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Higher Spin Fields and the Gelfand–Dickey Algebra

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Abstract. We show that in 2-dimensional field theory, higher spin algebras are contained in the algebra of formal pseudodifferential operators introduced by Gelfand and Dickey to describe integrable nonlinear differential equations in Lax form. The spin 2 and 3 algebras are discussed in detail and the generalization to all higher spins is outlined. This provides a conformal field theory approach to the representation theory of Gelfand–Dickey algebras.

1. Introduction

Recently Zamolodchikov investigated additional symmetries in 2-dimensional conformal field theory generated by higher spin local currents [1]. It is known that in two dimensions the independent components of the stress energy tensor $T(z)$, $\bar{T}(\bar{z})$, generate the (infinite) algebra of conformal transformations. The operator product expansion for the fields $T(z)$ has the form

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots, \tag{1}$$

where $\cdots$ denote all nonsingular terms. Introducing Fourier components $L_n (n \in \mathbb{Z})$, we obtain the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{2.3!}(n^3 - n)\delta_{n+m,0}. \tag{2}$$

Primary conformal fields $\varphi_\Delta(z)$ with conformal weight Δ are characterized by the operator product expansion

$$T(z)\varphi_\Delta(w) = \frac{\Delta}{(z-w)^2}\varphi_\Delta(w) + \frac{\partial\phi_\Delta(w)}{z-w} + \cdots. \tag{3}$$

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The complete set of local fields, occurring in a conformal theory consists of conformal families $[\varphi_\Delta]$ obtained by applying the operators L_n (with $n<0$) to the primary fields φ_Δ[2]. In quantum field theory, the set of all conformal families forms a closed operator algebra. The spin of a primary field φ is defined to be

$$s=\Delta-\bar{\Delta}, \tag{4}$$

where Δ and $\bar{\Delta}$ are the conformal weights appearing in the operator product expansions $T(z)\varphi(w,\bar{w})$ and $\bar{T}(\bar{z})\varphi(w,\bar{w})$, respectively. Bosonic fields correspond to integer values of s, while fermionic ones correspond to half-integer values. Interestingly, enough, primary fields W_s with weights $\Delta=s$, $\bar{\Delta}=0$ might arise in a given conformal field theory. Such fields signal the presence of additional symmetries in the theory, since for any arbitrary (but analytic) function $f(z)$, the currents

$$J_f(z):=f(z)\,W_s(z) \tag{5}$$

are conserved, i.e., $\partial_{\bar{z}}J_f(z)=0$. The simplest example ($s=1$) was considered in [3], where the additional infinite dimensional symmetry algebras were identified with Kac–Moody current algebras that arise in the conformal field theory of nonlinear σ-models with Wess–Zumino–Witten topological terms. However, one may consider higher spin algebras and try to construct new classes of conformal field theories associated with them.

Zamolodchikov investigated the spin 3 algebra by exploiting the operator product expansion for the fields $W_{s=3}(z)$ [1]. As it turns out, the operator algebra generated by the fields $T(z)$ and $W_3(z)$ is not a Lie algebra because its determining relations are quadratic. In terms of Fourier modes, the spin 3 algebra is described by the following commutation relations:

$$[L_n,W_m]=(2n-m)W_{m+n}, \tag{6a}$$

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{2.3!}(n^3-n)\delta_{n+m,0}, \tag{6b}$$

$$\begin{aligned}[W_n,W_m]=&\frac{c}{3.5!}n(n^2-1)(n^2-4)\delta_{n+m,0}+b^2(n-m)\Lambda_{n+m}\\&+(n-m)[\tfrac{1}{15}(n+m+2)(n+m+3)-\tfrac{1}{6}(n+2)(m+2)]L_{n+m}.\end{aligned} \tag{6c}$$

In Eq. (6), the following identifications have been made:

$$\Lambda_n=\sum_{k=-\infty}^{+\infty}:L_kL_{n-k}:+\tfrac{1}{5}x_nL_n, \tag{7a}$$

$$b^2=\frac{16}{22+5c}, \tag{7b}$$

$$x_{2l}=(1+l)(1-l);\quad x_{2l+1}=(2+l)(1-l). \tag{7c}$$

The highest weight representations of the operator algebra (6) were studied in [4], and as a result an infinite series of new conformal models that possess a global $\mathbb{Z}_3$ symmetry were constructed in two dimensions. Furthermore, it was found that these new (minimal) models with central charge

$$c = 2\left(1 - \frac{12}{p(p+1)}\right); \quad p = 4, 5, 6 \ldots \tag{8}$$

have a hidden relation with the $SL(3, \mathbb{R})$ Kac–Moody algebra, thus making it possible to prove the positivity theorem that guarantees their unitarity. The hope of obtaining a systematic description of all types of criticality in two dimensions by considering all higher spin algebras with integer (or half-integer) values of s was also expressed in [4].

