

Factorisation of Energy Dependent Schrödinger Operators: Miura Maps and Modified Systems

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Abstract. We consider the energy dependent Schrödinger operator $\mathbb{L} = \sum_{i=0}^N \lambda^i (\varepsilon_i \partial^2 + u_i)$, which we have previously shown to be associated with multi-Hamiltonian structures [2]. In this paper we use an unusual form of the Lax approach to derive by a *single construction* the time evolutions of the eigenfunctions of \mathbb{L} , the associated Hamiltonian operators and the Hamiltonian functionals. We then generalise the well known factorisation of standard Lax operators to the case of energy-dependent operators. The simple product of linear factors is replaced by a λ -dependent quadratic form. We thus generalise the resulting construction of Miura maps and modified equations. We show that for some of our systems there exists a sequence of N such modifications, the r^{th} modification possessing $(N-r+1)$ Hamiltonian structures.

1. Introduction

In a number of recent papers [1–4] we discussed the two generalised Schrödinger equations:

$$L_1 \psi \equiv \left(\sum_{i=0}^{N-1} \lambda^i (\varepsilon_i \partial^2 + u_i) \right) \psi = \lambda^N \psi, \quad (1.1a)$$

$$L_2 \psi \equiv \left(\partial^2 + \sum_{i=1}^N u_i \lambda^i \right) \psi = a^2 \psi, \quad (1.1b)$$

where a is a constant. We have shown that the isospectral flows of each of these spectral problems possess $(N+1)$ compatible Hamiltonian structures $\mathbf{B}_0, \dots, \mathbf{B}_N$. When $N=1$ these spectral problems give rise respectively to the KdV and Harry

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Dym hierarchies, both bi-Hamiltonian. When $N=2$, (1.1a) includes the dispersive water waves (DWW) and Ito hierarchies, both tri-Hamiltonian.

In the present paper we consider the Miura maps and modified equations associated with these spectral problems. In fact, we construct a sequence of N Miura maps and modifications for the N component system. The r^{th} modification possesses $(N-r+1)$ compatible Hamiltonian structures. The single Hamiltonian structure of the N^{th} modification is a constant coefficient, first order differential operator.

Our method of construction is an extension of the factorisation of differential operators [5–8]. To incorporate the spectral parameter we need to introduce a quadratic form in place of the pair of linear factors required for the simple KdV case. To each quadratic form there corresponds a different map. We exhibit one particular sequence of N quadratic forms, defined by matrices A_r , which enable each of our Hamiltonian structures to be (separately) brought to constant coefficient, first order form. When $\varepsilon_0=0$, the corresponding maps are invertible, so are just a change of co-ordinates. When $\varepsilon_0 \neq 0$, the maps are non-invertible and of Hamiltonian type, thus qualifying as genuine (see [8] and the definition below) Miura maps. Indeed, under the action of the Miura map M_r associated with A_r , the pre-images $\tilde{\mathbf{B}}_r, \dots, \tilde{\mathbf{B}}_N$ of $\mathbf{B}_r, \dots, \mathbf{B}_N$ are locally defined in terms of the modified variables, $\tilde{\mathbf{B}}_r$ being constant coefficient and first order. When $r=N$ only $\tilde{\mathbf{B}}_N$, given by (3.9b), is locally defined. The mapping M_N can be decomposed into a sequence of N maps M_N^k ($k=N, \dots, 1$), relating the k^{th} and $(k-1)^{\text{th}}$ modifications. We are thus led to the beautiful picture of Fig. 1.

For the remainder of this introduction we present some of the basic facts concerning Hamiltonian structures and Miura maps. Section 2 is concerned with the Hamiltonian formulation of the isospectral flows of (1.1). We add some proofs and bring some hindsight to these results. More importantly, we adopt here a completely different approach from that of [2]. In the present paper we start with the spectral problem (2.1a) and *in one construction* derive the time evolutions of the eigenfunctions of (2.1a), the associated Hamiltonian operators and the Hamiltonians. We employ an unusual form of the Lax approach, which leads to a surprisingly simple and elegant derivation of our results. Miura maps and modifications are introduced in Sect. 3. The results are so simple to present that we give the general formulae. As examples we have chosen the DWW equations and a 2-component extension of the Harry Dym equation. Since both of these are tri-Hamiltonian they each possess 2 modifications. Our second modification of the DWW equations is genuinely different from both of Kupershmidt's [9], in that there is no *invertible* transformation of co-ordinates connecting them.

Hamiltonian Property

In this paper we are concerned with systems of NLEEs [in $(1+1)$ -dimensions] which can be written in Hamiltonian form $u_t = \mathbf{B} \delta \mathcal{H}$, where \mathbf{B} is a Hamiltonian operator and $\delta \mathcal{H}$ the variational derivative of functional \mathcal{H} (all defined below). In the context of analysis and physics one would then deal with constants of motion and Poisson brackets in their integral form, respectively:

$$\mathbf{H} = \int \mathcal{H} dx, \quad \{\mathbf{K}, \mathbf{H}\} = \int \delta \mathcal{K} \mathbf{B} \delta \mathcal{H} dx, \quad (1.2)$$

which would involve particular boundary conditions on the functions $u_i(x)$. To avoid any such considerations it is customary to work within the framework of differential algebras. We briefly present a few of the basic facts below. For detailed discussions of the Hamiltonian theory of NLEEs, see [10–12].

Let A_u be the (differential) algebra of differential functions of $\mathbf{u} = (u_0, \dots, u_{N-1})^T$, that is the associative algebra of functions of u_k and (a finite number of) their x -derivatives, together with the derivation $\partial: A_u \rightarrow A_u$, $\partial \equiv \frac{\partial}{\partial x}$ (the total x -derivative). The algebra of differential operators on $A_u^N = A_u \times \dots \times A_u$ (N -times) will be denoted by $A_u^N[\partial]$. A skew-adjoint operator $\mathbf{B} \in A_u^N[\partial]$ ($\mathbf{B}^\dagger = -\mathbf{B}$, where \mathbf{B}^\dagger is the formal adjoint of \mathbf{B}) is Hamiltonian iff the (Poisson) bracket $\{\cdot, \cdot\}_B: A_u \times A_u \rightarrow A_u$,

$$\{G, H\}_B \equiv \delta G \mathbf{B} \delta H \quad (1.3)$$

satisfies the Jacobi identity (mod $\text{Im } \partial$). The Euler operator $\delta: A_u \rightarrow A_u^N$, $\delta \equiv (\delta_0, \dots, \delta_{N-1})^T$, $\delta_i = \frac{\delta}{\delta u_i}$, will sometimes be written as δ_u to distinguish it from its counterpart $\delta_v: A_v \rightarrow A_v^N$ on the algebra of differential functions in the modified variables v_i .

Remark. For two Hamiltonians to Poisson commute with respect to (1.3) means that the right-hand side of (1.3) is an exact derivative. In the analytic context the right-hand side is a boundary term which can only be “thrown away” with an appropriate choice of boundary condition.

For any $H \in A_u$ and $\mathbf{B} \in A_u^N[\partial]$ the formula $\partial_H \mathbf{u} = \mathbf{B} \delta H$ defines an evolutionary (meaning that $[\partial, \partial_H] = 0$) derivation ∂_H of A_u . When \mathbf{B} is Hamiltonian the commutativity of ∂_H and ∂_G is equivalent to $\{G, H\}_B \equiv 0 \pmod{\text{Im } \partial}$.

Since $\text{Ker } \delta = \text{Im } \partial$, $\delta: A_u \rightarrow A_u^N$ is essentially defined on the quotient space $\mathcal{A}_u = A_u / \text{Im } \partial$, and the Poisson bracket $\{G, H\}$ and evolutionary derivative ∂_H depend only upon the equivalence classes $\mathcal{G}, \mathcal{H} \in \mathcal{A}_u$ of $G, H \in A_u$. We will usually write $\frac{\partial}{\partial t}$ instead of ∂_H , when describing the Hamiltonian flow:

$$\mathbf{u}_t = \mathbf{B} \delta \mathcal{H}, \quad (1.4)$$

where t is the corresponding evolution parameter.

