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Stability of Shock Waves for the Broadwell Equations

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Abstract. For the Broadwell model of the nonlinear Boltzmann equation, there are shock profile solutions, i.e. smooth traveling waves that connect two equilibrium states. For weak shock waves, we prove asymptotic (in time) stability with respect to small perturbations of the initial data. Following the work of Liu [7] on shock wave stability for viscous conservation laws, the method consists of analyzing the solution as the sum of a shock wave, a diffusive wave, a linear hyperbolic wave and an error term. The diffusive and linear hyperbolic waves are approximate solutions of the fluid dynamic equations corresponding to the Broadwell model. The error term is estimated using a variation of the energy estimates of Kawashima and Matsumura [6] and the characteristic energy method of Liu [7].

1. Introduction

The Broadwell model for the nonlinear Boltzmann equation is

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)f_+ &= f_0^2-f_+f_-\,,\\ \frac{\partial}{\partial t}f_0 &= -\tfrac{1}{2}(f_0^2-f_+f_-)\,,\\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)f_- &= f_0^2-f_+f_-\,,\end{aligned}\tag{1.1}$$

in which f_+, f_0, f_- represent the densities of particles moving with speeds $1, 0, -1$ in the x direction. The physical significance of (1.1) is discussed in [2, 3]. Global existence for solutions of the initial value problem for (1.1) is proved in [10] and the fluid dynamic limit for (1.1) is analyzed in [3].

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Shock wave solutions of (1.1) are analyzed in [2, 4]. They are traveling wave solutions $f=(f_+,f_0,f_-)$ $(y=x-st)$ solving

$$\begin{aligned}(1-s)\frac{\partial}{\partial y}f_+ &= f_0^2-f_+f_-\,,\\ -s\frac{\partial}{\partial y}f_0 &= -\tfrac{1}{2}(f_0^2-f_+f_-)\,,\\ -(1+s)\frac{\partial}{\partial y}f_- &= f_0^2-f_+f_-\,,\end{aligned} \tag{1.2}$$

for $-\infty<y<\infty$ with conditions

$$\lim_{y\to-\infty} f(y)=g^{-\infty}\,,\qquad \lim_{y\to\infty} f(y)=g^{\infty}\,. \tag{1.3}$$

The limiting states $g^\infty=(g_+^\infty,g_0^\infty,g_-^\infty)$ and $g^{-\infty}=(g_+^{-\infty},g_0^{-\infty},g_-^{-\infty})$ are equilibria satisfying

$$g_+^\infty g_-^\infty=(g_0^\infty)^2\,,\qquad g_+^{-\infty}g_-^{-\infty}=(g_0^{-\infty})^2\,. \tag{1.4}$$

Moreover g^∞, $g^{-\infty}$, must be related by Rankine-Hugoniot and entropy conditions [4], i.e.

$$\begin{aligned}(1-s)g_+^\infty-4sg_0^\infty-(1+s)g_-^\infty &= (1-s)g_+^{-\infty}-4sg_0^{-\infty}-(1+s)g_-^{-\infty}\,,\\ (1-s)g_+^\infty+(1+s)g_-^\infty &= (1-s)g_+^{-\infty}+(1+s)g_-^{-\infty}\,,\\ s(g_+^\infty+4g_0^\infty+g_-^\infty) &< s(g_+^{-\infty}+4g_0^{-\infty}+g_-^{-\infty})\,.\end{aligned} \tag{1.5}$$

Solutions of (1.2)–(1.3) can be written explicitly as hyperbolic tangents. Such a solution is called a weak shock wave if $g^{-\infty}$ and g^∞ are close.

In this paper we prove asymptotic (in time) stability with respect to small perturbations in initial data for weak shock wave solutions of (1.1). The main result is the following theorem:

Theorem 1. *There is a number $\delta>0$ for which the following is true: Let $f_1(x-st)=(f_{1+},f_{10},f_{1-})(x\text{-}st)$ solve* (1.2), (1.3) *with*

$$|g_+^\infty-g_+^{-\infty}|^2+|g_0^\infty-g_0^{-\infty}|^2+|g_-^\infty-g_-^{-\infty}|^2<\delta^2\,. \tag{1.6}$$

Let $f_I(x)=(f_{I+},f_{I0},f_{I-})(x)$ be initial data that is uniformly bounded and satisfies

$$\int(|f_I-f_1|+|f_I-f_1|^2+|f_{Ix}-f_{1x}|^2+|f_{Ixx}-f_{1xx}|^2)dx<\delta^2\,. \tag{1.7}$$

Let $f(x,t)=(f_+,f_0,f_-)(x,t)$ solve (1.1) *with $f(x,t=0)=f_I(x)$. Then there is a finite number x_0 such that for $f_1=f_1(x+x_0,t)$,*

$$\begin{aligned}\sup_t \int_{-\infty}^{\infty} |f-f_1|^2+|f_x-f_{1x}|^2+|f_t-f_{1t}|^2dx&\\ +\int_0^\infty\int_{-\infty}^{\infty}|f-f_1|^2+|f_x-f_{1x}|^2+|f_t-f_{1t}|^2dxd\tau&\leqq c\delta^2\,,\end{aligned} \tag{1.8}$$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty}|f_x-f_{1x}|^2(x,t)dx=0\,. \tag{1.9}$$

Note. 1) The translation x_0 can be computed directly from the initial data, as in Eq. (3.4).