It is the purpose of this paper to provide a framework in which to study higher spin operator algebras in connection with the theory of integrable nonlinear differential equations and the hidden affine Kac–Moody symmetries they contain. In particular, we show that the spin 3 algebra (6) is contained in the Gelfand–Dickey algebra of formal pseudodifferential operators introduced to describe $1 + 1$ nonlinear differential equations, such as KdV and KP, in Hamiltonian (Lax) form. We shall also comment on the generalization of this approach to include all higher spin operator algebras. Incidentally, we mention that the Gelfand–Dickey algebra of formal pseudodifferential operators has been used in [5] to construct sheaves of Lie algebras on algebraic curves which provide globalized generalizations of the Virasoro algebra (in the sense that a central charge is associated to each point of a Riemann surface and each closed oriented curved on it). This constution has stimulated many attempts to develop the operator formalism for conformal field theories defined on higher genus Riemann surfaces (see for instance [6]), as well as clarify further the role that the diffeomorphism group of the circle plays in Polyakov's approach to string theory. The crucial ingredient is that the Virasoro algebra can be embedded in the Gelfand–Dickey algebra in a natural way (see also [7]). We shall return to this point later on.

Here is an outline of what follows. In Sect. 2 we review the basic theory of formal pseudodifferential operators and define the Gelfand–Dickey algebras of the type GD($SL(n)$). In Sects. 3 and 4 we study in detail the Virasoro and spin 3 operator algebras, respectively, using the Gelfand–Dickey bracket of the second kind. Finally (Sect. 5), we discuss the generalization to all higher spin fields and indicate possible applications of this approach to 2-dim conformal field theory and statistical mechanics.

2. The Algebra of Formal Pseudodifferential Operators

Let us now present some basic facts from the theory of formal pseudodifferential operators (see for instance [8]). First, consider the ring of all differential operators

$$L = u_n(z)\partial^n + u_{n-1}(z)\partial^{n-1} + \cdots + u_1(z)\partial + u_0(z). \tag{9}$$

(Here ∂ denotes the derivative with respect to z.) The multiplication law is provided by the Leibniz rule

$$(a_i(z)\partial^i)\circ(b_j(z)\partial^j) = \sum_k \binom{i}{k} a_i(z)\partial^k b_j(z)\partial^{i+j-k}, \tag{10}$$

which, from the point of view of one-dimensional quantum mechanics, is equivalent to the normal ordering prescription.[1] Even more important is the ring of formal pseudodifferential operators that consists of the formal series

$$\mathbb{L} = u_n(z)\partial^n + \cdots + u_1(z)\partial + u_0(z) + \sum_{k=-\infty}^{-1} u_k(z)\partial^k. \tag{11}$$

For notational purposes, it would be convenient to use the identifications

$$\mathbb{L}_+ = L; \quad \mathbb{L}_- = \sum_{k=-\infty}^{-1} u_k(z)\partial^k; \quad \operatorname{res}\mathbb{L} = u_{-1}(z), \tag{12}$$

where "res" stands for residue. The multiplication law for formal pseudodifferential operators is also provided by the Leibniz rule; for completeness, we mention that for all $k > 0$ the following identity is true:

$$\partial^{-k} \circ a(z) = \sum_{l=0}^{\infty} (-1)^l \frac{(k+l-1)!}{l!(k-1)!} \partial^l a(z) \partial^{-k-l}. \tag{13}$$

At this point notice that primary conformal fields have a natural interpretation in the ring of formal pseudodifferential operators. In particular, by introducing the bracket $[\mathbb{L}_1, \mathbb{L}_2] := \mathbb{L}_1 \circ \mathbb{L}_2 - \mathbb{L}_2 \circ \mathbb{L}_1$, we observe that

$$[\varepsilon_1(z)\partial, \varepsilon_2(z)\partial] = (\varepsilon_1(z)\varepsilon_2'(z) - \varepsilon_1'(z)\varepsilon_2(z))\partial \tag{14}$$

and

$$[\varepsilon(z)\partial, \varphi(z)\partial^k \bmod \partial^{k-1}] = (\varepsilon(z)\varphi'(z) - k\varepsilon'(z)\varphi(z))\partial^k \bmod \partial^{k-1}. \tag{15}$$

Equation (14) implies that the Lie algebra of vector fields on the circle is contained in the algebra of formal pseudodifferential operators, while (15) suggests that the operator $\varphi(z)\partial^k \bmod \partial^{k-1}$ represents a primary field with conformal weight $\Delta = -k$ (cf. Eq. (31)).