A system of evolution equations is said to be bi-Hamiltonian if there exist two Hamiltonian operators \mathbf{B}_0 and \mathbf{B}_1 and two Hamiltonians \mathcal{G} and \mathcal{H} such that

$$\mathbf{u}_t = \mathbf{B}_0 \delta \mathcal{G} = \mathbf{B}_1 \delta \mathcal{H}. \quad (1.5)$$

It is particularly interesting if the operator $\mathbf{B}_0 + \mathbf{B}_1$ is also Hamiltonian, in which case \mathbf{B}_0 and \mathbf{B}_1 are said to be compatible (in general the sum of the Poisson brackets would fail to satisfy the Jacobi identity). The importance of compatibility is that it enables us (under certain conditions) to construct an infinite hierarchy of (Poisson commuting) Hamiltonians. This important condition was first noticed by Magri [13]. To clarify this we introduce a definition:

Definition. A differential operator $\mathbf{D}: A_u^N \rightarrow A_u^N$ is said to be degenerate if there is a nonzero differential operator $\tilde{\mathbf{D}}: A_u^N \rightarrow A_u^N$ such that $\tilde{\mathbf{D}} \cdot \mathbf{D} \equiv 0$.

With this definition in hand it is now possible to state a useful lemma (see [11] for a proof).

Lemma. *If \mathbf{B}_0 and \mathbf{B}_1 are compatible Hamiltonian operators, with \mathbf{B}_0 non-degenerate, and*

$$\mathbf{B}_1 \delta \mathcal{G} = \mathbf{B}_0 \delta \mathcal{H}, \quad \mathbf{B}_1 \delta \mathcal{H} = \mathbf{B}_0 \mathbf{K}, \quad (1.6)$$

then there exists a function \mathcal{K} such that $\mathbf{K} = \delta \mathcal{K}$.

To prove the existence of an infinite hierarchy of Hamiltonians, \mathcal{H}_n , related to compatible Hamiltonian operators $\mathbf{B}_0, \mathbf{B}_1$, we need to check that three conditions hold:

- (i) \mathbf{B}_0 is non-degenerate.
- (ii) \exists an infinite sequence of vector functions $\mathbf{K}_0, \mathbf{K}_1, \dots$ satisfying $\mathbf{B}_1 \mathbf{K}_n = \mathbf{B}_0 \mathbf{K}_{n+1}$.
- (iii) \exists two function(al)s \mathcal{H}_0 and \mathcal{H}_1 s.t. $\mathbf{K}_0 = \delta \mathcal{H}_0, \mathbf{K}_1 = \delta \mathcal{H}_1$.

It then follows from the lemma that there exist function(al)s \mathcal{H}_n such that $\mathbf{K}_n = \delta \mathcal{H}_n \forall n \geq 0$.

Remark. Condition (ii) is not always easy to check, although it is for our systems. Indeed, it may not even be satisfied, as shown by an example of Kupershmidt [14].

For this construction, it is of no advantage for a system to be *more* than bi-Hamiltonian. However, the existence of multi-Hamiltonian structures does lead to a rich supply of (multi-)Hamiltonian modifications.

“A Remarkable Explicit Nonlinear Transformation”

Miura presented his famous transformation 20 years ago [15]. He showed that if:

$$u = -v_x - v^2 \quad (1.7a)$$

and v satisfies the MKdV equation:

$$v_t = v_{xxx} - 6v^2 v_x \quad (1.7b)$$

then u satisfies the KdV equation

$$u_t = u_{xxx} + 6uu_x. \quad (1.7c)$$

At the same time Miura et al. [16] used (Gardner’s generalisation of) this to prove the existence of an infinite number of conservation laws for both the KdV and MKdV equations and to derive the linear Schrödinger equation (1.1a) (with $N = 1$). It was later noticed that (1.7a) could be used to construct the second Hamiltonian structure of the KdV equation out of the single Hamiltonian structure of the MKdV equation. This is the property of most interest for this paper.

Equation (1.7b) can be written in Hamiltonian form

$$v_t = (-\partial) \frac{\delta \tilde{\mathcal{H}}}{\delta v}, \quad \tilde{\mathcal{H}} = \frac{1}{2}(v_x^2 + v^4). \quad (1.8a)$$

From the Miura map (1.7a) we have

$$u_t = (-\partial - 2v)v_t = (-\partial - 2v)(-\partial) \frac{\delta \tilde{\mathcal{H}}}{\delta v}. \quad (1.8b)$$

If we denote (1.7a) by $u = M[v]$, the operator $(-\partial - 2v)$ is the Fréchet derivative M' of M . Using the formula $\delta_v \tilde{\mathcal{H}} = (M')^\dagger \delta_u \mathcal{H}$, where $\mathcal{H}[u]$ is defined by $\tilde{\mathcal{H}}[v] = \mathcal{H} \circ M[v]$, we find

$$u_t = M'(-\partial)(M')^\dagger \delta_u \mathcal{H}. \quad (1.8c)$$

When M is given by (1.7a), Eq. (1.8c) can be written in terms of u and its derivatives to give the *local* equation (1.7c). However, for an arbitrary differential mapping $u = M[v]$ this process would take us out of the differential algebra setting, since:

- (i) there would not generally be a (*locally defined*) functional $\mathcal{H}[u]$ satisfying $\mathcal{H} \circ M = \tilde{\mathcal{H}}$.
- (ii) the differential operator $M'(-\partial)(M')^\dagger \in A_v[\partial]$ would not be an element of $A_u[\partial]$ since its coefficients would not be *locally defined* in terms of u and its derivatives.

It is a remarkable fact that neither of these problems arise for the mapping (1.7a), since taking $\mathcal{H} = \frac{1}{2}u^2$ we have

$$\mathcal{H} \circ M = \frac{1}{2}(v_x^2 + v^4) + (\frac{2}{3}v^3)_x \equiv \tilde{\mathcal{H}} \pmod{\text{Im } \partial} \quad (1.8d)$$

and

$$\begin{aligned} M'(-\partial)(M')^\dagger &= \partial^3 + 4(-v_x - v^2)\partial + 2(-v_{xx} - 2vv_x) \\ &= \partial + 4u\partial + 2u_x \in A_u[\partial]. \end{aligned} \quad (1.8e)$$

The Hamiltonian nature of the third order differential operator (1.8e) follows from that of $(-\partial)$ through the formula $M'(-\partial)(M')^\dagger$. Then (1.8c) is a Hamiltonian description of the KdV equation (1.7c).

Remark. The operator (1.8e) is the *second* Hamiltonian structure of the bi-Hamiltonian KdV equation. This construction gives no information regarding the existence of a first Hamiltonian structure.

In a more general algebraic setting, let $\mathbf{u} = (u_0, \dots, u_{N-1})^T$ and $\mathbf{v} = (v_0, \dots, v_{N-1})^T$ be the (respectively) unmodified and modified variables. Then

Definition. The mapping $\mathbf{u} = M[\mathbf{v}]$, $M \in A_v^N$, is a Miura map for Hamiltonian operator $\tilde{\mathbf{B}} \in A_v^N[\partial]$ if:

- (i) M is not invertible
- (ii) $\mathbf{B} = M'\tilde{\mathbf{B}}(M')^\dagger|_{\mathbf{u}=M[\mathbf{v}]} \in A_u^N[\partial]$.

This definition is adopted from [8].

Remark. The problem concerning the locally defined functional $\mathcal{H}[u]$ does not arise here since we use the map M to “push forward” Hamiltonian operator $\tilde{\mathbf{B}}$ and “pull back” arbitrary functionals $\mathcal{H}[u]$. This “pull back” is, of course, not surjective.

Remark. The Hamiltonian nature of the operator \mathbf{B} follows from that of $\tilde{\mathbf{B}}$ provided Miura map $\mathbf{u} = M[\mathbf{v}]$ is nondegenerate (see [8, 10] for a more detailed discussion of this). It is often the case that $\tilde{\mathbf{B}}$ is of much simpler form than \mathbf{B} , thus giving a simple proof of the Hamiltonian nature of \mathbf{B} . This is the case in example (1.8c) above.

Factorisation of Differential Operators

The relationship of Miura maps to the factorisation of differential operators is discussed in [5–8]. The map (1.7a) can be obtained from the Schrödinger operator

$$\mathbb{L} = \partial^2 + u \quad (1.9a)$$

through the identification:

$$\mathbb{L} = (\partial + v)(\partial - v). \quad (1.9b)$$

The spectral problem $\mathbb{L}\psi_1 = \lambda\psi_1$ for the KdV equation can be used to define that for the MKdV equation:

$$(\partial - v)\psi_1 = \lambda\psi_2, \quad (\partial + v)\psi_2 = \psi_1. \quad (1.9c)$$

This notion is easily extended to higher order Lax operators [6]. In [8] Kupershmidt and Wilson proved that the map obtained through the factorisation of the *general* Lax operator (but independent of spectral parameter) is genuinely Miura by their criteria.