2) The existence of a unique solution $f(x,t)$ for all time is guaranteed by the global existence theory for the Broadwell equation [10]. For a survey of the theory, see [9].

3) Through the Sobolev inequality, (1.8) and (1.9) imply that $\lim_{t\to\infty} \sup_x |f_1(x_0+x-st)-f(x,t)|=0$.

4) The solution produced in [10] may in general be exponentially growing. Our theorem shows that this is not the case for initial data close to a weak shock. An analogous result is proved in [1] for initial data with finite total mass. In that case the solution eventually decomposes into linear waves with characteristic speeds $\pm 1, 0$ of (1.1). In the present case, the initial state has infinite mass and the results of [1] are not applicable. The asymptotic behavior of solutions is approximated by the fluid dynamic limit with characteristic speeds related to the sound speed.

The proof of this theorem is partly based on the fluid dynamic approximation, i.e. the Chapman-Enskog expansion, for the difference between the solution f and the shock wave f_1. This approximation is valid for describing the nonlinear diffusion wave because the difference $f-f_1$ is small there. In the region of the shock, the difference $f-f_1$ consists mainly of a linear hyperbolic wave, which satisfies equations that are slightly different from the model Euler equations. Analogous stability results were proved by Liu for viscous conservation laws [7] and for the compressible Navier-Stokes equations [8]. Earlier results on stability for the Broadwell equations by Kawashima and Matsumura (abbreviated by KM) [6] and for viscous conservation laws by Goodman [5] and KM [6] are more restrictive in that they impose the constraint that the initial perturbation $f_I - f_1$ have no net (integral over x) mass or momentum, which precludes the diffusion wave.

Following Liu, the difference $f-f_1$ is decomposed into three parts: First there is a nonlinear diffusion wave f_2, which carries the net mass and momentum of $f-f_1$ and is an approximate solution of the model Navier-Stokes Eq. (2.14). The second part is a linear hyperbolic wave f_3 which corrects for the local mass and momentum errors in the diffusion wave but carries no net mass and momentum asymptotically in time. Selection of the correct linear hyperbolic Eq. (2.39), (2.40) for this wave is a crucial detail of this analysis. The third part is a remainder term f_4, which is estimated using a slight modification of the energy estimates of KM [6]. These energy estimates must be supplemented by estimates of the characteristic energy method [7] in regions where the diffusion wave is weakly expansive. Use of this method is the main difference between the present stability result and the result of KM [6]. In this paper the characteristic energy method is slightly simplified to use integration along the piecewise linear approximation of the characteristics. This was partly motivated by a suggestion from James Ralston.

The Broadwell equations are rewritten and the equations for diffusive waves and linear hyperbolic waves are derived in Sect. 2. The equation for the remainder and the error terms in that equation are described in Sect. 3. In Sect. 4 energy estimates are proved and in Sect. 5 the characteristic energy method is applied. The proof of Theorem 1 is summarized at the end of Sect. 5.

2. The Broadwell Equation and the Model Fluid Dynamic Equations

2A. The Broadwell Equation and Shock Profile

Rewrite the Broadwell Eq. (1.1) as

$$\begin{gathered}\varrho_t+m_x=0, \qquad m_t+z_x=0,\\ z_t+m_x=\tfrac{1}{8}\{(\varrho-z)^2-4(z^2-m^2)\}\equiv Q(f,f),\end{gathered} \tag{2.1}$$

in which $\varrho=f_+ +4f_0+f_-$ is the local mass density, $m=f_+-f_-$ is the local momentum and $z=f_+ +f_-$. For convenience denote now $f=(\varrho,m,z)$ and define the quadratic form Q to be the right-hand side of (2.1).

