1$. Denote

$$A_{\delta_0}(s) = A_{1\delta_0}(s)d_{01} + A_{2\delta_0}(s)d_{02}, \tag{1.13}$$

where $d_0/|d_0|=(d_{01},d_{02})$, $|d_0|$ is the norm of d_0. The main result of this paper is the following theorem.

Theorem 1.2. *The Floquet spectrum* $\mathrm{Spec}_k H$, $\forall k\in\mathbf{R}^2/L'$ *determines the following integrals*:

$$H_{\delta_0}(\mu)=\int\limits_0^{|\delta_0|^{-1}}\frac{ds}{\sqrt{\mu+4A_{\delta_0}(s)}}\tag{1.14}$$

for arbitrary $\delta_0\in S$ *and any* $\mu>-4\min_s A_{\delta_0}(s)$.

The proof of Theorem 2.1 is based on the study of the trace $\iint\limits_{T^2} G(x+Nd_0+md^{(0)},x,x_0)dx_1\,dx_2$, where $G(x,y,x_0)$ is the fundamental solution to the Schrödinger equation

$$i\frac{\partial G}{\partial x_0}=HG,\quad x_0>0,\tag{1.15}$$

$$G(x,y,0)=\delta(x-y).\tag{1.15'}$$

This trace is known once one knows all Floquet spectrums. The asymptotics of the trace when $x_0=\tau_0/N_1$, m and τ_0 are fixed and $N\to\infty$ has the following form:

$$\begin{aligned}&\iint\limits_{T^2} G\left(x+Nd_0+md^{(0)},x,\frac{\tau_0}{N_1}\right)dx\\&\quad=N_1\exp\left[-\frac{iN_1^3}{4\tau_0}-\frac{iN_1}{\sqrt{2\tau_0}}S_0(\sqrt{\tau_0/2})\right](a(\tau_0)+O(N_1^{-1})),\end{aligned}\tag{1.16}$$

where $N_1=N|d_0|+m(d_0/|d_0|\cdot d^{(0)})$. The explicit expression for $S_0(\tau_0)$ and $a(\tau_0)$ is given in Sect. 4. Note that $S_0(\tau_0), a(\tau_0)$ are spectral invariants and knowing $S_0(\tau_0)$ we can find the spectral invariant (1.14). The proof of (1.16) is quite technical and it is given in Sects. 3 and 4. In Sect. 2 we shall find sequence of approximative eigenvalues for the spectral problem (1.6), (1.7). Although the results of Sect. 2 are not used for the solution of the inverse problem they give an important information about the spectrum and suggest the form of the spectral invariants. Indeed it will be shown in Sect. 5 that the principal terms of the asymptotic expansion for the eigenvalues are indeed spectral invariants. In Sect. 5 we shall apply the asymptotics (1.16) to the solution of the inverse spectral problem. In particular we shall prove

Theorem 1.3. *Assume that* $A_k^{(t)}(x), V^{(t)}(x), k=1,2$, *are even and real analytic in* $x, x\in\mathbf{R}^2$, *and continuous in* t, $0\leqq t\leqq 1$. *Assume that* $A_k^{(t)}(x), k=1,2$ *satisfy a generic condition (as in Proposition 5.3). Let* $\vec{A}^{(t)}(x), V^{(t)}(x)$ *be an isospectral deformation of* $\vec{A}^{(0)}(x), V^{(0)}(x)$, i.e. $\mathrm{Spec}_0 H^{(t)}=\mathrm{Spec}_0 H^{(0)}$ *for* $0\leqq t\leqq 1$. *Then* $\vec{A}^{(t)}(x)=\vec{A}^{(0)}(x)$, $V^{(t)}(x)=V^{(0)}(x)$ *for* $0\leqq t\leqq 1$, i.e. $H^{(0)}$ *is spectrally rigid in the class of even real analytic vector and scalar potentials.*

2. Asymptotics of Eigenvalues

In this section we shall find a two-parameter sequence of approximate eigenvalues for the spectral problem (1.6), (1.7). Take arbitrary $\delta_0\in S$. Let $(d_0,d^{(0)})$ be the same basis of L as in Sect. 1, i.e. $d_0 \cdot \delta_0 = 0$, $d^{(0)} \cdot \delta_0 = 1$. Make the following orthonormal change of variables

$$s = \frac{\delta_0}{|\delta_0|} \cdot x, \quad t = \frac{d_0}{|d_0|} \cdot x. \tag{2.1}$$

Then

$$x = s\frac{\delta_0}{|\delta_0|} + t\frac{d_0}{|d_0|}. \tag{2.2}$$

At first we shall consider the eigenvalue problem for the following operation H_{δ_0} which we shall call the reduced operator in the direction δ_0:

$$H_{\delta_0}\psi = \left(-\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2} + 2iA_{\delta_0}(s)\frac{\partial}{\partial t} + C_{\delta_0}(s) \right)\psi = \lambda\psi, \tag{2.3}$$

where $A_{\delta_0}(s)$, $C_{\delta_0}(s)$ are periodic in s with the period $|\delta_0|^{-1}$. Note that the periodicity of ψ in x-variables implies

$$\psi\left(s + \frac{\delta_0 \cdot d}{|\delta_0|}, t + \frac{d_0 \cdot d}{|d_0|} \right) = \psi(s,t), \quad \forall d \in L. \tag{2.4}$$

We shall look for the eigenfunctions ψ in the following form:

$$\psi(s,t) = e^{i\xi t} w(s), \tag{2.5}$$

where ξ is large. Since $(d_0, d^{(0)})$ is a basis in L we have that $d = m_1 d_0 + m_2 d^{(0)}$, where $m_k \in \mathbf{Z}$, $k = 1, 2$ and (2.4) implies

$$w(s + m_2|\delta_0|^{-1}) = e^{-i\xi m_1 |d_0| - i\xi m_2 (d_0 \cdot d^{(0)})/|d_0|} w(s) \tag{2.6}$$

for any $m_1 \in \mathbf{Z}$, $m_2 \in \mathbf{Z}$. Therefore in order to ψ to be periodic we must have

$$\xi = \frac{2\pi}{|d_0|} n \quad \text{for some} \quad n \in \mathbf{Z} \tag{2.7}$$

and $w(s)$ satisfies the following Floquet boundary condition

$$w(s + m_2|\delta_0|^{-1}) = \exp\left[-i\xi_n m_2 \frac{d_0 \cdot d^{(0)}}{|d_0|} \right] w(s), \tag{2.8}$$

where $\xi_n = 2\pi n/|d_0|$ is fixed. Substituting (2.5) into (2.3) and cancelling $e^{i\xi_n t}$ we obtain

$$\xi_n^2 w - \frac{\partial^2 w}{\partial s^2} - 2A_{\delta_0}(s)\xi_n w + C_{\delta_0}(s) w = \lambda w(s). \tag{2.9}$$

Take

$$\lambda = \xi_n^2 - \mu\xi_n. \tag{2.10}$$

Then we obtain

$$\frac{\partial^2 w(s)}{\partial s^2} + (2A_{\delta_0}(s) - \mu)\xi_n w - C_{\delta_0}(s) w = 0. \tag{2.11}$$

Denote

$$h_n = \frac{1}{\sqrt{|\xi_n|}}. \tag{2.12}$$

Consider for definiteness the case when $\xi_n > 0$. The case when $\xi_n < 0$ can be treated analogously. After the division by $\xi_n = h_n^{-2}$ Eq. (2.11) takes the following form:

$$-h_n^2 \frac{\partial^2 w}{\partial s^2} + (\mu - 2A_{\delta_0}(s) + h_n^2 C_{\delta_0}(s))w = 0, \tag{2.13}$$

i.e. (2.13) for $h_n \to 0$ has the form of the semi-classical approximation in the quantum mechanics (see for example [6]). Let s_0 be an isolated local maximum of $A_{\delta_0}(s)$ and $s_0^- < s_0 < s_0^+$ be such that $A_{\delta_0}(s)$ is strictly increasing on (s_0^-, s_0) and is strictly decreasing on (s_0, s_0^+). For any μ such that $2A_{\delta_0}(s_0) - \varepsilon_0 \geqq \mu \geqq \max 2A_{\delta_0}(s_0^\mp) + \varepsilon_0$ consider the relation

$$\int\limits_{s_-(\mu)}^{s_+(\mu)} \sqrt{2A_{\delta_0}(s) - \mu}\, ds = t, \tag{2.14}$$

where $2A_{\delta_0}(s_\pm(\mu)) - \mu = 0$ and $\varepsilon_0 < A_{\delta_0}(s_0) - \max A_{\delta_0}(s_0^\mp)$. Denote by $\mu = \mu_0(t)$ the inverse function to (2.14) for $0 < C_1 \leqq t \leqq C_2$. It is known (see [6]) that there exists approximate eigenvalues μ_{mn} and approximate eigenfunctions $\varphi_m(s, h_n)$ such that

$$-h_n^2 \frac{\partial^2 \varphi_m}{\partial s^2} + (\mu_{mn} - 2A_{\delta_0}(s) + h_n^2 C_{\delta_0}(s))\varphi_m = O(h_n^N), \tag{2.15}$$

where $\mu_{mn} = \mu_0(\pi(m + 1/2)h_n) + \mu_{2mn}h_n^2 + \cdots + \mu_{Nmn}h_n^N, |\mu_{kmn}| \leqq C, 2 \leqq k \leqq N$. Note that (2.15) holds uniformly for all m such that

$$0 < C_1 \leqq \pi(m + \tfrac{1}{2})h_n \leqq C_2. \tag{2.16}$$

The relation (2.14) states that $\mu_{0mn} = \mu_0(m + 1/2)h_n)$ satisfies the Bohr-Sommerfeld quantization condition

$$\int\limits_{s_-(\mu_{0mn})}^{s_+(\mu_{0mn})} \sqrt{2A_{\delta_0} - \mu_{0mn}}\, ds = \pi(m + \tfrac{1}{2})h_n. \tag{2.17}$$

Let $\chi(s) \in C_0^\infty(\mathbf{R}^1)$ be such that $\chi(s) = 1$ for $s_-(\mu_{0mn}) - \varepsilon < s < s_+(\mu_{0mn}) + \varepsilon, \chi(s) = 0$ for $s > s_+(\mu_{0mn}) + 2\varepsilon$ and for $s < s_-(\mu_{0mn}) - 2\varepsilon, \varepsilon > 0$ is small. It is known that $\varphi_m(s, h_n) = O(\exp(-c/h_n))$ for $s < s_-(\mu_{0mn}) - \varepsilon$ and for $s > s_+(\mu_{0mn}) + \varepsilon$. Therefore $w_{mn} = \chi(s)\varphi_m(s, h_n)$ satisfies (2.15) on $[s_-(\mu_{0mn}) - 2\varepsilon, s_-(\mu_{0mn}) - 2\varepsilon + |\delta_0|^{-1}]$ and we use (2.8) to define w_{mn} for all $s \in \mathbf{R}^1$. Now we shall find a sequence of approximate eigenvalues for the general equation (1.1). Making the change of variables (2.1) we obtain

$$\begin{aligned} &-\frac{\partial^2 \hat{\psi}}{\partial s^2} - \frac{\partial^2 \hat{\psi}}{\partial t^2} + 2i(\hat{A}_1 \delta_{01} + \hat{A}_2 \delta_{02}) \frac{\partial \hat{\psi}}{\partial s} \\ &\qquad + 2i(\hat{A}_1 d_{01} + \hat{A}_2 d_{02}) \frac{\partial \hat{\psi}}{\partial t} + (\hat{A}_1^2 + \hat{A}_2^2 + \hat{V})\hat{\psi} = \lambda \hat{\psi}, \end{aligned} \tag{2.18}$$

where $\delta_0/|\delta_0| = (\delta_{01}, \delta_{02})$, $\hat{\varphi}(s,t)$ means the function $\varphi(x_1, x_2)$ written in the coordinates (s,t). Substitute in (2.18) $\hat{\psi} = e^{i\hat{\gamma}(s,t)}\hat{w}(s,t)$, where $\hat{\gamma}$ and $\hat{w}$ satisfy (2.4), i.e. $\gamma(x), w(x)$ are periodic with respect to the lattice L.

Cancelling $e^{i\hat{\gamma}}$ we obtain

$$-\frac{\partial^2\hat{w}}{\partial s^2}-\frac{\partial^2\hat{w}}{\partial t^2}+2i\left(\hat{A}_1\delta_{01}+\hat{A}_2\delta_{02}-\frac{\partial\hat{\gamma}}{\partial s}\right)\frac{\partial\hat{w}}{\partial s}+2i\left(\hat{A}_1 d_{01}+\hat{A}_2 d_{02}-\frac{\partial\hat{\gamma}}{\partial t}\right)\frac{\partial\hat{w}}{\partial t}$$
$$+\left[\hat{A}_1^2+\hat{A}_2^2-i\frac{\partial^2\hat{\gamma}}{\partial s^2}-i\frac{\partial^2\hat{\gamma}}{\partial t^2}+\left(\frac{\partial\hat{\gamma}}{\partial s}\right)^2+\left(\frac{\partial\hat{\gamma}}{\partial t}\right)^2\right.$$
$$\left.+\hat{V}-2(\hat{A}_1\delta_{01}+\hat{A}_2\delta_{02})\frac{\partial\hat{\gamma}}{\partial s}-2(\hat{A}_1 d_{01}+\hat{A}_2 d_{02})\frac{\partial\hat{\gamma}}{\partial t}\right]\hat{w}=\lambda\hat{w}. \tag{2.19}$$

We shall choose $\hat{\gamma}(s,t)$ such that

$$\frac{\partial\hat{\gamma}}{\partial t}=\hat{A}_1 d_{01}+\hat{A}_2 d_{02}-A_{\delta_0}(s), \tag{2.20}$$

$$\gamma_{\delta_0}(s)=0. \tag{2.20'}$$

Here $A_{\delta_0}(s)$ is the same as in (1.13) and $\gamma_{\delta_0}(s)$ is analogous to (1.12). Note that $\partial/\partial t = d_{01}\partial/\partial x_1 + d_{02}\partial/\partial x_2$. Therefore

$$\gamma(x_1,x_2)=\sum_{\delta\neq\delta_0}\sum_{n=-\infty}^{\infty}\frac{(a_{1\delta n}d_{01}+a_{2\delta n}d_{02})|d_0|}{2\pi in(\delta\cdot d_0)}e^{2\pi in(\delta\cdot x)}. \tag{2.21}$$

Using that $\operatorname{div}\vec{A}=0$ we have

$$\frac{\partial}{\partial s}(\hat{A}_1\delta_{01}+\hat{A}_2\delta_{02})+\frac{\partial}{\partial t}(\hat{A}_1 d_{01}+\hat{A}_2 d_{02})=0. \tag{2.22}$$

Since $\partial/\partial t\ \hat{A}_{k\delta_0}(s)=0$ we obtain using the decomposition (1.11) that

$$\frac{\partial}{\partial s}(A_{1\delta_0}(s)\delta_{01}+A_{2\delta_1}(s)\delta_{02})=0. \tag{2.23}$$

Since $\int\limits_0^{|\delta_0|^{-1}} A_{k\delta_0}(s)ds=0$ (see (1.5)) we have

$$A_{1\delta_0}(s)\delta_{01}+A_{2\delta_0}(s)\delta_{02}=0. \tag{2.24}$$

We shall often use the following decomposition

$$F(x_1,x_2)=F_{\delta_0}\left(\frac{\delta_0\cdot x}{|\delta_0|}\right)+F'(x_1,x_2), \tag{2.25}$$

where $F'(x_1,x_2)=\sum_{\delta\neq\delta_0}F_\delta(x\cdot\delta/|\delta|)$. In the coordinates (s,t) the decomposition (2.25) has the form

$$\hat{F}(s,t)=F_{\delta_0}(s)+\hat{F}'(s,t). \tag{2.26}$$

Note that the equation

$$\frac{\partial\hat{g}(s,t)}{\partial t}=\hat{F}(s,t) \tag{2.27}$$

has a unique solution $\hat{g}(s,t)$ such that $g(x_1, x_2)$ is periodic and

$$\hat{g}(s,t) = \hat{g}'(s,t), \quad \text{i.e.} \quad g_{\delta_0}(s) = 0 \tag{2.28}$$

if and only if

$$\hat{F}(s,t) = \hat{F}'(s,t) \quad \text{i.e.} \quad F_{\delta_0}(s) = 0, \tag{2.29}$$

and this solution has the form analogous to (2.21) in the x-coordinates.

We shall take $\lambda_{mn} = \xi_n^2 - \mu_{mn}\xi_n$, where ξ_n, μ_{mn} are the same as in (2.15) and we shall look for the approximate solution of (2.19) in the following form:

$$\hat{w} = e^{i\xi_n t}\hat{v}(s,t), \tag{2.30}$$

where

$$\hat{v}(s,t) = w_{mn}(s) + h_n \hat{v}_{1mn}(s,t) + h_n^2 \hat{v}_{2mn}(s,t), \tag{2.31}$$

$|\hat{v}_{kmn}(s,t)| \leqq C, k = 1, 2$, and $w_{mn}(s) = \chi(s)\varphi_m(s, h_n)$, φ_n is the same as in (2.15).