The commutator $[\mathbb{L}_1, \mathbb{L}_2]$ of two formal pseudodifferential operators $\mathbb{L}_1$ and $\mathbb{L}_2$ with degrees n_1 and n_2, respectively, has degree $n_1 + n_2 - 1$. Therefore, all formal pseudodifferential operators of negative degree only form an algebra with respect to $[\ ,\ ]$ also known as the Volterra algebra. For convenience, we drop the multiplication symbol $\circ$. From now on, we let

$$X = \sum_{i=1}^{\infty} \partial^{-i} x_i(z) \tag{16}$$

represent a generic element (symbol) of the Volterra algebra—usually, only a finite number of x_i's will be taken to be nonzero. Then we may define a pairing between the space of differential operators L and elements of the Volterra algebra by the formula [8]:

$$\langle L, X \rangle := \int dz \operatorname{res}(LX). \tag{17}$$

This enables us to think of the space of differential operators L as being the

[1] One may think of any such operator L as the normal ordered operator corresponding to the (classical) function $u_n(z)\bar{z}^n + u_{n-1}(z)\bar{z}^{n-1} + \cdots + u_1(z)\bar{z} + u_0(z)$ on the plane with canonical coordinates z and $\bar{z}$

(smooth) dual of the Volterra algebra. Since the residue of the commutator of any two (formal) pseudodifferential operators is always a total derivative, we have that

$$\langle L, [X_1, X_2] \rangle = \int dz \operatorname{res}([L, X_1]_+ X_2), \tag{18}$$

which defines the coadjoint representation of the Volterra algebra. For a given $L = u_n(z)\partial^n + \cdots + u_1(z)\partial + u_0(z)$, the righthand side of Eq. (17) involves only a finite number of terms, $\operatorname{res}(LX) = u_0(z)x_1(z) + \cdots + u_n(z)x_{n+1}(z)$, which makes the pairing $\langle \, , \rangle$ meaningful. We remark that the residue duality adopted here is very natural from the point of view of 2-dim conformal field theory, as it generalizes the notion of "in-out" duality considered in [12]. Indeed, the differential operator $u_n(z)\partial^n$ is the dual of $\partial^{-n-1}x_{n+1}(z) \bmod \partial^{-n-2}$, which is in agreement with the duality exhibited between Δ and $1-\Delta$ differentials (here $\Delta = -n$).

Let us now consider the space of all differential operators of (fixed) degree $n > 0$ with $u_n(z) = 1$. i.e.,

$$L = \partial^n + u_{n-1}(z)\partial^{n-1} + \cdots + u_1(z)\partial + u_0(z). \tag{19}$$

For any functional $f[u_0, \ldots, u_{n-1}]$, let X_f be the formal sum

$$X_f = \sum_{i=1}^{n} \partial^{-i} x_i(z), \quad \text{with } x_i = \frac{\delta f}{\delta u_{i-1}}. \tag{20}$$

We can use the coadjoint representation of the Volterra algebra in order to define the Poisson bracket

$$\{f, g\}_L^{(1)} := \int \operatorname{res}([L, X_f]_+ X_g) \tag{21}$$

for any two functional f, g of $u_0, \ldots, u_{n-1}$. Note that in Eq. (20) we have not included the $i = n+1$ term allowed by the residue duality (17). This is justified because, in any case, $[L, X_f]_+$ is a differential operator of degree at most $n-1$; and so in computing the residue of $[L, X_f]_+ X_g$, only terms with $1 \leqq i \leqq n$ will contribute.

The Poisson bracket (21) associated with the space of differential operators (19) is the Gelfand–Dickey bracket of the first kind introduced to describe integrable nonlinear differential equations in Lax form (see for instance [8]). However, it is known that integrable systems in 1 + 1 dimensions are bi-Hamiltonian in the sense that they can be equivalently described using two different kind of Poisson brackets. The Gelfand–Dickey bracket of the second kind is defined [7, 8] as

$$\{f, g\}_L^{(2)} := \int \operatorname{res}(V_{X_f}(L) X_g), \tag{22a}$$

where

$$V_{X_f}(L) = L(X_f L)_+ - (L X_f)_+ L. \tag{22b}$$

We note that the Gelfand–Dickey bracket of the second kind is more general than the first one, since an arbitrary shift of the form $L \to L + \lambda$ (where λ is a constant) yields

$$V_{X_f}(L) \to V_{X_f}(L) - \lambda [L, X_f]_+, \tag{23}$$

i.e., $[L, X]_+$ behaves like a "coboundary" of $V_X(L)$. For this reason, in what follows, we choose to work with the Gelfand–Dickey bracket of the second kind. At this point, we note that $V_X(L)$ may be viewed as the coadjoint action operator of the algebra of formal series (16) with respect to a new commutator $[\![\,,\,]\!]$. The latter is determined by the equation

$$\langle L, [\![X_1, X_2]\!]\rangle := \int \operatorname{res}(V_{X_1}(L)X_2), \tag{24}$$

thus generalizing (18). The algebra constructed this way from the space of differential operators (19), with $[\![\,,\,]\!]$ as its bracket, is called the Gelfand–Dickey algebra GD($GL(n)$), where $GL(n)$ denotes the general linear algebra in n dimensions.[2] It will be shown later on that this provides a natural generalization of the Virasoro algebra, while the associated operator $V_X(L)$ generalizes the coadjoint action operator of the Virasoro algebra (in the sense that $\mathrm{Ad}^*_{X_f}(L-\partial^n) = V_{X_f}(L)$).