The shock wave solutions of (2.1) [corresponding to solutions of (1.2), (1.3)] are traveling waves $(\varrho,m,z)(x,t)=(\varrho_1,m_1,z_1)(\xi=x\text{-}st)$ solving

$$\begin{gathered}-s\varrho_{1_\xi}+m_{1_\xi}=0, \qquad -sm_{1_\xi}+z_{1_\xi}=0,\\ -sz_{1_\xi}+m_{1_\xi}=\tfrac{1}{8}\{(\varrho_1-z_1)^2-4(z_1^2-m_1^2)\}\end{gathered} \tag{2.2}$$

with limiting values

$$\lim_{\xi\to\infty}(\varrho_1,m_1,z_1)=(\varrho_1^\infty,m_1^\infty,z_1^\infty), \qquad \lim_{\xi\to-\infty}(\varrho_1,m_1,z_1)=(\varrho_1^{-\infty},m_1^{-\infty},z_1^{-\infty}) \tag{2.3}$$

which are in equilibrium,

$$(\varrho_1^\infty-z_1^\infty)^2=4(z_1^\infty)^2-4(m_1^\infty)^2, \qquad (\varrho_1^{-\infty}-z_1^{-\infty})^2=4(z_1^{-\infty})^2-4(m_1^{-\infty})^2, \tag{2.4}$$

satisfy Rankine-Hugoniot conditions

$$-s\varrho_1^{-\infty}+m_1^{-\infty}=-s\varrho_1^\infty+m_1^\infty, \qquad -sm_1^{-\infty}+z_1^{-\infty}=-sm_1^\infty+z_1^\infty, \tag{2.5}$$

and satisfy an entropy condition

$$s\varrho_1^\infty<s\varrho_1^{-\infty}. \tag{2.6}$$

For any limiting states satisfying (2.4), (2.5), (2.6) a unique solution of (2.2), (2.3) is easily constructed [4].

For the sake of definiteness we assume that (ϱ_1,m_1,z_1) is a forward shock wave, which here just means that the shock speed s is positive. The speed s also satisfies the stability condition [4]

$$\lambda_2(\varrho_1^{-\infty},m_1^{-\infty})>s>\lambda_2(\varrho_1^\infty,m_1^\infty) \tag{2.7}$$

in which λ_1, λ_2, satisfying $\lambda_1<0<\lambda_2$, are the characteristic speeds of the model Euler equations [cf. (2.13)] described in the next section.

The explicit form of the shock wave is given by

$$\begin{pmatrix}\varrho_1\\ m_1\\ z_1\end{pmatrix}=-\tfrac{1}{2}\begin{pmatrix}\varrho^{-\infty}-\varrho^\infty\\ m^{-\infty}-m^\infty\\ z^{-\infty}-z^\infty\end{pmatrix}\tanh(\kappa(x+\bar{x}-st))+\tfrac{1}{2}\begin{pmatrix}\varrho^{-\infty}+\varrho^\infty\\ m^{-\infty}+m^\infty\\ z^{-\infty}+z^\infty\end{pmatrix}, \tag{2.8}$$

in which $\bar{x}$ is an arbitrary constant, with $\kappa=(1+3s^2)\,(16s)^{-1}(\varrho^{-\infty}-\varrho^\infty)$ [4].

2B. Model Fluid Equations

By assuming that $f=(\varrho, m, z)$ is a local Maxwellian satisfying $Q(f,f)=0$ at every (x,t) and by dropping the third equation of (2.1), the following model Euler equations are obtained:

$$\varrho_t+m_x=0, \qquad m_t+\hat{z}(\varrho, m)_x=0, \tag{2.9}$$

in which

$$\hat{z}(\varrho, m)=\varrho F(u) \tag{2.10}$$

with

$$u=m/\varrho, \qquad F(u)=(2\sqrt{1+3u^2}-1)/3. \tag{2.11}$$

The Euler Eqs. (2.9) can be rewritten as

$$\begin{pmatrix}\varrho\\ m\end{pmatrix}_t+\begin{pmatrix}0 & 1\\ a/b & m/b\end{pmatrix}\begin{pmatrix}\varrho\\ m\end{pmatrix}_x=0 \tag{2.9'}$$

with

$$a=(\varrho-\hat{z})/4, \qquad b=(\varrho+3\hat{z})/4. \tag{2.12}$$

The characteristic speeds of (2.9′) are

$$\begin{aligned}\lambda_1&=\frac{m-\sqrt{m^2+4ab}}{2b}=-\frac{2a}{m+\sqrt{m^2+4ab}},\\ \lambda_2&=\frac{m+\sqrt{m^2+4ab}}{2b}=\frac{2a}{-m+\sqrt{m^2+4ab}}.\end{aligned} \tag{2.13}$$

If the components f_+, f_0, f_- of the solution f are initially nonnegative, they remain nonnegative [1]. For such a solution $|u|\leqq 1$ (i.e., the average velocity is no larger than the molecular speed) and $a>0$, $b>0$. Thus $\lambda_1<0<\lambda_2$.