Substituting (2.30) into (2.19) and taking into account (2.20) and (2.24) we obtain

$$-\frac{\partial^2 \hat{v}}{\partial t^2} - \frac{\partial^2 \hat{v}}{\partial s^2} - 2i\xi_n \frac{\partial \hat{v}}{\partial t} + 2i\hat{A}_3'(s,t)\frac{\partial \hat{v}}{\partial s} + (\mu_{mn} - 2A_{\delta_0}(s))\xi_n \hat{v}$$

$$+ 2iA_{\delta_0}(s)\frac{\partial \hat{v}}{\partial t} + \hat{C}(s,t)\hat{v} = 0, \tag{2.32}$$

where

$$\hat{A}_3'(s,t) = \hat{A}_1\delta_{01} + \hat{A}_2\delta_{02} - \frac{\partial \hat{\gamma}}{\partial s}, \tag{2.33}$$

$$\hat{C}(s,t) = \hat{A}_1^2 + \hat{A}_2^2 - i\frac{\partial^2 \hat{\gamma}}{\partial s^2} - i\frac{\partial^2 \hat{\gamma}}{\partial t^2} + \left(\frac{\partial \hat{\gamma}}{\partial s}\right)^2 + \left(\frac{\partial \hat{\gamma}}{\partial t}\right)^2$$

$$- 2(\hat{A}_1\delta_{01} + A_2\delta_{02})\frac{\partial \hat{\gamma}}{\partial s} - 2(\hat{A}_1 d_{01} + \hat{A}_2 d_{02})\frac{\partial \hat{\gamma}}{\partial t} + \hat{V}. \tag{2.34}$$

Dividing by $\xi_n = 1/h_n^2$ and decompositing $\hat{C} = \hat{C}'(s,t) + C_{\delta_0}(s)$ we obtain from (2.32)

$$-2i\frac{\partial \hat{v}}{\partial t} - h_n^2\frac{\partial^2 \hat{v}}{\partial s^2} + (\mu_{mn} - 2A_{\delta_0}(s) + h_n^2 C_{\delta_0}(s))\hat{v} + 2ih_n^2 \hat{A}_3'(s,t)\frac{\partial \hat{v}}{\partial s}$$

$$+ h_n^2\left(-\frac{\partial^2 \hat{v}}{\partial t^2} + 2iA_{\delta_0}(s)\frac{\partial \hat{v}}{\partial t} + \hat{C}'(s,t)\hat{v}\right) = 0. \tag{2.35}$$

Denote by $\hat{H}_1$ the operator in the left-hand side of (2.35). We have

$$\hat{H}_1 w_{mn}(s) = r_{0mn} + 2ih_n^2 \hat{A}_3'(s,t)\frac{\partial w_{mn}(s)}{\partial s} + h_n^2 \hat{C}'(s,t) w_{mn}(s), \tag{2.36}$$

where $r_{0mn} = O(h_n^N)$. Note that $h_n \partial w_{mn}/\partial s = O(1)$. Denote by $\hat{v}_{11}'(s,t)$ the solution of the equation

$$\frac{\partial \hat{v}_{11}'}{\partial t} = \hat{A}_3'(s,t) \tag{2.37}$$

satisfying (2.28). Denote

$$\hat{v}_{1mn}(s,t) = -h_n \hat{v}'_{11}(s,t)\frac{\partial w_{mn}}{\partial s}. \tag{2.38}$$

Then $\hat{v}_{1mn} = O(1)$ and

$$\begin{aligned}\hat{H}_1(w_{mn} + h_n \hat{v}_{1mn}(s,t)) &= r_{0mn} + h_n^2 \hat{C}'(s,t) w_{mn}(s) - 2i\hat{A}'_3 \hat{v}'_{11} h_n^4 \frac{\partial^2 w_{mn}}{\partial s^2} \\ &\quad - h_n^2 \hat{v}'_{11} H_{1\delta_0}\left(\frac{\partial w_{mn}}{\partial s}\right) + 2h_n^4 \frac{\partial \hat{v}_{11}}{\partial s}\frac{\partial^2 w_{mn}}{\partial s^2} + r_{1mn},\end{aligned} \tag{2.39}$$

where $r_{1mn} = O(h_n^3)$ and $H_{1\delta_0} = -h_n^2(\partial^2/\partial s^2) + \mu_{mn} - 2A_{\delta_0}(s)$. Note that

$$H_{1\delta_0}\frac{\partial w_{mn}}{\partial s} = \frac{\partial}{\partial s} H_{1\delta_0} w_{mn} + 2A_{\delta_0 s}(s) w_{mn} = O(1). \tag{2.40}$$

Note also that

$$\hat{A}'_3 \hat{v}'_{11} = \frac{\partial \hat{v}'_{11}}{\partial t}\hat{v}'_{11} = \frac{1}{2}\frac{\partial}{\partial t}(\hat{v}'_{11})^2. \tag{2.41}$$

Denote by $\hat{v}'_{2k}(s,t), 1 \leqq k \leqq 4$, the solutions of the following equations:

$$\frac{\partial}{\partial t}\hat{v}'_{21}(s,t) = \hat{C}'(s,t), \quad \frac{\partial \hat{v}'_{22}(s,t)}{\partial t} = \hat{A}'_3 \hat{v}'_{11} = \frac{1}{2}\frac{\partial}{\partial t}(\hat{v}'_{11})^2, \tag{2.42}$$

$$\frac{\partial \hat{v}'_{23}}{\partial t} = \hat{v}'_{11}, \quad \frac{\partial \hat{v}'_{24}}{\partial t} = \frac{\partial \hat{v}'_{11}}{\partial s}. \tag{2.42'}$$

As before we require that all $\hat{v}'_{2k}(s,t)$ satisfy (2.28). Note that $\hat{v}'_{24} = \partial \hat{v}'_{23}/\partial s$ and $\hat{v}'_{22} = \hat{F}'_2(s,t) = 1/2(\hat{v}'_{11})^2 - F_{1\delta_0}(s)$, where $1/2(v'_{11})^2 = \hat{F}'_1(s,t) + F_{1\delta_0}(s)$ is the decomposition of $1/2(v'_{11})^2$ of the form (2.26). Denote

$$\begin{aligned}\hat{v}_{2mn}(s,t) &= \frac{i}{2}\hat{v}'_{21}(s,t) w_{mn}(s) + h_m^2 \hat{v}'_{22}\frac{\partial^2 w_{mn}}{\partial s^2} \\ &\quad - \frac{i}{2}\hat{v}'_{23} H_{1\delta_0}\left(\frac{\partial w_{mn}}{\partial s}\right) + i h_n^2 \hat{v}'_{24}\frac{\partial^2 w_{mn}}{\partial s^2}.\end{aligned} \tag{2.43}$$

Note that $\hat{v}_{2mn} = O(1)$. It is easy to check that

$$\hat{H}_1(w_{mn} + h_n \hat{v}_{1mn} + h_n^2 \hat{v}_{2mn}) = r_{2mn}, \tag{2.44}$$

where

$$r_{2mn} = O(h_n^3). \tag{2.45}$$

It follows from (2.44) and (2.45) that

$$(\hat{H} - \lambda_{mn})[e^{i\hat{\gamma}(s,t) + i\xi_n t}(w_{mn} + h_n \hat{v}_{1mn} + h_n^2 \hat{v}_{2mn})] = O(\xi_n r_{2mn}) = O(h_n) = O\left(\frac{1}{\sqrt{n}}\right), \tag{2.46}$$

where $\hat{H}$ is the operator H in (s,t) coordinates. Since $\hat{H}$ is self-adjoint we obtain from (2.46) the following result:

Theorem 2.1. *Let* $\lambda_{mn}=\xi_n^2-\mu_{mn}\xi_n$ be same as in (2.15) and let λ_{mn}^* be the exact eigenvalue of H closest to λ_{mn}. Then

$$|\lambda_{mn}^*-\lambda_{mn}|\leqq Ch_n=C\sqrt{\frac{|d_{02}|}{2\pi n}} \tag{2.47}$$

and (2.47) holds for all m satisfying (2.16).

3. The Green Function for the Time Dependent Schrödinger Equation

Let $G(x,y,x_0)$ be the Green function for the Schrödinger equation in $\mathbf{R}^2$:

$$i\frac{\partial G(x,y,x_0)}{\partial x_0}=HG(x,y,x_0),\quad x_0>0, \tag{3.1}$$

$$G(x,y,0)=\delta(x-y),\quad x=(x_1,x_2)\in\mathbf{R}^2,\quad y=(y_1,y_2)\in\mathbf{R}^2. \tag{3.1'}$$

Here x_0 is the time variable and $H=\sum_{k=1}^{2}(i\partial/\partial x_k+A_k(x))^2+V(x)$ is the same as in Sect. 1, i.e. $A_1(x), A_2(x), V(x)$ are C^∞ and L-periodic functions satisfying (1.2), (1.5).

Let $G_k(x,y,x_0)$, $k\in L'$ be the Green function satisfying (3.1), (3.1′) for $y\in T^2=\mathbf{R}^2/L$ and the Floquet boundary conditions

$$G_k(x+d,y,x_0)=e^{2\pi ik\cdot d}G_k(x,y,x_0),\quad \forall d\in L. \tag{3.3}$$

Then (cf. [3])

$$G_k(x,y,x_0)=\sum_{d\in L}e^{-2\pi id\cdot k}G(x+d,y,x_0), \tag{3.4}$$

where $G(x,y,x_0)$ is the same as in (3.1), (3.1′). The following trace formula holds (cf. [3])

$$\sum_{n=1}^{\infty}e^{-i\lambda_n(k)t}=\iint_{T^2}G_k(x,x,x_0)dx=\sum_{d\in L}e^{-2\pi id\cdot k}\iint_{T^2}G(x+d,x,x_0)dx, \tag{3.5}$$

where $\lambda_n(k)$ are the same as in (1.8). It follows (3.5) that knowing the Floquet spectrum of H for all $k\in\mathbf{R}^2/L'$ we can recover the following integrals for any $d\in L$:

$$\iint_{T^2}G(x+d,x,x_0)dx,\quad \forall d\in L. \tag{3.6}$$

Take arbitrary $\delta_0\in S$. As in Sect. 1 there is a basis $(d_0,d^{(0)})$ in L such that $d_0\cdot\delta_0=0, d^{(0)}\cdot\delta_0=1$. We shall identify $T^2=\mathbf{R}^2/L$ with the parallelogram spanned by d_0 and $d^{(0)}$. As in (2.1) make the change of variables

$$s=\frac{\delta_0}{|\delta_0|}\cdot x,\quad t=\frac{d_0}{|d_0|}\cdot x. \tag{3.7}$$

Let $\hat{G}(s,t,s',t',x_0)$ be the function $G(x,y,x_0)$ in the new coordinates where (s',t') is the image of $y=(y_1, y_2)$. We have

$$G(x+Nd_0+md^{(0)}, x, x_0)=\hat{G}(s+m|\delta_0|^{-1}, t+N_1, s, t, x_0), \tag{3.8}$$

where

$$N_1=N|d_0|+m\left(\frac{d_0}{|d_0|}\cdot d^{(0)}\right), \tag{3.9}$$

and therefore

$$\iint_{T^2} G(x+Nd_0+md^{(0)}, x, x_0)dx=\iint_{\hat{T}^2} \hat{G}(s+m|\delta_0|^{-1}, t+N_1, s, t, x_0)ds dt, \tag{3.10}$$

where $\hat{T}^2$ is the image of T^2 under the orthogonal transformation (3.7).

Our objective will be to find the asymptotics of the integral

$$\iint_{\hat{T}^2} \hat{G}\left(s+m|\delta_0|^{-1}, t+N_1, s, t, \frac{\tau_0}{N_1}\right)ds dt$$

as $N\to\infty$, m and τ_0 are fixed. We shall extensively use the stationary phase method and we choose the Schrödinger operator instead of the heat operator $\partial/\partial t+H$ because the application of the stationary phase method is easier for the Schrödinger operator. The Green function $\hat{G}(s,t,s',t',x_0)$ satisfies the following equation (cf. (2.18)):

$$\begin{aligned} i\frac{\partial}{\partial x_0}\hat{G}=&-\frac{\partial^2\hat{G}}{\partial s^2}-\frac{\partial^2\hat{G}}{\partial t^2}+2i\left(\hat{A}\cdot\frac{\delta_0}{|\delta_0|}\right)\frac{\partial\hat{G}}{\partial s}+2i\left(\hat{A}\cdot\frac{d_0}{|d_0|}\right)\frac{\partial\hat{G}}{\partial t}\\ &+(\hat{A}_1^2+\hat{A}_2^2+\hat{V})\hat{G}, \quad x_0>0, \end{aligned} \tag{3.11}$$

and the initial condition

$$\hat{G}(s,t,s',t',0)=\delta(s-s')\delta(t-t'). \tag{3.11'}$$

Here $\hat{A}(s,t)=(\hat{A}_1(s,t), \hat{A}_2(s,t))$. We shall find a good approximation for $\hat{G}(s,t,s',t',x_0)$ assuming that

$$\begin{aligned} &|s|\leqq m|\delta_0|^{-1}+C, \quad |s'|\leqq C, \quad |t|\leqq N_1+C, \quad |t'|\leqq C,\\ &0\leqq x_0\leqq C_1N_1^{-1}, C_1 \quad \text{is small.} \end{aligned} \tag{3.12}$$

Substitute in (3.11)

$$\hat{G}(s,t,s',t',x_0)=e^{i\hat{\gamma}(s,t)}\hat{g}(s,t,s',t',x_0), \tag{3.13}$$

where $\hat{\gamma}(s,t)$ is the same as in (2.20), (2.20'), (2.21). Then $\hat{g}$ satisfies the following equation (cf. (2.19)):

$$i\frac{\partial\hat{g}}{\partial x_0}=-\frac{\partial^2\hat{g}}{\partial s^2}-\frac{\partial^2\hat{g}}{\partial t^2}+2i\hat{A}_3'(s,t)\frac{\partial\hat{g}}{\partial s}+2iA_{\delta_0}(s)\frac{\partial\hat{g}}{\partial t}+\hat{C}(s,t)\hat{g}, \quad x_0>0, \tag{3.14}$$

$$\hat{g}(s,t,s',t',0)=e^{i\hat{\gamma}(s',t')}\delta(s-s')\delta(t-t'). \tag{3.14'}$$

In (3.14) (cf. (2.33), (2.34), (1.13)),

$$\hat{A}_3'(s,t)=\hat{A}\cdot\frac{\delta_0}{|\delta_0|}-\frac{\partial\hat{\gamma}}{\partial s}, \quad A_{\delta_0}(s)=A_{1\delta_0}(s)d_{01}+A_{2\delta_0}(s)d_{02}, \tag{3.15}$$

$$\hat{C}(s,t)=|\hat{A}|^2+\hat{V}-i\Delta\hat{\gamma}+|\nabla\hat{\gamma}|^2-2\left(\hat{A}\cdot\frac{\delta_0}{|\delta_0|}\right)\frac{\partial\hat{\gamma}}{\partial s}-2\left(\hat{A}\cdot\frac{d_0}{|d_0|}\right)\frac{\partial\hat{\gamma}}{\partial t}. \tag{3.16}$$

Denote the operator in the right-hand side of (3.14) by $\hat{H}_2$. Denote

$$\Lambda=(\xi^2+\varepsilon_0^4\eta^4+\varepsilon_0^{-8})^{1/4},\quad \varepsilon_0 \text{ is small.} \tag{3.17}$$

We shall look for the approximate solution of $(i\partial/\partial x_0-\hat{H}_2)v=0$ in the following form:

$$v=e^{-i\Lambda L(x_0\Lambda,s,t,s',t',\xi,\eta)}a(x_0\Lambda,s,t,\xi,\eta),\quad x_0>0. \tag{3.18}$$

Substituting (3.18) into (3.14) we obtain

$$\begin{aligned}&\left(i\frac{\partial}{\partial x_0}-\hat{H}_2\right)v\\&\quad=e^{-i\Lambda L}[(\Lambda^2L_\tau-\Lambda^2L_s^2-\Lambda^2L_t^2-i\Lambda\Delta L-2\Lambda A_{\delta_0}(s)L_t-2\Lambda\hat{A}_3'L_s-\hat{C})a\\&\qquad+(i\Lambda a_\tau-2i\Lambda L_s a_s-2i\Lambda L_t a_t-2iA_{\delta_0}(s)a_t-2i\hat{A}_3'a_s)+\Delta a],\end{aligned} \tag{3.19}$$

where

$$\tau=x_0\Lambda. \tag{3.20}$$

We shall choose L to satisfy the following eiconal equation:

$$L_\tau-L_s^2-L_t^2-2\Lambda^{-1}A_{\delta_0}(s)L_t-2\Lambda^{-1}\hat{A}_3'L_s=0 \tag{3.21}$$

with the initial condition

$$L(0,s,t,s',t',\xi,\eta)=(s-s')\eta\Lambda^{-1}+(t-t')\xi\Lambda^{-1}. \tag{3.22}$$

To find the solution of (3.21), (3.22) consider the system of equations for the bicharacteristics:

$$\frac{ds}{d\tau}=-2p-2\Lambda^{-1}\hat{A}_3'(s,t),\quad s(0,y,z,\xi,\eta)=y, \tag{3.23}$$

$$\frac{dp}{d\tau}=2\Lambda^{-1}A_{\delta_0 s}(s)q+2\Lambda^{-1}\hat{A}_{3s}'(s,t)p,\quad p(0,y,z,\xi,\eta)=\eta\Lambda^{-1}, \tag{3.24}$$

$$\frac{dt}{d\tau}=-2q-2\Lambda^{-1}A_{\delta_0}(s),\quad t(0,y,z,\xi,\eta)=z, \tag{3.25}$$

$$\frac{dq}{d\tau}=2\Lambda^{-1}\hat{A}_{3t}'(s,t)p,\quad q(0,y,z,\xi,\eta)=\xi\Lambda^{-1}. \tag{3.26}$$

We shall prove that the solution of (3.23)–(3.26) exists for all $0\leqq\tau<+\infty$. Denote

$$\begin{aligned}q_1&=q(\tau,y,z,\xi,\eta)-\xi\Lambda^{-1},\\t_1&=t(\tau,y,z,\xi,\eta)-z+2\xi\Lambda^{-1}\tau.\end{aligned} \tag{3.27}$$

Then the system (3.23)–(3.26) takes the following form:

$$\frac{ds}{d\tau} = -2p - 2\Lambda^{-1}\hat{A}'_3(s, z - 2\xi\Lambda^{-1}\tau + t_1), \quad s(0, y, z, \xi, \eta) = y, \tag{3.23'}$$

$$\frac{dp}{d\tau} = 2\xi\Lambda^{-2}A_{\delta_0 s}(s) + 2\Lambda^{-1}A_{\delta_0 s}(s)q_1$$

$$+ 2\Lambda^{-1}\hat{A}'_{3s}(s,t)p, \quad p(0, y, z, \xi, \eta) = \eta\Lambda^{-1}, \tag{3.24'}$$

$$\frac{dt_1}{d\tau} = -2q_1 - 2\Lambda^{-1}A_{\delta_0}(s), \quad t_1(0, y, z, \xi, \eta) = 0, \tag{3.25'}$$

$$\frac{dq_1}{d\tau} = 2\Lambda^{-1}\hat{A}'_{3t}(s, z - 2\xi\Lambda^{-1}\tau + t_1)p, \quad q_1(0, y, z, \xi, \eta) = 0. \tag{3.26'}$$

Since (3.24′), (3.26′) are linear with respect to p and q_1 with coefficients bounded by $C\Lambda^{-1}$, and since $|\xi|\Lambda^{-2} \leqq 1$ we have

$$|p| + |q - \xi\Lambda^{-1}| \leqq (C\tau + C|\eta|\Lambda^{-1}) \exp C\Lambda^{-1}\tau. \tag{3.28}$$

Using (3.28) we obtain from (3.23′), (3.25′),

$$|s - y| + |t - z + 2\xi\Lambda^{-1}\tau| \leqq C\Lambda^{-1}\tau + 4C(\tau^2 + |\eta|\Lambda^{-1}\tau) \exp C\Lambda^{-1}\tau. \tag{3.28'}$$

Estimates (3.28), (3.28′) imply that the solution of (3.23)–(3.26) exists for all $0 \leqq \tau < +\infty$. Differentiating (3.23′)–(3.26′) in y we obtain

$$\frac{d}{d\tau}s_y = -2p_y - 2\Lambda^{-1}\hat{A}'_{3s}(s,t)s_y - 2\Lambda^{-1}\hat{A}'_{3t}(s,t)t_{1y}, \quad s_y(0, y, z, \xi, \eta) = 1, \tag{3.29}$$

$$\frac{dp_y}{d\tau} = 2\xi\Lambda^{-2}A_{\delta_0 ss}(s)s_y + 2\Lambda^{-1}A_{\delta_0 ss}(s)q_1 s_y + 2\Lambda^{-1}A_{\delta_0 s}(s)q_{1y}$$

$$+ 2\Lambda^{-1}\hat{A}'_{3s}(s,t)p_y + 2\Lambda^{-1}p(\hat{A}'_{3ss}s_y + \hat{A}'_{3t}t_{1y}), \quad p_y(0, y, z, \xi, \eta) = 0, \tag{3.30}$$

$$\frac{dt_{1y}}{d\tau} = -2q_{1y} - 2\Lambda^{-1}A_{\delta_0 s}(s)s_y, \quad t_{1y}(0, y, z, \xi, \eta) = 0, \tag{3.31}$$

$$\frac{dq_{1y}}{d\tau} = 2\Lambda^{-1}\hat{A}'_{3t}(s,t)p_y + 2\Lambda^{-1}p(\hat{A}'_{3ts}s_y + \hat{A}'_{3tt}t_{1y}), \quad q_{1y}(0, y, z, \xi, \eta) = 0, \tag{3.32}$$

where $t_1 = t - z + 2\xi\Lambda^{-1}\tau$.