In Sect. 3 we generalise this construction to the case of the spectral dependent Schrödinger operators (1.1a, b).

2. The Spectral Problem

In this paper we consider a fairly general second order scalar spectral problem:

$$\mathbb{L}\psi \equiv (\varepsilon\partial^2 + u)\psi = 0, \quad (2.1a)$$

where ε and u depend polynomially upon the spectral parameter λ :

$$\varepsilon = \sum_0^N \varepsilon_i \lambda^i, \quad u = \sum_0^N u_i \lambda^i \quad (2.1b)$$

with ε_i being constant and u_i functions of x .

We look for time evolutions of the wave function ψ of the form:

$$\psi_t = \mathbb{P}\psi \equiv (\tfrac{1}{2}P\partial + Q)\psi, \quad (2.1c)$$

where P and Q are functions of u_i and their x -derivatives, and of the spectral parameter λ . A simple calculation leads to

$$\mathbb{L}_t - [\mathbb{P}, \mathbb{L}] = u_t + \varepsilon Q_{xx} - \tfrac{1}{2}Pu_x + \tfrac{1}{2}\varepsilon(P_{xx} + 4Q_x)\partial + \varepsilon P_x\partial^2. \quad (2.2a)$$

Evidently, we cannot expect the usual Lax equation to hold. However, the integrability conditions of (2.1a, c) imply that $(\mathbb{L}_t - [\mathbb{P}, \mathbb{L}])\psi = 0$ for eigenfunctions of (2.1a). To match the coefficient of ∂^2 we must take:

$$\mathbb{L}_t - [\mathbb{P}, \mathbb{L}] = P_x \mathbb{L}. \quad (2.2b)$$

This further implies that $P_{xx} + 4Q_x = 0$, so that (2.2b) takes the remarkably simple form:

$$u_t = (\tfrac{1}{4}\varepsilon\partial^3 + \tfrac{1}{2}(u\partial + \partial u))P \equiv JP. \quad (2.2c)$$

Remark. On the phase space defined by just one function u , the operator J , defined by (2.2c) is Hamiltonian, being (when $\varepsilon=1$) just the second Hamiltonian structure of the KdV equation. The operator J is the basic unit out of which all our Hamiltonian operators are built.

With ε and u defined by (2.1b), the operator J takes the form:

$$J = \sum_{k=0}^N J_k \lambda^k, \quad (2.3a)$$

where

$$J_k = \frac{1}{4} \varepsilon_k \partial^3 + \frac{1}{2} (u_k \partial + \partial u_k). \quad (2.3b)$$

Equation (2.2c) then takes the form

$$\sum_{k=0}^N \lambda^k u_{kt} = \left(\sum_{k=0}^N J_k \lambda^k \right) P. \quad (2.3c)$$

Substituting a polynomial expansion for P :

$$P = \sum_{k=0}^m P_{m-k} \lambda^k \quad (2.4a)$$

we find the recursion relations for the coefficients P_{m-k} and the equations of motion for u_i :

$$J_0 P_{k-N} + J_1 P_{k-N+1} + \dots + J_N P_k = 0, \quad k=0, \dots, m-1 \quad (2.4b)$$

$$\begin{aligned} u_{Nt_m} &= J_0 P_{m-N} + \dots + J_N P_m \\ u_{N-1t_m} &= J_0 P_{m-N+1} + \dots + J_{N-1} P_m \\ &\vdots \\ u_{0t_m} &= J_0 P_m \end{aligned} \quad (2.4c)$$

We prove below that, whenever $\varepsilon_N=0$, (2.4b) can be algebraically solved recursively for P_0, \dots, P_{m-1} , starting with $P_0 \in \text{Ker } J_N$. However, without further restriction, P_m is undetermined. There are just 2 cases:

Case 1. u_N constant. Take $u_N = -1$

Then, the first of equations (2.4c) takes the form (2.4b) with $k=m$. The remaining equations of (2.4c) are the equations of motion for u_0, \dots, u_{N-1} :

$$\begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}_{t_m} = \begin{bmatrix} 0 & & J_0 \\ & \ddots & \vdots \\ J_0 & \dots & J_{N-1} \end{bmatrix} \begin{bmatrix} P_{m-N+1} \\ \vdots \\ P_m \end{bmatrix}. \quad (2.5a)$$

We refer to this as the (generalised) KdV case. The operator $J_N = -\partial$ is constant coefficient.

Case 2. u_N non-constant

P_m is unconstrained. We may thus set $P_m=0$. The last of (2.4c) then implies $u_0 = -a^2$, a constant. The remaining equations of (2.4c) are the equations of

motion for u_1, \dots, u_N :

$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}_{t_m} = \begin{bmatrix} 0 & & J_0 \\ & \ddots & \vdots \\ J_0 & \dots & J_{N-1} \end{bmatrix} \begin{bmatrix} P_{m-N} \\ \vdots \\ P_{m-1} \end{bmatrix}. \quad (2.5b)$$

We refer to this as the (generalised) Harry Dym case. The operator $J_0 = \frac{1}{4} \varepsilon_0 \partial^3 - a^2 \partial$ is constant coefficient.

It is a remarkable fact that the scalar recursion relation (2.4b) can be written as an $N \times N$ matrix equation in exactly N different ways:

$$\mathbf{B}_n \mathbf{P}^{(k-1)} = \mathbf{B}_{n-1} \mathbf{P}^{(k)}, \quad n=1, \dots, N, \quad (2.6a)$$

where $\mathbf{P}^{(k)} = (P_{k-N+1}, \dots, P_k)^T$ and the matrix differential operators \mathbf{B}_n are determined by the following requirement: \mathbf{B}_n is skew adjoint and the n^{th} row of each matrix equation (2.6a) is just (2.4b), the remaining ones being identities. Explicitly, \mathbf{B}_n are:

$$\mathbf{B}_n = \left[\begin{array}{c|c} \begin{matrix} 0 & & J_0 \\ & \ddots & \vdots \\ J_0 & \dots & J_{n-1} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -J_{n+1} & \dots & -J_N \\ \vdots & \ddots & \vdots \\ -J_N & & 0 \end{matrix} \end{array} \right] \quad (2.6b)$$

and satisfy the formal relation $\mathbf{B}_n = \mathbf{R} \mathbf{B}_{n-1}$, where

$$\mathbf{R} = \mathbf{B}_1 \mathbf{B}_0^{-1} = \left[\begin{array}{c|c} 0 & \dots & 0 & -J_0 J_N^{-1} \\ \hline 1 & & 0 & -J_1 J_N^{-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -J_{N-1} J_N^{-1} \end{array} \right]. \quad (2.6c)$$

Our next step is to prove 3 basic facts:

1. The operators \mathbf{B}_n are each Hamiltonian and, furthermore, are mutually compatible.
2. For each $m \geq 0$, the recursion relation (2.4b) can be successively solved for all polynomial expansions (2.4a) subject to the condition $\varepsilon_N = 0$.
3. The vectors $\mathbf{P}^{(k)}$ given by (2.6a) are variational derivatives of a sequence of function(al)s \mathcal{H}_n (the Hamiltonians).

Then it follows from (2.6a) that the equations of motion (2.5) can be written in Hamiltonian form in $(N+1)$ distinct ways:

$$\mathbf{u}_{t_m} = \mathbf{B}_N \delta \mathcal{H}_m = \dots = \mathbf{B}_0 \delta \mathcal{H}_{m+N}. \quad (2.7)$$

The Operators \mathbf{B}_n

The algebraic form of the operators \mathbf{B}_n is so simple that (1) can be shown by direct calculation. We omit the details here, preferring to give an independent proof of the Hamiltonian character of the operators (2.6b), based on the existence of Miura maps, in Sect. 3.

