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New Bosonization and Conformal Field Theory over Z

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Dedicated to Professor M. Sato on his sixtieth birthday

Abstract. New formulation of bosonization is given so that it is defined over the ring **Z** of integers. The charge zero sector of the new boson Fock space is the completion of the coordinate ring of the universal Witt scheme. By using new bosonization, conformal field theory of free fermions over **Z** is given.

Introduction

String theory and conformal field theory have a deep connection with arithmetic geometry. (See for example, [ABMNV, B, BK, BM, FS] and references therein.) There are several attempts to generalize the theories in arithmetic directions: p-adic strings [FO], [V], adelic strings [FW, MA2], arithmetic bosonic strings [S, U], modular geometry of string theory and conformal field theory [F], conformal field theory over an arbitrary field which is related to automorphic representation [W]. (See also [MA1]. These papers are mainly based on the arithmetic properties of partition functions and correlation functions.

In the present paper we choose another approach to arithmetization of conformal field theory. Namely, conformal field theory of free fermions shall be realized arithmetico-geometrically so that the theory can be formulated over any commutative ring A with unity. If the ring A is the complex numbers **C**, we have the usual conformal field theory.

Conformal field theory of free fermions on compact Riemann surfaces has a deep connection with geometry of the moduli space $\mathscr{M}_g$ of compact Riemann surfaces of genus g (cf. [ABMNV, AGR, BMS, BS, EO, IMO, and KNTY]). Especially, the determinant line bundle $\lambda_{1/2}$ of spin bundles plays an essential role in the theory. The moduli space $\mathscr{M}_g$ is an algebraic variety defined over the ring of integers **Z** and $\lambda_{1/2}$ is a line bundle on the moduli space $\mathscr{M}_{g,4}$ of level 4 structure defined over $\mathbf{Z}[\frac{1}{2}]$. Therefore, it is natural to ask whether the theory can be formulated over the integers **Z** or at least over $\mathbf{Z}[\frac{1}{2}]$. The main purpose of the

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present paper is to show that this is indeed the case.

Let us explain briefly our theory. Conformal field theory of free fermions consists of three parts: geometry of Riemann surfaces with additional structures (dressed Riemann surfaces in the terminology of [KNTY]) ([ACKP, BS, KNTY]), the theory of the universal Grassmann manifold ([SS]) and physical theory of free fermions with bosonization ([AGR, EO, IMO, KNTY, VV]). To each dressed Riemann surface there corresponds a point of the universal Grassmann manifold and by the Plücker embedding the universal Grassmann manifold is embedded into the projective space $\mathbf{P}(\mathscr{F})$ associated with the fermion Fock space $\mathscr{F}$. In this way, to each dressed Riemann surface we associate a physical state (vacuum of the dressed Riemann surface) of fermions.

Thanks to modern algebraic geometry we can formulate the above procedure over $\mathbf{Z}$ (and over any commutative ring). To develop the theory over $\mathbf{Z}$ we need to generalize the notion of points so that even if we consider the theory over $\mathbf{Z}$ we need to consider the theory over any commutative ring. (See Appendix below.)

The important operators such as fermion operators, $\psi(z)$, $\bar{\psi}(z)$, current operators J_m are also defined over $\mathbf{Z}$ and Virasoro operators L_n are defined over $\mathbf{Z}[\frac{1}{2}]$. We can also modify the definition of Virasoro operators so that they are defined over $\mathbf{Z}$. Thus the usual conformal field theory can be already formulated over $\mathbf{Z}$.

The only non-trivial problem is bosonization, since we need the operator $\exp\left(\sum_{m=1}^{\infty} J_m t_m\right)$ which is not defined over $\mathbf{Z}$ but defined only over the rational numbers $\mathbf{Q}$. Difficulty is overcome by introducing new variables $x_1, x_2, x_3, \ldots$ such that

$$nt_n = \sum_{d|n} dx_d^{n/d},$$

where $t_1, t_2, t_3, \ldots$ are variables by which current operators have the forms

$$J_m = \frac{\partial}{\partial t_m}, \quad J_{-m} = mt_m,$$

for any positive integer m. The variables $x_1, x_2, x_3, \ldots$ are the coordinates of the universal Witt scheme W_∞. The zero sector of the new boson Fock space $\mathscr{H}(\mathbf{Z})$ over $\mathbf{Z}$ is $\mathbf{Z}[[x_1, x_2, x_3, \ldots]]$, the completion of the coordinate ring $\mathbf{Z}[x_1, x_2, x_3, \ldots]$ of the universal Witt scheme W_∞. Thus the current algebra and the modified Virasoro algebra with central charge 1 operates on $\mathbf{Z}[[x_1, x_2, x_3, \ldots]]$ and the Virasoro algebra with central charge 1 operates on $\mathbf{Z}[\frac{1}{2}][[x_1, x_2, x_3, \ldots]]$. We may also regard $\mathbf{Z}[x_1, x_2, x_3, \ldots]$ as the complex oriented cobordism ring Ω over a point. Thus the Virasoro algebra operates on the completion $\hat{\Omega}$ of Ω. This fact was pointed out by Morava. There is also a deep relationship between the new boson Fock space and formal groups. This will be discussed in the forthcoming paper.

Let us explain briefly the content of the present paper. In Sect. 1 we shall define the universal Grassmann manifold over a commutative ring A, which is a natural generalization of the one over the complex numbers $\mathbf{C}$ due to M. Sato. In Sect. 2 we shall define the fermion Fock space and fermion operators over a commutative ring A. Current operators and Virasoro operators will be also defined. In Sect. 3 by introducing new boson variables, new bosonization over $\mathbf{Z}$ will be defined. In Sect. 4 arithmetico-geometrical realization of conformal field theory of free fermions will be given. Here we need the language of schemes. We formulate the realization naively and our description is awkward from the viewpoint of algebraic geometry. The discussion in Sect. 4 shows that it is better to use the notion of functor or the language of adèles. The theory of the τ-function and correlation functions will be treated in the forthcoming paper.

For the reader not familiar with the language of schemes, in the Appendix we shall give a brief introduction to the theory of schemes.

1. Universal Grassmann Manifold

In this section we shall define the universal Grassmann manifold over a commutative ring A. For that purpose first we recall the definition of the universal Grassmann manifold over a field due to M. Sato.

Let k be a field. By $\mathscr{V} = k((\zeta))$ we mean a field of formal Laurent series $\sum_{n=N}^{+\infty} a_n\zeta^n$ in ζ with coefficients in the field k. We regard $\mathscr{V}$ as an infinite dimensional vector space over k. A filtration $\{F^m\mathscr{V}\}_{m\in\mathbf{Z}}$ of $\mathscr{V}$ is defined by

$$F^m\mathscr{V} = \left\{\sum_{i\geqq m} a_i\zeta^i\right\}.$$

Note that the filtration is decreasing

$$\dots \supset F^m\mathscr{V} \supset F^{m+1}\mathscr{V} \supset \dots$$

and $F^0\mathscr{V}$ is the formal power series $k[[\zeta]]$ with coefficients in k. For any vector subspace $U \subset \mathscr{V}$, we have a natural homomorphism

$$\alpha_U : U \to \mathscr{V}/F^0\mathscr{V}.$$

Definition 1.1. The universal Grassmann manifold $\mathrm{UGM}(k)$ over the field k consists of vector subspaces U of $\mathscr{V}$ which satisfy the following condition:

$$\dim_k \operatorname{Ker}\alpha_U < \infty, \quad \dim_k \operatorname{Coker}\alpha_U < \infty. \tag{1.1}$$

By $[U]$ we denote the point in $\mathrm{UGM}(k)$ corresponding to a subspace $U \subset \mathscr{V}$. For $[U]\in\mathrm{UGM}(k)$ we define the charge of U by

$$\chi(U) = \dim_k \operatorname{Ker}\alpha_U - \dim_k \operatorname{Coker}\alpha_U.$$

For each integer p we define

$$\mathrm{UGM}^{(p)}(k) = \{[U]\in\mathrm{UGM}(k) \mid \chi(U) = p\}.$$

$\mathrm{UGM}^{(p)}(k)$ is called the universal Grassmann manifold of charge p and has the structure of a proscheme (a projective limit of schemes) over k. Its tangent space at a point $[U]$ is given by

$$T_{[U]}\,\mathrm{UGM}^{(p)}(k) = \mathrm{Hom}_{k\text{-cont}}(U, \mathscr{V}/U), \tag{1.2}$$

where "k-cont" means a continuous homomorphism of k-vector spaces with respect to the topology induced by the filtration $\{F^m\mathscr{V}\}_{m\in\mathbf{Z}}$. The universal Grassmann manifold UGM(k) is a disjoint union of UGM$^{(p)}(k)$'s. For the details of the universal Grassmann manifold, we refer the reader to [SS, SN].

Next we define the universal Grassmann manifold over a commutative ring A. Let A be a commutative ring and (Spec A, $\mathcal{O}_A$) the corresponding affine scheme. Let an $\mathcal{O}_A$-module $\mathscr{V}(A)=\mathcal{O}_A((\zeta))$ consist of formal Laurent series with coefficients in $\mathcal{O}_A$. A filtration $\{F^m\mathscr{V}(A)\}_{m\in\mathbf{Z}}$ is defined by

$$F^m\mathscr{V}(A)=\left\{\sum_{i\geq m} a_i\zeta^i,\ a_i\in\mathcal{O}_A\right\}.$$

Then $F^m\mathscr{V}(A)$ is a free $\mathcal{O}_A$-submodule of $\mathscr{V}(A)$ and the filtration is decreasing. For an $\mathcal{O}_A$-submodule $\mathscr{U}$ of $\mathscr{V}(A)$ we have a natural $\mathcal{O}_A$-homomorphism

$$\alpha_{\mathscr{U}}:\mathscr{U}\to\mathscr{V}(A)/F^0\mathscr{V}(A).$$

Definition 1.2. The universal Grassmann manifold UGM(A) consists of locally free $\mathcal{O}_A$-submodules $\mathscr{U}$ of $\mathscr{V}(A)$ which satisfy the following condition:

$$\operatorname{Ker}\alpha_{\mathscr{U}} \text{ and } \operatorname{Coker}\alpha_{\mathscr{U}} \text{ are locally free } \mathcal{O}_A\text{-modules of finite rank.} \tag{1.3}$$

Precisely speaking, UGM is defined as a covariant functor from the category of commutative rings with unity to the category of sets and the functor is represented by a proscheme UGM defined over the integer ring $\mathbf{Z}$. Moreover UGM(A) in Definition 1.2 is the set of A-valued points of this proscheme. UGM(A) (precisely speaking UGM$\otimes A$) has the structure of an A-proscheme π:UGM$(A)\to$Spec A. For an element $\mathscr{U}$ of $\mathscr{V}(A)$, we define the charge of $\mathscr{U}$ by

$$\chi(\mathscr{U})=\operatorname{rank}_{\mathcal{O}_A}(\operatorname{Ker}\alpha_{\mathscr{U}})-\operatorname{rank}_{\mathcal{O}_A}(\operatorname{Coker}\alpha_{\mathscr{U}}).$$

Moreover, for each point $\mathfrak{p}\in$Spec A, the fibre $\pi^{-1}(\mathfrak{p})$ is the universal Grassmann manifold UGM$(k(\mathfrak{p}))$ over the residue field $k(\mathfrak{p})$.