Since (3.29)–(3.32) is a linear system in s_y, p_y, t_{1y}, q_{1y} with coefficients having the bounds $C + C\Lambda^{-1} + C\Lambda^{-1}(|p| + |q_1|)$ we obtain using (3.28) and assuming that $0 \leqq \tau \leqq \tau_+ < 1$,

$$\begin{aligned} |s_y - 1| + |p_y| + |t_y| + |q_y| &\leqq C\tau(\Lambda^{-1} + 1 + \Lambda^{-1}(|p| + |q_1|)) \exp C\Lambda^{-1}(|p| + |q_1|)\tau \\ &\leqq C\tau(1 + |\eta|\Lambda^{-2}) \exp C|\eta|\Lambda^{-2}\tau. \end{aligned} \tag{3.33}$$

We used in (3.33) that $t_y = t_{1y}, q_v = q_{1y}$. Note that $|\eta|\Lambda^{-1} \leqq \varepsilon_0^{-1}, \Lambda > \varepsilon_0^{-2}$. Therefore

$$|s_y - 1| + |p_y| + |t_y| + |q_y| \leqq C\tau(1 + \varepsilon_0^{-1}\Lambda^{-1}) \exp C\varepsilon_0^{-1}\Lambda^{-1}\tau \leqq C\tau \exp C\tau. \tag{3.33'}$$

Analogously we have

$$|s_z| + |p_z| + |t_z - 1| + |q_z| \leqq C\tau(1 + \varepsilon_0^{-1}\Lambda^{-1}) \exp C\varepsilon_0^{-1}\Lambda^{-1}\tau \leqq C_1\tau \exp C_1\tau. \tag{3.33''}$$

Since $\tau \leqq \tau_+$, where τ_+ is small we obtain that the Jacobian

$$\begin{vmatrix} s_y & t_y \\ s_z & t_z \end{vmatrix} \neq 0. \tag{3.34}$$

Therefore there exists functions

$$y = y(\tau, s, t, \xi, \eta), \quad z = z(\tau, s, t, \xi, \eta) \tag{3.35}$$

that are inverse to

$$s = s(\tau, y, z, \xi, \eta), \quad t = t(\tau, y, z, \xi, \eta). \tag{3.36}$$

Assuming that $L(\tau, s, t, s', t', \xi, \eta)$ is the solution of (3.20), (3.21) we have (see for example [2]):

$$\begin{aligned} &\frac{d}{d\tau} L(\tau, s(\tau, y, z, \xi, \eta), t(\tau, y, z, \xi, \eta), s', t', \xi, \eta) \\ &\quad = L_\tau + L_s \frac{ds}{d\tau} + L_t \frac{dt}{d\tau} = L_s^2 + L_t^2 + 2\Lambda^{-1} A_{\delta_0} L_t + 2\Lambda^{-1} \hat{A}_3' L_s \\ &\quad\quad + L_s(-2p - 2\Lambda^{-1}\hat{A}_3') + L_t(-2q - 2\Lambda^{-1} A_{\delta_0}) = -p^2 - q^2, \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} p(\tau, y, z, \xi, \eta) &= L_s(\tau, s(\tau, y, z, \xi, \eta), t(\tau, y, z, \xi, \eta), s', t', \xi, \eta), \\ q(\tau, y, z, \xi, \eta) &= L_t(\tau, s(\tau, y, z, \xi, \eta), t(\tau, y, z, \xi, \eta), s', t', \xi, \eta). \end{aligned} \tag{3.38}$$

Then the solution of (3.20), (3.21) is given by the following formula (cf. [2]):

$$\begin{aligned} L = ((y(\tau, s, t, \xi, \eta) - s')\eta' &+ (z(\tau, s, t, \xi, \eta) - t')\xi' \\ &- \int_0^\tau (p^2(\tau', y(\tau, s, t, \xi, \eta), z(\tau, s, t, \xi, \eta), \xi, \eta) \\ &+ q^2(\tau', y(\tau, s, t, \xi, \eta), z(\tau, s, t, \xi, \eta), \xi, \eta)))d\tau', \end{aligned} \tag{3.39}$$

where $\xi' = \xi\Lambda^{-1}, \eta' = \eta\Lambda^{-1}$.

Note that if (s, t, p, q) is a solution of (3.23)–(3.26) then $(s + m|\delta_0|^{-1}, t + N_1, p, q)$ is also a solution of (3.23)–(3.26) with initial data $(y + m|\delta_0|^{-1}, z + N_1, \xi', \eta')$. So that the uniqueness of the Cauchy problem implies

$$\begin{aligned} s(\tau, y, z, \xi, \eta) + m' &= s(\tau, y + m', z + N_1, \xi, \eta), \\ t(\tau, y, z, \xi, \eta) + N_1 &= t(\tau, y + m', z + N_1, \xi, \eta), \end{aligned} \tag{3.40}$$

where $m' = m|\delta_0|^{-1}, N_1 = N|d_0| + m(d_0/|d_0|)\cdot d^{(0)}$. Therefore

$$\begin{aligned} y(\tau, s, t, \xi, \eta) + m' = y(\tau, s + m', t + N_1, \xi, \eta), z + N_1 \\ = z(\tau, s + m', t + N_1, \xi, \eta). \end{aligned} \tag{3.41}$$

It follows from (3.40), (3.41) that

$$L(\tau, s+m', t+N_1, s', t', \xi, \eta) = L(\tau, s, t, s', t', \xi, \eta) + m'\eta' + N_1\xi'. \tag{3.42}$$

Therefore all partial derivatives of L in s and t are periodic, i.e. (2.4) holds. Also all partial derivatives of $s(\tau, y, z, \xi, \eta)$, $t(\tau, y, z, \xi, \eta)$ are periodic in (y, z) and all partial derivatives of $y(\tau, s, t, \xi, \eta)$, $z(\tau, s, t, \xi, \eta)$ are periodic in (s, t). ■

Now we shall solve the transport equations, i.e. the equations for $a \approx a_0 + a_1 + \cdots$. We have from (3.19)

$$ia_{0\tau} - 2i(L_s + \Lambda^{-1}\hat{A}_3')a_{0s} - 2i(L_t + \Lambda^{-1}A_{\delta_0}(s))a_{0t} - i(\Delta L - i\Lambda^{-1}\hat{C})a_0 = 0, \tag{3.43}$$

$$a_0(0, s, t, \xi, \eta) = 1. \tag{3.43'}$$

Denote

$$\hat{a}_0(\tau, y, z, \xi, \eta) = a_0(\tau, s(\tau, y, z, \xi, \eta), t(\tau, y, z, \xi, \eta), \xi, \eta). \tag{3.44}$$

Then (3.43), (3.43') takes the following form

$$\frac{d}{d\tau}\hat{a}_0(\tau, y, z, \xi, \eta) = (\Delta L - i\Lambda^{-1}\hat{C})\hat{a}_0, \quad \hat{a}_0(0, y, z, \xi, \eta) = 1. \tag{3.45}$$

Therefore

$$\hat{a}_0(\tau, y, z, \xi, \eta) = \exp\left(\int_0^\tau (\Delta L - i\Lambda^{-1}\hat{C}(s(\tau', y, z, \xi, \eta), t(\tau', y, z, \xi, \eta)))d\tau'\right). \tag{3.46}$$

The equation for a_1 has the following form:

$$a_{1\tau} - 2(L_s + \Lambda^{-1}\hat{A}_3')a_{1s} - 2(L_t + \Lambda^{-1}A_{\delta_0}(s))a_{1t} - (\Delta L - i\Lambda^{-1}\hat{C})a_1 = i\Lambda^{-1}\Delta a_0, \tag{3.47}$$

$$a_1(0, s, t, \xi, \eta) = 0. \tag{3.47'}$$

Analogously one can write equations for $a_k, k \geqq 2$.

Lemma 3.1. *Functions $a_k, k \geqq 0$, are periodic in (s,t) and satisfy the following estimates*:

$$|a_k| \leqq C_k \tau^k \Lambda^{-k}, \quad k \geqq 0, \tag{3.48}$$

$$\left|\frac{\partial^{\alpha+\beta} a_k}{\partial s^\alpha \partial t^\beta}\right| \leqq C_{k\alpha\beta}\tau^k\Lambda^{-k}, \quad k \geqq 0, \quad \alpha \geqq 0, \quad \beta \geqq 0. \tag{3.49}$$

Proof. The periodicity of $a_k(\tau, s, t, \xi, \eta)$ in (s, t) follows from (3.40), (3.41) and (3.42). To prove (3.48), (3.49) we shall need a more precise estimate of the solutions of (3.23)–(3.26) for large $\Lambda = (\xi^2 + \varepsilon_0^4\eta^4 + \varepsilon_0^{-8})^{1/4}$.

It follows from (3.26') and (3.28) that

$$q_1 = O(\Lambda^{-1}\tau^2) + O(\eta\Lambda^{-2}\tau), \tag{3.50}$$

where $q_1 = q - \xi\Lambda^{-1}$. Using (3.50) and (3.25') we get

$$t_1 = O(\Lambda^{-1}\tau) + O(\eta\Lambda^{-2}\tau^2), \tag{3.51}$$

where $t_1 = t - z + 2\xi\Lambda^{-1}\tau$. Here and below in this section $w_1 = O(w)$ means

$|w_1| \leqq C|w|$, where C is independent of ξ, η and ε_0. In the case when $|w_1| \leqq C'|w|$, where C' is independent of ξ, η but may depend on ε_0 we shall write $w_1 = O'(w)$. Note that $|\eta|\Lambda^{-1} \leqq \varepsilon_0^{-1}$. So that

$$q = \xi\Lambda^{-1} + O(\Lambda^{-1}\tau^2) + O(\varepsilon_0^{-1}\Lambda^{-1}\tau) = \xi\Lambda^{-1} + O'(\Lambda^{-1}), \tag{3.50'}$$

$$t = z - 2\xi\Lambda^{-1}\tau + O'(\Lambda^{-1}). \tag{3.51'}$$

It follows from (3.33'),

$$|s_y - 1| + |p_y| + |t_y| + |q_y| = O(\tau).$$

Moreover integrating (3.32) and using (3.33') and (3.28) we obtain

$$q_y = q_{1y} = O(\tau\Lambda^{-1}) + O(\tau\eta\Lambda^{-2}) = O'(\Lambda^{-1}). \tag{3.52}$$

Substituting (3.52) in (3.31) we get

$$t_y = t_{1y} = O(\tau\Lambda^{-1}) + O(\tau^2\eta\Lambda^{-2}) = O'(\Lambda^{-1}). \tag{3.53}$$

Analogously differentiating (3.23')–(3.23') in z we obtain

$$q_z = q_{1z} = O(\tau\Lambda^{-1}) + O(\tau\eta\Lambda^{-2}), \quad t_z - 1 = t_{1z} = O(\tau\Lambda^{-1}) + O(\tau^2\eta\Lambda^{-2}). \tag{3.54}$$

Also analogously to (3.33'),

$$|s_z| + |p_z| \leqq C(\Lambda^{-1} + |\eta|\Lambda^{-2}) = O'(\Lambda^{-1}). \tag{3.55}$$

For the inverse functions (3.35) we have from (3.28')

$$|y(\tau, s, t, \xi, \eta) - s| + |z - t - 2\xi\Lambda^{-1}\tau| \leqq C(\tau + |\eta|\Lambda^{-1}\tau) = C' \tag{3.56}$$

and

$$|y_s| \leqq C, \quad |y_t| \leqq C(\Lambda^{-1} + |\eta|\Lambda^{-2}), \tag{3.57}$$

$$|z_s| + |z_t - 1| \leqq C(\Lambda^{-1} + |\eta|\Lambda^{-2}). \tag{3.58}$$

To prove (3.57), (3.58) one should substitute (3.35) into (3.36) and differentiate in s and t using (3.33'), (3.33''), (3.53), (3.54), (3.55). Now we are ready to estimate the derivatives of L. We have from (3.39),

$$L_s = y_s\eta' + z_s\xi' - 2\int_0^\tau p(p_y y_s + p_z z_s)d\tau' - 2\int_0^\tau q(q_y y_s + q_z z_s)d\tau',$$

$$L_t = y_t\eta' + z_t\xi' - 2\int_0^\tau p(p_y y_t + p_z z_t)d\tau' - 2\int_0^\tau q(q_y y_t + q_z z_t)d\tau'. \tag{3.59}$$

Using (3.52)–(3.58) and taking into account that $|\xi'|\Lambda^{-1} = |\xi|\Lambda^{-2} \leqq 1$ we obtain

$$L_s = O'(1), \quad L_t = \xi' + O'(1). \tag{3.60}$$

Differentiating the system (3.23')–(3.26') successively in y and z we obtain that higher derivatives of s, p, t_1, q_1 satisfy estimates analogous to (3.52)–(3.58). Therefore we shall have that

$$\left|\frac{\partial^{\alpha+\beta}L}{\partial s^\alpha \partial t^\beta}\right| \leqq C'_{\alpha\beta}, \quad \forall\alpha, \quad \forall\beta, \tag{3.61}$$

where $C'_{\alpha\beta}$ are independent of ξ, η. In particular ΛL is bounded. Therefore (3.46) implies that

$$|a_0| \leqq C', \quad |\Lambda a_0| \leqq C', \tag{3.62}$$

and (3.47), (3.47′) gives

$$|a_1| \leqq C'\tau\Lambda^{-1}. \tag{3.63}$$

Analogously one can estimate $a_k, k \geqq 2$, to complete the proof of (3.48), (3.49). ■

Now we shall estimate derivatives of L in ξ and η. Substituting in (3.39) $\tau = x_0\Lambda, q = \xi\Lambda^{-1} + q_1, z = t + 2\xi\Lambda^{-1}\tau + z_1$, we obtain

$$\begin{aligned}\Lambda L &= (y - s')\eta + (t + 2\xi\Lambda^{-1}\tau + z_1 - t')\xi - \int_0^\tau \Lambda p^2 d\tau' - \int_0^\tau \Lambda(\xi\Lambda^{-1} + q_1)^2 d\tau' \\ &= \xi^2 x_0 + (t - t')\xi + (y - s')\eta + z_1\xi - \int_0^\tau \Lambda p^2 d\tau' - \int_0^\tau (2\xi q_1 + \Lambda q_1^2) d\tau'.\end{aligned} \tag{3.64}$$

It follows from (3.51) that

$$z_1 = O(\Lambda^{-1}\tau) + O(\eta\Lambda^{-2}, \tau^2) = O'(\Lambda^{-1}). \tag{3.65}$$

Note that

$$\Lambda_\xi = \tfrac{1}{2}\xi\Lambda^{-3} = O(\Lambda^{-1}), \quad \Lambda_\eta = \varepsilon_0^4\eta^3\Lambda^{-3} = O(\varepsilon_0), \tag{3.66}$$

$$\left|\frac{\partial^{\alpha+\beta}\Lambda}{\partial\xi^\alpha\partial\eta^\beta}\right| \leqq C_{\alpha\beta}\Lambda^{1-2\alpha-\beta}, \quad \forall\alpha \geqq 0, \quad \forall\beta \geqq 0. \tag{3.66′}$$

Changing in (3.23′)–(3.26′) τ to $x_0\Lambda$, multiplying by Λ and then differentiating in η we obtain

$$\frac{ds_\eta}{dx_0} = -\Lambda p_\eta - \Lambda_\eta p - 2(\hat{A}'_{3s}s_\eta + \hat{A}'_{3t}t_{1\eta}), \tag{3.67}$$

$$\begin{aligned}\frac{dp_\eta}{dx_0} &= 2\xi\Lambda^{-1}A_{\delta_0 s}(s)s_\eta + 2(\xi\Lambda^{-1})_\eta A_{\delta_0}(s) + 2A_{\delta_0 s}q_{1\eta} + 2\hat{A}'_3 p_\eta \\ &\quad + 2q_1 A_{\delta_0 ss}(s)s_\eta + 2p(\hat{A}'_{3ss}s_\eta + \hat{A}'_{3st}t_{1\eta}),\end{aligned} \tag{3.68}$$

$$\frac{dt_{1\eta}}{dx_0} = -2\Lambda q_{1\eta} - 2\Lambda_\eta q_1 - 2A_{\delta_0 s}(s)s_\eta, \tag{3.69}$$

$$\frac{dq_{1\eta}}{\mathrm{d}x_0} = 2p_\eta\hat{A}3t + 2p(\hat{A}'_{3ts}s_\eta + \hat{A}'_{3tt}t_{1\eta}) \tag{3.70}$$

with the initial data $s_\eta = t_{1\eta} = q_{1\eta} = 0$ for $\tau = 0$ and $p_\eta(0, y, z, \xi, \eta) = \partial/\partial\eta(\eta\Lambda^{-1}) = O(\Lambda^{-1})$. Note that $t = z - 2\xi x_0 + t_1$. Note that (3.67)–(3.70) is a linear system with respect to $s_\eta, p_\eta, t_{1\eta}, q_\eta$ with coefficients bounded by $C\Lambda$, since $|\eta|\Lambda^{-1} \leqq \varepsilon_0^{-1} \leqq \Lambda$. Moreover since $\Lambda_\eta = O(\varepsilon_0)$ and $\eta\Lambda^{-1} = O(\varepsilon_0^{-1})$ we have that $\Lambda_\eta q_1 = O(1), \Lambda_\eta p = O(1)$. Therefore all nonhomogeneous terms in (3.67)–(3.70), i.e. terms that does not contain $s_\eta, p_\eta, t_{1\eta}, q_{1\eta}$ have order $O(1)$. Therefore we obtain analogously to (3.33),

$$|s_\eta|+|p_\eta|+|t_{1\eta}|+|q_{1\eta}|\leqq (C\Lambda^{-1}+Cx_0)\exp C\Lambda x_0\leqq C\Lambda^{-1}\exp C\tau\leqq C_1\Lambda^{-1}, \tag{3.71}$$

where C_1 is independent of ξ,η and ε_0.