The Recursion

The sequence of equations (2.4b) can be used to recursively define P_k in terms of differential functions of u_i if, for each k ,

$$\sum_{i=0}^{N-1} J_i P_{k-N+i} \in \text{Im } J_N. \quad (2.8)$$

Rather than check this at each step we prefer to use a different method. We prove that there exists a formal infinite series solution:

$$\mathcal{P} = \sum_{n=0}^{\infty} P_n \lambda^{-n} \quad (2.9a)$$

of the equation $J\mathcal{P}=0$, i.e.

$$\frac{1}{4} \varepsilon \mathcal{P}_{xxx} + u \mathcal{P}_x + \frac{1}{2} u_x \mathcal{P} = 0 \quad (2.10a)$$

with P_n being differential functions of u_0, \dots, u_N . Then:

$$P = (\lambda^m \mathcal{P})_+, \quad m \geq 0, \quad (2.9b)$$

where $(\)_+$ means only terms with non-negative (for the KdV case) or positive (for the Harry Dym case) powers of λ , is of the form (2.4a) and gives us a solution of (2.4b).

Equation (2.10a), when multiplied by \mathcal{P} , can be integrated to give:

$$u \mathcal{P}^2 + \varepsilon \left(\frac{1}{2} \mathcal{P} \mathcal{P}_{xx} - \frac{1}{4} \mathcal{P}_x^2 \right) = C(\lambda), \quad (2.10b)$$

where $C(\lambda)$ is a λ -dependent constant of integration. With u and ε given by (2.1b), $\varepsilon_N=0$, we set $C(\lambda)=C\lambda^N$ and sequentially calculate P_n . The first two terms are

$$P_0 = \sqrt{\frac{C}{u_N}}, \quad P_1 = \frac{1}{2u_N} \left(-u_{N-1} P_0 - \frac{1}{2} \varepsilon_{N-1} P_{0xx} + \frac{1}{4} \varepsilon_{N-1} \frac{P_{0x}^2}{P_0} \right). \quad (2.10c)$$

Since the leading term in the coefficients of λ^{N-n} is $2P_0 P_n$ we can *always* solve for P_n in terms of previously calculated (differential) expressions in u_0, \dots, u_N .

In the following discussion we concentrate on the (generalized) KdV choice, so that $u_N = -1$, and (2.10c), with $C = -4$, gives us

$$P_0 = 2, \quad P_1 = u_{N-1}. \quad (2.10d)$$

Variational Derivatives

To establish the Hamiltonian character of the corresponding flows (2.5a) we must prove that $\mathbf{P}^{(k)}$, defined by (2.6a), are variational derivatives of some functionals \mathcal{H}_k :

$$\mathbf{P}^{(k)} = \delta \mathcal{H}_k. \quad (2.11a)$$

From (2.10d) it follows that $\mathbf{P}^{(0)} = \delta \mathcal{H}_0$, $\mathbf{P}^{(1)} = \delta \mathcal{H}_1$ with

$$\mathcal{H}_0 = 2u_{N-1}, \quad \mathcal{H}_1 = 2u_{N-2} + \frac{1}{2}u_{N-1}^2. \quad (2.11b)$$

Formula (2.11a) thus follows from (2.6a) and the lemma of the introduction.

We summarise these results as follows:

Theorem. *There exists an infinite sequence of isospectral flows of (2.1a) which can be represented as the integrability conditions (2.2) of (2.1a) and (2.1c) with P defined by (2.9). These equations are Hamiltonian with respect to the $(N+1)$ mutually compatible Hamiltonian operators (2.6):*

$$\mathbf{u}_{t_m} = \mathbf{B}_{N-l} \delta \mathcal{H}_{m+l}, \quad l = 0, \dots, N, m = 0, 1, \dots \quad (2.12)$$

All the flows (2.12) commute.

λ -Expansion of the Riccati Equation

In the paper [2] we used an alternative method of generating an infinite sequence of Hamiltonians for the isospectral flows of the energy dependent Schrödinger spectral problem (2.1). A formal power series solution:

$$y = \sum_{r=-\infty}^s y_r \xi^r \quad (2.13)$$

of the Riccati equation

$$\varepsilon(y_x + y^2) + u = 0 \quad (2.14)$$

associated with the linear problem (2.1) through $y = \frac{\psi_x}{\psi}$, $\xi^2 = \lambda$, gives us an infinite sequence of conserved quantities y_r (half of which are trivial). To prove that the resulting (nontrivial) Hamiltonians \mathcal{H}_k are compatible with the multi-Hamiltonian ladder

$$\mathbf{B}_n \delta \mathcal{H}_k = \mathbf{B}_{n-l} \delta \mathcal{H}_{k+l} \quad (2.15)$$

we consider Eq. (2.14) as defining a change of variables $u = F[y] = -\varepsilon(y_x + y^2)$. The usual transformation properties of variational derivatives give

$$\frac{\delta}{\delta y} = (F')^\dagger \frac{\delta}{\delta u}. \quad (2.16a)$$

Acting on y we obtain

$$\varepsilon(\mathcal{P}_x - 2y\mathcal{P}) = 1, \quad (2.16b)$$

where $\mathcal{P} = \frac{\delta y}{\delta u}$. Differentiating (2.16b) twice and making use of (2.14) we arrive at

Eq. (2.10a). Since $\mathcal{P} = \frac{\delta y}{\delta u}$ differs from $\frac{\delta y}{\delta u_n}$ only by a factor of λ^n this proves that the variational derivatives of Hamiltonians obtained from the Riccati equation (2.14) satisfy (2.10a). Thus the Hamiltonians \mathcal{H}_k themselves satisfy (2.15). The advantage

of using the Riccati equation (2.14) and the expansion (2.13) instead of (2.10a) and (2.9b), respectively, is that we end up with Hamiltonians rather than just their variational derivatives thus rendering the lemma of Sect. 1 superfluous.

Remark. The apparent inconsistency of power series expansions (2.13) in the powers of ξ and (2.9a) in the powers of $\lambda = \xi^2$ is resolved by the fact that half of y_r are trivial so that the corresponding variational derivatives vanish. Thus we are effectively left with a power series expansion in the powers of $\xi^2 = \lambda$ only.

Inverse Hierarchy

In [2] we discussed the inverse hierarchy which exist whenever $\varepsilon_0 = 0$. This corresponds to having P polynomial in λ^{-1} rather than in λ . However, in the more general context of this paper the inverse hierarchy can be related to the previously discussed direct hierarchy by a simple transformation.

Dividing (2.1a) by λ^N we obtain

$$\sum_{i=0}^N \lambda^{i-N} (\varepsilon_i \partial^2 + u_i) \psi = 0. \quad (2.17a)$$

With $\lambda \rightarrow \lambda^{-1}$, $u_i \rightarrow u_{N-i}$, $\varepsilon_i \rightarrow \varepsilon_{N-i}$, (2.17a) is transformed into

$$\sum_{i=0}^N \lambda^i (\varepsilon_i \partial^2 + u_i) \psi = 0, \quad (2.17b)$$

which is identical to (2.1a). Thus the direct Harry Dym type hierarchies are transformed by the above substitutions into inverse KdV type ones. The condition $\varepsilon_0 = 0$ translates (after the above transformation) into $\varepsilon_N = 0$ and thus implies the existence of the inverse hierarchy.

3. Factorisation of Spectral Dependent Operators

In this section we construct Miura maps which relate isospectral flows of (2.1) to their modifications. To achieve this we generalise the factorisation approach described in the introduction. However, it is not enough to just choose v to be polynomial in λ in order to obtain u as a polynomial. We replace the factorisation (1.9b) by a quadratic form. The N modifications mentioned in the introduction arise from a sequence of N such quadratic forms. To achieve unity between the N component KdV and Harry Dym cases we once again work with $(N+1)$ components to begin with, but later specialise to each of these two cases.

Basic Formulae

We denote the modified variables by $v = (v_0, \dots, v_N)^T$.

Define

$$\begin{aligned} l_k &= \alpha_k \partial + v_k, \alpha_k \text{ constants}, \quad k = 0, \dots, N, \\ l &= (l_0, \dots, l_N). \end{aligned} \quad (3.1)$$

Let A be any constant, λ -dependent, $(N+1) \times (N+1)$ matrix, and use this to define a λ -dependent second order differential operator by the quadratic form $lA(-l^\dagger)$, the λ -dependence being derived purely from that of A . Equating this to our operator \mathbb{L} of (2.1) gives rise to a map between functions v_i and u_i . Different choices of A give rise to different maps.