Put

$$\mathscr{D}_z(A)=A\left[z,z^{-1},\frac{d}{dz}\right], \tag{1.4}$$

where $z=\zeta^{-1}$. Then for each element $P\in\mathscr{D}_z(A)$, there exists an integer n such that

$$P(F^m\mathscr{V}(A))\subset F^{m+n}\mathscr{V}(A)$$

for any $m\in\mathbf{Z}$. This implies, by (1.2), P defines a regular vector field on each fibre $\pi^{-1}(\mathfrak{p})$, and moreover defines a regular relative vector field $\theta(P)$, that is, an element of $H^0(\mathrm{UGM}(A), \Theta_{\mathrm{UGM}(A)/\mathrm{Spec}\,A})$.

For the later purpose we introduce a new notation. Let $\mathbf{Z}_h$ be the set of integers shifted by half. We put

$$e^{\mu}=\zeta^{\mu+1/2},\quad \mu\in\mathbf{Z}_h.$$

Then, to any point $[\mathscr{U}]\in$UGM(A) we associate a set $M(\mathscr{U})$ of integers shifted by half defined by

$$M(\mathscr{U})=\{\nu\in\mathbf{Z}_h | F^{\nu+1/2}\mathscr{U}/F^{\nu+3/2}\mathscr{U}\neq 0\},$$

where

$$F^m\mathscr{U} = F^m\mathscr{V}(A)\cap\mathscr{U}.$$

We call $M(\mathscr{U})$ the Maya-diagram associated to $\mathscr{U}$. A subset M of $\mathbf{Z}_h$ is a Maya-diagram if and only if there exist an integer p and a mapping

$$\mu:\{p-1/2, p-3/2, \dots, p-(2n+1)/2, \dots\} \to \mathbf{Z}_h$$

such that

$$\mu(p-1/2) > \mu(p-3/2) > \mu(p-5/2) > \dots,$$
$$M = \{\mu(p-1/2), \mu(p-3/2), \mu(p-5/2), \dots\}$$

and there exists a positive integer n_0 such that

$$\mu\left(p-\frac{2n+1}{2}\right) = p - \frac{2n+1}{2}$$

for all $n \geqq n_0$. The number p is called the charge of the Maya-diagram M.

Let $M(\mathscr{U})$ be the Maya-diagram associated to a point $[\mathscr{U}]\in \mathrm{UGM}(A)$. Let p be the charge of $M(\mathscr{U})$. Then we have

$$M(\mathscr{U}) = \{\mu(p-1/2), \mu(p-3/2), \mu(p-5/2), \dots\}$$

by a suitable map μ.

Definition 1.3. A frame $\{\xi_{p-1/2}, \xi_{p-3/2}, \dots\}$ is a basis of $\mathscr{U}$ as an $\mathscr{O}_A$-module satisfying the following conditions.

1. $\xi_\nu \in F^{\mu(\nu)+1/2}\mathscr{U}$ for all $\nu\in\mathbf{Z}_h$ with $\nu < p$.
2. There exists ν_0 such that $\xi_\nu \equiv e^\nu \bmod F^{\nu+3/2}\mathscr{U}$ for all $\nu \leqq \nu_0$.

By Definition 1.2, there always exists a frame of $\mathscr{U}$ for any point $[\mathscr{U}]\in\mathrm{UGM}(A)$.

Note that the Maya-diagram $M(\mathscr{U}\otimes k(\mathfrak{p}))$ with $\mathfrak{p}\in \operatorname{Spec} A$ coincides with $M(\mathscr{U})$.

2. Fermion Fock Space

We use the same basis $\{e^\mu\}_{\mu\in\mathbf{Z}_h}$ of $\mathscr{V}(A)$ introduced in the previous section. To each Maya-diagram $M = \{\mu(p-\frac{1}{2}), \mu(p-\frac{3}{2}), \dots\}$ of charge p, the symbol $|M\rangle$ is defined by

$$|M\rangle = e^{\mu(p-1/2)} \wedge e^{\mu(p-3/2)} \wedge \cdots. \tag{2.1}$$

The symbol $|M\rangle$ is regarded as a pure state of free fermions of charge p.

Definition 2.1. The fermion Fock space $\mathscr{F}_p(A)$ of charge p over a commutative ring A is a direct product

$$\mathscr{F}_p(A) = \prod_{M\in\mathscr{M}_p} A|M\rangle,$$

where $\mathscr{M}_p$ is the set of Maya-diagrams of charge p. The fermion Fock space $\mathscr{F}(A)$ over A is a direct sum

$$\mathscr{F}(A) = \bigoplus_{p\in\mathbf{Z}} \mathscr{F}_p(A).$$

For each fermion Fock space $\mathscr{F}_p(A)$ of charge p we define the projective space $\mathbf{P}(\mathscr{F}_p(A))$ by

$$\mathbf{P}(\mathscr{F}_p(A)) = \{\mathscr{F}_p(A) - \{0\}\}/A^\times,$$

where $A^\times$ is the set of invertible elements of A. We put

$$\mathbf{P}(\mathscr{F}(A)) = \coprod_{p\in\mathbf{Z}} \mathbf{P}(\mathscr{F}_p(A)).$$

For any point $[\mathscr{U}]\in \mathrm{UGM}(A)$, we choose a frame $\xi = \{\xi_{p-1/2}, \xi_{p-3/2}, \dots\}$ as in Definition 1.3. Then the infinite wedge product

$$\xi_{p-1/2} \wedge \xi_{p-3/2} \wedge \cdots$$

is written as a linear combination of $|M\rangle$'s, where $M\in\mathscr{M}_p$, and defines a point of $\mathbf{P}(\mathscr{F}_p(A))$. The point is uniquely determined by the point $[\mathscr{U}]$ and does not depend on the choice of a frame.

Lemma 2.2. *The mapping*

$$\left.\begin{array}{ll} \lambda: \mathrm{UGM}^{(p)}(A) \to \mathbf{P}(\mathscr{F}_p(A)) & \\ \qquad [\mathscr{U}] \quad \mapsto \xi_{p-1/2} \wedge \xi_{p-3/2} \wedge \cdots & \end{array}\right\}$$

is an embedding and the image of λ is characterized by the Plücker relations.

The embedding λ is called the Plücker embedding.

We introduce the dual fermion Fock space $\bar{\mathscr{F}}_p(A)$ of charge p as follows. For a Maya-diagram $M = \{\mu(p-\frac{1}{2}),\ \mu(p-\frac{3}{2}), \dots\}$ of charge p, we define the dual symbol by

$$\langle M| = \cdots \wedge \bar{e}_{\mu(p-3/2)} \wedge \bar{e}_{\mu(p-1/2)},$$

where $\{\bar{e}_\mu\}_{\mu\in\mathbf{Z}_h}$ is a dual basis with pairing

$$\langle \bar{e}_\mu, e^\nu\rangle = \delta_\mu^\nu. \tag{2.2}$$

Then the dual fermion Fock spaces $\bar{\mathscr{F}}_p(A)$ and $\bar{\mathscr{F}}(A)$ are defined by

$$\bar{\mathscr{F}}_p(A) = \bigoplus_{M\in\mathscr{M}_p} A\langle M|,$$

$$\bar{\mathscr{F}}(A) = \bigoplus_{p\in\mathbf{Z}} \bar{\mathscr{F}}_p(A).$$

The pairing (2.1) induces a natural pairing

$$\begin{array}{l} \bar{\mathscr{F}}_p(A) \times \mathscr{F}_p(A) \to A \\ \qquad (\langle\psi|, |\psi'\rangle) \mapsto \langle\psi|\psi'\rangle, \end{array}$$

which can be naturally extended to a pairing

$$\bar{\mathscr{F}}(A) \times \mathscr{F}(A) \to A.$$

We also use the following notation:

$$|p\rangle = e^{p-1/2} \wedge e^{p-3/2} \wedge e^{p-5/2} \wedge \cdots,$$

$$\langle p| = \cdots \wedge \bar{e}_{p-5/2} \wedge \bar{e}_{p-3/2} \wedge \bar{e}_{p-1/2}.$$

Definition 2.3. For each half integer $\mu \in \mathbf{Z}_h$, the fermion operators ψ_μ, $\bar{\psi}_\mu$ operating on $\mathscr{F}(A)$ by the left (respectively on $\bar{\mathscr{F}}(A)$ by the right) is defined by

$$\psi_\mu = e^\mu \rfloor \qquad (\text{respectively} \wedge \bar{e}_\mu),$$
$$\bar{\psi}_\mu = e^{-\mu} \wedge \qquad (\text{respectively} \rfloor \bar{e}_{-\mu}),$$

where $\rfloor$ means the interior product.

These fermion operators have the following anticommutation relations.

$$[\psi_\mu, \psi_\nu]_+ = 0, \quad [\bar{\psi}_\mu, \bar{\psi}_\nu]_+ = 0, \quad [\psi_\mu, \bar{\psi}_\nu]_+ = \delta_{\mu+\nu,0}.$$

Moreover, we have

$$\psi_\mu|p\rangle = 0 \quad (\mu > p), \qquad \langle p|\psi_\mu = 0 \quad (\mu < p),$$
$$\bar{\psi}_\mu|p\rangle = 0 \quad (\mu > -p), \quad \langle p|\bar{\psi}_\mu = 0 \quad (\mu < -p).$$

The normal ordering $:\psi_\nu\bar{\psi}_\mu:$ is defined by

$$:\psi_\nu\bar{\psi}_\mu: = \begin{cases} -\bar{\psi}_\mu\psi_\nu, & \text{if} \quad \nu > 0 \quad \text{and} \quad \mu < 0, \\ \psi_\nu\bar{\psi}_\mu, & \text{otherwise.} \end{cases}$$

Put

$$\psi(z) = \sum_{\mu \in \mathbf{Z}_h} \psi_\mu z^{-\mu-1/2},$$
$$\bar{\psi}(z) = \sum_{\mu \in \mathbf{Z}_h} \bar{\psi}_\mu z^{-\mu-1/2}.$$

Definition 2.4. For a differential operator $P \in \mathscr{D}_z(A)$ (see (1.4)), the second quantized operator $\Phi(P)$ is defined by

$$\Phi(P) = \mathrm{Res}_{z=0}(:P(\psi(z))\bar{\psi}(z):dz),$$

where $\mathrm{Res}_{z=0}$ means taking the coefficient of dz/z.