Using (3.71) we obtain from (3.70).

$$|q_{1\eta}|\leqq Cx_0\Lambda^{-1}+C|p|x_0\Lambda^{-1}\leqq C\tau\Lambda^{-2}+C\tau|\eta|\Lambda^{-3}. \tag{3.72}$$

Also (3.72), (3.50) and (3.69) give

$$|t_{1\eta}|\leqq Cx_0(\tau\Lambda^{-1}+\tau|\eta|\Lambda^{-2})+Cx_0\Lambda^{-1}\leqq C\tau\Lambda^{-2}+C\tau^2|\eta|\Lambda^{-3}. \tag{3.73}$$

Analogously changing in (3.23′)–(3.26′) τ to $x_0\Lambda$ multiplying by Λ and differentiating in ξ we shall obtain a linear system with respect to $s_\xi, p_\xi, t_{1\xi}, q_{1\xi}$ with the initial data $0, O(\eta\Lambda^{-3}), 0, 0$ respectively. Analogously to the estimates of the solutions of (3.67)–(3.70) we obtain

$$\begin{aligned}|s_\xi|+|p_\xi|+|t_{1\xi}|+|q_{1\xi}|&\leqq C|\eta|\Lambda^{-3}+Cx_0(|\eta|\Lambda^{-2}+\Lambda^{-1})\leqq C\Lambda^{-2}+C|\eta|\Lambda^{-3}\\&\leqq C(1+\varepsilon_0^{-1})\Lambda^{-2}\leqq C'\Lambda^{-2},\end{aligned} \tag{3.74}$$

where C' means a constant independent of ξ and η but dependent on ε_0.

Next we shall estimate $\partial y/\partial\xi$, $\partial y/\partial\eta$, $\partial z/\partial\xi$, $\partial z/\partial\eta$, where y, z are the same as in (3.35) with $\tau=x_0\Lambda$. Substituting in (3.35) $\tau=x_0\Lambda$ and the functions (3.36) we obtain differentiating in η:

$$0=\frac{\partial z}{\partial\eta}+z_s\frac{\partial s}{\partial\eta}+z_t\frac{\partial t}{\partial\eta}.$$

Therefore using (3.71), (3.73) and (3.58) we obtain

$$\frac{\partial z}{\partial\eta}=O(\Lambda^{-2})+O(\eta\Lambda^{-3}). \tag{3.75}$$

Analogously

$$\frac{\partial z_1}{\partial\xi}=O(\Lambda^{-3})+O'(\Lambda^{-1})(O'(\Lambda^{-2})+O(\Lambda^{-1}))=O'(\Lambda^{-2}). \tag{3.75′}$$

$$y_\eta=O(\Lambda^{-1}),\quad y_\xi=O(\Lambda^{-2})+O(|\eta|\Lambda^{-3})=O'(\Lambda^{-2}). \tag{3.76}$$

Denote by $\chi(\tau)$ a C_0^∞ function such that

$$\chi(\tau)=0\quad\text{for}\quad|\tau|>\tau_+,\quad \chi(\tau)=1\quad\text{for}\quad|\tau|<\tau_-. \tag{3.77}$$

Lemma 3.2. *Given m one can choose parameters $\tau_+,\tau_-,\varepsilon_0,\tau_0$ such that*

$$\left|\frac{\partial}{\partial\xi}(\Lambda L)\right|+\left|\frac{\partial}{\partial\eta}(\Lambda L)\right|\geqq C_0>0\quad\text{on}\quad\operatorname{supp}\chi'(x_0\Lambda), \tag{3.78}$$

for N sufficiently large, where

$$x_0=\frac{\tau_0}{N_1},\quad \Lambda=(\xi^2+\varepsilon_0^4\eta^4+\varepsilon_0^{-8})^{1/4},\quad N_1=N|d_0|+m\frac{d_0}{|d_0|}\cdot d^{(0)}.$$

Proof. It follows from (3.64) and estimates (3.74), (3.75′) and (3.76) that

$$
\begin{aligned}
(\Lambda L)_\xi &= 2\xi x_0 + (t-t') + y_\xi \eta + z_1 + z_{1\xi}\xi - \Lambda^2 \int_0^{x_0} 2p(p_\xi + p_y y_\xi + p_z z_\xi) dx_0' \\
&\quad - (\Lambda^2)_\xi \int_0^{x_0} p^2 dx_0' - 2(\xi\Lambda)_\xi \int_0^{x_0} q_1 dx_0' - 2\xi\Lambda \int_0^{x_0} (q_{1\xi} + q_{1y} y_\xi + q_{1z} z_\xi) dx_0' \\
&\quad - (\Lambda^2)_\xi \int_0^{x_0} q_1^2 dx_0' - \Lambda^2 \int_0^{x_0} 2q_1(q_{1\xi} + q_{1y} y_\xi + q_{1z} z_\xi) dx_0' \\
&= 2\xi x_0 + (t-t') + O'(\Lambda^{-1}) + O'(1). \qquad (3.79)
\end{aligned}
$$

Note that

$$|t-t'| \leqq N_1 + O(1). \qquad (3.80)$$

Consider the region where

$$|\xi| > \varepsilon_0 \Lambda^2. \qquad (3.81)$$

Then taking into account that $x_0\Lambda > \tau_-$ on $\operatorname{supp} \chi'(x_0\Lambda)$ we obtain

$$|(\Lambda L)_\xi| \geqq 2|\xi| x_0 - N_1 - O'(1) > \frac{2|\xi|}{\Lambda}\tau_- - N_1 - O'(1) > 2\varepsilon_0\tau_-\Lambda - N_1 - O'(1). \qquad (3.82)$$

Since $x_0 = \tau_0/N_1$ we have that

$$N_1 = \frac{\tau_0}{x_0} < \frac{\tau_0\Lambda}{\tau_-} \quad \text{on} \quad \operatorname{supp}\chi'(x_0\Lambda). \qquad (3.83)$$

Therefore choosing τ_0 such that

$$\tau_0 < 2\varepsilon_0 \tau_-^2, \qquad (3.84)$$

we obtain for N_1 large enough

$$|(\Lambda L)_\xi \geqq 2\varepsilon_0\tau_-\Lambda - \frac{\tau_0\Lambda}{\tau_-} - O'(1) = \frac{(2\varepsilon_0\tau_-^2 - \tau_0)}{\tau_-}\Lambda - O'(1) \geqq C\Lambda. \qquad (3.85)$$

Note that $x_0\Lambda = (\tau_0\Lambda/N_1) > \tau_-$. Therefore $\Lambda \to \infty$ as $N_1 \to \infty$.

Now consider the region where $|\xi| < \varepsilon_0\Lambda^2$. We have that

$$\Lambda^4 = \xi^2 + \varepsilon_0^4\eta^4 + \varepsilon_0^{-8} < \varepsilon_0^2\Lambda^4 + \varepsilon_0^4\eta^4 + \varepsilon_0^{-8}.$$

Therefore

$$|\eta| \geqq \frac{1}{2\varepsilon_0}\Lambda \qquad (3.86)$$

in the region $|\xi| < \varepsilon_0\Lambda^2$ assuming that $\varepsilon_0 < 1/2$ and N_1 is large. Analogously to (3.27) denote

$$p_1 = p(\tau, y, z, \xi, \eta) - \eta\Lambda^{-1}, \qquad (3.87)$$

$$s_1 = s(\tau, y, z, \xi, \eta) - y + 2\eta\Lambda^{-1}\tau. \qquad (3.87')$$

Then the system (3.23')–(3.26') will have the following form:

$$\frac{ds_1}{d\tau} = -2p_1 - 2\Lambda^{-1}\hat{A}_3'(y - 2\eta\Lambda^{-1}\tau + s_1, z - 2\xi\Lambda^{-1}\tau + t_1), \tag{3.23''}$$

$$\frac{dp_1}{d\tau} = 2\xi\Lambda^{-2}A_{\delta_0 s}(s) + 2\Lambda^{-1}A_{\delta_0 s}(s)q_1 + 2\Lambda^{-1}\hat{A}_{3s}'(s,t)p_1 + 2\eta\Lambda^{-2}\hat{A}_{3s}'(s,t), \tag{3.24''}$$

$$\frac{dt_1}{d\tau} = -2q_1 - 2\Lambda^{-1}A_{\delta_0}(s), \tag{3.25''}$$

$$\frac{dq_1}{d\tau} = 2\Lambda^{-1}\hat{A}_{3t}'p_1 + 2\eta\Lambda^{-2}\hat{A}_{3t}', \tag{3.26''}$$

with zero initial conditions for $\tau = 0$.

It follows from (3.23″)–(3.26″) that

$$|s_1| + |p_1| \leqq |C|\xi|\Lambda^{-2} + C|\eta|\Lambda^{-2} \leqq C\varepsilon_0, \tag{3.88}$$

since we consider the region where $|\xi| \leqq \varepsilon_0\Lambda^2$ and $|\eta|\Lambda^{-2} \leqq \varepsilon_0^{-1}\Lambda^{-1} \leqq \varepsilon_0$. Analogously to (3.52), (3.53), (3.54), (3.57), (3.58) we have

$$|s_{1y}| + p_{1y}| \leqq C\varepsilon_0, \tag{3.89}$$

$$y = s - 2\eta\Lambda^{-1}\tau + y_1(\tau, s, t, \xi, \eta), \quad |y_1| \leqq C\varepsilon_0, \tag{3.90}$$

$$|y_{1s}| \leqq C\varepsilon_0. \tag{3.90'}$$

Finally analogously to (3.71) we have

$$|s_{1\eta}| + |p_{1\eta}| \leqq C\varepsilon_0\Lambda^{-1} \tag{3.91}$$

$$|y_{1\eta}| \leqq C\varepsilon_0\Lambda^{-1}. \tag{3.91'}$$

It follows from (3.64) that

$$\begin{aligned}\Lambda L &= \xi^2 x_0 + (t - t')\xi + (s + 2\eta\Lambda^{-1}\tau + y_1 - s')\eta + z_1\xi \\ &\quad - \int_0^\tau \Lambda(\eta\Lambda^{-1} + p_1)^2 d\tau' - \int_0^\tau (2\xi q_1 + \Lambda q_1^2) d\tau' \\ &= \xi^2 x_0 + (t - t')\xi + \eta^2 x_0 + (s - s')\eta + y_1\eta + z_1\xi \\ &\quad - 2\int_0^{x_0} \eta\Lambda p_1 dx_0' - \Lambda^2 \int_0^{x_0} p_1^2 dx_0' - 2\int_0^{x_0} \xi\Lambda q_1 dx_0' - \Lambda^2 \int_0^{x_0} q_1^2 dx_0'.\end{aligned} \tag{3.92}$$

Therefore

$$\begin{aligned}(\Lambda L)_\eta &= 2\eta x_0 + (s - s') + y_1 + y_{1\eta}\eta + z_{1\eta}\xi - 2(\eta\Lambda)_\eta \int_0^{x_0} p_1 dx_0' \\ &\quad - 2\eta\Lambda \int_0^{x_0} (p_{1\eta} + p_{1y}y_\eta + p_{1z}z_\eta) dx_0' - (\Lambda^2)_\eta \int_0^{x_0} p_1^2 dx_0' \\ &\quad - \Lambda^2 \int_0^{x_0} 2p_1(p_{1\eta} + p_{1y}y_\eta + p_{1z}z_\eta) dx_0' - 2(\xi\Lambda)_\eta \int_0^{x_0} q_1 dx_0' \\ &\quad - 2\xi\Lambda \int_0^{x_0} (q_{1\eta} + q_{1y}y_\eta + q_{1z}z_\eta) dx_0' - (\Lambda^2)_\eta \int_0^{x_0} q_1^2 dx_0' \\ &\quad - \Lambda^2 \int_0^{x_0} 2q_1(q_{1\eta} + q_{1y}y_\eta + q_{1z}z_\eta) dx_0'.\end{aligned} \tag{3.93}$$

It follows from the estimates (3.88)–(3.91′), (3.72)–(3.76), and since $|\xi| \leqq \varepsilon_0 \Lambda^2$ that

$$(\Lambda L)_\eta = 2\eta x_0 + (s - s') + O(1), \tag{3.94}$$

where $|O(1)| < C$, C is independent of ξ, η and ε_0. Therefore in the region $|\xi|\Lambda^{-2} \leqq \varepsilon_0$ we have

$$|(\Lambda L)_\eta \geqq 2|\eta|x_0 - |s - s'| - O(1) \geqq \frac{2|\eta|\tau_-}{\Lambda} - m' - O(1) \geqq \frac{\tau_-}{\varepsilon_0} - m' - O(1). \tag{3.95}$$

We assume that ε_0 is such that

$$\frac{\tau_-}{\varepsilon_0} - m' - O(1) > 1. \tag{3.96}$$

Therefore $|(\Lambda L)_\eta| > 1$ in the region $|\xi| \leqq \varepsilon_0 \Lambda^{-2}$. ∎

Successively differentiating (3.23′)–(3.26′) with respect to ξ and η we can estimate higher derivatives of s, t, y, z and therefore the higher derivatives of L. We shall have that L and $a_k, k \geqq 0$, satisfy estimates of the form $|\partial^{\alpha+\beta}/(\partial\xi^\alpha \partial\eta^\beta) b| \leqq C'_{\alpha\beta}\Lambda^{m-\alpha-\beta}$, $\alpha \geqq 0, \beta \geqq 0$. Therefore the differentiation in ξ and η decreases the order with respect to Λ.

Now we are ready to construct the Green function $\hat{G}(s, t, s', t', x_0)$ of $(i\partial/\partial x_0 - \hat{H})$. Denote

$$\hat{G}_{\mu 1}(s, t, s', t' x_0) = e^{i\hat{\gamma}(s,t)} \hat{g}_{\mu 1}(s, t, s', t', x_0), \tag{3.97}$$

where

$$\hat{g}_{\mu 1}(s, t, s', t', x_0) = \frac{e^{-i\hat{\gamma}(s',t')}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(x_0 \Lambda) e^{-i\Lambda L}(a_0 + a_1 + \cdots + a_\mu) d\xi d\eta, \tag{3.97′}$$

where L is the same as in (3.39) and $a_k, k \geqq 0$, satisfy the transport equations (3.43), (3.43′), (3.47), (3.47′) and etc. Note that

$$\begin{aligned} \hat{g}_{\mu 1}(s, t, s', t', 0) &= \frac{e^{-i\hat{\gamma}(s',t')}}{(2\pi)^2} \iint e^{-i(s-s')\eta - i(t-t')\xi} d\xi d\eta \\ &= e^{-i\hat{\gamma}(s',t')} \delta(s - s') \delta(t - t'). \end{aligned} \tag{3.98}$$

Also we have

$$\left(i\frac{\partial}{\partial x_0} - \hat{H}_2\right)\hat{g}_{\mu 1} = \frac{e^{-i\hat{\gamma}(s',t')}}{(2\pi)^2} \iint (r_{\mu_1} + r_{\mu_2}) e^{-i\Lambda L} d\xi d\eta, \tag{3.99}$$

where

$$r_{\mu 1} = i\Lambda\chi'(x_0\Lambda)(a_0 + a_1 + \cdots + a_\mu), \tag{3.100}$$

$$r_{\mu 2} = O(\tau^\mu \Lambda^{-\mu}), \quad \mu \text{ is large.} \tag{3.100′}$$

Since $|(\Lambda L)_\xi + |(\Lambda L)_\eta| \geqq C_0$ on $\operatorname{supp} \chi'(x_0\Lambda)$ we obtain repeatedly integrating by parts in ξ and η and using that the differentiation in ξ and η decreases the order in Λ:

$$\int r_{\mu 1} e^{-iL} d\xi d\eta = \int r_{\mu 3} e^{-iL} d\xi d\eta, \tag{3.101}$$

where

$$|r_{\mu 3}| \leqq C_\mu \Lambda^{-2\mu}. \tag{3.102}$$

Since $x_0\Lambda \geqq \tau_-$ on supp r_3 we have that

$$|r_{\mu 3}| \leqq C_\mu \Lambda^{-2\mu} \leqq \frac{C_\mu}{\tau_-^\mu} x_0^\mu \Lambda^{-\mu}. \tag{3.102}$$

Note thåt $r_{\mu 1}, r_{\mu 2}, r_{\mu 3}$ and $\Lambda L-(s-s')\eta-(t-t')\xi$ are periodic in (s,t) and independent of s', t'. We shall find

$$g_{\mu 2} = \frac{e^{-i\hat{\gamma}(s',t')}}{(2\pi)^2} \iint r_{\mu 4} e^{-i(s-s')\eta - i(t-t')\xi - ix_0(\xi^2+\eta^2)} d\xi d\eta,$$

such that $r_{\mu 4}$ is periodic in (s,t) and

$$\left(i\frac{\partial}{\partial x_0} - \hat{H}_2\right) g_{\mu 2} = -\frac{e^{-i\hat{\gamma}(s',t')}}{(2\pi)^2} \iint (r_{\mu 2}+r_{\mu 3}) e^{-i\Lambda L} d\xi d\eta, \tag{3.103}$$

$$g_{\mu 2}(s,t,s',t',0) = 0. \tag{3.103'}$$

Then $r_{\mu 4}(s,t,\xi,\eta,x_0)$ will satisfy

$$\begin{aligned} &i\frac{\partial r_{\mu 4}}{\partial x_0} - 2i\eta\frac{\partial r_{\mu 4}}{\partial s} - 2i\xi\frac{\partial r_{\mu 4}}{\partial t} - 2iA_{\delta_0}(s)\frac{\partial r_{\mu 4}}{\partial t} - 2i\hat{A}_3'\frac{\partial r_{\mu 4}}{\partial s} \\ &\quad + \Delta r_{\mu 4} - (\hat{C} + 2A_{\delta_0}(s)\xi + 2\hat{A}_3'\eta) r_{\mu 4} = -(r_{\mu 3}+r_{\mu 2}) e^{-i\Lambda L + iL_0}, \end{aligned} \tag{3.104}$$

$$r_{\mu 4}(s,t,\xi,\eta,0) = 0, \tag{3.104'}$$

where $L_0 = x_0(\xi^2+\eta^2) + (s-s')\eta + (t-t')\xi$.