Next, we restrict ourselves to $G = SL(n)$ that arises as a special case of the GD($GL(n)$) algebra. The GD($SL(n)$) algebra is constructed from the space of all differential operators (19) with $u_{n-1} = 0$, i.e.,

$$L = \partial^n + u_{n-2}(z)\partial^{n-2} + \cdots + u_0(z), \tag{25}$$

and as we shall see, it is most suitable for describing higher spin operator algebras. In analogy with the GD($GL(n)$) algebra, we define the bracket of the second kind by Eq. (22). But since $u_{n-1} = 0$, the $i = n$ term in the formal sum (20) is meaningless unless it is taken to be zero. In such case, straightforward calculation shows that the operator $V_{X_f}(L)$ involves a term of degree $n-1$ and so the choice $u_{n-1} = 0$ does not seem to be invariant. This problem is resolved if one considers

$$X_f = \sum_{i=1}^{n-1} \partial^{-i} \frac{\delta f}{\delta u_{i-1}} + \partial^{-n} x_n \tag{26}$$

with x_n determined by the requirement $\operatorname{res}[L, X_f] = 0$—as it turns out, $\operatorname{res}[L, X_f]$ is the coefficient of the term with degree $n-1$ in $V_{X_f}(L)$. Therefore, the GD($SL(n)$) algebra is well defined provided that x_n is chosen appropriately. In what follows we show that the Virasoro, as well as higher spin operator algebras, are described in terms of GD($SL(n)$) for all $n = 2, 3, \ldots$.

3. $L = \partial^2 + u$ and Spin 2

Following the general construction described above, we consider

$$X_f = \partial^{-1} \frac{\delta f}{\delta u} + \partial^{-2} x_2 \tag{27}$$

with x_2 determined by the requirement $\operatorname{res}[L, X_f] = 0$. Explicit calculation shows that $\operatorname{res}[\partial^2 + u, X_f] = 2x_2' - (\delta f/\delta u)''$ and so the GD($SL(2)$) algebra is well-defined provided that

$$x_2' = \frac{1}{2}\left(\frac{\delta f}{\delta u}\right)''. \tag{28}$$

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[2] Gelfand–Dickey algebras labeled by more general simple Lie algebras G, GD(G), have been introduced in [9] in connection with generalized hierarchies of nonlinear differential equations in $1+1$ dimensions

Furthermore, we find that in this case

$$V_{X_f}(L) = -\frac{1}{2}\left(\frac{\delta f}{\delta u}\right)''' - 2u\left(\frac{\delta f}{\delta u}\right)' - u'\left(\frac{\delta f}{\delta u}\right). \tag{29}$$

Therefore, the Gelfand–Dickey bracket of the second kind takes the form

$$\{f(u(z)), g(u(z'))\}^{(2)}_{\partial^2+u} = \int d\tilde{z}\frac{\delta f(u(z))}{\delta u(\tilde{z})}\mathcal{O}_{u(\tilde{z})}\frac{\delta g(u(z'))}{\delta u(\tilde{z})}, \tag{30a}$$

where

$$\mathcal{O}_{u(\tilde{z})} = \tfrac{1}{2}\partial^3_{\tilde{z}} + u(\tilde{z})\partial_{\tilde{z}} + \partial_{\tilde{z}}u(\tilde{z}). \tag{30b}$$

The Poisson bracket (30) has already been used in the Hamiltonian formulation of the KdV equation, as well as in 2-dim conformal field theory (see [10] and references therein). This is the bracket between any two functionals f, g defined on the dual of the Virasoro algebra. Moreover, the operator $\mathcal{O}_{u(\tilde{z})}$ describes the conformal variation of the quadratic differentials $u(\tilde{z})$ and hence the coadjoint action of the Virasoro algebra. The commutation relations between the (coordinate) functionals u are easily found to be

$$\{u(z), u(z')\}^{(2)}_{\partial^2+u} = (u(z) + u(z'))\partial_z\delta(z-z') + \tfrac{1}{2}\partial^3_z\delta(z-z'). \tag{31}$$

This result is not surprising at all because the space of differential operators $L = \partial^2 + u$ is known to be isomorphic with the (smooth) dual of the Virasoro algebra (31). Recall that under arbitrary reparametrizations of the circle $z \to \sigma(z)$, the operators $\partial^2 + u$ transform as

$$\partial^2 + u \to \sigma'^{-3/2}(\partial^2 + {}^{\sigma}u)\sigma'^{-1/2}, \tag{32a}$$

where

$${}^{\sigma}u(z) = \sigma'^2 u(\sigma(z)) + \frac{1}{2}\left(\frac{\sigma'''}{\sigma'} - \frac{3}{2}\left(\frac{\sigma''}{\sigma'}\right)^2\right). \tag{32b}$$

Also, it is known that the space of quadratic differentials is isomorphic with the (smooth) dual of the algebra of vector fields on the circle, and so the desired correspondence is established with the aid of densities of weight $-\frac{1}{2}$ [10].