A direct calculation shows

$$\Phi(P) = \mathrm{Res}_{z=0}(:\psi(z)(P^+\bar{\psi}(z)):dz),$$

where the adjoint operator P^+ of P is defined by the following rules:

1. $z^+ = z, \left(\dfrac{d}{dz}\right)^+ = -\dfrac{d}{dz},$
2. $(PQ)^+ = Q^+P^+$ for any $P, Q \in \mathscr{D}_z(A)$.

Theorem 2.5. (i) *For any $P \in \mathscr{D}_z(A)$, we have*

$$[\Phi(P), \psi(z)] = P\psi(z), \quad [\Phi(P), \bar{\psi}(z)] = -P^+\bar{\psi}(z).$$

(ii) *For P, $Q \in \mathscr{D}_z(A)$ we have*

$$[\Phi(P), \Phi(Q)] = \Phi([Q, P]) + c(P, Q)id,$$

where

$$c(P, Q) = \mathrm{Res}_{z=0}\left(\mathrm{Res}_{w=0}P\left(w, \frac{d}{dw}\right)\frac{1}{w-z}Q\left(z, \frac{d}{dz}\right)\frac{1}{w-z}dw\right)dz.$$

Note that $c(P, Q)$ is an element of $H^2(\mathscr{D}_z(A), A)$. By the same reason as in [KNTY, Sect. 3, 3.24, 3.25], $\Phi(P)$ induces a vector field $\bar{\theta}(P)$ on $\mathscr{F}^{\times}(A)=\oplus_p \mathscr{F}_p^{\times}(A)$, $\mathscr{F}_p^{\times}(A)=\mathscr{F}_p(A)-\{0\}$ and $\tilde{\theta}(P)$ induces a vector field $\theta(P)$ on $\mathbf{P}(\mathscr{F}(A))=\coprod_p \mathbf{P}(\mathscr{F}_p(A))$. Moreover, $\theta(P)$ preserves the Plücker relations and coincides with the image of $\theta(P)$ defined in the previous section by the Plücker embedding.

The current $J(z)$ is defined by

$$J(z)=:\bar{\psi}(z)\psi(z):.$$

The current $J(z)$ has the formal expansion

$$J(z)=\sum_{n\in\mathbf{Z}} J_n z^{-n-1},$$

where

$$J_n=-\Phi(z^n).$$

The current operators J_n have the following commutation relations

$$[J_n, J_m]=n\delta_{n+m,0}.$$

If 2 is invertible in A (e.g. $A=\mathbf{Z}[\frac{1}{2}]$), the energy momentum tensor $T(z)$ is defined by

$$T(z)=\tfrac{1}{2}:\frac{d\psi(z)}{dz}\bar{\psi}(z)-\psi(z)\frac{d\bar{\psi}(z)}{dz}:.$$

$T(z)$ has the formal expansion

$$T(z)=\sum_{n\in\mathbf{Z}} L_n z^{-n-2},$$

where

$$L_n=\Phi\left(z^n\left(z\frac{d}{dz}+\frac{1}{2}(n+1)\right)\right).$$

The Virasoro operators L_n form the Virasoro algebra. In our case we have

$$[L_n, L_m]=(n-m)L_{n+m}+\tfrac{1}{12}(m^3-m)\delta_{n+m,0}.$$

If 2 is not invertible, we define the modified energy momentum tensor $\tilde{T}(z)$ by

$$T(z)=:\frac{d\psi(z)}{dz}\bar{\psi}(z)-\psi(z)\frac{d\bar{\psi}(z)}{dz}:.$$

The modified Virasoro operator $\tilde{L}_n$ is defined by

$$\tilde{L}_n=\begin{cases}\Phi\left(z^n\left(z\dfrac{d}{dz}+\dfrac{1}{2}(n+1)\right)\right) & \text{if} \quad n \text{ is odd,}\\[2ex] \Phi\left(z^n\left(2z\dfrac{d}{dz}+(n+1)\right)\right) & \text{if} \quad n \text{ is even.}\end{cases}$$

Then $\tilde{L}_n$'s form a Lie algebra over any ring A.

3. Bosonization over the Ring of Integers

To define a new bosonization, first we recall the usual bosonization. The boson Fock space $\mathscr{H}_T(\mathbf{Q})$ over the rational numbers $\mathbf{Q}$ is defined by

$$\mathscr{H}_T(\mathbf{Q}) = \mathbf{Q}[[t_1, t_2, \dots]] \otimes \mathbf{Q}[u, u^{-1}] = \oplus_{m\in\mathbf{Z}} \mathscr{H}_{T,m}(\mathbf{Q})u^m,$$

where t_i's are infinite numbers of variables. We define the degree and the charge of these variables as follows:

$$\begin{aligned} \deg t_i &= i, & \deg u &= 0, \\ \operatorname{charge} t_i &= 0, & \operatorname{charge} u^m &= m. \end{aligned}$$

The usual bosonization $B: \mathscr{F}(\mathbf{Q}) \to \mathscr{H}_T(\mathbf{Q})$ is defined by

$$B|\Psi\rangle = \sum_{n\in\mathbf{Z}} \langle n|\exp\left(\sum_{m=1}^{\infty} J_m t_m\right)|\Psi\rangle u^n \tag{3.1}$$

for any $|\Psi\rangle \in \mathscr{F}(\mathbf{Q})$.

The bosonization B has the following properties.

1. B is an isomorphism.
2. B preserves the charge and the degree.
3. For any linear operator $P: \mathscr{F}(\mathbf{Q}) \to \mathscr{F}(\mathbf{Q})$, put $P_B = BPB^{-1}$. Then we have the following:

$$J_B = \sum_{n\in\mathbf{Z}} a_n z^{-n-1},$$

where we have

$$\begin{cases} a_n = \dfrac{\partial}{\partial t_n}, \quad a_{-n} = n t_n \quad \text{for} \quad n > 0, \\ a_0 = u\dfrac{\partial}{\partial u}. \end{cases} \tag{3.2}$$

4. For any integer k, the Vertex operator $V_k(z)$ of charge k is defined by

$$V_k(z) = \exp\left(k\sum_{n=1}^{\infty} t_n z^n\right) u^k \exp\left(ku\log z\frac{\partial}{\partial u} - k\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\right). \tag{3.3}$$

Then we have

$$\begin{cases} \psi_B(z) = V_{-1}(z), \\ \bar{\psi}_B(z) = V_1(z). \end{cases} \tag{3.4}$$

Now we introduce new variables $x_1, x_2, x_3, \dots$ as follows:

$$n t_n = \sum_{d|n} d x_d^{n/d}. \tag{3.5}$$

For example, we have

$$\begin{aligned} t_1 &= x_1, \\ 2t_2 &= x_1^2 + 2x_2, \\ 3t_3 &= x_1^3 + 3x_3, \\ 4t_4 &= x_1^4 + 2x_2^2 + 4x_4. \end{aligned}$$

We can express x_d as a polynomial of t_i's with coefficients in the rational numbers $\mathbf{Q}$. For example, we have

$$x_1 = t_1,$$
$$x_2 = t_2 - \tfrac{1}{2}t_1^2,$$
$$x_3 = t_3 - \tfrac{1}{3}t_1^3,$$
$$x_4 = t_4 - \tfrac{1}{2}t_2^2 + \tfrac{1}{2}t_2t_1^2 - \tfrac{3}{8}t_1^4.$$

Now we introduce a new boson Fock space $\mathscr{H}(\mathbf{Q})$ over $\mathbf{Q}$ by

$$\mathscr{H}(\mathbf{Q}) = \mathbf{Q}[[x_1, x_2, \dots]] \otimes \mathbf{Q}[u, u^{-1}],$$

where

$$\begin{cases} \deg x_i = i, & \deg u = 0, \\ \operatorname{charge} x_i = 0, & \operatorname{charge} u^n = n. \end{cases} \tag{3.6}$$

Then there is a ring isomorphism

$$W^*: \mathscr{H}_T(\mathbf{Q}) \to \mathscr{H}(\mathbf{Q}),$$

defined by

$$W^*(t_n) = \sum_{d|n} \frac{d}{n} x_d^{n/d},$$
$$W^*(u) = u. \tag{3.7}$$

Thus we have the mapping

$$\mathscr{F}(\mathbf{Q}) \xrightarrow{B} \mathscr{H}_T(\mathbf{Q}) \xrightarrow{W^*} \mathscr{H}(\mathbf{Q}).$$

We regard the fermion Fock space $\mathscr{F}(\mathbf{Z})$ over the integer ring $\mathbf{Z}$ as a subspace (lattice) of $\mathscr{F}(\mathbf{Q})$. The image $W^* \circ B(\mathscr{F}(\mathbf{Z}))$ is characterized as follows.

Theorem 3.1. *We have*

$$W^* \circ B(\mathscr{F}(\mathbf{Z})) = \mathbf{Z}[[x_1, x_2, \dots]] \otimes \mathbf{Z}[u, u^{-1}].$$

Hence $W^* \circ B(\mathscr{F}(\mathbf{Z}))$ is a $\mathbf{Z}$-subring of $\mathscr{H}(\mathbf{Q})$.

To prove the theorem we need several notations. The Schur polynomials $p_j(t_1, t_2, \dots, t_j) (j = 1, 2, \dots)$ are defined by

$$\exp\left(\sum_{n=1}^{\infty} t_n z^n\right) = \sum_{j=0}^{\infty} p_j(t) z^j,$$

where we put

$$p_0(t) \equiv 1.$$

Put

$$\tilde{p}_j(x_1, \dots, x_j) = p_j(W^*(t_1), \dots, W^*(t_j)).$$

Lemma 3.2.