Denote $\hat{H}_3 = \hat{H}_2 + 2i\eta(\partial/\partial s) + 2i\xi(\partial/\partial t) - 2A_{\delta_0}(s)\xi - 2\hat{A}_3'\eta$, where $\hat{H}_2$ is the same as in (3.103) or (3.14). Since $\hat{H}_3$ is self-adjoint we obtain taking the imaginary part of the scalar product of (3.104) with $r_{\mu 4}$ and then integrating in x_0:

$$\|r_{\mu 4}(x_0)\|^2 \leqq \int_0^{x_0} \|(r_{\mu 2}+r_{\mu 3}) e^{-i\Lambda L + iL_0}\| \, \|r_{\mu 4}(y_0)\| dy_0, \tag{3.105}$$

where $\|v\|$ is the L_2-norm over $\hat{T}^2$ (see (3.10)). Therefore

$$\begin{aligned} \max_{0 \leqq y_0 \leqq x_0} \|r_{\mu 4}(y_0)\| &\leqq \int_0^{x_0} \|(r_{\mu 2}+r_{\mu 3}) e^{-i\Lambda L + iL_0}\| dy_0 \\ &\leqq x_0 \max_{0 \leqq y_0 \leqq x_0} \|(r_{\mu 3}+r_{\mu 2}) e^{-i\Lambda L + iL_0}\|_0. \end{aligned} \tag{3.106}$$

To prove the existence of *a* periodic $r_{\mu 4}$ one should replace $\hat{H}_3$ by $\hat{H}_3 + i\varepsilon\Delta$. Then there exists periodic $r_{\mu 4\varepsilon}$ such that

$$\left(i\frac{\partial}{\partial x_0} - \hat{H}_3 - i\varepsilon\Delta\right) r_{\mu 4\varepsilon} = -(r_{\mu 2}+r_{\mu 3}) e^{i\Lambda L + iL_0}, \tag{3.107}$$

$$r_{\mu 4\varepsilon | x_0 = 0} = 0, \tag{3.107'}$$

since (3.107) is a parabolic equation. Moreover the estimate (3.106) holds for $r_{\mu 4\varepsilon}$, $\forall \varepsilon > 0$.

Taking a weak limit as $\varepsilon\to 0$ we prove the existence of $r_{\mu 4}$. It follows from (3.100′), (3.102′) that

$$\|(r_{\mu 2}+r_{\mu 3})e^{-i\Lambda L+iL_0}\| = \|r_{\mu 2}+r_{\mu 3}\| \leqq C\tau^{\mu}\Lambda^{-\mu} \leqq Cx_0^{\mu/2}\Lambda^{-\mu/2}. \tag{3.108}$$

Therefore

$$\hat{G}_{\mu 1}+\hat{G}_{\mu 2} = e^{i\hat{\gamma}(s,t)}(\hat{g}_{\mu 1}+\hat{g}_{\mu 2}) \tag{3.109}$$

is the Green function satisfying (3.1), (3.2).

4. The Asymptotics of the Trace of the Green Function

In this section we shall find the asymptotics as $N\to\infty$ of the trace

$$\iint_{T^2} G\left(x+Nd_0+md^{(0)},x,\frac{\tau_0}{N_1}\right)dx = \iint_{\hat{T}^2}\hat{G}\left(s+m|\delta_0|^{-1},t+N_1,s,t,\frac{\tau_0}{N_1}\right)ds\,dt, \tag{4.1}$$

where G and $\hat{G}$ are the same as in (3.10), $N_1=N|d_0|+m(d_0/|d_0|)\cdot d^{(0)}, N\to\infty, m$ and τ_0 are fixed. We have $\hat{G}=\hat{G}_{\mu 1}+\hat{G}_{\mu 2}$, where $\hat{G}_{\mu 1},\hat{G}_{\mu 2}$ are the same as in (3.109). Using the Cauchy–Schwartz inequality and (3.108) we obtain

$$\begin{aligned}
&\left|\iint_{\hat{T}^2}\hat{G}_{\mu 2}\left(s+m|\delta_0|^{-1},t+N_1,s,t,\frac{\tau_0}{N_1}\right)ds\,dt\right| \\
&\leqq \iint_{\hat{T}^2}|\hat{G}_{\mu 2}|\,ds\,dt \leqq \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\iint_{\hat{T}^2}|r_{\mu 4}|\,ds\,dt\,d\xi\,d\eta \\
&\leqq \frac{|\hat{T}^2|}{(2\pi)^2}\iint_{-\infty}^{\infty}\|r_{\mu 4}\|\,d\xi\,d\eta \\
&\leqq \frac{|\hat{T}^2|}{(2\pi)^2}\iint_{-\infty}^{\infty}Cx_0^{\mu/2+1}\Lambda^{-\mu/2}d\xi\,d\eta \leqq C_1N_1^{-\mu/2-1},
\end{aligned} \tag{4.2}$$

since $x_0=\tau_0/N_1$. In (4.2) $|\hat{T}^2|=|d_0||\delta_0|^{-1}$ is the area of $\hat{T}^2$. Now we shall consider

$$\begin{aligned}
&\iint_{\hat{T}^2}\hat{G}_{\mu 1}\left(s+m|\delta_0|^{-1},t+N_1,s,t,\frac{\tau_0}{N_1}\right)ds\,dt \\
&= \sum_{k=0}^{\mu}\iint_{\hat{T}^2}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\chi(x_0\Lambda)e^{-i\Lambda L}a_k(x_0\Lambda,s,t,\xi,\eta)\,d\xi\,d\eta\,ds\,dt.
\end{aligned}$$

Changing the order of integration and making the change of variables (3.36) we obtain

$$\begin{aligned}
&\iint_{\hat{T}^2}\hat{G}_{\mu 1}\left(s+m|\delta_0|^{-1},t+N_1,s,t,\frac{\tau_0}{N_1}\right)ds\,dt \\
&= \sum_{k=0}^{\mu}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\left(\iint_{\hat{T}_1^2}\chi(x_0\Lambda)e^{-i\Lambda\hat{L}}\hat{a}_k(x_0\Lambda,y,z,\xi,\eta)\left|\frac{\mathscr{D}(s,t)}{\mathscr{D}(y,z)}\right|dy\,dz\right)d\xi\,d\eta,
\end{aligned} \tag{4.3}$$

where $\hat{L}$, $\hat{a}_k$ are the functions L, a_k in the new coordinates and $\hat{T}_1^2$ is the image of $\hat{T}^2$ under the map (3.36). It follows from (3.40) that $\hat{T}_1^2$ is a fundamental domain with respect to the lattice $L: \hat{T}_1 = \mathbf{R}^2/L$. Therefore we can replace $\hat{T}_1^2$ by $\hat{T}^2$ since all functions in (4.3) are periodic with respect to L.

As usually we consider the integrals in ξ and η as the ascillatory integrals, i.e. one should introduce a cutoff function $\chi(\varepsilon\sqrt{\xi^2+\eta^2})$ and consider the weak limit as $\varepsilon \to 0$. We have (see (3.39)):

$$\hat{L}(\tau, y, z, \xi, \eta) = (y + m' - s(\tau, y, z, \xi, \eta))\eta' + (z + N_1 - t(\tau, y, z, \xi, \eta))\xi'$$
$$- \int_0^\tau p^2(\tau', y, z, \xi, \eta)dt' - \int_0^\tau q^2(\tau', y, z, \xi, \eta)d\tau', \tag{4.4}$$

where $\xi' = \xi\Lambda^{-1}, \eta' = \eta\Lambda^{-1}, \tau = x_0\Lambda, m' = m|\delta_0|^{-1}$, $\Lambda = (\xi^2 + \varepsilon_0^4\eta^4 + \varepsilon_0^{-8})^{1/4}$, $x_0 = \tau_0/N_1$. Using (3.64) and that $t_1 = -2\int_0^\tau q_1(\tau', y, z, \xi, \eta)d\tau' - 2\Lambda^{-1}\int_0^\tau A_{\delta_0}(s)d\tau'$ (see (3.25′)) we obtain:

$$\Lambda\hat{L}(\tau, y, z, \xi, \eta) = \xi^2 x_0 + N_1\xi + (y + m' - s(x_0\Lambda, y, z, \xi, \eta))\eta$$
$$- \xi t_1(x_0\Lambda, y, z, \xi, \eta) - \Lambda^2 \int_0^{x_0} p^2(x_0'\Lambda, y, z, \xi, \eta)dx_0'$$
$$- 2\xi\Lambda \int_0^{x_0} q_1(x_0'\Lambda, y, z, \xi, \eta)dx_0' - \Lambda^2 \int_0^{x_0} q_1^2 dx_0'$$
$$= \xi^2 x_0 + N_1\xi + (y + m' - s)\eta + 2\xi \int_0^{x_0} A_{\delta_0}(s)dx_0'$$
$$- \Lambda^2 \int_0^{x_0} p^2(x_0'\Lambda, y, z, \xi, \eta)dx_0' - \Lambda^2 \int_0^{x_0} q_1^2 dx_0', \tag{4.5}$$

where $q = \xi\Lambda^{-1} + q_1, t = z - 2\xi x_0 + t_1$. Analogously to (3.79) we have

$$(\Lambda\hat{L})_\xi = 2\xi x_0 + N_1 + O'(\Lambda^{-1}), \tag{4.6}$$
$$(\Lambda\hat{L})_{\xi\xi} = 2x_0 + O'(\Lambda^{-2}), \tag{4.7}$$

where $O'(\Lambda^{-k})$, $k = 1, 2$, means $|O'(\Lambda^{-k})| \leqq C'\Lambda^{-k}$, C' is independent of ξ, η and N_1. It follows from (4.6) that $|(\Lambda\hat{L})_\xi| \geqq C$ unless

$$-\frac{N_1 + C'}{2x_0} < \xi < -\frac{(N_1 - C')}{2x_0}. \tag{4.8}$$

For ξ satisfying (4.8) we have

$$(\Lambda L\hat{\Lambda})_{\xi\xi} = 2x_0(1 + O'(N_1^{-1})), \tag{4.9}$$

since $\Lambda^2 \geqq |\xi| \geqq CN_1/x_0$ when ξ belongs to the region (4.8).

Since $|(\Lambda\hat{L})_\xi| \geqq C$ outside of the region (4.8) we have integrating by parts in ξ that the contribution of the complement to the region (4.8) is $O(N_1^{-\mu})$, $\forall\mu > 0$. Therefore the main contribution as $N_1 \to \infty$ will come from ξ satisfying (4.8). We shall find a more precise asymptotics of $\Lambda\hat{L}$ and its derivatives assuming that ξ satisfies (4.8) and $N_1 \to \infty$. It follows from (3.26′) that

$$q_1(\tau,y,z,\xi,\eta)=2\Lambda^{-1}\int_0^\tau \hat{A}'_{3t}(s(\tau',y,z,\xi,\eta),z-2\xi\Lambda^{-1}\tau'+t_1(\tau',y,z,\xi,\eta))$$
$$\cdot p(\tau',y,z,\xi,\eta)d\tau'. \tag{4.10}$$

We have

$$\frac{d}{d\tau}\hat{A}'_3(s,t)=\hat{A}'_{3s}\frac{ds}{d\tau}+\hat{A}_{3t}(s,t)\left(-2\xi\Lambda^{-1}+\frac{dt_1}{d\tau}\right). \tag{4.11}$$

Note that (see (3.25′), (3.50)) $dt_1/d\tau=O'(\Lambda^{-1})$. So that

$$-2\xi\Lambda^{-1}+\frac{dt_1}{d\tau}=(-2\xi+O'(1))\Lambda^{-1}. \tag{4.12}$$

Therefore

$$\hat{A}'_{3t}(s,t)=\frac{1}{-2\xi\Lambda^{-1}+\dfrac{dt_1}{d\tau}}\left(\frac{d}{d\tau}\hat{A}'_3(s,t)-\hat{A}'_{3s}\frac{ds}{d\tau}\right). \tag{4.13}$$

Substituting (4.13) into (4.10) and integrating by parts we obtain

$$q_1(\tau,y,z,\xi',\eta')=\int_0^\tau\frac{2}{2\xi-\dfrac{dt_1}{d\tau}\Lambda}\frac{ds}{d\tau}p\hat{A}'_{3s}d\tau'-\int_0^\tau\hat{A}'_3(s,t)\frac{d}{d\tau}\frac{2p}{\left(-2\xi+\dfrac{dt_1}{s\tau}\Lambda\right)}d\tau'$$
$$+\frac{2A'_3(s,t)p}{-2\xi+\dfrac{dt_1}{d\tau}\Lambda}-\frac{2\hat{A}'_3(y,z)\eta\Lambda^{-1}}{-2\xi-2A_{\delta_0}(y)}. \tag{4.14}$$

It follows from (4.14) and the estimates of Sect. 3 that

$$q_1(\tau,y,z,\xi\eta)=O'\left(\frac{1}{\xi}\right), \tag{4.15}$$

$$q_{1\xi}(x_0\Lambda,y,z,\xi,\eta)=O'(\xi^{-1}\Lambda^{-1}), \tag{4.16}$$

$$q_{1\eta}(x_0\Lambda,y,z,\xi,\eta)=O'(\xi^{-1}\Lambda^{-1}). \tag{4.17}$$

Substituting (4.15) into (3.25′) we get

$$t_1=-2\int_0^\tau q_1d\tau'-2\Lambda^{-1}\int_0^\tau A_{\delta_0}(s)d\tau'=-2\Lambda^{-1}\int_0^\tau A_{\delta_0}(s)d\tau'+O(\xi^{-1}), \tag{4.18}$$

$$|t_{1\xi}|=O(\xi^{-1}\Lambda^{-1}). \tag{4.19}$$

Also substituting (4.15) onto (3.23′) and (3.24′) we obtain

$$s=y-2\int_0^t p(\tau',y,z,\xi,\eta)d\tau'-2\Lambda^{-1}\int_0^\tau\hat{A}'_3(s,z-2\xi\Lambda^{-1}\tau'+t_1)d\tau', \tag{4.20}$$

$$p=\eta\Lambda^{-1}+2\xi\Lambda^{-2}\int_0^\tau A_{\delta_0s}(s)d\tau'+O'(\Lambda^{-1}\xi^{-1})+2\Lambda^{-1}\int_0^\tau\hat{A}'_{3s}(s,t)pd\tau'. \tag{4.21}$$

There exists periodic $\hat{v}'_{11}(s,t)$ such that $\partial/\partial t\,\hat{v}'_{11}=\hat{A}'_3(s,t)$ (see (2.37)). Analogously to (4.11) we obtain

$$\hat{A}'_3 = \frac{\partial \hat{v}'_{11}}{\partial t} = \left(\frac{dt}{d\tau}\right)^{-1}\left(\frac{d}{d\tau}\hat{v}'_{11} - \frac{\partial \hat{v}'_{11}}{\partial s}\frac{ds}{d\tau}\right), \tag{4.22}$$

where $dt/d\tau = -2\xi\Lambda^{-1} + dt_1/d\tau$. Substituting (4.22) into (4.20) and integrating by parts we get

$$\begin{aligned} s = {} & y - 2\int_0^\tau p(\tau', y, z, z, \xi, \eta)d\tau' + 2\Lambda^{-1}\int_0^\tau \left(\frac{dt}{d\tau}\right)^{-1}\frac{ds}{d\tau}\frac{\partial \hat{v}'_{11}(s,t)}{\partial s}d\tau' \\ & + 2\Lambda^{-1}\int_0^\tau \left(\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1}\right)\hat{v}'_{11}d\tau' - 2\Lambda^{-1}\left(\frac{dt}{d\tau}\right)^{-1}\hat{v}'_{11}(s,t) \\ & - 2(2\xi + 2A_{\delta_0}(y))^{-1}\hat{v}'_{11}(y,z) = y - 2\int_0^\tau p\,d\tau' + O'\left(\frac{1}{\xi}\right). \end{aligned} \tag{4.23}$$

Analogously using that $\hat{A}'_{3s} = \partial/\partial t\,\hat{A}'_4(s,t)$, where $\hat{A}'_4(s,t)$ is periodic we obtain

$$\hat{A}_{3s} = \left(\frac{dt}{d\tau}\right)^{-1}\left(\frac{d}{d\tau}\hat{A}'_4 - \hat{A}'_{4s}(s,t)\frac{ds}{d\tau}\right). \tag{4.24}$$