The following quantities occur frequently below:

Define

$$\mathcal{V}_{kn} = -\alpha_k v_{nx} - \alpha_n v_{kx} - 2v_k v_n, \quad (3.2)$$

and

$$A_r = \begin{bmatrix} 1 & \dots & \lambda^{r-1} & & \\ \vdots & & \ddots & & \\ \lambda^{r-1} & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}, \quad r=0, \dots, N+1. \quad (3.3)$$

Making use of the formula $l_k l_n^\dagger + l_n l_k^\dagger = -2\alpha_k \alpha_n \partial^2 - \mathcal{V}_{kn}$ one can easily see that the identification:

$$\mathbb{L} = lA_r(-l^\dagger) \quad (3.4)$$

gives rise to the equations:

$$\varepsilon_k = \sum_{i=0}^k \alpha_{k-i} \alpha_i, \quad k=0, \dots, r-1, \quad (3.5a)$$

$$\varepsilon_k = \sum_{i=0}^{N-k} \alpha_{k+i} \alpha_{N-i}, \quad k=r, \dots, N, \quad (3.5b)$$

$$u_k = \frac{1}{2} \sum_{i=0}^k \mathcal{V}_{i, k-i}, \quad k=0, \dots, r-1, \quad (3.6a)$$

$$u_k = \frac{1}{2} \sum_{i=0}^{N-k} \mathcal{V}_{k+i, N-i}, \quad k=r, \dots, N. \quad (3.6b)$$

Remark. The formulae (3.5) are not a priori consistent. For instance, for $r > 1$, there is no choice of α_k which would give $\varepsilon_0 = 0, \varepsilon_1 = 1$. Thus, Ito's equation is ruled out of consideration here. Such inconsistencies are, however, exceptional.

When these formulae are consistent, Eqs. (3.6) define a differential mapping from v_i to u_i , sometimes invertible, sometimes not. We are particularly interested in the latter, since these define genuine Miura maps (in the sense of the restrictive definition given by Kupershmidt and Wilson [8], and in the introduction). The importance of these maps (both invertible and Miura) is in their action upon Hamiltonian operators. We must, therefore, reduce to one of our N -component systems before proceeding.

Generalised KdV Systems

Here we have $\varepsilon_N=0$, $u_N=-1$. The formulae (3.5), (3.6) corresponding to A_{N+1} do not admit this reduction, so we must restrict attention to A_0, \dots, A_N . We must have $\alpha_N=0$ and $v_N=\pm 1$ (we choose $v_N=-1$). Thus, $l_N=-1$, $\mathcal{V}_{NN}=-2$, $\mathcal{V}_{Nk}=2v_k$, so that (3.5b), (3.6b) reduce to

$$\varepsilon_k = \sum_{i=1}^{N-k-1} \alpha_{k+i} \alpha_{N-i}, \quad k=r, \dots, N-1, \quad (3.5c)$$

$$u_k = 2v_k + \frac{1}{2} \sum_{i=1}^{N-k-1} \mathcal{V}_{k+i, N-i}, \quad k=r, \dots, N-1, \quad (3.6c)$$

and (3.6c) defines an invertible map between v_r, \dots, v_{N-1} and u_r, \dots, u_{N-1} [subject to the consistency of (3.5c)]. In particular when $r=0$ the map $(v_0, \dots, v_{N-1}) \rightarrow (u_0, \dots, u_{N-1})$ is invertible.

Generalised Harry Dym Systems

Here we have $\varepsilon_N=0$, $u_0=-a^2$ (constant). The formulae corresponding to A_0 do not admit this reduction, so we must restrict attention to A_1, \dots, A_{N+1} . We must have $\alpha_N=0$, $v_0=\pm a$ (we choose $v_0=-a$). Thus $l_0=\alpha_0\partial-a$ and (3.5b), (3.6a) reduce to

$$\varepsilon_k = \sum_{i=1}^{N-k-1} \alpha_{k+i} \alpha_{N-i}, \quad k=r, \dots, N-1, \quad (3.5d)$$

$$u_k = -\alpha_0 v_{kx} + 2av_k + \frac{1}{2} \sum_{i=1}^{k-1} \mathcal{V}_{i, k-i}, \quad k=1, \dots, r-1. \quad (3.6d)$$

We will present our results in the context of the generalised KdV reduction. Most of our formulae can, however, be used in both cases. We shall point out the differences where they occur, and present a Harry Dym type example.

Subject only to the consistency of (3.5a, c) we have the following important proposition.

Proposition 1. *Under the change of variables $\mathbf{u}=M_r[\mathbf{v}]$ defined by (3.6a, c), the Hamiltonian operator $\mathbf{B}_r \in A_u^N[\partial]$, given by (2.6b), is the image:*

$$\mathbf{B}_r = M'_r \tilde{\mathbf{B}}_r (M'_r)^\dagger|_{u=M_r[v(r)]} \quad (3.7a)$$

of the constant, first order Hamiltonian operator $\tilde{\mathbf{B}}_r \in A_v^N[\partial]$:

$$\tilde{\mathbf{B}}_r = \frac{1}{4} \left[\begin{array}{c|c} \begin{array}{cc} 0 & -\partial \\ \hline -\partial & 0 \end{array} & \begin{array}{c} 0 \\ \hline 0 \end{array} \\ \hline \begin{array}{c} 0 \\ \hline 0 \end{array} & \begin{array}{cc} 0 & \partial \\ \hline \partial & 0 \end{array} \end{array} \right] \quad (3.7b)$$

where the diagonal blocks are respectively $r \times r$ and $(N-r) \times (N-r)$.

We will sometimes denote by $\mathbf{v}^{(r)}$ the modified variables corresponding to the map $\mathbf{u} = M_r[\mathbf{v}^{(r)}]$.

Proof. The Fréchet derivative of the mapping M_r is given by

$$M'_r = \begin{bmatrix} \begin{array}{c|c} \begin{array}{c} m_0 \\ \vdots \\ m_{r-1} \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & \begin{array}{c} m_N \dots m_{r+1} \\ \vdots \\ m_N \end{array} \end{array} \quad , \quad (3.8a)$$

where

$$m_k = -\alpha_k \partial - 2v_k, \quad k=0, \dots, N \quad (3.8b)$$

(giving $m_N = 2$ when $\alpha_N = 0$, $v_N = -1$). To obtain (3.7a) we use

$$m_k \partial m_n^\dagger + m_n \partial m_k^\dagger = -2(\alpha_k \alpha_n \partial^3 + \mathcal{V}_{kn} \partial + \partial \mathcal{V}_{kn}). \quad (3.8c)$$

The formulae (3.6a, c) then give the result.

Remark. The analogous proposition holds in the Harry Dym case. The only change is that (3.6b, d) defines now the $(r-1)^{\text{th}}$ mapping $\mathbf{u} = M_{r-1}[\mathbf{v}]$. This corresponds to the fact that we must now consider the set $\mathcal{A}_1, \dots, \mathcal{A}_{N+1}$ (instead of $\mathcal{A}_0, \dots, \mathcal{A}_N$) as defining the factorisation (3.4b).

Remark. Using (3.8c) one can easily check that the factorisation (3.4) has its counterpart on the level of the third order operator J : $J = m(-\frac{1}{4}\partial) \Lambda m^\dagger$, where $m = (m_0, \dots, m_N)$. The factorisation of J seems to be, in fact, more fundamental for our purposes than that of \mathbb{L} . It survives the super extension of the energy dependent Schrödinger operator discussed in [17].

We already have seen that since $\varepsilon_N = 0$, the map (3.6b) between v_r, \dots, v_{N-1} and u_r, \dots, u_{N-1} takes the invertible form (3.6c). Similarly when $\varepsilon_0 = 0$, the map (3.6a) becomes invertible:

$$u_k = -2v_0 v_k - \alpha_k v_{0x} + \frac{1}{2} \sum_{i=1}^{k-1} \mathcal{V}_{i, k-i}, \quad k=0, \dots, r-1, \quad (3.6e)$$

so that the whole map $\mathbf{u} = M_r[\mathbf{v}]$ is invertible for each $r=0, \dots, N$. In this case $\alpha_0 = 0$ and $m_0 = -2v_0$, so that M'_r is easily inverted.

We now concentrate on the case $\varepsilon_0 \neq 0$, so that the map (3.6a, c) is a genuine Miura map (for $r > 0$). For clarity, we choose the most interesting case [2] of $\varepsilon_0 = 1$, $\varepsilon_i = 0$ for $i \geq 1$.