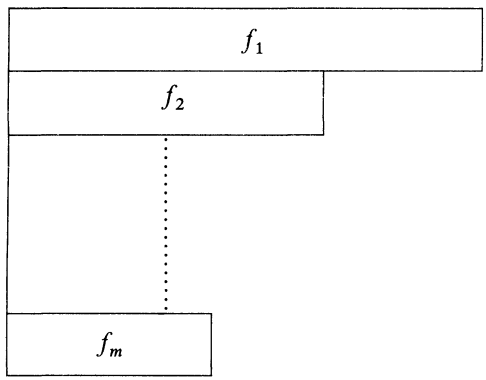
$$\tilde{p}_j(x_1, x_2, \dots, x_j) = \sum_{\substack{d_1 + 2d_2 + \cdots + jd_j = j \\ d_i \geq 0}} x_1^{d_1} x_2^{d_2} \cdots x_j^{d_j}.$$

Proof. The result is clear from the following:

$$\exp\left(\sum_{n=1}^{\infty} t_n z^n\right) = \exp\left(\sum_{n=1}^{\infty} \sum_{d|n} \frac{d}{n} x_d^{n/d} z^n\right)$$
$$= \exp\left(\sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x_d^m z^{md}\right) = \prod_{d=1}^{\infty} (1 - x_d z^d)^{-1}. \tag{3.8}$$

Q.E.D.

For a Young diagram Y of signature $(f_1, \ldots, f_m)$, $f_1 \geqq f_2 \geqq \cdots \geqq f_m$,



we define the Schur function $\chi_Y(t)$ by

$$\chi_Y(t) = \det(p_{f_i - i + j}(t))_{1 \leq i, j \leq m}. \tag{3.9}$$

The modified Schur function $\tilde{\chi}_Y(x)$ is defined by

$$\tilde{\chi}_Y(x) = \det(\tilde{p}_{f_i - i + j}(x))_{1 \leq i, j \leq m}. \tag{3.10}$$

By the definition of the fermion Fock space of charge p,

$$\psi_{j_1} \cdots \psi_{j_r} \bar{\psi}_{i_s} \cdots \bar{\psi}_{i_1} |0\rangle, \quad p = s - r, \quad i_1 < \cdots < i_s, \quad j_1 < \cdots < j_r$$

are a basis of $\mathscr{F}_p(A)$ for any ring A.

Lemma 3.3. *If A is a field of characteristic* 0 (e.g. $A = \mathbf{Q}, \mathbf{C}$), *then we have*

$$B(\psi_{j_1} \cdots \psi_{j_r} \bar{\psi}_{i_s} \cdots \bar{\psi}_{i_1} |0\rangle) = (-1)^{j_1 + \cdots + j_r - r/2 + p(p-1)/2} \chi_Y(t) u^p,$$

where $p = s - r$, and where Y is the Young diagram of signature

$$(-i_1 - p + 1/2, \ldots, -i_2 - p + 3/2, \ldots, -i_s - p + s - 1/2, \underbrace{r, \ldots, r}_{-j_r - 1/2},$$
$$\underbrace{r - 1, \ldots, r - 1}_{j_r - j_{r-1} - 1}, \ldots, \underbrace{1, \ldots, 1}_{j_2 - j_1 - 1}).$$

For the proof, see [DJKM].

Lemma 3.4. *$B\mathscr{F}(\mathbf{Z})$ is naturally a ring.*

Proof. By Lemma 3.3 it suffices to prove that $B\mathscr{F}_0(\mathbf{Z})$ is a ring. This follows from the fact that for Schur functions $\chi_Y(t)$ and $\chi_{Y'}(t)$ there exist Schur functions $\chi_{Y''}(t)$ and integers $m_{Y,Y',Y''}$ such that

$$\chi_Y(t)\chi_{Y'}(t)=\sum_{Y''} m_{Y,Y',Y''}\chi_{Y''}(t). \quad \text{Q.E.D.}$$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Since we have

$$B\mathscr{F}_0(\mathbf{Z})=\mathbf{Z}[\chi_Y(t)]_{Y\in\mathscr{Y}}=\mathbf{Z}[p_1,p_2,\dots],$$

where $\mathscr{Y}$ is the set of Young diagrams, it suffices to show

$$\mathbf{Z}[[\tilde{p}_1,\tilde{p}_2,\dots]]=\mathbf{Z}[[x_1,x_2,\dots]].$$

By induction on n, we prove $x_n\in\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]$. This is clear for $n=1$, since $\tilde{p}_1=x_1$. Assume that $x_j\in\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]$ for $j\leqq n-1$. By Lemma 3.2,

$$\tilde{p}_n(x_1,\dots,x_n)=x_n+\sum_{\substack{d_1+2d_2+\cdots+(n-1)d_{n-1}=n\\ d_i\geqq 0}} x_1^{d_1}x_2^{d_2}\cdots x_{n-1}^{d_{n-1}}.$$

The second term of the right-hand side is in $\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]$ by Lemma 3.4 and the induction hypothesis. Hence, $x_n\in\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]$. This implies $\mathbf{Z}[x_1,x_2,\dots]\subset\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]$. The opposite inclusion is clear. Thus we have

$$\mathbf{Z}[\tilde{p}_1,\tilde{p}_2,\dots]=\mathbf{Z}[x_1,x_2,\dots].$$

Now by the completion by means of the degrees, we have

$$\mathbf{Z}[[\tilde{p}_1,\tilde{p}_2,\dots]]=\mathbf{Z}[[x_1,x_2,\dots]]. \quad \text{Q.E.D.}$$

Definition 3.5. The boson Fock space $\mathscr{H}(A)$ on a ring A is defined by

$$\mathscr{H}(A)=A[[x_1,x_2,\dots]]\otimes_A A[u,u^{-1}],$$

where

$$\deg x_i=i, \qquad \deg u=0$$
$$\text{charge } x_i=0, \quad \text{charge } u^n=n.$$

Definition 3.6. The bosonization $\tilde{B}:\mathscr{F}(A)\to\mathscr{H}(A)$ is an A-linear map defined by

$$\tilde{B}(\psi_{j_1}\cdots\psi_{j_r}\bar{\psi}_{i_s}\cdots\bar{\psi}_{i_1}|0\rangle=(-1)^{j_1+\cdots+j_r-r/2+p(p-1)/2}\tilde{\chi}_Y(x)u^p,$$

where $p=s-r$, and where $\tilde{\chi}_Y(x)$ is defined by (3.10).

By Theorem 3.1 we obtain the following corollary.

Corollary 3.7. *The bosonization $\tilde{B}:\mathscr{F}(A)\to\mathscr{H}(A)$ is an isomorphism.*

To express the Vertex operators $V_k(z)$ by means of new coordinates, we need to define the universal Witt vector ring $W_\infty(A)$. Put

$$\begin{cases} s_n=\sum_{d|n}\dfrac{d}{n}y_d^{n/d}, \\ s'_n=\sum_{d|n}\dfrac{d}{n}y_d'^{n/d}. \end{cases} \tag{3.11}$$

There are polynomials

$$A_n(y_1, y_2, \ldots, y_n; y'_1, y'_2 \ldots, y'_n) \in \mathbf{Z}[y_1, y_2, \ldots, y_n, y'_1, y'_2 \ldots, y'_n],$$
$$M_n(y_1, y_2, \ldots, y_n; y'_1, y'_2 \ldots, y'_n) \in \mathbf{Z}[y_1, y_2, \ldots, y_n, y'_1, y'_2 \ldots, y'_n], \quad n = 1, 2, \ldots,$$

such that

$$s_n + s'_n = \sum_{d|n} \frac{d}{n} A_d(y_1, y_2, \ldots, y_d; y'_1, y'_2 \ldots, y'_d)^{n/d},$$

$$s_n \cdot s'_n = \sum_{d|n} \frac{d}{n} M_d(y_1, y_2, \ldots, y_d; y'_1, y'_2 \ldots, y'_d)^{n/d}, \quad n = 1, 2, 3, \ldots .$$

Thus, we can introduce a new addition $+_w$ and a new multiplication $\cdot_w$ on A^∞ by

$$\begin{aligned}(a_1, a_2, \ldots) +_w (b_1, b_2, \ldots) &= (A_1(a_1, a_2), A_2(a_1, a_2; b_1, b_2), \ldots,\\ &\qquad A_n(a_1, \ldots, a_n; b_1, \ldots, b_n), \ldots),\\ (a_1, a_2, \ldots) \cdot_w (b_1, b_2, \ldots) &= (M_1(a_1, a_2), M_2(a_1, a_2; b_1, b_2), \ldots,\\ &\qquad M_n(a_1, \ldots, a_n; b_1, \ldots, b_n), \ldots).\end{aligned}$$

Then $W_\infty(A) = (A^\infty, +_w, \cdot_w)$ is a commutative ring and is called the universal Witt ring over the ring A. The unity of $W_\infty(A)$ is

$$(1, 0, 0, 0, 0, \ldots),$$

and the zero is

$$(0, 0, 0, 0, 0, \ldots)$$

(cf. [M, Lecture 26]). If we use the language of schemes, there exists a commutative ring scheme W_∞ defined over $\mathbf{Z}$, called the universal Witt scheme, whose A-valued points $W_\infty(A)$ is the universal Witt ring defined above. Then the ring $\mathbf{Z}[x_1, x_2, \ldots]$ is the coordinate ring of the universal Witt scheme and $\mathbf{Z}[[x_1, x_2, \ldots]]$ is the completion of the coordinate ring with respect to the degree defined in (3.6).

By $[1/z]$ we denote the element $(1/z, 0, 0, \ldots)$ in the ring $W_\infty(A[1/z])$. By the s-coordinates defined in (3.11), this element corresponds to the element $(1/z, 1/2z, \ldots, 1/nz^n, \ldots)$. By $\imath$ we mean the inverse of the addition of $W_\infty(A)$, that is,

$$\imath(a_1, a_2, \ldots) +_w (a_1, a_2, \ldots) = (0, 0, 0, \ldots).$$

For any element $\mathbf{a} = (a_1, a_2, \ldots) \in W_\infty(A)$ and any positive integer k, by $k\mathbf{a}$ we mean

$$\underbrace{\mathbf{a} +_w \mathbf{a} +_w \cdots +_w \mathbf{a}}_{k},$$

and $-k\mathbf{a}$ means $\imath(k\mathbf{a})$. Note that $\imath^2 = 1$ and $\imath(k\mathbf{a}) = k(\imath\mathbf{a})$.