A better approximation of (2.1) is given by the model Navier-Stokes equations, which are obtained from (2.1) through the Chapman-Enskog expansion [3] as

$$\varrho_t+m_x=0, \qquad m_t+\tilde{z}(\varrho, m)_x=0, \tag{2.14}$$

in which

$$\tilde{z}(\varrho, m)=\hat{z}(\varrho, m)-v(u)u_x. \tag{2.15}$$

The viscosity function v is

$$v(u)=2(1-F(u))(1+3u^2)^{-3/2}. \tag{2.16}$$

The Euler equations (2.9) have discontinuous shock wave solutions, while the Navier-Stokes equations (2.14) have smooth shock profile solutions, which approximate the shock wave solutions of the Broadwell equation (2.1) if the shocks are weak.

Just as for the real Navier-Stokes equations [8], the shock waves are compressive; that is the associated characteristic speed λ_2 (for forward shocks) decreases across the shock. As a consequence, λ_1, λ_2, a and b defined by (2.13) are strictly monotone across a viscous shock wave for (2.14). For a weak shock wave of (2.1) KM [6] showed that

$$\frac{\partial}{\partial \xi}(\lambda_1)<0, \quad \frac{\partial}{\partial \xi}(\lambda_2)<0, \tag{2.17}$$

in which λ_1, λ_2 are evaluated at the shock wave $f=f_1$. Also since the shock wave is forward (with approximate speed λ_2), and since the shock wave decays exponentially away from its center, it satisfies

$$\begin{gathered}|\partial_t f_1+\lambda_2 \partial_x f_1|<c\delta|\lambda_{2x}|, \\ |\chi_+(f_1-f_1(\infty))|+|\chi_-(f_1-f_1(-\infty))|+|\partial_x f_1|+|\partial_t f_1|<c|\lambda_{2x}|<c\delta e^{-\gamma|x-st|},\end{gathered} \tag{2.18}$$

in which $\chi_+=1$ for $x>st$, $\chi_+=0$ for $x<st$, $\chi_-=1-\chi_+$, and γ is some constant.

2C. *Diffusion Waves*

Liu [7] showed that, for a viscous conservation law such as (2.14), a small perturbation of a constant evolves approximately as a diffusion wave. For the forward shock wave solution (ϱ_1, m_1, z_1) of (2.2), the diffusion wave (ϱ_2, m_2) moves backward, and so it is a perturbation of the limiting state $(\varrho_1^{-\infty}, m_1^{-\infty}, z_1^{-\infty})$. As in [7], the diffusion wave solves

$$\varrho_{2t}+m_{2x}=e_1, \tag{2.19}$$

$$m_{2t}+\tilde{z}_{2x}=e_2, \tag{2.20}$$

in which

$$\tilde{z}_2=\tilde{z}(\varrho_2+\varrho_1^{-\infty}, m_2+m_1^{-\infty})-z_1^{-\infty}. \tag{2.21}$$

From now on we write $f_2 \equiv (\varrho_2, m_2, \tilde{z}_2)$. The error terms e_1, e_2 are chosen as in Sect. 3 of [7] so that (2.19), (2.20) is equivalent to Burger's equation.

To be precise, f_2 is uniquely determined by the following properties:

(i) $(\varrho_2+\varrho_1^{-\infty}, m_2+m_1^{-\infty})$ lies on the integral curve of the right eigenvector r_1 through $(\varrho_1^{-\infty}, m_1^{-\infty})$,

(ii) $\tilde{\lambda}_1 \equiv \lambda_1(\varrho_2+\varrho_1^{-\infty}, m_2+m_1^{-\infty})-\lambda_1^{-\infty}$ is a self-similar solution of the Burgers equation, i.e.

$$\begin{aligned}\tilde{\lambda}_1(x,t)=&[(\exp(\kappa\delta/2\sqrt{\alpha})-1)(t+1)^{-1/2}\exp(-y^2)] \\ &\times\left[2\sqrt{\pi/\alpha}+(\exp(\kappa\delta/2\sqrt{\alpha})-1)\int_y^\infty \alpha^{-1/2}e^{-\xi^2}d\xi\right]^{-1},\end{aligned} \tag{2.22}$$

$$y \equiv (x-(t+1)\lambda_1^{-\infty})/2\sqrt{\alpha(t+1)}. \tag{2.23}$$

The quantities r_1 and $\alpha \equiv \alpha_+(\varrho_1^{-\infty}, m_1^{-\infty})$ are defined in (2.26), (2.28) below. κ has value 1 or -1 for diffusion wave with positive or negative mass. Here $\kappa\delta$ replaces δ in [7], since we now take $\delta>0$.