Miura Maps

Let $\alpha_0 = 1$, $\alpha_i = 0$ for $i \geq 1$, so that $\varepsilon_0 = 1$, $\varepsilon_i = 0$ for $i \geq 1$. In this case the formulae (3.5a, c) with $r > 0$ are always consistent. However, the map M_0 defined by (3.6c) is

still valid even though Eqs. (3.5c) are sometimes inconsistent for $r=0$ [see remark following (3.15)]. The map $\mathbf{u}=M_0[\mathbf{v}]$, corresponding to \mathcal{A}_0 , is invertible, whilst those corresponding to all other $\mathcal{A}_r(r>0)$ are genuine Miura maps. In fact, (3.6b) defines an invertible map whilst (3.6a) is the genuine Miura part. Thus the upper block of \mathcal{A}_r is the important part when discussing genuine Miura maps. We therefore consider the map M_N corresponding to \mathcal{A}_N . In this case the Miura map $\mathbf{u}=M_N[\mathbf{v}]$ is given purely by (3.6a). The Fréchet derivative (3.8a) is then:

$$M'_N = \begin{bmatrix} m_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ m_{N-1} & \dots & & m_0 \end{bmatrix} \quad (3.9a)$$

with $m_0 = -\partial - 2v_0$, $m_i = -2v_i$, $i \geq 1$, and the constant coefficient operator

$$\tilde{\mathbf{B}}_N = \frac{1}{4} \begin{bmatrix} 0 & -\partial \\ & \ddots \\ -\partial & 0 \end{bmatrix} \quad (3.9b)$$

is mapped onto \mathbf{B}_N of (2.6b). It is easy to see that the pre-image of \mathbf{B}_r , for $r < N$, is non-local.

The Miura map $\mathbf{u}=M_N[\mathbf{v}^{(N)}]$ can be decomposed into N primitive ones. Define a sequence of maps $\mathbf{u}^{(k)}=M_N^{k+1}[\mathbf{u}^{(k+1)}]$ by:

$$u_k^{(k)} = \frac{1}{2} \sum_{i=0}^k \mathcal{U}_{i,k-i}^{(k+1)}, \quad u_i^{(k)} = u_i^{(k+1)} \quad \text{for } i \neq k \quad (3.10a)$$

where $\mathcal{U}_{ij}^{(k)}$ is given by (3.2) but with v_i replaced by $u_i^{(k)}$. We can write $\mathbf{u} \equiv \mathbf{u}^{(0)} = M_N[\mathbf{u}^{(N)}] \equiv M_N[\mathbf{v}^{(N)}]$ as the composition of these maps:

$$\mathbf{u}^{(0)} = M_N[\mathbf{u}^{(N)}] = M_N^1 \circ M_N^2 \circ \dots \circ M_N^N[\mathbf{u}^{(N)}].$$

The Fréchet derivative M'_N is thus the product of N Fréchet derivatives: $M'_N = (M_N^1)' \times \dots \times (M_N^N)'$,

$$(M_N^k)' = \begin{bmatrix} 1 & & & 0 \\ 0 & \ddots & & \\ & & \ddots & 1 \\ m_{k-1} & \dots & & m_0 \\ & & 0 & 1 \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \quad (3.10b)$$

with m_{k-1}, \dots, m_0 on the k^{th} row. It is a very simple calculation to see that the product of these matrices is just (3.9a).

Remark. Each of these maps is non-invertible.

Non-Degeneracy

It was remarked in the introduction that whenever a Miura map M is non-degenerate (injective), then the Hamiltonian nature of the operator

$$\mathbf{B} = M' \tilde{\mathbf{B}}(M')^\dagger|_{u=M[v]}$$

follows from that of the operator $\tilde{\mathbf{B}}$. To check the non-degeneracy of a Miura map it is enough to check that its Fréchet derivative is formally invertible (as a matrix). In our case it is an easy matter to *explicitly* write down the inverses of (3.9a) and (3.10b) in terms of m_k and m_k^{-1} , thus proving the non-degeneracy of M_N and M_N^k . The same is true of all the other Miura maps discussed in this paper. It is thus possible to deduce the existence of a sequence of Hamiltonian operators for each of our modifications, as described below.

Starting with $\mathbf{B}_N^N = \tilde{\mathbf{B}}_N$ of (3.9b) define $\mathbf{B}_N^k \in A_{u^{(k)}}^N[\partial]$ inductively by:

$$\mathbf{B}_N^{k-1} = (M_N^k)' \mathbf{B}_N^k ((M_N^k)')^\dagger|_{u^{(k-1)} = M_N^k[u^{(k)}]}, \quad k = N, \dots, 1. \quad (3.10c)$$

\mathbf{B}_N^0 is just our original \mathbf{B}_N of (2.6b).

Direct calculation shows that, as indicated by the notation, each \mathbf{B}_N^k is *locally* defined in terms of the variables $u^{(k)}$. Thus, each of the maps M_N^k is a genuine, Hamiltonian Miura map.

Let $M^{(r)} = M_N^1 \circ M_N^2 \circ \dots \circ M_N^r$. Explicitly, this has the form

$$u_k^{(0)} = \frac{1}{2} \sum_{i=0}^k \mathcal{U}_{i,k-i}^{(r)}, \quad k = 0, \dots, r-1, \quad (3.11a)$$

$$u_k^{(0)} = u_k^{(r)}, \quad k = r, \dots, N-1. \quad (3.11b)$$

The choice of A corresponding to the map $M^{(r)}$ is:

$$A^{(r)} = \left[\begin{array}{c|c} \begin{array}{ccc} 1 & \dots & \lambda^{r-1} \\ \vdots & & \vdots \\ \lambda^{r-1} & & 0 \end{array} & 0 \\ \hline 0 & \begin{array}{ccc} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \frac{1}{2}\lambda^r & \dots & \lambda^N \end{array} \end{array} \right]. \quad (3.11c)$$

Formula (3.11a) is identical to (3.6a). Since (3.11b) is the identity map and (3.6c) is invertible, $M^{(r)}$ and M_r differ only by the invertible map:

$$\begin{aligned} u_k^{(r)} &= v_k^{(r)}, & k &= 0, \dots, r-1, \\ u_k^{(r)} &= 2v_k^{(r)} + \frac{1}{2} \sum_{i=1}^{N-k-1} \mathcal{V}_{k+i, N-i}^{(r)}, & k &= r, \dots, N-1. \end{aligned} \quad (3.12)$$

Thus, there exists an operator $\mathbf{B}_r^r \in A_{\mathbf{u}^{(r)}}^N[\partial]$ such that $(M^{(r)})' \mathbf{B}_r^r ((M^{(r)})')^\dagger|_{\mathbf{u}^{(0)} = M^{(r)}[\mathbf{u}^{(r)}]} = \mathbf{B}_r^0 = \mathbf{B}_r$. The operators \mathbf{B}_r^r and $\tilde{\mathbf{B}}_r$ [given by (3.7b)] differ only through the invertible map (3.12). Explicitly:

$$\mathbf{B}_r^r = \begin{bmatrix} 0 & -\frac{1}{4}\partial & & \\ & \ddots & & \\ -\frac{1}{4}\partial & & 0 & \\ \hline & & & 0 \\ 0 & & & -J_{r+1} \dots -J_N \\ & & & \vdots \\ & & & -J_N \\ & & & 0 \end{bmatrix}. \quad (3.13)$$

In fact, we have more.