Now we express the vertex operator $V_k(z)$ by means of the variables $(x_1, x_2, \ldots)$. First note that

$$\begin{aligned}&\left\{\exp\left(-k \sum_{n=1}^{\infty} \frac{1}{nz^n} \frac{\partial}{\partial t_n}\right)\right\} f(t_1, t_2, \ldots, t_n \cdots)\\ &\quad = f(t_1 - k/z, t_2 - k/2z^2, \ldots, t_n - k/nz^n, \ldots).\end{aligned}$$

Hence, in the new variables $x_1, x_2, \ldots$, this operator is expressed in the form

$$T^*_{k\iota[1/z]}g(x_1, x_2, \ldots) = g((x_1, x_2, \ldots) +_w k\iota[1/z]).$$

We define $M^*_{z^k}$ by

$$M^*_{z^k}h(u) = h(z^k u).$$

Note that

$$\left\{\exp\left(ku\log(z)\frac{\partial}{\partial u}\right)\right\}h(u) = h(z^k u).$$

Put

$$\tilde{V}_k(z) = \prod_{d=1}^{\infty}(1 - x_d z^d)^{-k} u^k \circ M^*_{z^k} \circ T^*_{k\iota[1/z]}, \tag{3.12}$$

where $\prod_{d=1}^{\infty}(1 - x_d z^d)^{-k} u^k$ operates on $\mathscr{H}(A)$ by the multiplication. By (3.3) and (3.8) $\tilde{V}_k(z)$ is the vertex operator expressed by the variables $x_1, x_2, \ldots$. Thus we have the following proposition.

Proposition 3.8. *On the boson Fock space $\mathscr{H}(A)$, the fermion operators are expressed in the following forms*:

$$\tilde{B}\psi(z)\tilde{B}^{-1} = \tilde{V}_{-1}(z),$$
$$\tilde{B}\bar{\psi}(z)\tilde{B}^{-1} = \tilde{V}_1(z).$$

The current operators are expressed by the following forms.

Proposition 3.9. *For a positive integer n, we have*

$$\tilde{B}J_n\tilde{B}^{-1} = \sum_{m=1}^{\infty} b_{nm,n}\frac{\partial}{\partial x_{nm}},$$
$$\tilde{B}J_{-n}\tilde{B}^{-1} = \sum_{d|n} d x_d^{n/d},$$

where

$$b_{n,n} = 1,$$
$$b_{nm,n} = \sum_{\substack{d_1|d_2|\cdots|d_r|m \\ 1=d_1<d_2<\cdots<d_r<m}} (-1)^r x_{nd_1}^{(d_2/d_1)-1} x_{nd_2}^{(d_3/d_2)-1} \cdots x_{nd_r}^{(m/d_r)-1}.$$

Proof. Consider a matrix B of infinite size whose (nm, n)-components $b_{nm,n}$ are defined as above and other (k, n)-components $b_{k,n}$ are zero. Then by a direct calculation we have $B = A^{-1}$, where the matrix A is defined by

$${}^t\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, \ldots\right) = {}^tA^t\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_n}, \ldots\right)$$

by using (3.5). Q.E.D.

Remark 3.10. The above proof of Theorem 1 shows that as a new boson Fock space over a ring A we may choose $A[[p_1, p_2, \ldots]] \otimes A[u, u^{-1}]$, where p_i is the

i^{th} Schur polynomial. This is related to a λ-ring and formal groups associated with Jacobians of curves. This point of view will be discussed in our forthcoming paper.

4. Curves and their Moduli Space

In this section, we connect the theory developed in the previous sections with geometry of algebraic curves. The present section corresponds to Sects. 2C) and D) of [KNTY] in which the theory over the complex numbers $\mathbf{C}$ is treated. Below we shall formulate the theory over a commutative ring A with unity. Since we need to use the language of schemes, the reader who is not familiar with scheme theory is recommended to read the appendix first where we shall give a brief introduction to scheme theory. Though the theory below seems complicated, it is essentially the same as that over the complex numbers $\mathbf{C}$.

By a curve $\pi: C \to \operatorname{Spec}(A)$ of genus g over a commutative ring A we mean that π is a proper smooth morphism of schemes whose generic geometric fiber is a non-singular algebraic curve of genus g. Hence for any point $\mathfrak{p} \in \operatorname{Spec}(A)$, the fiber is a non-singular curve of genus g defined over the residue field $k(\mathfrak{p})$. Let $Q: \operatorname{Spec}(A) \to C$ be a section of π, i.e., Q is a morphism of schemes with $\pi \cdot Q = id$. By I_Q we denote the ideal sheaf of the image $Q(\operatorname{Spec}(A))$. Note that there is a homomorphism $\pi^*: \mathcal{O}_A \to \mathcal{O}_C$ which defines an $\mathcal{O}_A$-algebra structure on $\mathcal{O}_C$ via the morphism π. There is a canonical $\mathcal{O}_A$-algebra isomorphism

$$u_0: \mathcal{O}_C / I_Q \simeq \mathcal{O}_A.$$

The ideal sheaf I_Q is an invertible $\mathcal{O}_C$-module, since π is smooth and of relative dimension 1. Therefore the conormal bundle $N_Q^* = I_Q / I_Q^2$ of $Q(\operatorname{Spec}(A))$ in C may be regarded as an invertible $\mathcal{O}_A$-module via the morphism π. Assume that N_Q^* is a free $\mathcal{O}_A$-module. This means that there is an element $s \in \Gamma(\mathcal{U}, I_Q)$ such that $\bar{s} \equiv s \bmod I_Q^2$ generates N_Q^* as an $\mathcal{O}_A$-module, where $\mathcal{U}$ is an affine neighborhood of the section $Q(\operatorname{Spec}(A))$.

Lemma 4.1. *Under the above hypothesis, there is an $\mathcal{O}_A$-algebra isomorphism*

$$\mathcal{O}_C / I_Q^{n+1} \simeq \mathcal{O}_A[\zeta]/(\zeta^{n+1}).$$

Proof. For any $a \in \mathcal{O}_{C,x}$, $x \in Q$ $(\operatorname{Spec}(A))$, there is a unique expansion

$$a \equiv a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n \bmod I_Q^{n+1},$$

where $a_i \in \mathcal{O}_{A,\pi(x)}$. The mapping

$$a \mapsto a_0 + a_1 \zeta + a_2 \zeta^2 + \cdots + a_n \zeta^n$$

is a desired $\mathcal{O}_A$-algebra homomorphism. Q.E.D.

Note that if A is a principal ideal domain, N_Q^* is always $\mathcal{O}_A$-free.

Definition 4.2. Fix a positive integer $l \geqq 3$. For any commutative ring A, the set $\mathscr{C}_{1,g,l}(A)$ is defined by

$$\mathscr{C}_{1,g,l}(A) = \{(\pi: C \to \operatorname{Spec}(A), Q, \sigma_l)\},$$

where π is a curve of genus g over A, Q is a section of π, and σ_l is a level l structure of π. (For the definition of the level l structure, see [OS, Sect. 1].)

2) For any integer $n \geqq 2$ and any commutative ring A, the set $\mathscr{C}_{n,g,l}(A)$ is defined by

$$\mathscr{C}_{n,g,l}(A) = \{(\pi: C \to \operatorname{Spec}(A), Q \text{ with } N_Q^* \text{ free as an } \mathcal{O}_A\text{-module},$$
$$u_n: \mathcal{O}_C/I_Q^{n+1} \simeq \mathcal{O}_A[\zeta]/(\zeta^{n+1}))\},$$

where u_n is an $\mathcal{O}_A$-algebra homomorphism.

Note that for any $n > m$, there is a natural mapping

$$f_{m,n}(A): \mathscr{C}_{n,g,l}(A) \to \mathscr{C}_{m,g,l}(A).$$

Let $\pi: \mathscr{C}_{g,l}(A) \to \mathscr{M}_{g,l}(A)$ be the universal family of curves over the moduli space of curves of genus g with level l structure. $\pi: \mathscr{C}_{g,l} \to \mathscr{M}_{g,l}$ is defined over the ring $\mathbf{Z}[1/l]$.

Theorem 4.3. 1) *For any commutative ring A, $\mathscr{C}_{1,g,l}(A)$ is equal to the set $\mathscr{C}_{g,l}(A)$ of A-valued points of $\mathscr{C}_{g,l}$.*

2) *For any integer $n \geqq 2$, there exists a scheme $\mathscr{C}_{g,l}^{(n)}$ with morphism $f_n: \mathscr{C}_{g,l}^{(n)} \to \mathscr{C}_{g,l}$ such that for any principal ideal domain A, the set of A-valued points of $\mathscr{C}_{g,l}^{(n)}$ is equal to $\mathscr{C}_{n,g,l}(A)$. Moreover, f_n is a principal bundle with structure group $\mathscr{G}_n$, where $\mathscr{G}_n(A) = \operatorname{Aut}_A(A[[\zeta]]/(\zeta^{n+1}))$.*

Proof. 1) The first part is clear from the definition.

2) Put $\mathscr{C} = \mathscr{C}_{g,l}$, $\mathscr{M} = \mathscr{M}_{g,l}$. Let $\Delta: \mathscr{C} \to \mathscr{C} \times_{\mathscr{M}} \mathscr{C}$ be the diagonal mapping and I_Δ the ideal sheaf of $\Delta(\mathscr{C})$ in $\mathscr{C} \times_{\mathscr{M}} \mathscr{C}$. We let $\{\mathscr{U}_\alpha\}_{\alpha \in A}$ be the all maximal affine open sets of $\mathscr{C}$ such that $(I_\Delta/I_\Delta^2)|\mathscr{U}_\alpha$ is $\mathcal{O}_{\mathscr{U}_\alpha}$-free. For each $x \in A$, we fix an $\mathcal{O}_{\mathscr{U}_\alpha}$-algebra isomorphism

$$u_{\alpha,n}: \mathcal{O}_{p^{-1}(\mathscr{U}_\alpha)}/I^{n+1}_{\Delta|p^{-1}(\mathscr{U}_\alpha)} \simeq \mathcal{O}_{\mathscr{U}_\alpha}[\zeta]/(\zeta^{n+1}),$$

where $p: \mathscr{C} \times_{\mathscr{M}} \mathscr{C} \to \mathscr{C}$ is the projection to the first factor.