For the construction of the diffusion wave from Burger's equation, the viscosity matrix of (2.14) must be expressed in the basis of left and right eigenvectors of the convection matrix of (2.9'). The viscosity matrix is

$$V=\begin{pmatrix} 0 & 0 \\ -amb^{-3} & a\varrho b^{-3} \end{pmatrix}. \tag{2.24}$$

The left eigenvectors $l_\pm$ and right eigenvectors $r_\pm$ of the convection matrix are

$$l_1=((-m-\sqrt{m^2+4ab})/2b,1), \qquad l_2=((-m+\sqrt{m^2+4ab})/2b,1), \tag{2.25}$$

$$\begin{aligned} r_1&=(-2b/\sqrt{m^2+4ab},1-m/\sqrt{m^2+4ab})^\dagger, \\ r_2&=(2b/\sqrt{m^2+4ab},\ 1+m/\sqrt{m^2+4ab})^\dagger. \end{aligned} \tag{2.26}$$

In the coordinate system r_1, r_2, the viscosity matrix becomes

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} V(r_1 r_2)=\begin{pmatrix} \alpha_+ & \alpha_- \\ \alpha_+ & a_- \end{pmatrix} \tag{2.27}$$

with

$$\alpha_\pm=\frac{\pm m(2b-\varrho)+\varrho\sqrt{m^2+4ab}}{\sqrt{m^2+4ab}}\cdot\frac{a}{b^3}. \tag{2.28}$$

Using the facts that $a>0$, $b>0$ and $|F(u)|<1$, it is easily shown that the diagonal elements of (2.26) are positive, i.e. $\alpha_\pm>0$, which is needed for the construction of the diffusion wave f_2.

Define

$$\Omega_1\equiv\{(x,t),t\geqq 0,x\leqq 0\}, \qquad \Omega_2\equiv\{(x,t),t\geqq 0,x\geqq 0\}.$$

It follows as in Sect. 5 of [7] that the set Ω_1 can be divided into two parts Ω_+ and Ω_- such that the characteristics for (ϱ_2, m_2) are compressive in Ω_- and weakly expansive in Ω_+, i.e.

$$\frac{\partial\lambda_1(\varrho_2,m_2)}{\partial x}<0 \quad \text{in} \quad \Omega_-, \qquad \frac{\partial\lambda_1(\varrho_2,m_2)}{\partial x}>0 \quad \text{in} \quad \Omega_+. \tag{2.29}$$

Moreover $\Omega_\pm$ are characterized by (with y defined in (2.23))

$$\begin{aligned} \Omega_+(\Omega_-)&\equiv\{(x,t)\in\Omega_1: y\geqq y_1(y\leqq y_1)\} \quad \text{if} \quad \kappa=1, \\ \Omega_+(\Omega_-)&\equiv\{(x,t)\in\Omega_1: y\leqq y_1(y\geqq y_1)\} \quad \text{if} \quad \kappa=-1, \end{aligned} \tag{2.30}$$

for some constant y_1, as shown in Fig. 1. Define also $\Omega_\pm(t)=\Omega_\pm\cap\{(x,\tau):0<\tau<t\}$, $\Omega_i(t)=\Omega_i\cap\{(x,\tau):0\leqq\tau<t\}$ for $i=1,2$.

The nonlinear diffusion wave satisfies the following bounds (cf. (3.8) in [7]):

$$|f_2|\simeq|\varrho_2|+|m_2|+|z_2|+|u_2|\leqq c\delta(t+1)^{-1/2}e^{-y^2}, \tag{2.31}$$

$$|f_{2x}|\leqq c\delta(t+1)^{-1}(|y|+1)e^{-y^2}, \tag{2.32}$$

$$|f_{2t}|\leqq c\delta(t+1)^{-3/2}(|y|^2+1)e^{-y^2}, \tag{2.33}$$

$$|f_{2xt}|\leqq c\delta(t+1)^{-2}(|y|^3+1)e^{-y^2}, \tag{2.34}$$

$$|f_{2xx}|\leqq c\delta(t+1)^{-3/2}(|y|^2+1)e^{-y^2}, \tag{2.35}$$

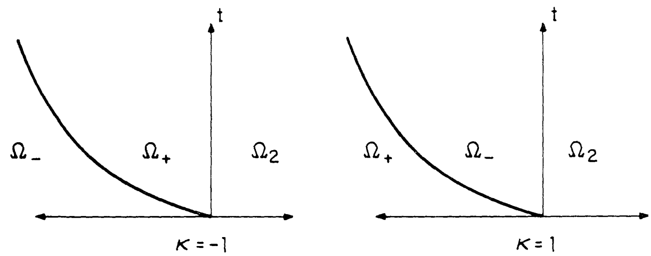

Fig. 1. Regions Ω_+, Ω_-, Ω_2 in (x,t) for the two cases $\kappa=1$ and $\kappa=-1$

in which y is defined by (2.23). Moreover since f_2 is determined through its eigenvalue λ_1, it follows that

$$|\partial_x f_2| < c|\partial_x \lambda_1|, \qquad |\partial_t f_2| < c|\partial_t \lambda_1|, \tag{2.36}$$

$$(\partial_t + \lambda_1 \partial_x) f_2 = O(1)\partial_{xx}\lambda_1 . \tag{2.37}$$