Proposition 2. *There exist local Hamiltonian operators \mathbf{B}_k^r such that $(M^{(r)})' \mathbf{B}_k^r ((M^{(r)})')^\dagger = \mathbf{B}_k^0 \equiv \mathbf{B}_k$ for $k=r, \dots, N$. These constitute $(N-r+1)$ compatible Hamiltonian structures for the r^{th} modification. The sequence of modified Hamiltonians is defined by $\mathcal{H}_n^r = \mathcal{H}_n \circ M^{(r)}$ and the r^{th} modified hierarchy is written as:*

$$\mathbf{u}_{t_n}^{(r)} = \mathbf{B}_{N-k}^r \delta \mathcal{H}_{n+k}^r, \quad k=0, \dots, N-r, n=0, 1, \dots \quad (3.14)$$

We can represent these modifications and their Hamiltonian structures schematically as follows:

$$\begin{array}{ccccccc} \mathbf{u} \equiv \mathbf{u}^{(0)} & \xleftarrow{M_N^1} & \mathbf{u}^{(1)} & \xleftarrow{M_N^2} & \dots & \xleftarrow{M_N^r} & \mathbf{u}^{(r)} \xleftarrow{M_{N+1}^r} \dots \xleftarrow{M_N^N} \mathbf{u}^{(N)} \equiv \mathbf{v}^{(N)} \\ \mathbf{B}_N \equiv \mathbf{B}_N^0 & \longleftarrow & \mathbf{B}_N^1 & \dots & \longleftarrow & \mathbf{B}_N^r & \longleftarrow \dots \longleftarrow \mathbf{B}_N^N \\ \vdots & & & & & & \\ \mathbf{B}_r \equiv \mathbf{B}_r^0 & \longleftarrow & \mathbf{B}_r^1 & \dots & \longleftarrow & \mathbf{B}_r^r & \\ \vdots & & & & & & \\ \mathbf{B}_0 \equiv \mathbf{B}_0^0 & & & & & & \end{array}$$

Fig. 1

Remark. If we wish to study the r^{th} modified hierarchy it is more convenient to transform each of the Hamiltonian structures $\mathbf{B}_r^r, \dots, \mathbf{B}_N^r$ into the equivalent set $\tilde{\mathbf{B}}_r^r, \dots, \tilde{\mathbf{B}}_N^r$, obtained by the action of the invertible map (3.12). In these co-ordinates, $\tilde{\mathbf{B}}_r^r$ is just $\tilde{\mathbf{B}}_r$ of (3.7b), so that the lowest Hamiltonian structure is constant and first order.

Hamiltonian Property of the Operators \mathbf{B}_n

The Miura maps and invertible transformations described above give us an easy proof that those differential operators \mathbf{B}_r which are related to the constant coefficient operators $\tilde{\mathbf{B}}_r$ are Hamiltonian. Unfortunately, this does not apply to all the operators \mathbf{B}_n (with an arbitrary choice of ε) since Eqs. (3.5a, b) may not have any solutions. It is easily seen, however, that when $\varepsilon_0 \varepsilon_N \neq 0$ the problem of inconsistency does not arise and (3.5a, b) can be solved for $\alpha_0, \dots, \alpha_N$, whatever choice of

$\varepsilon_1, \dots, \varepsilon_{N-1}$. Thus the (non-invertible) map

$$\begin{aligned} u_k &= \frac{1}{2} \sum_{i=0}^k \mathcal{V}_{i, k-i}, & k=0, \dots, r-1, \\ u_k &= -\alpha_N v_{kx} + 2v_k + \frac{1}{2} \sum_{i=1}^{N-k-1} \mathcal{V}_{k+i, N-i}, & k=r, \dots, N-1, \end{aligned} \quad (3.15)$$

which differs from (3.6a,c) only by the terms involving $\alpha_n = \sqrt{\varepsilon_N} \neq 0$, is a Hamiltonian Miura map transforming the constant coefficient Hamiltonian operator $\tilde{\mathbf{B}}_r$ into \mathbf{B}_r . This gives us a proof that all the operators \mathbf{B}_n with $\varepsilon_0 \varepsilon_N \neq 0$ are indeed Hamiltonian, and this property survives the limit $\varepsilon_0 \rightarrow 0$ and/or $\varepsilon_N \rightarrow 0$, even though the map (3.15) itself becomes, in general, singular. Thus, whatever the choice of ε , the operators \mathbf{B}_n are all Hamiltonian.

Remark. In [2] we noticed that in the particular case when just one ε_k , say ε_r , is different from zero, \mathbf{B}_r can be transformed (by an invertible change of variables) into a constant coefficient form. In the present set up such a transformation is provided by (3.6a,c) or (3.15) with all the α_k (including α_r) equal to zero. That choice of α_k leads to $\varepsilon = 0$ rather than to $\varepsilon = \varepsilon_r \lambda^r$. However, \mathbf{B}_r does not depend on ε_r , so that this transformation is still valid for the one operator \mathbf{B}_r , even though it cannot be used with any of the others. This simple quadratic map is the inverse of the complicated transformation constructed out of Hamiltonians in [2].

Modified Spectral Problem

Generalising the derivation of (1.9c) we can use the factorisation (3.4) to obtain the spectral problem corresponding to each of our modifications (3.11). The first modification uses $\mathcal{A}^{(1)}$ [given by (3.11c)] which leads to:

$$(\partial + u_0^{(1)})(\partial - u_0^{(1)})\psi_1 + (\lambda u_1^{(1)} + \dots + \lambda^{N-1} u_{N-1}^{(1)})\psi_1 = \lambda^N \psi_1. \quad (3.16a)$$

Defining ψ_2 by:

$$(\partial - u_0^{(1)})\psi_1 = \lambda \psi_2 \quad (3.16b)$$

we find:

$$(\partial + u_0^{(1)})\psi_2 = (-u_1^{(1)} - \dots - \lambda^{N-2} u_{N-1}^{(1)} + \lambda^{N-1})\psi_1. \quad (3.16c)$$

Equations (3.16b,c) constitute a 2×2 matrix spectral problem for the first modification. The spectral problems for the remaining modifications are obtained from this one in succession by a series of substitutions and gauge transformations. This is illustrated by the example given below.

Example: Dispersive Water Waves. We illustrate the above construction on the simplest nontrivial example, $N=2$. The resulting DWW hierarchy is tri-Hamiltonian, the Hamiltonian operators being

$$\begin{aligned} \mathbf{B}_0 &= \begin{pmatrix} -\frac{1}{2}u_1\partial - \frac{1}{2}\partial u_1 & \partial \\ \partial & 0 \end{pmatrix}, & \mathbf{B}_1 &= \begin{pmatrix} \frac{1}{4}\partial^3 + \frac{1}{2}u_0\partial + \frac{1}{2}\partial u_0 & 0 \\ 0 & \partial \end{pmatrix}, \\ \mathbf{B}_2 &= \begin{pmatrix} 0 & \frac{1}{4}\partial^3 + \frac{1}{2}u_0\partial + \frac{1}{2}\partial u_0 \\ \frac{1}{4}\partial^3 + \frac{1}{2}u_0\partial + \frac{1}{2}\partial u_0 & \frac{1}{2}u_1\partial + \frac{1}{2}\partial u_1 \end{pmatrix}. \end{aligned} \quad (3.17)$$

The first nontrivial flow $\mathbf{u}_{t_1} = \mathbf{B}_2 \delta \mathcal{H}_1$ ($\mathcal{H}_1 = 2u_0 + \frac{1}{2}u_1^2$) is

$$\begin{aligned} u_{0t_1} &= \frac{1}{4}u_{1xx} + \frac{1}{2}u_1u_{0x} + u_0u_{1x}, \\ u_{1t_1} &= u_{0x} + \frac{3}{2}u_1u_{1x}. \end{aligned} \quad (3.18)$$

The Miura maps $\mathbf{u} = M_2^1[\mathbf{w}]$, $\mathbf{w} = M_2^2[\mathbf{v}]$ (we denote $\mathbf{u}^{(1)}$ by \mathbf{w} here) are given by:

$$u_0 = -w_{0x} - w_0^2, \quad u_1 = w_1, \quad (3.19a)$$

$$w_0 = v_0, \quad w_1 = -v_{1x} - 2v_0v_1. \quad (3.19b)$$

Their superposition $\mathbf{u} = M_2[\mathbf{v}] = M_2^1 \circ M_2^2[\mathbf{v}]$ is

$$u_0 = -v_{0x} - v_0^2, \quad u_1 = -v_{1x} - 2v_0v_1. \quad (3.19c)$$

To compare the above Miura maps with those introduced by Kupershmidt in [9] we perform an invertible change of variables

$$\begin{aligned} q &= u_0 + \frac{1}{4}u_1^2 - \frac{1}{2}u_{1x} = F_1[u_0, u_1], \\ r &= u_1 = F_2[u_0, u_1]. \end{aligned} \quad (3.20)$$

The map (3.20) transforms the Hamiltonian operators (3.17) into $\bar{\mathbf{B}}_n = F' \mathbf{B}_n (F')^\dagger|_{q,r}$, $n=0, 1, 2$, given by

$$\begin{aligned} \bar{\mathbf{B}}_0 &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad \bar{\mathbf{B}}_1 = \frac{1}{2} \begin{pmatrix} q\partial + \partial q & -\partial^2 + r\partial \\ +\partial^2 + \partial r & 2\partial \end{pmatrix}, \\ \bar{\mathbf{B}}_2 &= \frac{1}{4} \begin{pmatrix} (r-\partial)(q\partial + \partial q) + (q\partial + \partial q)(r+\partial) & (r-\partial)^2\partial + 2(q\partial + \partial q) \\ \partial(r+\partial)^2 + 2(q\partial + \partial q) & 2(r\partial + \partial r) \end{pmatrix}. \end{aligned} \quad (3.21)$$

The flow (3.18), when written in the new coordinates (3.20), reads

$$\begin{aligned} q_{t_1} &= \frac{1}{2}(-q_x + 2qr)_x, \\ r_{t_1} &= \frac{1}{2}(r_x + 2q + r^2)_x, \end{aligned} \quad (3.22)$$

which is just the standard form of DWW equation.