Put $\mathscr{U}_{\alpha,n} = \mathscr{U}_\alpha \times \mathscr{G}_n$. If $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \neq \emptyset$, then we patch together $\mathscr{U}_{\alpha,n}$ and $\mathscr{U}_{\beta,n}$ by $u_{\beta,n} \circ u_{\alpha,n}^{-1}$. In this way we have a scheme $\mathscr{C}_{g,l}^{(n)}$ with a morphism $f_n: \mathscr{C}_{g,l}^{(n)} \to \mathscr{C}_{g,l}$ induced by the projection $p_\alpha: \mathscr{U}_\alpha \times \mathscr{G}_n \to \mathscr{U}_\alpha$.

Let $q: \operatorname{Spec}(A) \to \mathscr{C}_{g,l}^{(n)}$ be an A-valued point. Since $\mathscr{U}_{\alpha,n}$, $\alpha \in A$, is an affine open covering, $q(\operatorname{Spec}(A))$ is contained in, say, $\mathscr{U}_{\alpha,n}$. Put $p^{-1}(q(\operatorname{Spec}(A))) = C$, $\pi: C \to \operatorname{Spec}(A)$, $Q = \Delta_n \circ q$, where Δ_n is the diagonal mapping $\mathscr{C}_{g,l}^{(n)} \to \mathscr{C}_{g,l}^{(n)} \times \mathscr{C}_{g,l}^{(n)}$. Since $I_\Delta/I_\Delta^2|\mathscr{U}_\alpha$ is $\mathcal{O}_{\mathscr{U}_\alpha}$-free, $I_Q/I_Q^2 = q^*(I_\Delta/I_\Delta^2)$ is $\mathcal{O}_A$-free. Moreover, q determines an element $\beta_q \in \mathscr{G}_n(A) = \operatorname{Aut}(\mathcal{O}_A[\zeta]/(\zeta^{n+1}))$. Put $u_n = \beta_q \circ q^*(u_{\alpha,n})$. Then $(\pi, Q, u_n) \in \mathscr{C}_{n,g,l}(A)$. Conversely, to any element of $\mathscr{C}_{n,g,l}(A)$, by a similar way we can always associate an A-valued point of $\mathscr{C}_{g,l}^{(n)}$. Q.E.D.

Definition 4.4. The proscheme $\hat{\mathscr{C}}_{g,l}^{(n)}$ is defined by

$$\hat{\mathscr{C}}_{g,l} = \varprojlim_n \mathscr{C}_{g,l}^{(n)}.$$

The tangent spaces of $\mathscr{C}_{g,l}^{(n)}$ and $\hat{\mathscr{C}}_{g,l}$ are given as follows. Let $X_n = \{\pi: C \to \operatorname{Spec} A, \mathrm{Q}$ with free $\mathcal{O}_A$-module $N_Q^*, u_n\}$ correspond to an A-valued point of $\mathscr{C}_{g,l}^{(n)}$.

The infinitesimal deformation of X_{n-1} is given by $H^1(C, \Theta_{C/A}(-nQ))$. By using $\mathcal{O}_A$-algebra homomorphism u_n, we have an exact sequence

$$0 \to \Theta_{C/A}(-nQ) \to \Theta_{C/A}((m-n)Q)$$
$$\to \mathcal{O}_A \zeta^{n-m} \frac{d}{d\zeta} \oplus \mathcal{O}_A \zeta^{n-m+1} \frac{d}{d\zeta} \oplus \cdots \oplus \mathcal{O}_A \zeta^{n-1} \frac{d}{d\zeta} \to 0,$$

where $\Theta_{C/A}$ is the sheaf of relative vector fields of $\pi: C \to \mathrm{Spec}(A)$.

If $m > n + 2g - 1$, then we have

$$T_{X_{n-1}} \mathscr{C}_{g,l}^{(n)} \simeq H^1(C, \Theta_{C/A}(-nQ))$$
$$\simeq (A\zeta^{n-m} \oplus \cdots \oplus A\zeta^{n-1}) \frac{d}{d\zeta} \Big/ H^0(C, \Theta_{C/A}((m-n)Q))$$
$$\simeq \zeta^{-(m-n)} (A \oplus A\zeta \oplus \cdots \oplus A\zeta^{m-1}) \frac{d}{d\zeta} \Big/ H^0(C, \Theta_{C/A}((m-n)Q)).$$

Note that we can take the projective limit of both sides since $\Theta_{C/A}(-n'Q) \hookrightarrow \Theta_{C/A}(-nQ)$ for $n' > n$. Now fixing $m - n = k > 2g - 1$ and taking $n \to \infty$, we have

$$T_X \hat{\mathscr{C}}_{g,l} \simeq \zeta^{-k} A[[\zeta]] \frac{d}{d\zeta} \Big/ H^0(C, \Theta_{C/A}(kQ)).$$

This holds for any $k > 2g - 1$, hence taking $k \to \infty$, we finally obtain

$$T_X \hat{\mathscr{C}}_{g,l} \simeq A((\zeta)) \frac{d}{d\zeta} \Big/ H^0(C, \Theta_{C/A}(*Q)).$$

The right-hand side is independent of g. Hence each element of $\mathbf{Z}((z^{-1}))(d/dz), z = \zeta^{-1}$, induces a tangent vector at each $X \in \hat{\mathscr{C}}_{g,l}(A)$.

Next let us assume that for a point $X = (\pi: C \to \mathrm{Spec}(A), Q, u) \in \hat{\mathscr{C}}_{g,l}(A)$, there exists an invertible sheaf $\mathscr{L}$ on C such that $\mathscr{L}^{\otimes 2} = \omega_{C/A}$, where $\omega_{C/A}$ is the relative dualizing sheaf of $\pi: C \to \mathrm{Spec}(A)$. For any positive integer m, we have an exact sequence

$$0 \to \pi_* \mathscr{L}((m-n)Q) \to \pi_* \mathscr{L}(mQ) \to \mathscr{L} \otimes I_Q^{-m} \otimes \mathcal{O}_C / I_Q^n \to R^1 \pi_* \mathscr{L}((m-n)Q) \to 0.$$

Since $\sqrt{d\zeta}$ gives a trivialization of $\mathscr{L} \otimes \mathcal{O}_C / I_Q^n$ as $\mathcal{O}_A$-module up to sign $\pm$, via u_{n-1} we have a natural mapping

$$\pi_* \mathscr{L}(mQ) \to \zeta^{-m} \mathcal{O}_A[\zeta]/(\zeta^n) \sqrt{d\zeta},$$

and taking $n \to \infty$, we have

$$\sigma_m(X): \pi_* \mathscr{L}(mQ) \to z^{m+1} \mathcal{O}_A[[z^{-1}]] \sqrt{dz},$$

where $z = 1/\zeta$.

Lemma 4.5. *$\sigma_m(X)$ is injective and* Coker $\sigma_m(X)$ *is a locally free $\mathcal{O}_A$-module.*

Proof. For injectivity, we have

$$\mathrm{Ker}\, \sigma_m(X) = \bigcap_{n=1}^{\infty} \pi_* \mathscr{L}((m-n)Q)) = 0.$$

If $m-n<g-1$, then $\pi_*\mathscr{L}((m-n)Q)=0$. Hence

$$0\to\pi_*\mathscr{L}(mQ)\to\mathscr{L}\otimes I_Q^{-m}\otimes\mathcal{O}_C/I_Q^n\to R^1\pi_*\mathscr{L}((m-n)Q)\to 0.$$

The sheaf $R^1\pi_*\mathscr{L}((m-n)Q)$ is locally free of rank $n-m+g-1$ for $n\gg 0$. Hence

$$\operatorname{Coker}\sigma_m(X)=\varprojlim_n R^1\pi_*\mathscr{L}((m-n)Q)$$

which is locally free $\mathcal{O}_A$-module. Q.E.D.

Taking $m\to\infty$, we obtain

$$\sigma(X):\pi_*\mathscr{L}(*Q)\to\mathcal{O}_A((z^{-1}))\sqrt{dz}. \tag{4.1}$$

Corollary 4.6. *$\sigma(X)$ is injective and the image defines a point of* UGM(A)

Proposition 4.7. *There exist a finite étale covering $\pi:\tilde{\mathscr{M}}_{g,l}\to\mathscr{M}_{g,l}$ defined over* $\mathbf{Z}[1/2l]$ *and an invertible sheaf $\mathscr{L}$ on $\tilde{\mathscr{C}}_{g,l}=\mathscr{C}_{g,l}\times_{\mathscr{M}_{g,l}}\tilde{\mathscr{M}}_{g,l}$ with $\mathscr{L}^{\otimes 2}=\omega_{\tilde{\mathscr{C}}_{g,l}/\tilde{\mathscr{M}}_{g,l}}$.*

We denote the pull back of the sheaf $\mathscr{L}$ to $\tilde{\hat{\mathscr{C}}}_{g,l}=\hat{\mathscr{C}}_{g,l}\times_{\mathscr{M}_{g,l}}\tilde{\mathscr{M}}_{g,l}$ by the same letter $\mathscr{L}$.

Corollary 4.8 *There exists a morphism*

$$\sigma:\tilde{\hat{\mathscr{C}}}_{g,l}\to \mathrm{UGM}^{(0)}$$

such that $\sigma(X)$ is given by (4.1).

Corollary 4.9. *For any element $p(z)(d/dz)\in\mathbf{Z}[1/2l][z,z^{-1}]$, we have*

$$\sigma_*\left(p(z)\frac{d}{dz}\right)=p(z)\frac{d}{dz}+\frac{1}{2}p'(z).$$

Appendix. Brief Introduction to Scheme Theory

We begin with the definition of a ring. A ring R is a set with addition $+$ and multiplication $\cdot$ such that R with addition is an additive group and multiplication is associative and distributive over addition, that is, for all a, b, $c\in R$, we have

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c,$$
$$a\cdot(b+c)=a\cdot b+a\cdot c,$$
$$(b+c)\cdot a=b\cdot a+c\cdot a.$$

Moreover, if we have

$$a\cdot b=b\cdot a,$$

for all $a,b\in R$, the ring R is called a *commutative* ring. If there exists an element $e\in R$ such that

$$a\cdot e=e\cdot a=a,$$

for all $a\in R$, the element e is called *unity* of R. In the following unity is written by 1 and we write ab instead of $a\cdot b$ for multiplication.

For example, the set $\mathbf{Z}$ of integers is a commutative ring with respect to the usual addition and multiplication and 1 is unity. The set $2\mathbf{Z}$ of all even integers is a commutative ring but has not unity. *In the following we only consider a commutative ring with unity.* Such a ring is often called a *unitary* commutative ring.