Using (6.12), (6.13), (3.5) of [7], the error terms e_1, e_2 satisfy the bound

$$|e_i| \leqq c\delta(1+t)^{-3/2}(|y|^2+1)e^{-y^2} . \tag{2.38}$$

2D. Linear Hyperbolic Wave

A linear hyperbolic wave $(\varrho_3, m_3, \tilde{z}_3(\varrho_3, m_3))$ is needed as in [7] to compensate for the errors e_1, e_2 in (2.19), (2.20). The equations for ϱ_3, m_3 are

$$\begin{gathered} \varrho_{3t} + m_{3x} = -e_1 , \qquad m_{3t} + \tilde{z}_3(\varrho_3, m_3)_x = -e_2 , \\ (\varrho_3, m_3)(x,t) \to 0 \quad \text{as} \quad t \to \infty , \end{gathered} \tag{2.39}$$

in which

$$\tilde{z}_3(\varrho_3, m_3) = \frac{A}{B}\varrho_3 + \frac{M}{B} m_3 , \tag{2.40}$$

with A, B, M depending on (ϱ_1, m_1, z_1) and $(\varrho_2, m_2))$ as

$$\begin{gathered} A = \tfrac{1}{4}\{(\varrho_1 + \varrho_2) - (z_1 + \tilde{z}_2)\} , \qquad B = \tfrac{1}{4}\{(\varrho_1 + \varrho_2) + 3(z_1 + \tilde{z}_2)\} , \\ M = m_1 + m_2 . \end{gathered} \tag{2.41}$$

The form of $\tilde{z}_3$ is chosen so that $f_2 + f_3$, together with the shock wave f_1, forms an accurate approximate solution [cf. (3.11)]. Clearly from (2.19), (2.20), and (2.39),

$$\frac{d}{dt}\int_{-\infty}^{\infty} (\varrho_2 + \varrho_3)(x,t)\,dx = 0 , \qquad \frac{d}{dt}\int_{-\infty}^{\infty} (m_2 + m_3)(x,t)\,dx = 0 \tag{2.42}$$

for $t \geqq 0$. Thus if the net mass and momentum of the initial perturbation is contained in $(\varrho_2, m_2) + (\varrho_3, m_3)$, it will remain there for all time. Moreover the construction in Sect. 3 of (ϱ_2, m_2) and (ϱ_3, m_3) guarantees that

$$\int_{-\infty}^{\infty} \begin{pmatrix} \varrho_2 + \varrho_3 \\ m_2 + m_3 \end{pmatrix} (x,t)\,dx = \bar{c}\delta r_1^{-\infty} \tag{2.43}$$

in which $\bar{c}$ is a constant and $r_1^{-\infty}=r_1(\varrho_1^{-\infty}, m_1^{-\infty})$ is the right eigenvector defined in (2.26).

As shown in [7] the error terms (e_1, e_2) have a good form and decay rate. The initial data is appropriately chosen as in [7] so that $f_3 \to 0$ as $t \to \infty$. It follows that $f_3 \equiv (\varrho_3, m_3, \tilde{z}_3)$, with ϱ_3, m_3 solving (2.39), has the following decay properties (cf. Theorems 7.5, 7.6 of [7])

$$|f_3(x,t)| \leqq c\delta\{[1+t+ty^2]^{-1}+[1+t+t^{1/2}|y|]^{-3/2}\}, \tag{2.44}$$

$$\begin{aligned}&|f_{3t}|+|f_{3x}(x,t)|+|f_{3xx}(x,t)|\\&\leqq c\delta\{[1+t+ty^2]^{-3/2}+[1+t+t^{1/2}|y|]^{-2}+|t+1|^{-3/2}(|f_{1x}|+|f_{2x}|)\}\,.\end{aligned} \tag{2.45}$$

3. Wave Decomposition of the Solution

The equations are reformulated here in terms of ϱ, m, z. Consider initial data $f_I=(\varrho_I, m_I, z_I)(x)$ that is a perturbation of a given forward shock wave solution $(\varrho_1, m_1, z_1)(\xi)$, i.e.

$$(\varrho_I, m_I, z_I)(x)=(\varrho_1, m_1, z_1)(x)+(\varrho', m', z')(x,0) \tag{3.1}$$

with $(\varrho', m', z')(\cdot, 0) \in L^1(x)$ and $\lim\limits_{x\to\pm\infty}(\varrho', m', z')(x,0)=0$. The solution $(\varrho, m, z)(x,t)$ of the Broadwell equation with this initial data is written as

$$(\varrho, m, z)(x,t)=(\varrho_1, m_1, z_1)(x-st)+(\varrho', m', z')(x,t)\,. \tag{3.2}$$