Superposition of the Miura map (3.19a) with (3.20), giving:

$$\begin{aligned} q &= -w_{0x} - \frac{1}{2}w_{1x} - w_0^2 + \frac{1}{4}w_1^2, \\ r &= w_1, \end{aligned} \quad (3.23)$$

is easily seen to be equivalent to Kupershmidt's first modification of DWW hierarchy [9]. The second Miura map (3.19b), however, is not equivalent to either of Kupershmidt's second modifications. The corresponding modified DWW hierarchy is given by:

$$\mathbf{v}_{t_1} = \mathbf{B}_2^2 \delta \mathcal{H}_1^2, \quad (3.24)$$

where $\mathcal{H}_1^2[\mathbf{v}] = \mathcal{H}_1 \circ M_2[\mathbf{v}]$. The first nontrivial flow is

$$\begin{aligned} v_{0t_1} &= (\frac{1}{4}v_{1xx} + \frac{1}{2}v_1v_{0x} - v_0^2v_1)_x, \\ v_{1t_1} &= (v_0 - v_0v_1^2 - \frac{1}{2}v_1v_{1x})_x. \end{aligned} \quad (3.25a)$$

An invertible change of variables:

$$h_0 = v_0 + \frac{1}{2} v_{1x}, \quad h_1 = v_1, \quad (3.25b)$$

brings (3.25a) into a more symmetric form:

$$\begin{aligned} h_{0t_1} &= \left(\frac{1}{2} h_{0x} + \left(\frac{1}{2} h_1 - h_1^2\right) h_{0x} - \frac{1}{2} h_1 h_{1xx} - \frac{1}{4} h_1^2 + \frac{1}{4} h_1 h_{1x}^2 + \frac{1}{4} h_1^2 h_{1xx} - h_1 h_0^2\right)_x, \\ h_{1t_1} &= \left(-\frac{1}{2} h_{1x} + h_0 + \frac{1}{6} (h_1^3)_x - \frac{1}{4} (h_1^2)_x - h_0 h_1^2\right)_x. \end{aligned} \quad (3.25c)$$

The spectral problem for the first modification (3.19a) is given by (3.16), which, in this case, takes the form:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} w_0 & \lambda \\ \lambda - w_1 & -w_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (3.26a)$$

Writing (3.26a) in the variables (v_0, v_1) [using (3.19b)] and gauge transforming with $T = \begin{pmatrix} 1 & 0 \\ -v_1 & 1 \end{pmatrix}$ we obtain the spectral problem for the second modification:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} v_0 + \lambda v_1 & \lambda \\ \lambda(1 - v_1^2) & -v_0 - \lambda v_1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (3.26b)$$

Example: 2-Component Harry Dym System [4]. This corresponds to the Harry Dym type reduction of the $N=2$, $\varepsilon=1$, Schrödinger linear problem (2.1). The 3 Hamiltonian operators of the hierarchy are given by:

$$\begin{aligned} \mathbf{B}_0 &= \begin{pmatrix} -\frac{1}{2} u_1 \partial - \frac{1}{2} \partial u_1 & -\frac{1}{2} u_2 \partial - \frac{1}{2} \partial u_2 \\ -\frac{1}{2} u_2 \partial - \frac{1}{2} \partial u_2 & 0 \end{pmatrix}, \\ \mathbf{B}_1 &= \begin{pmatrix} \frac{1}{4} \partial^3 - a^2 \partial & 0 \\ 0 & -\frac{1}{2} u_2 \partial - \frac{1}{2} \partial u_2 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & \frac{1}{4} \partial^3 - a^2 \partial \\ \frac{1}{4} \partial^3 - a^2 \partial & \frac{1}{2} u_1 \partial + \frac{1}{2} \partial u_1 \end{pmatrix}. \end{aligned} \quad (3.27)$$

The factorisation (3.4) with

$$A_3 = \begin{bmatrix} 1 & \lambda & \lambda^2 \\ \lambda & \lambda^2 & 0 \\ \lambda^2 & 0 & 0 \end{bmatrix} \quad (3.28a)$$

gives us the Miura map $\mathbf{u} = M_2[v]$:

$$\begin{aligned} u_1 &= -v_{1x} + 2av_1, \\ u_2 &= -v_{2x} + 2av_2 - v_1^2, \end{aligned} \quad (3.28b)$$

relating \mathbf{B}_2 to the constant coefficient operator $\tilde{\mathbf{B}}_2$. The map (3.28b) can be decomposed as follows:

$$u_1 = -w_{1x} + 2aw_1, \quad u_2 = w_2, \quad (3.28c)$$

$$w_1 = v_1, \quad w_2 = -v_{2x} + 2av_2 - v_1^2. \quad (3.28d)$$

The transformation (3.28c) with $a=0$ is equivalent to the Miura map presented in [4].

The factorisation (3.4) with

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & \lambda^2 \end{bmatrix} \quad (3.29a)$$

gives us an invertible transformation

$$u_1 = -2v_1v_2, \quad u_2 = -v_2^2, \quad (3.29b)$$

transforming \mathbf{B}_0 into its constant coefficient form $\tilde{\mathbf{B}}_0$.

4. Conclusions

In this paper we have introduced two basic methods, which we have applied to the linear spectral problem (2.1a). The first of these was to use an unusual form of the Lax approach so that in *one construction* we obtain the time evolution (2.1c), the Hamiltonian operators (2.6b) and the Hamiltonians (2.11). The second was to use the quadratic form (3.4) as a generalisation of the factorisation (1.9) as a means of producing the Miura (or invertible) maps (3.6).

The method we give for constructing the time evolutions (2.1c) is applicable to a much broader class of spectral problems than the usual Gelfand-Dikii approach. For instance, the latter cannot easily be applied to energy dependent operators. However, even in the most standard cases, such as the KdV hierarchy, our approach gives a very simple and elegant construction of the associated Hamiltonian operators. Applied to the usual third order Lax operator our method immediately gives the two Hamiltonian structures for the Boussinesq hierarchy. It is not, however, guaranteed that multi-component, multi-Hamiltonian extensions will exist. In the Boussinesq case, for example, it seems that only a 4 component, tri-Hamiltonian extension is possible [18]. On the other hand, Kupershmidt's non-standard Lax operators [9], when written in purely differential form, can be extended to energy dependent versions. His representation of the DWW equations can be extended in this way [12], but the resulting system is gauge equivalent to a subclass of the systems discussed in this paper.

The importance of the existence of our Miura and invertible maps is (at least) two-fold. First, by relating each Hamiltonian structure to a constant coefficient, first order operator, these maps give us a simple, direct proof that our operators are indeed Hamiltonian. More importantly, however, they give us the remarkable chain of modifications depicted in Fig. 1. This is a direct generalisation of the known, simplest cases of the (single component) KdV and (two component) DWW equations. Another important aspect is our method of construction. Whilst this can be reduced to the standard factorisation method for the KdV ($N=1$) case, it is a new construction for the energy dependent linear problem. It should be remarked that the origin of the Miura maps for the DWW equations was previously unknown.

In [17] we generalise all the results of the present paper to the "super" case. This places a variety of previously known, but disparate, examples into a single framework. Contrary to Kupershmidt's expectation that modifications do not survive super extensions [19], our construction *does* still work in this case.

The Hamiltonian structures of this paper have recently been derived by an r -matrix approach [20, 21]. An interesting open problem is the construction of our Miura maps by Lie algebraic means.

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