The value of the central charge of the Virasoro algebra (31) is $c = 6$ (cf. Eq. (2)). This is because we chose to work with the differential operator $L = \partial^2 + u$. Having chosen $\lambda\partial^2 + u$ instead (with constant $\lambda = c/6$), the Gelfand–Dickey bracket of the second kind would have been described by (30) with $\mathcal{O}_u = c/12\partial^3 + u\partial + \partial u$. This way, we conclude that the GD($SL(2)$) algebra provides a realization of the Virasoro algebra. We also remark that for $\partial^2 + u$ the Gelfand–Dickey bracket of the first kind is

$$\{f(u(z)), g(u(z'))\}^{(1)}_{\partial^2+u} = \int d\tilde{z}\frac{\delta f(u(z))}{\delta u(\tilde{z})}(-2\partial_{\tilde{z}})\frac{\delta g(u(z'))}{\delta u(\tilde{z})}. \tag{33}$$

This gives only coboundary contributions ($\sim \partial_z\delta(z-z')$) to the commutation relations of the Virasoro algebra, as expected from Eq. (23) that relates the Gelfand–Dickey brackets of the first and second kind in the general case.

4. $L = \partial^3 + u_1\partial + u_0$ and Spin 3

For any functional $f[u_0, u_1]$, let us consider the formal sum (cf. Eq. (26))

$$X_f = \partial^{-1}\frac{\delta f}{\delta u_0} + \partial^{-2}\frac{\delta f}{\delta u_1} + \partial^{-3}x_3. \tag{34}$$

In this case we find that $\operatorname{res}[\partial^3 + u_1\partial + u_0, X_f] = 3x_3' + (\delta f/\delta u_0)''' - 3(\delta f/\delta u_1)'' + (u_1(\delta f/\delta u_0))'$ and so, according to the general theory, the GD($SL(3)$) algebra is well defined provided that x_3 satisfies the equation

$$3x_3' = 3\left(\frac{\delta f}{\delta u_1}\right)'' - \left(\frac{\delta f}{\delta u_0}\right)''' - \left(u_1\frac{\delta f}{\delta u_0}\right)'. \tag{35}$$

Furthermore, we find by a straightforward (but lengthy) calculation that $V_{X_f}(L)$ is given by

$$V_{X_f}(\partial^3 + u_1\partial + u_0) = \tilde{u}_1\partial + \tilde{u}_0, \tag{36a}$$

where

$$\begin{aligned}\tilde{u}_1 = {} & 2\left(\frac{\delta f}{\delta u_0}\right)'''' - 5\left(\frac{\delta f}{\delta u_1}\right)''' + 2\left(u_1\frac{\delta f}{\delta u_0}\right)'' - u_1\left(\frac{\delta f}{\delta u_1}\right)' - u_0\left(\frac{\delta f}{\delta u_0}\right)' \\ & - \left(u_1\frac{\delta f}{\delta u_1}\right)' - 2\left(u_0\frac{\delta f}{\delta u_0}\right)' + 3x_3'',\end{aligned} \tag{36b}$$

$$\begin{aligned}\tilde{u}_0 = {} & \left(\frac{\delta f}{\delta u_0}\right)''''' - 2\left(\frac{\delta f}{\delta u_1}\right)'''' + \left(u_1\frac{\delta f}{\delta u_0}\right)''' + u_1\left(\frac{\delta f}{\delta u_0}\right)''' + x_3''' - 2u_0\left(\frac{\delta f}{\delta u_1}\right)' \\ & + u_0\left(\frac{\delta f}{\delta u_0}\right)'' + u_1\left(u_1\frac{\delta f}{\delta u_0}\right)' + u_1x_3' - \left(u_0\frac{\delta f}{\delta u_1}\right)' - 2u_1\left(\frac{\delta f}{\delta u_1}\right)'' - \left(u_0\frac{\delta f}{\delta u_0}\right)'',\end{aligned} \tag{36c}$$

and x_3 satisfies the requirement (35). The equation above describes the coadjoint action of the GD($SL(3)$) algebra in the sense that $\mathrm{Ad}^*_{X_f}(L - \partial^3) := V_{X_f}(\partial^3 + u_1\partial + u_0) = \tilde{u}_1\partial + \tilde{u}_0$. Comparison with Eq. (26) suggests that (36) generalizes the coadjoint action of the Virasoro algebra in a very interesting way.

To be more precise, let us examine first the transformation properties of the operators $\partial^3 + u_1(z)\partial + u_0(z)$ under arbitrary reparametrizations of the circle $z \to \sigma(z)$. Explicit computation yields

$$\partial^3 + u_1(z)\partial + u_0(z) \to \sigma'^{-2}(\partial^3 + {}^{\sigma}u_1(z)\partial + {}^{\sigma}u_0(z))\sigma'^{-1}, \tag{37a}$$

where

$${}^{\sigma}u_1(z) = \sigma'^2 u_1(\sigma(z)) + 2S_\sigma(z), \tag{37b}$$

$${}^{\sigma}u_0(z) = \sigma'^3 u_0(\sigma(z)) + \sigma'\sigma'' u_1(\sigma(z)) + S_\sigma'(z). \tag{37c}$$