From (2.1) the solution has two time-invariant quantities, mass and momentum, i.e.

$$\int_{-\infty}^{\infty}\begin{pmatrix}\varrho'\\ m'\end{pmatrix}(x,t)dx=\int_{-\infty}^{\infty}\begin{pmatrix}\varrho'\\ m'\end{pmatrix}(x,0)dx \tag{3.3}$$

for all $t>0$. For a weak forward shock wave the jump $[\varrho_1, m_1] =(\varrho_1^{+\infty}, m_1^{+\infty})-(\varrho_1^{-\infty}, m_1^{-\infty})$ is nearly equal to the right eigenvector $r_2(\varrho_1^{-\infty}, m_1^{-\infty})$ defined in (2.26). It follows that $[\varrho_1, m_1]$ and $r_1(\varrho_1^{-\infty}, m_1^{-\infty})^\dagger$ are linearly independent and that the net perturbed mass and momentum can be written as a linear combination of them, i.e. for some constants $\bar{c}$ and x_0,

$$\int_{-\infty}^{\infty}(\varrho', m')(x,t)dx=\bar{c}\delta r_1(\varrho_1^{-\infty}, m_1^{-\infty})^\dagger+x_0[\varrho_1, m_1]\,. \tag{3.4}$$

Note that

$$\int_{-\infty}^{\infty}\{(\varrho_1, m_1)(x+x_0-st)-(\varrho_1, m_1)(x-st)\}dx=x_0[\varrho_1, m_1]\,. \tag{3.5}$$

Thus for convenience we can take x_0 in (3.4) to be zero after replacing $(\varrho_1, m_1, z_1)(x-st)$ in (3.1), (3.2) by $(\varrho_1, m_1, z_1)(x+x_0-st)$.

With x_0 set to zero this way and $\bar{c}$ defined by (3.4), the solution (ϱ, m, z) is decomposed into the shock wave, diffusion wave, linear hyperbolic wave and remainder, i.e.

$$\begin{aligned}(\varrho, m, z)(x,t)=&(\varrho_1, m_1, z_1)(x-st)+\sum_{i=2}^{3}(\varrho_i, m_i, \tilde{z}_i(\varrho_i, m_i))(x,t)\\&+(\varrho_*, m_*, z_*)(x,t)\,.\end{aligned} \tag{3.6}$$

The diffusion wave $(\varrho_2, m_2, \tilde{z}_2)$ satisfies (2.19), (2.20) the linear hyperbolic wave $(\varrho_3, m_3, \tilde{z}_3)$ satisfies (2.39), and together they satisfy (2.43).

Combine the equations for $(\varrho_i, m_i, z_i)(i=1,2,3)$ together with Eq. (2.1) for (ϱ, m, z) to obtain the following equation for (ϱ_*, m_*, z_*):

$$\begin{gathered} \varrho_{*t} + m_{*x} = 0, \qquad m_{*t} + z_{*x} = 0, \\ z_{*t} + m_{*x} = G \equiv Q(f,f) - (z_1 + \tilde{z}_2 + \tilde{z}_3)_t - (m_1 + m_2 + m_3)_x. \end{gathered} \tag{3.7}$$

The initial data for $f_* = (\varrho_*, m_*, z_*)$ is defined through (3.6) since the initial values of (ϱ_i, m_i, z_i) $i=1,2,3$ are already chosen. Because of (1.7) f_* satisfies

$$\int |f_*| + |f_*|^2 + |f_{*x}|^2 + |f_{*xx}|^2 \, dx \leqq \delta^2$$

for $t=0$. Because of (2.43) and (3.4) with $x_0 = 0$, f_* has no net mass or momentum, i.e.

$$\int_{-\infty}^{\infty} (\varrho_*, m_*)(x,t) \, dx = 0. \tag{3.8}$$

We wish to show that $(\varrho, m, z) \to (\varrho_1, m_1, z_1)$ as $t \to \infty$. Since $(\varrho_i, m_i, z_i) \to 0$ for $i=2,3$, we need only show that $(\varrho_*, m_*, z_*) \to 0$.