Definition A.1. A subset $\mathfrak{I}$ of a ring R is called an *ideal*, if for all elements $a, b \in \mathfrak{I}$ and any element $t \in R$, we have

$$a + b \in \mathfrak{I}, \quad ta \in \mathfrak{I}.$$

Note that for an ideal $\mathfrak{I}$, if $a \in \mathfrak{I}$, then $-a \in \mathfrak{I}$. Let $a_1, a_2, \ldots, a_n$ be elements of the ring R and let $\mathfrak{I}$ be the set defined by

$$\mathfrak{I} = \{b_1 a_1 + \cdots + b_n a_n \,|\, b_i \in R, i = 1, 2, \ldots, n\}.$$

Then $\mathfrak{I}$ is an ideal of R. The ideal $\mathfrak{I}$ is said to generated by $a_1, \ldots, a_n$ and written by $(a_1, a_2, \ldots, a_n)$.

Let us consider a non-zero ideal of $\mathfrak{N}$ of $\mathbf{Z}$. Define the integer n by

$$n = \min \{m \,|\, m \in \mathfrak{N}, m > 0\}.$$

Note that if $m \in \mathfrak{N}$, then $-m \in \mathfrak{N}$. Take a positive integer $l \in \mathfrak{N}$. Then we have

$$l = l_1 n + l_2 \quad 0 \leqq l_2 < n.$$

By definition, $l_2 \in \mathfrak{N}$. Then by the definition of the integer n, we have $l_2 = 0$. This means that the ideal $\mathfrak{N}$ is generated by the integer n. Thus any ideal of $\mathbf{Z}$ is generated by a single element.

For an ideal $\mathfrak{I}$ of a ring R the *quotient ring* $R/\mathfrak{I}$ is defined as follows. Introduce an equivalence relation $\sim$ in R by

$$a \sim b \Leftrightarrow a - b \in \mathfrak{I}.$$

Hence for an element $a \in R$ the equivalence class $\bar{a}$ containing a is $a + \mathfrak{I}$. Now the set $R/\mathfrak{I}$ consists of all the equivalence classes in R and the addition and multiplication are given by

$$\bar{a} + \bar{b} = \overline{a + b},$$
$$\bar{a}\bar{b} = \overline{ab}.$$

It is easy to show that this is well defined and $R/\mathfrak{I}$ is a ring with respect to these operations.

Let R and S be rings. A map $\varphi: R \to S$ is called a *homomorphism* of rings, if

$$\varphi(a + b) = \varphi(a) + \varphi(b),$$
$$\varphi(ab) = \varphi(a)\varphi(b),$$
$$\varphi(1) = 1,$$

for all a, $b \in R$. Then the *kernel* $\operatorname{Ker} \varphi$ of φ is defined by

$$\operatorname{Ker} \varphi = \{a \in R \,|\, \varphi(a) = 0\}.$$

Also the *image* $\operatorname{Im} \varphi$ of φ is defined by

$$\operatorname{Im} \varphi = \{\varphi(a) \,|\, a \in R\}.$$

Then $\operatorname{Ker}\varphi$ is an ideal of R and $\operatorname{Im}\varphi$ is a subring of S. Moreover, the quotient ring $R/\operatorname{Ker}\varphi$ is isomorphic to $\operatorname{Im}\varphi$.

A non-zero element a of a ring R is called a *zero divisor*, if there exists a non-zero element b of R with $ab=0$. A ring containing no zero divisors is called a *domain*. For example, let $n=pq$ be the product of two prime numbers p, q and let us consider the quotient ring $\mathbf{Z}/(n)$. Then the classes $\bar{p}$ and $\bar{q}$ are non-zero in $R/(n)$ but $\bar{p}\bar{q}=\bar{0}$. Hence $\bar{p}$ and $\bar{q}$ are zero divisors. On the other hand $\mathbf{Z}/(p)$ has no zero divisors, hence is a domain (actually, $\mathbf{Z}/(p)$ is a field with p elements).

Definition A.2. An ideal $\mathfrak{J}$ of a ring R is called a *prime ideal*, if the quotient ring $R/\mathfrak{J}$ is a domain. An ideal $\mathfrak{J}$ is called a *maximal ideal*, if any ideal of R containing $\mathfrak{J}$ is either $\mathfrak{J}$ or R.

It is easily shown that an ideal $\mathscr{I}$ is maximal if and only if the quotient ring $R/\mathfrak{J}$ is a field. Hence a maximal ideal is a prime ideal. For example, an ideal (n) of the ring $\mathbf{Z}$ of integers is prime if and only if n is a prime number or zero. Moreover an ideal (n) of $\mathbf{Z}$ is maximal if and only if n is a prime number. Let us give other examples.

Example A.3. The ring $\mathbf{C}[x]$ consists of polynomials in x with coefficients in the complex numbers $\mathbf{C}$. Let $\mathfrak{J}$ be a non-zero ideal of R. Let $f(x)$ be a non-zero element of $\mathfrak{J}$ which has the minimal degree. Then any element $g(x)$ of $\mathfrak{J}$ is written by

$$g(x)=g_1(x)f(x)+g_2(x), \quad \deg g_2(x)<\deg f(x).$$

But $g_2(x)\in\mathfrak{J}$ and $f(x)$ has the minimal degree among non-zero polynomials in R, $g_2(x)=0$. Hence the ideal $\mathfrak{J}$ is generated by a single element $f(x)$. Let

$$f(x)=(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_n)^{m_n}$$

be the factorization of $f(x)$. By the similar argument to that in the case of $\mathbf{Z}$, an ideal $(f(x))$ is prime if and only if $f(x)$ is irreducible, hence of degree 1 or $f(x)=0$. Moreover, $(f(x))$ is maximal if and only if $f(x)$ is of degree 1. Thus there is a one to one correspondence between the complex numbers and the set of maximal ideals in $\mathbf{C}[x]$ given by $a\mapsto(x-a)$.

A domain R any of whose ideal is generated by a single element is called a *principal ideal domain*. The above examples show that the ring $\mathbf{Z}$ of integers and the polynomial ring $\mathbf{C}[x]$ are principal ideal domains. Next let us give more complicated examples.

Example A.4. Let p be a prime number. As we stated above $\mathbf{F}_p=\mathbf{Z}/(p)$ is a field with p-elements. The ring $\mathbf{F}_p[x]$ consists of polynomials with coefficients in the field $\mathbf{F}_p$. Then any ideal $\mathfrak{J}$ of $\mathbf{F}_p[x]$ is generated by a single polynomial $f(x)$ and an ideal $(f(x))$ is a prime ideal if and only of the polynomial $f(x)$ is irreducible. Contrary to the above Example A.3 there are infinitely many irreducible polynomials of degree >1. For example, if $p\equiv 3 \bmod 4$, then x^2+1 is irreducible.

The polynomial ring $\mathbf{C}[x,y]$ of two variables with coefficients in $\mathbf{C}$ is also a domain but not principal. For example, (x,y) is a maximal ideal which has two generators.

Example A.5. Let $R=\mathbf{Z}[x]$ be a ring consisting of polynomials with coefficients in integers. Let p be a prime number. Then (p) is a prime ideal. Also an ideal $(p,f(x))$ is prime if and only if $f(x) \bmod p$ is irreducible in $\mathbf{F}_p[x]$.

A subset S of a ring R is called a multiplicative subset, if $1\in S$ and it is closed by the multiplication, that is, $a,b\in S$ implies $ab\in S$. The localization $S^{-1}R$ of R with respect to the multiplicative subset S is the ring

$$R_S=\left\{\frac{a}{s}\middle| a\in R, s\in S\right\},$$

where the equality is defined by

$$\frac{a}{s_1}=\frac{b}{s_2}\Leftrightarrow s_3(s_2a-s_1b)=0 \quad \text{for some} \quad s_3\in S,$$

and the addition and the multiplication are defined by the usual formulas about fractions. Note that $S^{-1}R=0$ if and only if $0\in S$. The natural map $\phi: R\to S^{-1}R$ defined by $\phi(a)=a/1$ is a homomorphism of rings.

Let f be an element of a ring R. Then the set $\{f^m: m \text{ is a non-negative integer}\}$ is a multiplicative subset of R. In this case the localization is often denoted by R_f. Let $\mathfrak{p}$ be a prime ideal of a ring R. Then the subset $S=R-\mathfrak{p}$ is a multiplicative subset. In this case the localization $S^{-1}R$ is denoted by $R_\mathfrak{p}$. The ring $R_\mathfrak{p}$ is a *local ring*, that is, it has only one maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Note that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a quotient field of $R/\mathfrak{p}$.

Now we are ready to define a scheme. First we define an affine scheme.

Let A be a commutative ring with unity. We denote by $\operatorname{Spec} A$ the set of all prime ideals of A. We set $X=\operatorname{Spec} A$. For an ideal $\mathfrak{A}$ of A, we denote by $V(\mathfrak{A})$ the set of all prime ideals of A which contain $\mathfrak{A}$. Then the family $\mathscr{T}=\{V(\mathfrak{A})|\mathfrak{A}$ is an ideal of $A\}$ enjoys the following properties.

(i) $V(0))=\operatorname{Spec} A$, $V(A)=\varnothing$ (the empty set).

(ii) For ideals $\mathfrak{A}_\lambda(\lambda\in\Lambda)$ of A, $V\left(\sum_{\lambda\in\Lambda}\mathfrak{A}_\lambda\right)=\bigcap_{\lambda\in\Lambda}V(\mathfrak{A}_\lambda)$.

(iii) For ideals $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$, $V(\mathfrak{A}_1\cap\cdots\cap\mathfrak{A}_n)=V(\mathfrak{A}_1)\cup\cdots U\, V(\mathfrak{A}_n)$.