Here, $S_\sigma(z) = (\sigma'''/\sigma') - (3/2)(\sigma''/\sigma')^2$ denotes the Schwartzian derivative of the diffeomorphism σ. Note that $u_1(z)$ transforms (up to the Schwartzian term) as a quadratic differential, while (37c) suggests the presence of a spin 3 field in the formalism. In particular, using (37b), (37c), we have that

$$ {}^{\sigma}u_0(z) - \tfrac{1}{2}{}^{\sigma}u_1'(z) = \sigma'^3\left(u_0(\sigma(z)) - \frac{1}{2\sigma'} u_1'(\sigma(z)) \right), \tag{38} $$

and so the combination $u_0(z) - \frac{1}{2}u_1'(z)$ transforms as a conformal field of weight 3. In establishing these results, we found it most useful to think of the operators $L = \partial^3 + u_1\partial + u_0$ as acting on densities of weight -1 rather than on scalar functions. In fact, as we shall demonstrate later on, there is a natural generalization of the transformations (32) and (37) to any differential operator $L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0$ viewed as acting on densities of weight $-(n-1)/2$.

At the moment, we would like to investigate the GD($SL(3)$) algebra in more detail, since Eq. (37b) indicates that the Virasoro algebra is contained as a subalgebra in GD($SL(3)$). For this, we make use of the Gelfand–Dickey bracket of the second kind that is associated with the space of differential operators $\partial^3 + u_1\partial + u_0$. Recalling Eq. (36) for the coadjoint action, we obtain the following expression for the bracket:

$$ \begin{aligned} &\{f(u_1(z), u_0(z)), g(u_1(z'), u_0(z'))\}^{(2)}_{\partial^3+u_1\partial+u_0} \\ &= \int d\tilde{z} \frac{\delta g}{\delta u_1(\tilde{z})}\left[\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)'''' - 2\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)''' + \left(u_1(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)'' - u_1(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' \right. \\ &\quad \left. - u_0(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)' - \left(u_1(\tilde{z})\frac{\delta f}{\delta u_1(\tilde{z})}\right)' - 2\left(u_0(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)' \right] \\ &\quad + \frac{\delta g}{\delta u_0(\tilde{z})}\left[\frac{2}{3}\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''''' - \left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'''' + \frac{2}{3}\left(u_1(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' \right. \\ &\quad + \tfrac{2}{3}u_1(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' - u_1(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'' - 2u_0(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' + u_0(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)'' \\ &\quad \left. + \tfrac{2}{3}u_1(\tilde{z})\left(u_1(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)' - \left(u_0(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)'' - \left(u_0(\tilde{z})\frac{\delta f}{\delta u_1(\tilde{z})}\right)' \right]. \end{aligned} \tag{39} $$

Note that the bracket between the (coordinate) functionals u_1 is equal to

$$ \{u_1(z), u_1(z')\}^{(2)}_{\partial^3+u_1\partial+u_0} = (u_1(z) + u_1(z'))\partial_z\delta(z-z') + 2\partial_z^3\delta(z-z'), \tag{40} $$

which shows that the Virasoro algebra is, indeed, contained in GD($SL(3)$). However, thanks to all the terms that appear in Eq. (39), this is not the whole story. Further calculation shows that for $w(z) := u_0(z) - \frac{1}{2}u_1'(z)$ the following is true:

$$ \{u_1(z), w(z')\}^{(2)}_{\partial^3+u_1\partial+u_0} = (w(z) + 2w(z'))\partial_z\delta(z-z'), \tag{41} $$

i.e., $w(z)$ transforms as a local field of conformal weight 3. Moreover, we find that the Gelfand–Dickey bracket for the fields w is given by

$$ \begin{aligned} \{w(z), w(z')\}^{(2)}_{\partial^3+u_1\partial+u_0} = &-\tfrac{1}{6}\partial_z^5\delta(z-z') - \tfrac{1}{3}(u_1^2(z) + u_1^2(z'))\partial_z\delta(z-z')) \\ &- \tfrac{5}{12}(u_1(z) + u_1(z'))\partial_z^3\delta(z-z') \\ &+ \tfrac{1}{4}(u_1''(z) + u_1''(z'))\partial_z\delta(z-z'). \end{aligned} \tag{42} $$

Consequently, $u_1(z)$ and $w(z)$ form an algebra with quadratic determining relations

(40)–(42). Here, the central charge of the Virasoro algebra (40) is $c=24$. Note, however, that all other values of c are easily obtained by considering $L=\lambda\partial^3+u_1\partial+u_0$ with $\lambda=c/24$.