Substituting (4.24) into (4.21) and integrating by parts we have

$$\begin{aligned} p = {} & \eta\Lambda^{-1} + 2\xi\Lambda^{-2}\int_0^\tau A_{\delta_0 s}(s)d\tau' + O'(\Lambda^{-1}\xi^{-1}) \\ & - 2\Lambda^{-1}\int_0^\tau \left(\frac{dt}{d\tau}\right)^{-1}\frac{ds}{d\tau}p\hat{A}'_{4s}(s,t)d\tau' \\ & - 2\Lambda^{-1}\int_0^\tau \frac{d}{d\tau}\left(\left(\frac{dt}{d\tau}\right)^{-1}p\right)\hat{A}'_4(s,t)d\tau' + 2\Lambda^{-1}\left(\frac{dt}{s\tau}\right)^{-1}p\hat{A}'_4(s,t) \\ & - 2(-2\xi - 2A_{\delta_0}(y))^{-1}\eta\Lambda^{-1}\hat{A}'_4(y,z) \\ = {} & \eta\Lambda^{-1} + 2\xi\Lambda^{-2}\int_0^\tau A_{\delta_0 s}(s)d\tau' + O'(\xi^{-1}). \end{aligned} \tag{4.25}$$

In (4.21)–(4.25) we assumed that ξ satisfies (4.8). Denote by $s_0(\tau, y, \eta', \xi\Lambda^{-2})$, $p_0(\tau, y, \eta', \xi\Lambda^{-2})$ the solution of following system:

$$\frac{ds_0}{d\tau} = -2p_0, \quad s_0(0, y, z, \eta', \xi\Lambda^{-2}) = y, \tag{4.26}$$

$$\frac{dp_0}{d\tau} = 2\xi\Lambda^{-2}A_{\delta_0 s}(s_0), \quad p_0(0, y, z, \eta', \xi\Lambda^{-2}) = \eta'. \tag{4.26'}$$

Replacing (4.26), (4.26′) by the equivalent system of integral equations and subtracting from (4.23), (4.25) we obtain

$$s(\tau, y, z, \xi, \eta) = s_0 + O'\left(\frac{1}{\xi}\right), \tag{4.27}$$

$$p(\tau, y, z, \xi, \eta) = p_0 + O'\left(\frac{1}{\xi}\right). \tag{4.27'}$$

Denote

$$\bar{\eta}=|\xi|^{-1/2}\eta,\quad \bar{\tau}=|\xi|^{1/2}\Lambda^{-1}\tau=|\xi|^{1/2}x_0,\quad \bar{p}_0=|\xi|^{-1/2}\Lambda p_0,\quad \bar{s}_0=s_0, \tag{4.28}$$

where ξ belongs to the region (4.8), $\xi<0$. Then $\bar{s}_0,\bar{p}_0$ satisfy the following equations:

$$\frac{d\bar{s}_0}{d\bar{\tau}}=-2\bar{p}_0,\quad \bar{s}_0(0,y,\bar{\eta})=y, \tag{4.29}$$

$$\frac{d\bar{p}_0}{d\bar{\tau}}=-2A_{\delta_0 s}(\bar{s}_0),\quad \bar{p}_0(0,y,\bar{\eta})=\bar{\eta}. \tag{4.29'}$$

Substituting (4.27), (4.27′) into (4.5) and using (4.28) we obtain

$$\begin{aligned}\Lambda\hat{L}(x_0\Lambda,y,z,\xi,\eta)&=\xi^2x_0+N_1\xi+(y+m'-\bar{s}_0(|\xi|^{1/2}x_0,y,\bar{\eta}))|\xi|^{1/2}\bar{\eta}\\&\quad-|\xi|^{1/2}\int_0^{x_0|\xi|^{1/2}}(\bar{p}_0^2(\tau',y,\bar{\eta})+2A_{\delta_0}(\bar{s}_0))d\tau'\\&\quad+O'(\Lambda\xi^{-1})=\xi^2x_0+N_1\xi+|\xi|^{1/2}\hat{S}(\bar{\tau},y,\bar{\eta})\\&\quad+\hat{S}_1(x_0,y,z,\xi,\eta),\end{aligned} \tag{4.30}$$

where

$$\hat{S}(\bar{\tau},y,\bar{\eta})=(y+m'-\bar{s}_0)\bar{\eta}-\int_0^{\bar{\tau}}(\bar{p}_0^2+2A_{\delta_0}(\bar{s}_0))d\tau',\quad \hat{S}_1=O'(\Lambda\xi^{-1}). \tag{4.30'}$$

Remark 4.1. Since $\partial s_0/\partial y\ (\bar{\tau},y,\bar{\eta})\neq 0$ for $0\leqq\bar{\tau}\leqq\bar{\tau}_+,\bar{\tau}_+$ is small, there exists a function $y=y_0(\bar{\tau},s,\bar{\eta})$ inverse to $s_0(\bar{\tau},y,\bar{\eta})$. Analogously to (3.39) we have that the function

$$S(\bar{\tau},s,\bar{\eta})=\hat{S}(\bar{\tau},y_0(\bar{\tau},s,\bar{\eta}),\bar{\eta})$$

is the solution of the eiconal equation

$$S_{\bar{\tau}}-S_s^2+2A_{\delta_0}(s)=0 \tag{4.31}$$

with the initial condition

$$S(0,s,\bar{\eta})=s\bar{\eta}.\quad \blacksquare \tag{4.31'}$$

We shall find the partial derivatives of $\hat{S}(\bar{\tau},y,\bar{\eta})$ in y and $\bar{\eta}$. We have

$$\frac{\partial\hat{S}}{\partial\bar{\eta}}=(y+m'-\bar{s}_0)-\bar{s}_{0\bar{\eta}}\bar{\eta}-\int_0^{\bar{\tau}}(2\bar{p}_0\bar{p}_{0\bar{\eta}}+2A_{\delta_0 s}(\bar{s}_0)\bar{s}_{0\bar{\eta}})d\tau', \tag{4.32}$$

where $\bar{\tau}=x_0|\xi|^{1/2}$.

Using (4.29) and integrating by parts we obtain

$$\begin{aligned}\int_0^{\bar{\tau}}2\bar{p}_0\frac{\partial\bar{p}_0}{\partial\bar{\eta}}d\tau'&=-\int_0^{\bar{\tau}}\bar{p}_0\frac{d}{d\tau'}\frac{\partial\bar{s}_0}{\partial\bar{\eta}}d\tau'=-\bar{p}_0\frac{\partial\bar{s}_0}{\partial\bar{\eta}}\bigg|_0^{\bar{\tau}}\\&\quad+\int_0^{\bar{\tau}}\frac{d\bar{p}_0}{d\tau'}\frac{\partial\bar{s}_0}{\partial\bar{\eta}}d\tau'=-\bar{p}_0\frac{\partial\bar{s}_0}{\partial\bar{\eta}}-2\int_0^{\bar{\tau}}A_{\delta_0 s}(\bar{s}_0)\frac{\partial\bar{s}_0}{\partial\bar{\eta}}d\tau',\end{aligned}$$

since $\partial\bar{s}_0/\partial\bar{\eta}=0$ for $\tau'=0$. Therefore

$$\frac{\partial \hat{S}}{\partial \bar{\eta}} = (y + m' - \bar{s}_0) + (\bar{p}_0 - \bar{\eta})\frac{\partial \bar{s}_0}{\partial \bar{\eta}}. \tag{4.33}$$

Analogously using (4.29′) we obtain

$$\frac{\partial \hat{S}}{\partial y} = \bar{\eta} - \frac{\partial \bar{s}_0}{\partial y}\bar{\eta} - \int_0^{\bar{\tau}} \left(2\bar{p}_0 \frac{\partial \bar{p}_0}{\partial y} + 2A_{\delta_0 s}(\bar{s}_0)\frac{\partial \bar{s}_0}{\partial y} \right) d\tau'$$

$$= \bar{\eta} - \frac{\partial \bar{s}_0}{\partial y}\bar{\eta} + \int_0^{\bar{\tau}} \left(\bar{p}_0 \frac{d}{d\tau}\left(\frac{\partial \bar{s}_0}{\partial y}\right) - 2A_{\delta_0 s}(\bar{s}_0)\frac{\partial \bar{s}_0}{\partial y} \right) d\tau'$$

$$= \bar{\eta} - \frac{\partial \bar{s}_0}{\partial y}\bar{\eta} + \left(\bar{p}_0 \frac{\partial \bar{s}_0}{\partial y} \right)\Bigg|_0^{\tau} = \bar{\eta} - \frac{\partial \bar{s}_0}{\partial y}\bar{\eta} + \bar{p}_0\frac{\partial \bar{s}_0}{\partial y} - \bar{\eta} = (\bar{p}_0 - \bar{\eta})\frac{\partial \bar{s}_0}{\partial y}, \tag{4.34}$$

since $\bar{p}_0(0, y, \bar{\eta}) = \bar{\eta}, \partial \bar{s}_0/\partial y|_{\tau=0} = 1$.

Since $\partial \bar{s}_0/\partial y \neq 0$ for $0 \leqq \bar{\tau} \leqq \bar{\tau}_+, \bar{\tau}_+$ is small, we have that $\partial \hat{S}/\partial y = \partial \hat{S}/\partial \bar{\eta} = 0$ if and only if

$$\bar{p}_0(\bar{\tau}_m, y, \bar{\eta}) = \bar{\eta}, \bar{s}_0(\bar{\tau}_m, y, \bar{\eta}) = y + m|\delta_0|^{-1}. \tag{4.35}$$

If $m = 0$ then $(y, \bar{\eta})$ satisfying (4.35) belongs to a periodic orbit $P_{\bar{\tau}_0}^{(0)}$ of (4.29), (4.29′) with the period $\bar{\tau}_0$. Note that any point of $P_{\bar{\tau}_0}^{(0)}$ will also satisfy (4.35) and therefore it will be a critical point of $\hat{S}(\bar{\tau}_0, y, \bar{\eta})$. When $m \neq 0$ then $(y, \bar{\eta})$ belongs to the trajectory $P_{\bar{\tau}_m}^{(m)}$ that describes the whirling motion of the pendulum

$$\frac{d^2 \bar{s}_0}{d\bar{\tau}^2} - 4A_{\delta_0 s}(\bar{s}_0) = 0. \tag{4.36}$$

Also any point of $P_{\bar{\tau}_m}^{(m)}$ will be a critical point of $\hat{S}(\bar{\tau}_m, y, \bar{\eta})$. Note that (4.36) is equaivalent to (4.29), (4.29′) and that

$$P_{\bar{\tau}_m}^{(m)} = P_{\bar{\tau}_1}^{(1)}, \quad \bar{\tau}_m = m\bar{\tau}_1 \quad \text{for any} \quad m \neq 1. \tag{4.37}$$

Since $\partial \hat{S}/\partial y = \partial \hat{S}/\partial \bar{\eta} = 0$ on $P_{\bar{\tau}_m}^{(m)}$, we have that the restriction of $\hat{S}(\bar{\tau}_m, y, \bar{\eta})$ to $P_{\bar{\tau}_m}^{(m)}$ does not depend on $(y, \bar{\eta}) \epsilon P_{\bar{\tau}_m}^{(m)}$ and depends only on $\bar{\tau}_m$. We shall denote this restriction by $S_0(\bar{\tau}_m)$. To use the stationary phase method we shall need to compute $\partial^2 \hat{S}(\bar{\tau}_m, y, \bar{\eta})/\partial \bar{\eta}^2$ on $P_{\bar{\tau}_m}^{(m)}$. It follows from (4.33) that

$$\frac{\partial^2 \hat{S}}{\partial \bar{\eta}^2} = -\frac{\partial \bar{s}_0}{\partial \bar{\eta}} + \left(\frac{\partial \bar{p}_0}{\partial \bar{\eta}} - 1 \right)\frac{\partial \bar{s}_0}{\partial \bar{\eta}} + (\bar{p}_0 - \bar{\eta})\frac{\partial^2 \bar{s}_0}{\partial \bar{\eta}^2} = \left(\frac{\partial \bar{p}_0}{\partial \bar{\eta}} - 2 \right)\frac{\partial \bar{s}_0}{\partial \bar{\eta}}, \tag{4.38}$$

since $\bar{p}_0 - \bar{\eta} = 0$ on $P_{\bar{\tau}_m}^{(m)}$. Let $\bar{\eta} = \psi(\bar{\tau}_m, y)$ be the equation of $P_{\bar{\tau}_m}^{(m)}$, $m \neq 0$. Then differentiating in y the equations

$$\bar{s}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) = y + m|\delta_0|^{-1}, \quad p(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) = \psi(\bar{\tau}_m, y), \tag{4.39}$$

we obtain

$$\bar{s}_{0y} + \bar{s}_{0\bar{\eta}}\psi_y = 1, \quad \bar{p}_{0y} + \bar{p}_{0\bar{\eta}}\psi_y = \psi_y. \tag{4.40}$$

Therefore eliminating ψ_y we get

$$1 - \bar{p}_{0\bar{\eta}} - \bar{s}_{0y} - (\bar{s}_{0\bar{\eta}}\bar{p}_{0y} - \bar{s}_{0y}\bar{p}_{0\bar{\eta}}) = 0. \tag{4.41}$$

Since $(\bar{s}_{0y}, \bar{p}_{0y})$, $(\bar{s}_{0\bar{\eta}}, \bar{p}_{0\bar{\eta}})$ are the solution of the same system

$$\frac{d}{d\bar{\tau}} V_1 = -2V_2, \quad \frac{dV_2}{d\bar{\tau}} = -2A_{\delta_0 ss}(\bar{s}_0)V_1, \tag{4.42}$$

we have that the Wronskian

$$\bar{s}_{0y}\bar{p}_{0\bar{\eta}} - \bar{s}_{0\bar{\eta}}\bar{p}_{0y} = 1.$$

Therefore

$$\bar{p}_{0\bar{\eta}} + \bar{s}_{0y} = 2. \tag{4.43}$$

So that

$$\frac{\partial^2 \hat{S}}{\partial \bar{\eta}^2} = \bar{s}_{0y}\bar{s}_{0\bar{\eta}}, \tag{4.44}$$

and therefore $\partial^2\hat{S}/\partial\bar{\eta}^2 \neq 0$ for small $\bar{\tau} \neq 0$.

We make in (4.3) the change of variables $\eta = |\xi|^{1/2}\bar{\eta}$ and apply the stationary phase method in $(y, \bar{\eta})$. Outside of a neighbourhood of $P_{\bar{\tau}_m}^{(m)}$ we have $|\partial\hat{S}/\partial\bar{\eta}| + |\partial\hat{S}/\partial y| \geqq C$, and therefore the integration by parts gives the contribution of order $O(|\xi|^{-\mu_1})$, $\forall \mu_1$. Note that $|\xi| \geqq CN_1^2$ in the region (4.8). In the neighbourhood of $P_{\bar{\tau}_m}^{(m)}$ we shall use the stationary phase method in $\bar{\eta}$ for fixed y and then integrate in y, z and ξ. We obtain from (4.3),

$$\iint_{\hat{T}^2} \hat{G}_{\mu_1}\left(s + m|\delta_0|^{-1}, t + N_1, s, t, \frac{\tau_0}{N_1}\right) ds\,dt$$

$$= \sum_{k=0}^{\mu} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \chi_1(2\xi x_0 + N_1) \iint_{\hat{T}^2} |\xi|^{1/2}$$

$$\cdot\left(\int_{-\infty}^{\infty} \chi(x_0 \Lambda) e^{-i(\xi^2 x_0 + N_1\xi) - i|\xi|^{1/2}\hat{S}(x_0|\xi|^{1/2}, y, \bar{\eta})} \cdot e^{-i\hat{S}_1} \hat{a}_k(x_0\Lambda, y, z, \xi, |\xi|^{1/2}\bar{\eta}) \left|\frac{\mathscr{D}(s,t)}{\mathscr{D}(y,z)}\right| d\bar{\eta}\right)$$

$$\cdot dy\,dz\,d\xi + O(N_1^{-\mu})$$

$$= \sum_{k=0}^{\mu} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \iint_{\hat{T}^2} \chi_1(2\xi x_0 + N_1)|\xi|^{1/2}$$

$$\cdot e^{-i(\xi^2 x_0 + N_1\xi)} e^{-i|\xi|^{1/2} S_0(x_0|\xi|^{1/2}) - i\pi/4} \frac{2\sqrt{\pi}}{\sqrt{\bar{s}_{0y}\bar{s}_{0\eta}}} |\xi|^{-1/4}$$

$$\cdot a_{k1}(x_0, \xi, y, z)\,dy\,dz\,d\xi + O(N_1^{-\mu}), \tag{4.45}$$

where $\chi_1(2\xi x_0 + N_1)$ is a cutoff function with the support in the region (4.8), $a_{k1} = O'(1/\xi^k)$,

$$a_{01}(x_0, \xi, y, z) = \hat{a}_0(x_0\Lambda_0, y, z, \xi, \bar{\eta}_0|\xi|^{1/2}) \left|\frac{\mathscr{D}(s,t)}{\mathscr{D}(y,z)}\right|_{\bar{\eta}=\bar{\eta}_0},$$

$\bar{\eta}_0 = \psi(\bar{\tau}_m, y)$, $\bar{\tau}_m = x_0|\xi|^{1/2}$, Λ_0 is equal to Λ at $\eta = |\xi|^{1/2}\bar{\eta}_0$, $\hat{a}_0(x_0\Lambda, y, z, \xi, \eta)$ is the same as in (3.46). Note that $\hat{S}_1 = O'(\eta\xi^{-1}) = O'(\bar{\eta}|\xi|^{-1/2})$ and we do not consider

$e^{-i\hat{S}_1}$ as a part of the phase function. Note also that $\chi(x_0\Lambda_0)=1$ for τ_0 small where $x_0=\tau_0/N_1$.