Therefore, the set $\mathfrak{J}$ satisfies the axioms of closed sets of a topological space. Hence, the set $\mathfrak{J}$ defines a topology of X, which is called the Zariski topology. Another way to introduce the Zariski topology is to define a basis of open sets. For any element $f\in A$, the set $D(f)$ is defined by

$$D(f)=\operatorname{Spec} A-V((f))=\{\mathfrak{p}\in\operatorname{Spec} A\,|\, f\notin\mathfrak{p}\}.$$

Then $D(f)$ is an open set and the family $\{D(f)\}_{f\in A}$ is a basis of open sets of $\operatorname{Spec} A$. Indeed, if $V(\mathfrak{A})$ is a closed set and $\mathfrak{p}\notin V(\mathfrak{A})$, then $\mathfrak{p}\not\supseteq\mathfrak{A}$, hence there is an element $f\in\mathfrak{A}$, $f\notin\mathfrak{p}$. Then $\mathfrak{p}\in D(f)$ and $D(f)\cap V(\mathfrak{A})=\varnothing$. Hence, we have

$$\operatorname{Spec} A-V(\mathfrak{A})=\bigcup_{f\notin\mathfrak{A}}D(f).$$

For a prime ideal $\mathfrak{p}$ of A, we denote by $A_\mathfrak{p}$ the localization of A at $\mathfrak{p}$. For an open set U of X, a ring $\mathcal{O}_X(U)$ is defined by the set consisting of maps $s: U\to\prod_{\mathfrak{p}\in U}R_\mathfrak{p}$

with $s(\mathfrak{p})\in R_{\mathfrak{p}}$ for each $\mathfrak{p}\in U$ such that for each $\mathfrak{p}$ of U there exist a neighbourhood W of $\mathfrak{p}$ contained in U and elements a, f of A such that for each $\mathfrak{q}\in W$, $f\notin\mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{p}}$. The ring $\mathcal{O}_X(U)$ is nothing but the set of "regular functions" on U. For open sets U, W with $W \subset U$ we have a natural restriction homomorphism $\mathrm{Res}_{WU}:\mathcal{O}_X(U)\to\mathcal{O}_X(W)$.

For example, if $U = D(f)$, then $\mathcal{O}_X(D(f)) = A_f$. Put $W = D(fg)$, $g\in A$. Then $W \subset U$, and $\mathrm{Res}_{WU}:\mathcal{O}_X(U)\to\mathcal{O}_X(W) = A_{fg}$ is given by

$$\frac{a}{f^n}\to\frac{ag^n}{(fg)^n}.$$

A sheaf $\mathcal{F}$ of rings (respectively abelian groups) over a topological space X consists of the data

(1) for every open set $U \subset X$, a ring (respectively abelian group) $\mathcal{F}(U)$, and
(2) for every inclusion $W \subset U$ of open sets of X, a homomorphism $\rho_{WU}:\mathcal{F}(U)\to\mathcal{F}(W)$ of rings (respectively abelian groups) which enjoys the following properties:
   (a) $\mathcal{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,
   (b) $\rho_{UU} = \mathrm{id}$ (the identity map),
   (c) If $W \subset V \subset U$ are inclusions of open sets, then $\rho_{WU} = \rho_{WV}\circ\rho_{VU}$,
   (d) if U is an open set, $\{W_i\}_{i\in I}$ is an open covering of U and $s\in\mathcal{F}(U)$ is an element such that $\rho_{W_iU}(s) = 0$ for all i, then $s = 0$,
   (e) If U is an open set, $\{W_i\}_{i\in I}$ is an open covering of U and $s\in\mathcal{F}(W_i)$ is given for each i such that $\rho_{W_i\cap W_jW_i}(s_i) = \rho_{W_i\cap W_jW_j}(s_j)$ for all i and j, then there is an element $s\in\mathcal{F}(U)$ such that $p_{W_iU}(s) = s_i$ for all i.

We can show that $\mathcal{O}_X$ is a sheaf of rings on the topological space $X = \mathrm{Spec}\,A$. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of Spec A. A pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ of rings on X is called a *ringed space*. The ringed space $(\mathrm{Spec}\,A, \mathcal{O}_A)$ is called an *affine scheme*. For the affine scheme the stalk $\mathcal{O}_{A,\mathfrak{p}}$ of a point $\mathfrak{p}\in\mathrm{Spec}\,A$ is a local ring. In this case a ringed space is called a *local ringed space*. A *morphism* of ringed spaces $\varphi:(X, \mathcal{O}_X)\to(Y, \mathcal{O}_Y)$ is a collection of a continuous mapping $\varphi: X\to Y$ and ring homomorphisms

$$\psi_U:\mathcal{O}_Y(U)\to\mathcal{O}_X(\varphi^{-1}(U))$$

for every open set $U \subset Y$ which commute with restriction homomorphisms of sheaves.

Let $\varphi: A\to B$ be a ring homomorphism. For each prime ideal $\mathfrak{p}$ of the ring B, $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of A. Hence there is a natural map $\varphi^a:\mathrm{Spec}\,B\to\mathrm{Spec}\,A$. It is easy to show that the map φ^a is a continuous map of topological spaces. For an element $f\in A$, φ induces a natural ring homomorphism $A_f\to B_{\varphi(f)}$. These facts imply that $\varphi: A\to B$ induces a *morphism* of ringed space ${}^t\varphi:(\mathrm{Spec}\,B, \mathcal{O}_B)\to(\mathrm{Spec}\,A, \mathcal{O}_A)$. Conversely, it is shown that any morphism of affine schemes is induced from a ring homomorphism. In this way we can show that the category of affine schemes and the category of commutative rings with unity are anti-equivalent.

Let R be a ring. An *R-valued point* of an affine scheme $X = (\mathrm{Spec}\,A, \mathcal{O}_A)$ is a morphism $\psi:(\operatorname{Spec} R, \mathcal{O}_R) \to (\operatorname{Spec} A, \mathcal{O}_A)$. By $X(R)$ we denote the set of R-valued points of X. Note that $X(R) = \operatorname{Hom}_{\text{ring}}(A, R)$ by the above consideration. The notion of R-valued points is a generalization of the usual notion of points.

Example A.6. Put $A = \mathbf{C}[x]$ and $X = (\operatorname{Spec} A, \mathcal{O}_A)$. Let $\psi: A \to \mathbf{C}$ be a ring homomorphism. Then $\operatorname{Ker} \psi$ is a prime ideal $(x - \alpha)$ for some $\alpha \in \mathbf{C}$. Hence we have $X(\mathbf{C}) = \mathbf{C}$. Note that there is no ring homomorphism $\mathbf{C}[x] \to \mathbf{Z}$. Hence $X(\mathbf{Z}) = \varnothing$.

Put $B = \mathbf{Z}[x]$ and $Y = (\operatorname{Spec} B, \mathcal{O}_B)$. Then we have $Y(\mathbf{Z}) = \mathbf{Z}$ and $Y(\mathbf{C}) = \mathbf{C}$.

Finally we define a *scheme*. A scheme $(X, \mathcal{O}_X)$ is a ringed space such that there exists an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X and that $(U_\lambda, \mathcal{O}_X | U_\lambda)$ is an affine scheme. That is, a scheme is obtained by patching together affine schemes. A morphism of schemes is that of ringed spaces. It is easily shown that a morphism of schemes is induced from morphisms of affine schemes. An R-valued point of a scheme $(X, \mathcal{O}_X)$ is a morphism $(\operatorname{Spec} R, \mathcal{O}_R) \to (X, \mathcal{O}_X)$.

Example A.7 1) Put $A = \mathbf{C}[x]$, $B = \mathbf{C}[y]$ and consider affine schemes $X_1 = (\operatorname{Spec} A, \mathcal{O}_A)$ and $X_2 = (\operatorname{Spec} B, \mathcal{O}_B)$. We patch together X_1 and X_2 by identifying open sets $D_1 = D((x))$ in X_1 and $D_2 = D((y))$ in X_2 by a ring isomorphism

$$\begin{gathered} \mathbf{C}[x, 1/x] \to \mathbf{C}[y, 1/y], \\ x \mapsto 1/y. \end{gathered}$$

The scheme thus obtained is the one-dimensional projective space $\mathbf{P}^1_{\mathbf{C}}$ over $\mathbf{C}$. The set $\mathbf{P}^1(\mathbf{C})$ of $\mathbf{C}$-valued points is the usual one-dimensional complex projective space (or Riemann sphere). Namely, $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$.

2) Let R be a ring and put $A = R[x]$, $B = R[y]$, $X_1 = (\operatorname{Spec} A, \mathcal{O}_A)$, $X_2 = (\operatorname{Spec} B, \mathcal{O}_B)$. Again we patch together X_1 and X_2 by identifying open sets $D_1 = D((x))$ in X_1 and $D_2 = D((y))$ in X_2 by a ring isomorphism

$$\begin{gathered} R[x, 1/x] \to R[y, 1/y], \\ x \mapsto 1/y. \end{gathered}$$

In this way we obtain the one-dimensional projective scheme $\mathbf{P}^1_R$ over the ring R. The natural inclusions $R \hookrightarrow R[x]$ and $R \hookrightarrow R[y]$ defines a morphism $\phi: \mathbf{P}^1_R \to (\operatorname{Spec} R, \mathcal{O}_R)$ which is called the structure morphism. The set $\mathbf{P}^1(R)$ of R-valued points of $\mathbf{P}^1$ is equal to $R \cup \{\infty\}$ as a set.

For affine schemes $X = (\operatorname{Spec} A, \mathcal{O}_A)$ and $Y = (\operatorname{Spec} B, \mathcal{O}_B)$ the *product* $X \times Y$ is the scheme $(\operatorname{Spec} C, \mathcal{O}_C)$, where $C = A \otimes_Z B$. Let $\varphi: X = (\operatorname{Spec} A, \mathcal{O}_A) \to Z = (\operatorname{Spec} R, \mathcal{O}_R)$ and $\psi: Y = (\operatorname{Spec} B, \mathcal{O}_B) \to Z = (\operatorname{Spec} R, \mathcal{O}_R)$ be morphisms of affine schemes. The *fibre product* $X \times_Z Y$ of X and Y over Z is the scheme $W = (\operatorname{Spec} D, \mathcal{O}_D)$, where $D = A \otimes_R B$. The notion of the product and the fibre product of affine schemes is easily generalized to that of schemes by using the fact that a scheme is covered by affine schemes. Let $\varphi: X \to Y$ be a morphism of schemes and x is a point of Y. Hence there is an affine scheme $U = (\operatorname{Spec} E, \mathcal{O}_E)$ which is isomorphic to an open set of Y containing the point x. Let $\mathfrak{p}$ be the prime ideal of E corresponding to the point x. The *fibre* $\varphi^{-1}(x)$ of φ over the point x is the fibre product $X \times_U V$, where V is the scheme $(\operatorname{Spec} K, \mathcal{O}_K)$, K is the field $E_\mathfrak{p}/\mathfrak{p}E_\mathfrak{p}$

which is the quotient field of $E/\mathfrak{p}$ and a natural ring homomorphism $f: E \to E/\mathfrak{p} \to K$ induces a natural morphism $\psi: V \to Y$.

To develop scheme theory further we need the notion of functors and cohomology theory of sheaves. For the details we refer the reader to [H, M, Lecture 3, and S].

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