For completeness, we mention that in this case the Gelfand–Dickey bracket of the first kind is equal to

$$\{f,g\}^{(1)}_{\partial^3+u_1\partial+u_0}=3\int d\tilde{z}\left(\frac{\delta g}{\delta u_0(\tilde{z})}\partial_{\tilde{z}}\frac{\partial f}{\delta u_1(\tilde{z})}+\frac{\delta g}{\delta u_1(\tilde{z})}\partial_{\tilde{z}}\frac{\delta f}{\delta u_0(\tilde{z})}\right). \tag{43}$$

Shifting $u_0(z)$ (and hence $w(z)$) by a constant is equivalent to modifying the Gelfand–Dickey bracket of the second kind by the expression (43). It can be readily checked that such modifications do not alter the commutation relations (40)–(42) at all. On the other hand, we may shift $u_1(z)$ by a constant (which is equivalent to redefining the vacuum expectation value of the stress-energy tensor of the theory). For $u_1(z)\to u_1(z)-1$, the commutation relations (40)–(42) become

$$\{u_1(z),u_1(z')\}=(u_1(z)+u_1(z'))\partial_z\delta(z-z')+2\partial_z^3\delta(z-z')-2\partial_z\delta(z-z'), \tag{44a}$$

$$\{u_1(z),w(z')\}=(w(z)+2w(z'))\partial_z\delta(z-z'), \tag{44b}$$

$$\begin{aligned}\{w(z),w(z')\}=&-\tfrac{1}{6}(\partial_z^5\delta(z-z')-5\partial_z^3\delta(z-z')+4\partial_z\delta(z-z'))\\&-\tfrac{1}{3}(u_1^2(z)+u_1^2(z'))\partial_z\delta(z-z')+\tfrac{2}{3}(u_1(z)+u_1(z'))\partial_z\delta(z-z')\\&-\tfrac{5}{12}(u_1(z)+u_1(z'))\partial_z^3\delta(z-z')+\tfrac{1}{4}(u_1''(z)+u_1''(z'))\partial_z\delta(z-z').\end{aligned} \tag{44c}$$

(Here, for convenience, we have dropped the labels of the Gelfand–Dickey bracket.)

Although the algebra (44) is not a Lie algebra, it has a very natural interpretation in the context of 2-dim conformal field theory. Comparison with the commutation relations (6) shows that the spin 3 operator algebra is a particular representation of (44). To illustrate this result in more detail, we point out that the quadratic terms appearing on the right-hand side of Eq. (44c) need to be regularized upon quantization. An appropriate regularization (which is also consistent with the algebra commutation relations) is acquired by assigning

$$\begin{aligned}&(\Lambda(z)+\Lambda(z'))\partial_z\delta(z-z')+\tfrac{1}{10}(\partial_z^5-5\partial_z^3+4\partial_z)\delta(z-z')-[\tfrac{2}{5}(u_1(z)+u_1(z'))\partial_z\\&\quad-\tfrac{1}{4}(u_1(z)+u_1(z'))\partial_z^3+\tfrac{3}{20}(u_1''(z)+u_1''(z'))\partial_z]\delta(z-z')\end{aligned}$$

to the classical quantity $(u_1^2(z)+u_1^2(z'))\partial_z\delta(z-z')$. In the expression above, $\Lambda(z)$ is essentially the normal ordered operator that represents $u_1^2(z)$ with Fourier modes given by Eq. (7). This way, the right-hand side of Eq. (44c) assumes the form

$$\begin{aligned}&-\tfrac{1}{5}(\partial_z^5-5\partial_z^3+4\partial_z)\delta(z-z')-\tfrac{1}{3}(\Lambda(z)+\Lambda(z'))\partial_z\delta(z-z')\\&\quad-\tfrac{1}{10}[5(u_1(z)+u_1(z'))\partial_z^3-3(u_1''(z)+u_1''(z'))\partial_z-8(u_1(z)+u_1(z'))\partial_z]\delta(z-z'),\end{aligned}$$

and so Zamolodchikov's spin 3 operator algebra is a well defined representation of (44) with $W(z)\leftrightarrow(i/\sqrt{3})w(z)$, $T(z)\leftrightarrow u_1(z)$, and $c=24$. In general, if we consider $L=(c/24)\partial^3+u_1\partial+u_0$, the operator algebra (6) will result for all values of the central charge c.

5. Higher Spin Fields and Conclusions

Next, we discuss briefly the relation between higher spin n operator algebras and the Gelfand–Dickey algebras $\mathrm{GD}(SL(n))$ constructed from the space of differential operators $L = \partial^n + u_{n-2}(z)\partial^{n-2} + \cdots + u_0(z)$. It is relatively easy to show that in this case the Gelfand–Dickey bracket of the second kind for the (coordinate) functionals u_{n-2} is equal to

$$\{u_{n-2}(z), u_{n-2}(z')\}_L^{(2)} = (u_{n-2}(z) + u_{n-2}(z'))\partial_z\delta(z-z') + \frac{c_n}{12}\partial_z^3\delta(z-z') \tag{45}$$

with $c_n = n(n-1)(n+1)$, which in turn implies that the Virasoro algebra is a subalgebra of $\mathrm{GD}(SL(n))$ (see also ref. [5, 7]). Alternatively, this result is established using the transformation properties of the operators $\partial^n + u_{n-2}(z)\partial^{n-2} + \cdots + u_0(z)$ under arbitrary reparametrizations $z \to \sigma(z)$. Explicit calculation shows that

$$\partial^n + u_{n-2}(z)\partial^{n-2} + \cdots + u_0(z) \to \sigma'^{-(n+1)/2}(\partial^n + {}^\sigma u_{n-2}(z)\partial^{n-2} + \cdots + {}^\sigma u_0(z))\sigma'^{-(n-1)/2} \tag{46a}$$