Finally changing in (4.45) $\xi=N_1^2\xi'-N_1^2/2\tau_0$ and applying the stationary phase method in ξ' we obtain

$$\iint_{\hat{T}^2}\hat{G}_{\mu_1}\left(s+m|\delta_0|^{-1},t+N_1,s,t,\frac{\tau_0}{N_1}\right)ds\,dt$$
$$=\sum_{k=0}^{\mu}\frac{1}{(2\pi)^2}\iint_{\hat{T}^2}\iint_{-\infty}^{\infty}\chi_1(2\tau_0N_1\xi')$$
$$\cdot\exp\left[-i\frac{N_1^3}{4\tau_0}-i\tau_0N_1^3\xi'^2\right]a_{k2}\left(\tau_0,y,z,N_1^2\xi'-\frac{N_1^2}{2\tau_0}\right)$$
$$\cdot\frac{N_1^2 2\sqrt{\pi}}{\sqrt{\bar{s}_{0y}\bar{s}_{0\eta}}}d\xi'\,dy\,dz+O(N_1^{-\mu})$$
$$=\sum_{k=0}^{\mu}\frac{1}{(2\pi)^2}\iint_{\hat{T}^2}\exp\left[-i\frac{N_1^3}{4\tau_0}-i\frac{N_1}{\sqrt{2\tau_0}}S_0\left(\sqrt{\frac{\tau_0}{2}}\right)-i\frac{\pi}{2}\right]\frac{(2\sqrt{\pi})^2N_1}{(2\tau_0^3)^{1/4}\sqrt{\bar{s}_{0y}\bar{s}_{0\bar{\eta}}}}$$
$$\cdot a_{k3}(\tau_0,y,z,N_1^{-1})dy\,dz+O(N_1^{-\mu}), \tag{4.46}$$

where

$$a_{k2}(\tau_0,y,z,\xi)=\exp(-i|\xi|^{1/2}S_0(x_0|\xi|^{1/2}))\cdot|\xi|^{1/4}a_{k1}(x_0,\xi,y,z),\quad a_{k3}=O\left(\frac{1}{N_1^k}\right),$$
$$a_{03}(\tau_0,y,z,N_1^{-1})=\hat{a}_0(x_0\bar{\Lambda}_0,y,z,\bar{\xi}_0,\bar{\eta}_0|\bar{\xi}_0|^{1/2})\mathscr{D}_0. \tag{4.47}$$

Also in (4.46) $\bar{\xi}_0=-N_1^2/2\tau_0,\bar{\eta}_0=\psi(\bar{\tau}_m,y),\bar{\tau}_m=\sqrt{\tau_0/2},\bar{\Lambda}_0=(|\bar{\xi}_0|^2+\varepsilon_0^4|\bar{\xi}_0|^2\bar{\eta}_0^4+\varepsilon_0^{-8})^{1/4}$, $\mathscr{D}_0$ is the value of the Jacobian

$$\mathscr{D}(\tau,y,z,\xi,\eta)=\begin{vmatrix}s_y(\tau,y,z,\xi,\eta), s_z\\ t_t(\tau,y,z,\xi,\eta), t_z\end{vmatrix}$$

at $\tau=x_0\bar{\Lambda}_0,\xi=\bar{\xi}_0,\eta=|\bar{\xi}_0|^{1/2}\psi(\bar{\tau}_m,y)$. Note that (s_y,t_y) are the solutions of the following system (cf. (3.29), (3.31)):

$$\frac{d}{d\tau}s_y=-2p_y-2\Lambda^{-1}\hat{A}'_{3s}s_y-2\Lambda^{-1}\hat{A}'_{3t}t_y,$$
$$\frac{d}{d\tau}t_y=-2q_y-2\Lambda^{-1}A_{\delta_0s}(s)s_y. \tag{4.48}$$

Also we have (see (3.38))

$$p_y=L_{ss}s_y+L_{st}t_y,\quad q_y=L_{ts}s_y+L_{tt}t_y. \tag{4.49}$$

Therefore

$$\frac{d}{d\tau}s_y=(-2L_{ss}-2\Lambda^{-1}\hat{A}'_{3s})s_y+(-2L_{st}-2\Lambda^{-1}\hat{A}'_{3t})t_y,$$
$$\frac{d}{d\tau}t_y=(-2L_{ts}-2\Lambda^{-1}A_{\delta_0s}(s))s_y-2L_{tt}t_y,\quad s_y|_{\tau=0}=1,\quad t_y|_{\tau=0}=0. \tag{4.50}$$

Functions (s_z, t_z) satisfy the same system (4.50) with the initial conditions $s_z|_{\tau=0}=0$, $t_z|_{\tau=0}=1$. Therefore $\mathscr{D}(\tau, y, z, \xi, \eta)$ is the Wronskian of the system (4.50). The well-known formula for the Wronskian gives

$$\mathscr{D}=\exp\int_0^\tau(-2L_{ss}-2\Lambda^{-1}A'_{3s}-2L_{tt})d\tau'. \tag{4.51}$$

Comparing (4.51) with (3.46) we obtain

$$a_{03}(\tau_0, y, z, N_1^{-1})=\mathscr{D}_0^{-1/2}(1+O(\bar{\Lambda}_0^{-1}))\mathscr{D}_0=\mathscr{D}_0^{1/2}(1+O(N_1^{-1})). \tag{4.52}$$

Using (3.53), (3.54), (3.55), (4.27) we obtain

$$\mathscr{D}_0=\begin{vmatrix}\dfrac{\partial\bar{s}_0(\bar{\tau}_m, y, \bar{\eta}_0)}{\partial y}+O(N_1^{-1}), & O(N_1^{-1})\\ O(N_1^{-1}), & 1+O(N_1^{-1})\end{vmatrix}=\frac{\partial\bar{s}_0(\bar{\tau}_m, y, \bar{\eta}_0)}{\partial y}+O(N_1^{-1}). \tag{4.53}$$

So that

$$a_{03}(\tau_0, y, z, N_1^{-1})=\left(\frac{\partial\bar{s}_0(\bar{\tau}_m, y, \bar{\eta}_0)}{\partial y}\right)^{1/2}+O(N_1^{-1}). \tag{4.54}$$

Therefore we have proved the following theorem:

Theorem 4.1. *The trace* $\iint_{\hat{T}^2}\hat{G}(s+m|\delta_0|^{-1}, t+N_1, s, t, \tau_0/N_1)dsdt$ *has the following asymptotics as* $N_1\to\infty$:

$$\iint_{\hat{T}^2}\hat{G}\left(s+m|\delta_0|^{-1}, t+N_1, s, t, \frac{\tau_0}{N_1}\right)dsdt=-\frac{iN_1|d_0|}{\pi(2\tau_0^3)^{1/4}}$$
$$\cdot\exp\left(-i\frac{N_1^3}{4\tau_0}-i\frac{N_1}{\sqrt{2\tau_0}}S_0\left(\sqrt{\frac{\tau_0}{2}}\right)\right)\left(\int_0^{|\delta_0|^{-1}}\frac{dy}{\sqrt{\bar{s}_{0\bar{\eta}}(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y))}}+O\left(\frac{1}{N_1}\right)\right), \tag{4.55}$$

where $\bar{\tau}_m=\sqrt{\tau_0/2}$.

We used that in the parallelogram $\hat{T}^2$ y changes from 0 to $|\delta_0|^{-1}$ and $-y|\delta_0||d_0|\leqq z\leqq -y|\delta_0||d_0|+|d_0|$. Therefore the contribution of the integration in z to the principal term in (4.55) will be $|d_0|$.

Remark 4.2. Repeating the integration by parts in (4.14), (4.18), (4.23), (4.25) and computing more terms by the stationary phase method in (4.45) and (4.46) we can obtain an explicit expression for more terms in the asymptotic expansion (4.55). This will give new spectral invariants.

5. Spectral Invariants

It follows from (4.55) that $S_0(\bar{\tau}_m)$ and $\int_0^{|\delta_0|^{-1}}(s_{0\bar{\eta}})^{-1/2}dy$ are spectral invariants. We have

$$S_0(\bar{\tau}_m) = \hat{S}(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)), \tag{5.1}$$

where $\bar{\eta}_0 = \psi(\bar{\tau}_m, y)$ is the equation of $P_{\bar{\tau}_m}^{(m)}$. Since $\partial\hat{S}/\partial\bar{\eta} = 0$ on $P_{\bar{\tau}_m}^{(m)}$, we obtain

$$\frac{dS_0(\bar{\tau}_m)}{d\bar{\tau}} = \frac{\partial\hat{S}}{\partial\bar{\tau}} + \frac{\partial\hat{S}}{\partial\bar{\eta}}\psi_{\bar{\tau}} = \frac{\partial\hat{S}}{\partial\bar{\tau}}. \tag{5.2}$$

Therefore using (4.30′) and (4.29) we obtain

$$\frac{dS_0(\bar{\tau}_m)}{d\bar{\tau}} = -\bar{s}_{0\bar{\tau}}\bar{\eta}_0 - (\bar{p}_0^2 + 2A_{\delta_0}(\bar{s}_0)) = 2\bar{p}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y))\psi(\bar{\tau}_m, y)$$
$$-(\bar{p}_0^2 + 2A_{\delta_0}(\bar{s}_0)) = \bar{p}_0^2(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) - 2A_{\delta_0}(\bar{s}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y))). \tag{5.3}$$

We used in (5.3) that $\bar{p}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) = \psi(\bar{\tau}_m, y)$ (see (4.39)). Denote

$$E(\bar{\tau}, y, \eta) = 2(\bar{p}_0^2(\bar{\tau}, y, \eta) - 2A_{\delta_0}(\bar{s}_0(\bar{\tau}, y, \eta)) = \frac{1}{2}\left(\frac{d\bar{s}_0}{d\bar{\tau}}\right)^2 - 4A_{\delta_0}(\bar{s}_0). \tag{5.4}$$

It follows from (4.36) that

$$\frac{dE(\bar{\tau}, y, \eta)}{d\bar{\tau}} = \frac{d\bar{s}_0}{d\bar{\tau}}\left(\frac{d^2\bar{s}_0}{d\bar{\tau}^2} - 4A_{\delta_0 s}(\bar{s}_0)\right) = 0. \tag{5.5}$$

Therefore $E(\bar{\tau}, y, \eta)$ is constant on $P_{\bar{\tau}_m}^{(m)}$, i.e.

$$E(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) = E(0, y, \psi(\bar{\tau}_m, y)) = 2(\psi^2(\bar{\tau}_m, y) - 2A_{\delta_0}(y)). \tag{5.6}$$

It follows from (5.3) that

$$\frac{dS_0(\bar{\tau}_m)}{d\bar{\tau}} = \tfrac{1}{2}E(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)), \tag{5.7}$$

and therefore $E(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y))$ is independent of y. Denote

$$E_0(\bar{\tau}_m) = E(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)). \tag{5.8}$$

It follows from (5.7) that $E_0(\bar{\tau}_m)$ is also a spectral invariant. We have

$$\psi(\bar{\tau}_m, y) = -\sqrt{\tfrac{1}{2}E_0(\bar{\tau}_m) + 2A_{\delta_0}(y)}, \tag{5.9}$$

and we take the negative sign of the square root since we consider $\bar{\tau}_1 > 0$. For $\bar{\tau}_1 < 0$ one should take the positive sign. We also assume that

$$E_0(\bar{\tau}_m) > -4\min_y A_{\delta_0}(y). \tag{5.10}$$

Note that $\bar{\eta}_0 = \psi(\bar{\tau}_m, y) \to -\infty$ as $E_0(\bar{\tau}_m) \to +\infty$ and vice versa. It follows from (5.4) and (5.9) that

$$\frac{d\bar{s}_0}{d\bar{\tau}} = \sqrt{2E_0(\bar{\tau}_m) + 8A_{\delta_0}(\bar{s}_0)}. \tag{5.11}$$

Therefore

$$\frac{d\bar{s}_0}{\sqrt{2E_0(\bar{\tau}_m) + 8A_{\delta_0}(\bar{s}_0)}} = d\bar{\tau}. \tag{5.12}$$

We have $\bar{s}_0(0, y, \psi(\bar{\tau}_m, y)) = y$, $\bar{s}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y)) = y + m|\delta_0|^{-1}$. So that integrating from $\bar{\tau} = 0$ to $\bar{\tau} = \bar{\tau}_m$ we get

$$\int_y^{y+m|\delta_0|^{-1}} \frac{ds}{\sqrt{2E_0(\bar{\tau}_m) + 8A_{\delta_0}(s)}} = \bar{\tau}_m. \tag{5.13}$$

Since $A_{\delta_0}(s)$ has the period $|\delta_0|^{-1}$, and since $\bar{\tau}_m = m\bar{\tau}_1$ we obtain

$$\frac{m}{\sqrt{2}} \int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{E_0(\bar{\tau}_m) + 4A_{\delta_0}(s)}} = m\bar{\tau}_1.$$

Also $E_0(\bar{\tau}_m) = E_0(\bar{\tau}_1)$, where $\bar{\tau}_1$ corresponds to $m = 1$. Therefore

$$\int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{E_0(\bar{\tau}_1) + 4A_{\delta_0}(s)}} = \sqrt{2}\bar{\tau}_1. \tag{5.14}$$

Note that $E_0(\bar{\tau}_1)$ is a decreasing function of $\bar{\tau}_1$ since differentiating in $\bar{\tau}_1$ we obtain

$$\int_0^{|\delta_0|^{-1}} \frac{(-\frac{1}{2})E_0'(\bar{\tau}_1)ds}{(E_0(\bar{\tau}_1) + 4A_{\delta_0}(s))^{3/2}} = \sqrt{2}, \tag{5.15}$$

i.e. $E_0'(\bar{\tau}_1) < 0$. Also $\bar{\tau}_1 \to 0$ as $E_0(\bar{\tau}_1) \to +\infty$ and vice versa. Denote by $H(\mu)$ the inverse function

$$E_0(H(\mu)) = \mu. \tag{5.16}$$

Then

$$\int_0^{|\delta_0|^{-1}} \frac{ds}{\sqrt{\mu + 4A_{\delta_0}(s)}} = \sqrt{2}H(\mu). \tag{5.17}$$

Denote $A_+ = \max_s A_{\delta_0}(s)$, $A_- = \min_s A_{\delta_0}(s)$. It follows from (5.17) that $H(\mu)$ is analytic in μ for all $\mu \in \mathbf{C} \setminus (-\infty, -4A_-]$. Therefore by the analytic continuation we can recover $H(\mu)$ for any $\mu \in \mathbf{C} \setminus (-\infty, -4A_-]$ from the values of $H(\mu)$ for large positive μ. Take any $\mu_0 < -4A_-$. We have

$$\int_0^{|\delta_0|^{-1}} (\mu_0 + i\varepsilon + 4A_{\delta_0}(s))^{-1/2} ds = \int_{\mu_0 > -4A_{\delta_0}(s)} + \int_{\mu_0 < -4A_{\delta_0}(s)} . \tag{5.18}$$

Note that

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_{\mu_0 > -4A_{\delta_0}(s)} (\mu_0 + i\varepsilon + 4A_{\delta_0}(s))^{-1/2} ds \\ &= \lim_{\varepsilon \downarrow 0} \int_{\mu_0 > -4A_{\delta_0}(s)} (\mu_0 - i\varepsilon + 4A_{\delta_0}(s))^{-1/2} ds \\ &= \int_{\mu_0 > -4A_{\delta_0}(s)} (\mu_0 + 4A_{\delta_0}(s))^{-1/2} ds, \end{aligned} \tag{5.19}$$

where we took the branch of the square root $\sqrt{z}$ which is positive for positive z. Also

$$\lim_{\varepsilon \downarrow 0} \int_{\mu_0 < -4A_{\delta_0}(s)} (\mu_0 + i\varepsilon + 4A_{\delta_0}(s))^{-1/2} ds = \int_{\mu_0 < -4A_{\delta_0}(s)} (-i)(-4A_{\delta_0}(s) - \mu_0)^{-1/2} ds, \tag{5.20}$$

$$\lim_{\varepsilon\downarrow 0} \int_{\mu_0<-4A_{\delta_0}(s)} (\mu_0 - i\varepsilon + 4A_{\delta_0}(s))^{-1/2} ds = \int_{\mu_0<-4A_{\delta_0}(s)} i(-4A_{\delta_0}(s) - \mu_0)^{-1/2} ds. \tag{5.20'}$$

Therefore

$$\begin{aligned} H_1(\mu_0) &= \lim_{\varepsilon\downarrow 0} (H(\mu_0 + i\varepsilon) - H(\mu_0 - i\varepsilon)) \\ &= -i\sqrt{2} \int_{\mu_0<-4A_{\delta_0}(s)} (-4A_{\delta_0}(s) - \mu_0)^{-1/2} ds \end{aligned} \tag{5.21}$$

is a spectral invariant. Also we have

$$\begin{aligned} H(\mu_0 + i0) &= \frac{1}{\sqrt{2}} \int_0^{|\delta_0|^{-1}} (\mu_0 + i0 + 4A_{\delta_0}(s))^{-1/2} ds \\ &= \frac{1}{\sqrt{2}} \int_{\mu_0>-4A_{\delta_0}(s)} + \frac{1}{\sqrt{2}} \int_{\mu_0<-4A_{\delta_0}(s)} \\ &= \frac{1}{\sqrt{2}} \int_{\mu_0>-4A_{\delta_0}(s)} (\mu_0 + 4A_{\delta_0}(s))^{-1/2} ds \\ &\quad + \frac{(-i)}{\sqrt{2}} \int_{\mu_0<-4A_{\delta_0}(s)} (-4A_{\delta_0}(s) - \mu_0)^{1/2} ds. \end{aligned} \tag{5.22}$$

Therefore since we already know $H(\mu + i0)$ and $H_1(\mu)$ we can recover

$$H_2(\mu_0) = \int_{\mu_0>-4A_{\delta_0}(s)} (4A_{\delta_0}(s) + \mu_0)^{-1/2} ds. \tag{5.23}$$

Note that the spectral invariant (5.23) is similar to (2.14). Note also that one can obtain the spectral invariant (5.21) by studying the asymptotics of the trace (4.1) when $m = 0$, but this asymptotics is harder to obtain than in the case $m \neq 0$. Now we shall consider the spectral invariant $\int_0^{|\delta_0|^{-1}} (\bar{s}_{0\bar{\eta}})^{-1/2} dy$.

Proposition 5.1. *The integral* $\int_0^{|\delta_0|^{-1}} (\bar{s}_{0\bar{\eta}})^{-1/2} dy$ *can be recovered from* $H(\mu)$.