with

$${}^\sigma u_{n-2}(z) = \sigma'^2 u_{n-2}(\sigma(z)) + \frac{n(n-1)(n+1)}{12} S_\sigma(z), \tag{46b}$$

which generalizes the transformations (32) and (37) for all values of n. For clarity we note that under arbitrary reparametrization $z \to \sigma(z)$, the form of the operators $L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0$ may not be preserved due to the occurrence of the term $-(n(n-1)/2)(\sigma''/\sigma'^{n+1})\partial^{n-1}$ of degree $n-1$. This will definitely be the case if we assume that the operators L act on scalar functions. However, if we think of them as acting on densities of weight $-(n-1)/2$ (as illustrated by (46a)), all terms of degree $n-1$ cancel, which makes the choice $u_{n-1} = 0$ consistent. We point out that this is equivalent to the requirement $\mathrm{res}[X_f, L] = 0$ imposed for well-definiteness of the $\mathrm{GD}(SL(n))$ algebras earlier on.

Further study of the transformation (46a) shows that up to Schwartzian terms, $u_{n-2}(z)$, $u_{n-3}(z), \ldots, u_0(z)$ (or appropriate combinations of them) behave as conformal fields of weight $2, 3, \ldots, n$, respectively. For instance, for $n = 4$ we find that

$${}^\sigma u_1(z) = \sigma'^3 u_1(\sigma(z)) + 2\sigma'\sigma'' u_2(\sigma(z)) + 5S'_\sigma(z), \tag{47a}$$

$${}^\sigma u_0(z) = \sigma'^4 u_0(\sigma(z)) + \tfrac{3}{2}\sigma'^2\sigma'' u_1(\sigma(z)) + \frac{3}{2}\left(\frac{\sigma''^2}{\sigma'^2} + S_\sigma(z)\right)\sigma'^2 u_2(\sigma(z)) + \tfrac{3}{2}(S''_\sigma(z) + \tfrac{3}{2}S_\sigma^2(z)). \tag{47b}$$

In analogy with the $n = 2, 3$ cases that we have already investigated, the Gelfand–Dickey bracket for these fields would provide us with an algebra equivalent to the spin n operator algebra of 2-dim conformal field theory. This way, we identify the space of differential operators $L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0$ with the (smooth) dual of the spin n operator algebra. Using the coadjoint action operator (22b), we may extend the methods of ref. [10] to all higher spin operator algebras and study the geometry of the resulting coadjoint orbits. Such orbits might be found useful for investigating the geometric role of higher spin fields. We intend to study this problem in more detail elsewhere.

Moreover, highest weight representations of all spin n operator algebras are expected to be of paramount importance in quantum conformal field theory, as well as in statistical mechanics in 2 dimensions. According to the general philosophy of 2-dim critical phenomena, the investigation of critical singularities reduces to the problem of finding appropriate conformally invariant quantum field theory solutions (see [2,4] and references therein). Such conformal (minimal) models correspond to degenerate Verma module representations of the underlying symmetry algebra for discrete values of the central charge c. For the spin 2 algebra, the corresponding discrete values of c are given by $1-(6/p(p+1))$ with $p=3,4,5,\ldots$ [11], while for the spin 3 operator algebra by Eq. (8). In generalizing these results to higher spin operator algebras, we realize that unitary representations of the GD($SL(n)$) algebras play an important role. Taking into account the embedding of the Virasoro algebra in GD($SL(n)$) described by Eqs. (45) and/or (46), we find (as a result of some preliminary computations) that the following set of discrete values

$$c=(n-1)\left(1-\frac{n(n+1)}{p(p+1)}\right); \quad p=n+1,\, n+2,\ldots \tag{48}$$

is associated with (minimal) models of spin n operator algebras. Details of the underlying calculations will be presented in [12]. However, it is worth mentioning here that the representations of affine Kac–Moody algebras are closely related with the theory of integrable nonlinear diferential equations and hence with the Gelfand–Dickey algebras (see, for instance, [13] and references therein). In particular, the GD($SL(n)$) algebra may be viewed as a reduction of the $SL(n)$ Kac–Moody algebra. This seems to provide the basic ingredient for understanding hidden relations between (minimal) conformal models of spin n operator algebras and highest weight representations of the $SL(n)$ Kac–Moody algebras in favor of showing the positivity theorem that guarantees the unitarity of these models. Along these lines, it would be most interesting to generalize the Goddard–Kent–Olive construction [11] to all values of n in a well prescribed way.

Finally, we note that it is possible to extend our results to superconformal (and more generally to higher half-integer spin) operator algebras by using the supersymmetric generalization of the Gelfand–Dickey algebra of formal pseudodifferential operators introduced in [14] to study the super KP hierarchy of nonlinear differential equations. This will be the subject of future publications [12]. For completeness we mention that certain results somewhat related to ours have also been discussed by others [15].

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Note added in proof. Recently I became aware of reference [16] where Gelfand–Dickey algebras have also been used in the study of higher spin operator algebras.

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