Proof. Integrating (5.12) for 0 to $\bar{\tau}$ we obtain

$$\int_y^{\bar{s}_0(\bar{\tau},y,\bar{\eta})} (E(\bar{\tau}, y, \bar{\eta}) + 4A_{\delta_0}(s'))^{-1/2} ds' = \sqrt{2}\bar{\tau}, \tag{5.24}$$

where $E(\bar{\tau}, y, \bar{\eta}) = E(0, y, \bar{\eta}) = 2\bar{\eta}^2 - 4A_{\delta_0}(y)$. Therefore

$$\int_y^{\bar{s}_0(\bar{\tau},y,\bar{\eta})} (2\bar{\eta}^2 - 4A_{\delta_0}(y) + 4A_{\delta_0}(s'))^{-1/2} ds' = \sqrt{2}\bar{\tau}. \tag{5.25}$$

Differentiating (5.25) in $\bar{\eta}$ when y and $\bar{\tau}$ are fixed we obtain

$$\begin{aligned} &\bar{s}_{0\bar{\eta}}(\tau, y, \bar{\eta})(2\bar{\eta}^2 - 4A_{\delta_0}(y) + 4A_{\delta_0}(\bar{s}_0(\tau, y, \bar{\eta})))^{-1/2} \\ &+ \int_y^{\bar{s}_0(\bar{\tau},y,\bar{\eta})} (-\tfrac{1}{2})(2\bar{\eta}^2 - 4A_{\delta_0}(y) + 4A_{\delta_0}(s'))^{-3/2} \cdot 4\bar{\eta}\, ds' = 0. \end{aligned} \tag{5.26}$$

Substituting in (5.26) $\bar{\tau}=\bar{\tau}_m$, $\bar{\eta}=\psi(\bar{\tau}_m, y)$ we obtain

$$\bar{s}_{0\bar{\eta}}(\bar{\tau}_m, y, \bar{\eta}_0)(2\bar{\eta}_0^2-4A_{\delta_0}(y)+4A_{\delta_0}(y+m|\delta_0|^{-1}))^{-1/2} - \int_y^{y+m|\delta_0|^{-1}} 2\bar{\eta}_0(E_0(\bar{\tau}_m)+4A_{\delta_0}(s'))^{-3/2}ds'=0, \tag{5.27}$$

where $\bar{\eta}_0=\psi(\bar{\tau}_m, y)$ and we used that $\bar{s}_0(\bar{\tau}_m, y, \psi(\bar{\tau}_m, y))=y+m|\delta_0|^{-1}, E_0(\bar{\tau}_m)=2\bar{\eta}_0^2-4A_{\delta_0}(y)$. Therefore

$$\bar{s}_{0\bar{\eta}}(\tau_m, y, \bar{\eta}_0)=m\sqrt{2}(E_0(\bar{\tau}_m)+4A_{\delta_0}(y))\int_0^{|\delta_0|^{-1}}(E_0(\bar{\tau}_m)+4A_{\delta_0}(s'))^{-3/2}ds'. \tag{5.28}$$

Note that

$$\int_0^{|\delta_0|^{-1}}(\mu+4A_{\delta_0}(s'))^{-3/2}ds'=-2\sqrt{2}H'(\mu). \tag{5.29}$$

Therefore

$$\int_0^{|\delta_0|^{-1}}(\bar{s}_{0\bar{\eta}}(\tau_m, y, \bar{\eta}_0))^{-1/2}dy=(m\sqrt{2}(-2\sqrt{2}H'(E_0(\bar{\tau}_m)))^{-1/2} \cdot \int_0^{|\delta_0|^{-1}}(E_0(\bar{\tau}_m)+4A_{\delta_0}(y))^{-1/2}dy=\frac{1}{\sqrt{2m}}(-H'(E_0(\bar{\tau}_m)))^{-1/2}H(E_0(\bar{\tau}_m)), \tag{5.30}$$

i.e. knowing $H(\mu)$ we can recover $\int_0^{|\delta_0|^{-1}}(\bar{s}_{0\bar{\eta}})^{-1/2}dy$. ■

We shall assume that $A_{\delta_0}(s)$ satisfies the following "generic" condition:

$$\text{If } s_+ \text{ is the point such that } A_{\delta_0}(s_+)=\max A_{\delta_0}(s)=A_+, \text{ then } A'_{\delta_0}(s_+)=0,\ A''_{\delta_0}(s_+)<0 \text{ and } A_{\delta_0}(s)<A_{\delta_0}(s_+) \text{ for } |s-s_+|<|\delta_0|^{-1}. \tag{5.31}$$

Let $s_1<s_+<s_2$ be such that $A_{\delta_0}(s)$ is increasing on the interval (s_1, s_+), decreasing on (s_+, s_2) and $A_{\delta_0}(s_1)=A_{\delta_0}(s_2)$. Therefore there exists inverse functions $s_1(\mu)$ and $s_2(\mu)$ such that

$$4A_{\delta_0}(s_k(\mu))=-\mu, \quad k=1,2, \tag{5.32}$$

where

$$-4A_+\leqq\mu\leqq-4A_{\delta_0}(s_1)=-4A_{\delta_0}(s_2). \tag{5.33}$$

Consider the spectral invariant $H_2(\mu)$ for μ satisfying (5.33). We have

$$\begin{aligned} H_2(\mu)&=\int_{\mu>-4A_{\delta_0}(s)}(\mu+4A_{\delta_0}(s))^{-1/2}ds=\int_{s_1(\mu)}^{s_+}+\int_{s_+}^{s_2(\mu)} \\ &=-\int_{-4A_+}^{\mu}(\mu-\nu)^{-1/2}s'_1(\nu)d\nu+\int_{-4A_+}^{\mu}(\mu-\nu)^{-1/2}s'_2(\nu)d\nu \\ &=-\int_{-4A_+}^{\mu}\frac{s'_1(\nu)-s'_2(\nu)}{\sqrt{\mu-\nu}}d\nu. \end{aligned} \tag{5.34}$$

Solving the Abel equation (5.34) we shall find $s_1'(\mu)-s_2'(\mu)$, $-4A_+\leqq\mu\leqq-4A_{\delta_0}(s_1)$. Since $s_1(-4A_1)=s_2(-4A_+)=s_+$ we can recover also $s_1(\mu)-s_2(\mu)$ by the integration. ■

We have

$$4A_+-4A_{\delta_0}(s)=-2A''_{\delta_0}(s_+)(s-s_+)^2+O((s-s_+)^3)=B^2(s-s_+), \tag{5.35}$$

where $B(t)\in C^\infty$ for $s_1-s_+<t<s_2-s_+$ and $B'(0)\neq 0$. Therefore there exists a C^∞ function $b(t)$ that is the inverse to $B(t)$:

$$B(b(t))=t. \tag{5.36}$$

Since $B^2(t)=4A_+-4A_{\delta_0}(s_++t)$ we have

$$4A_{\delta_0}(s_++b(t))=4A_+-t^2. \tag{5.37}$$

Comparing (5.37) with (5.32) we get

$$\begin{aligned} s_++b(t)&=s_1(\mu), \quad t=-\sqrt{4A_++\mu},\\ s_++b(t)&=s_2(\mu), \quad t=\sqrt{4A_++\mu}. \end{aligned} \tag{5.37}$$

So that

$$\begin{aligned} s_1(\mu)&=s_++b(-\sqrt{4A_++\mu}),\\ s_2(\mu)&=s_++b(\sqrt{4A_++\mu}). \end{aligned} \tag{5.38}$$

It follows from (5.38) that

$$s_2(\mu)-s_1(\mu)=b(\sqrt{4A_++\mu})-b(-\sqrt{4A_++\mu}), \tag{5.39}$$

where $-4A_+\leqq\mu\leqq-4A_{\delta_0}(s_1)$. Therefore knowing $s_2(\mu)-s_1(\mu)$ we can recover the odd part of the function $b(t)$.

Proposition 5.2. *Assume that A_{δ_0} is an even function satisfying (5.31) and having only one maximum and one minimum on $[0,|\delta|^{-1}]$. Then the spectral invariant $H_2(\mu)$ determines $A_{\delta_0}(s)$ up to the position of the maximum point.*

Proof. Since $A_{\delta_0}(s)$ has only one maximum and one minimum we have that $s_1(\mu)$ and $s_2(\mu)$ are defined on $[-4A_+,-4A_-]$. If $A_{\delta_0}(s)$ is even then $b(t)$ is odd and therefore

$$s_2(\mu)-s_+=-(s_1(\mu)-s_+), \quad s_2(\mu)-s_1(\mu)=2s_2(\mu)-2s_+, \tag{5.40}$$

so that we can recover $s_1(\mu)-s_+$ and $s_2(\mu)-s_+$. Therefore knowing $s_k(\mu)-s_+$, $k=1,2$, we can recover $A_{\delta_0}(s)$ up to the position of the point s_+. Since $A_{\delta_0}(s)$ is even there are only two possibilities: either $s_+=0$ or $s_+=1/2|\delta_0|^{-1}$. ■

Proposition 5.3. *Assume that $A_{\delta_0}(s)$ is even and real analytic. Assume that $A_{\delta_0}(s)$ has m extremal values $A_-=A_1<A_2<\dots<A_m=A_+$, i.e. if $A'_{\delta_0}(s_k)=0$, then $A_{\delta_0}(s_k)$ is equal to one of A_p, $1\leqq p\leqq m$. Assume also that $A_{\delta_0}(0)\neq A_{\delta_0}(s_k)$ for $s_k\in(0,|\delta_0|^{-1})$ and $A''_{\delta_0}(0)\neq 0$. Then there is at most 2m even real analytic functions having the same spectral invariant $H_2(\mu)$.*

Proof. If $A_{\delta_0}(s)$ is real analytic then $s_1(\mu), s_2(\mu)$ and $b(t)$ are also analytic. Also $H_2(\mu)$ is a piece-wise analytic function of μ which is analytic on $[-4A_+, -4A_-]$ except $\mu = -4A_k, k = 1, 2, \dots, m$. Suppose $A_{\delta_0}(0) = A_p$ and $s = 0$ is a local maximum of $A_{\delta_0}(s)$. In a neighbourhood of A_p we have $H_2(\mu) = H_{2p}(\mu) + R_{2p}(\mu)$, where $R_{2p}(\mu)$ is analytic in a neighbourhood of A_p and $H_{2p}(\mu)$ has the form (5.34) with A_+ replaced by A_p and s_+ replaced by $s = 0$. Analogously to the proof of Proposition 5.2 one can recover $A_{\delta_0}(s)$ from $H_{2p}(\mu)$ in a small neighbourhood of $s = 0$. Since $A_{\delta_0}(s)$ is analytic this determines $A_{\delta_0}(s)$ for all s. When $s = 0$ is a local minimum we reach the same conclusion by using the spectral invariant $H_1(\mu)$ instead of $H_2(\mu)$. Take arbitrary A_p. Assume that $A_p = \tilde{A}_\delta(0)$, where $\tilde{A}_{\delta_0}(s)$ is even analytic function in a neighbourhood of $s = 0$. As before we can recover $\tilde{A}_{\delta_0}(s)$ using the spectral invariants $H_2(\mu)$ or $H_1(\mu)$. In general this function $\tilde{A}_{\delta_0}(s)$ may have no analytic and periodic extension to all $s \in \mathbf{R}^1$. Even if such extension exists the corresponding spectral invariant $\tilde{H}_2(\mu)$ may not coincide with $H_2(\mu)$ for all $\mu \in [-4A_+, -4A_-]$. In any case there are at most $2m$ functions $\tilde{A}_{\delta_0}(s)$ having the same spectral invariant $H_2(\mu)$ since for any $\tilde{A}_{\delta_0}(s)$ the function $\tilde{A}_{\delta_0}(s + 1/2|\delta_0|^{-1})$ has the same spectral invariants. ■

Now we shall consider the problem of recovering the scalar potential from the spectral invariants assuming that the vector potential $\vec{A}(x)$ is already known. The lowest order term in the expansion (4.55) containing $V(x)$ has the following form:

$$-i|\bar{\xi}_0|^{-1/2} \int_0^{|\delta_0|^{-1}} (\bar{s}_{0\bar{\eta}})^{-1/2} \left(\int_0^{\bar{\tau}_m} V_{\delta_0}(\bar{s}_0(\tau, y, \psi(\bar{\tau}_m, y)) d\tau) \right) dy. \tag{5.41}$$

Indeed it follows from (3.46), (4.47), (4.52), (4.53) that

$$\hat{a}_0(\tau_0, y, z, \bar{\xi}_0, \bar{\eta}_0|\bar{\xi}_0|^{1/2}) = (\bar{s}_{0y}(\bar{\tau}_m, y, \bar{\eta}_0))^{-1/2} \Bigg(1 - i\bar{\Lambda}_0^{-1} \int_0^{x_0\bar{\Lambda}_0} \hat{C}(s(\tau', y, z, \bar{\xi}_0, \bar{\eta}_0|\bar{\xi}_0|^{1/2}),$$

$$t(\tau', y, z, \bar{\xi}_0, \bar{\eta}_0|\bar{\xi}_0|^{1/2})) d\tau' + O(\bar{\Lambda}_0^{-1}) \Bigg),$$

where $O(\bar{\Lambda}_0^{-1})$ consists of terms either independent of $V(x)$ or having order $O(\bar{\Lambda}_0^{-2})$ and $\hat{C}$ is the same as in (3.16). Representing $\hat{C} = \hat{C}'(s, t) + C_{\delta_0}(s)$ and integrating by parts $\int_0^{x_0\bar{\Lambda}_0} \hat{C}' d\tau$ analogously to (4.14), (4.23) we obtain that the main contribution is given by $\int_0^{\bar{\tau}_m} C_{\delta_0}(\bar{s}_0(\tau', y, \bar{\eta}_0)) d\tau'$. Therefore it follows from (4.46) that the principal term depending on $V(x)$ has the form (5.41).

We shall simplify (5.41). Since $\bar{s}_{0\bar{\tau}} > 0$ we have changing variables $s = \bar{s}_0(\tau, y, \psi(\bar{\tau}_m, y))$ for fixed y and using (5.12),

$$\int_0^{\bar{\tau}_m} V_{\delta_0}(\bar{s}_0(\tau, y, \psi(\bar{\tau}_m, y)) d\tau = \int_y^{y + m|\delta_0|^{-1}} V_{\delta_0}(s) \frac{ds}{\sqrt{2E_0(\bar{\tau}_m) + 8A_{\delta_0}(s)}}$$

$$= \int_0^{|\delta_0|^{-1}} \frac{m}{\sqrt{2}} V_{\delta_0}(s)(E_0(\bar{\tau}_m) + 4A_{\delta_0}(s))^{-1/2} ds. \tag{5.42}$$

Note that (5.42) is independent of y. Therefore (5.41) can be written in the following form:

$$-i|\bar{\xi}_0|^{-1/2}\int_0^{|\delta_0|^{-1}}(\bar{s}_{0\bar{\eta}})^{-1/2}dy\int_0^{|\delta_0|^{-1}}\frac{m}{\sqrt{2}}V_{\delta_0}(s)(E_0(\bar{\tau}_m)+4A_{\delta_0}(s))^{-1/2}ds. \tag{5.43}$$

Since $\int_0^{|\delta_0|^{-1}}(s_{0\bar{\eta}})^{-1/2}dy$ is already known (see the Proposition 5.1) we can recover

$$H_3(\mu)=\int_0^{|\delta_0|^{-1}}V_{\delta_0}(s)(\mu+4A_{\delta_0}(s))^{-1/2}ds. \tag{5.44}$$

Analogously to (5.21), (5.23) we can recover also

$$H_4(\mu)=\int_{\mu<-4A_{\delta_0}(s)}V_{\delta_0}(s)(-\mu-4A_{\delta_0}(s))^{-1/2}ds, \tag{5.45}$$

$$H_5(\mu)=-\int_{\mu>-4A_{\delta_0}(s)}V_{\delta_0}(s)(\mu+4A_{\delta_0}(s))^{-1/2}ds. \tag{5.46}$$

In (5.45), (5.46) $\mu<-4A_-$. Let $s_+, s_1(\mu)$, $s_2(\mu)$ be the same as in (5.32), (5.34). Analogously to (5.34) we have

$$H_5(\mu)=-\int_{-4A_+}^{\mu}\frac{V_{\delta_0}(s_1(v))s_1'(v)-V_{\delta_0}(s_2(v))s_2'(v)}{\sqrt{\mu-v}}dv. \tag{5.47}$$

Solving the Abel equation (5.47) we can recover

$$V_{\delta_0}(s_1(v))s_1'(v)-V_{\delta_0}(s_2(v))s_2'(v).$$

Assume that $V_{\delta_0}(s)$ and $A_{\delta_0}(s)$ are even functions and $s=0$ is a local maximum. Then $s_2(\mu)=-s_1(\mu)$, $s_2'(\mu)=-s_1'(\mu)$ and

$$V(s_2(\mu))=V(-s_1(\mu))=V(s_1(\mu)).$$

Therefore

$$V_{\delta_0}(s_1(\mu))s_1'(\mu)-V_{\delta_0}(s_2(\mu))s_2'(\mu)=2V_{\delta_0}(s_1(\mu))s_1'(\mu),$$

and we can recover $V_{\delta_0}(s)$ in the neighbourhood of $s=0$.

Theorem 5.1. *Let $\vec{A}^{(t)}(x)=(A_1^{(t)}(x),A_2^{(t)}(x))$ and $V^{(t)}(x)$ be continuous families of even real analytic vector and scalar potentials, $0\leqq t\leqq 1$. Assume that the lattice L satisfies the condition (1.10) and $\vec{A}^{(0)}(x)$ for any $\delta_0\in S<L'$ satisfies the same generic condition as in the Proposition 5.3. Assume that the periodic spectrum of $H^{(t)}$ is independent of t, $0\leqq t\leqq 1$, where $H^{(t)}$ is the Schrödinger operator corresponding to $\vec{A}^{(t)}(x)$ and $V^{(t)}(x)$. Then $\vec{A}^{(t)}(x)=\vec{A}^{(0)}(x)$, $V^{(t)}(x)=V^{(0)}(x)$ for all $t\in(0,1]$,* i.e. *there is a rigidity of isospectral deformations.*

Proof. It follows from Proposition 5.3 that for any δ_0 there is only a finite number of $\tilde{A}_{\delta_0}(s)$ having the same spectral invariants as $A_{\delta_0}^{(0)}(s)$. Since $A_{\delta_0}^{(t)}(s)$ depends continuously on t we have that $A_{\delta_0}^{(t)}(s)=A_{\delta_0}^{(0)}(s)$ for all t. Knowing $\vec{A}(x)$ we can recover uniquely the even $V_{\delta_0}(s)$ in a neighbourhood of $s=0$. Since $V_{\delta_0}(s)$ is analytic it will uniquely determine $V_{\delta_0}(s)$ for all s. Therefore $V_{\delta_0}^{(t)}(s)=V_{\delta_0}^{(0)}(s)$, $\forall\delta_0$. ■

Remark 5.1. Computing the next term in the asymptotic expansion (4.55) (see Remark 4.2) one can show that $\int_0^{|\delta_0|^{-1}} C_{\delta_0}(s)(\mu+4A_{\delta_0}(s))^{-1/2}ds$ is also a spectral invariant.

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