First we rearrange the right-hand side G in (3.7). Decompose G as

$$G = H + 2Q(f_1 + f_2 + f_3, f_*) + Q(f_*, f_*), \tag{3.9}$$

in which $f_i = (\varrho_i, m_i, z_i)$, $f_* = (\varrho_*, m_*, z_*)$. The parts of G containing f_* are

$$\begin{gathered} Q(f_1 + f_2, f_*) = \tfrac{1}{2}(A\varrho_* + Mm_* - Bz_*), \\ Q(f_3, f_*) = \tfrac{1}{8}(\varrho_3 - \tilde{z}_3)\varrho_* + \tfrac{1}{2} m_3 m_* - \tfrac{1}{8}(\varrho_3 + 3\tilde{z}_3) z_*, \\ Q(f_*, f_*) = \Gamma(\varrho_*, m_*, z_*) \equiv \tfrac{1}{8}\{(\varrho_* - z_*)^2 - 4(z_*^2 - m_*^2)\}, \end{gathered} \tag{3.10}$$

in which A, B, M are defined in (2.41).

For the linear hyperbolic wave, $z_3 = \tilde{z}_3(\varrho_3, m_3)$ was chosen in (2.40) so that

$$Q(f_1 + f_2, f_3) = 0. \tag{3.11}$$

Also use the equation for f_1 to find that the inhomogeneous part of G is $H = H_1 + H_2 + H_3$, in which

$$H_1 = Q(f_1^{-\infty} + f_2, f_1^{-\infty} + f_2) - (z_1^{-\infty} + \tilde{z}_2)_t - (m_1^{-\infty} + m_2)_x, \tag{3.12}$$

$$H_2 = 2Q(f_1 - f_1^{-\infty}, f_2), \tag{3.13}$$

$$H_3 = Q(f_3, f_3) - \tilde{z}_{3t} - m_{3x}, \tag{3.14}$$

since $Q(f_1^{-\infty}, f_1^{-\infty}) = (z_1^{-\infty})_t = (m_1^{-\infty})_x = 0$. The term H_1 is the error in the third equation of (2.1) in which f is replaced by a solution $f_2 + f_1^{-\infty}$ of the Chapman-Enskog expansion (i.e. a solution of the model Navier-Stokes equations). A straightforward calculation in Appendix A shows that

$$H_1 = -(F_2 - u_2 F_2')e_1 - F_2' e_2 - F_2'(v_2 u_{2x})_x + (v_2 u_{2x})_t - \tfrac{3}{8}(v_2 u_{2x})^2, \tag{3.15}$$

in which $F_2 = F(u_1^{-\infty} + u_2)$, $v_2 = v(u_1^{-\infty} + u_2)$.

Because of (3.8) it is natural to introduce

$$(\phi, \psi)(x,t) = \int_{-\infty}^{x} (\varrho_*, m_*)(y,t)dy \tag{3.16}$$

with $(\phi,\psi)(x=\pm\infty,t)=(0,0)$. Integrate the first two equations of (3.7) and eliminate z_* to rewrite this system as

$$\phi_t + \psi_x = 0, \tag{3.17}$$

$$\begin{aligned}\psi_t + (A/B)\phi_x + (M/B)\psi_x = B^{-1}(&\psi_{xx} - \psi_{tt} - \Gamma(\phi_x, \psi_x, -\psi_t)\\ &- H - K(\phi_x, \psi_x, \psi_t)),\end{aligned} \tag{3.18}$$

in which $H = H_1 + H_2 + H_3$ is defined in (3.12)–(3.14), Γ is defined in (3.10) and

$$K(\phi_x, \psi_x, \psi_t) = Q(f_3, f_*) = \tfrac{1}{8}(\varrho_3 - z_3)\phi_x + \tfrac{1}{2}m_3\psi_x - \tfrac{1}{8}(\varrho_3 + 3z_3)\psi_t. \tag{3.19}$$

4. Energy Estimates

In this section energy estimates are derived for the system (3.17), (3.18). These estimates are the same as the estimates of Kawashima and Matsumura [6], except for small changes caused by the inclusion of f_2 in A, B, M and the terms H and K. However, because of these small changes, the energy estimates do not close: on the weakly expansive region Ω_+ of the diffusive wave f_2, a second set of characteristic energy estimates are needed and are derived in the next section.

First change variables from (x,t) to (ξ,t) with $\xi = x - st$ and s the shock speed. Note that f_2 is not a traveling wave with speed s; thus in the present problem A, B, M depend on t as well as ξ, in contrast to [6]. Rewrite (3.17), (3.18) as

$$L_1(\phi,\psi) = 0, \tag{4.1}$$

$$L_2(\phi,\psi) = -\Gamma_1, \tag{4.2}$$

in which

$$\begin{aligned}L_1(\phi,\psi) &\equiv \phi_t - s\phi_\xi + \psi_\xi,\\ L_2(\phi,\psi) &\equiv (\psi_t - s\psi_\xi)_t - s(\psi_t - s\psi_\xi)_\xi - \psi_{\xi\xi} + A\phi_\xi + (M - sB)\psi_\xi + B\psi_t,\\ \Gamma_1 &\equiv \Gamma(\phi_\xi, \psi_\xi, -(\psi_t - s\psi_\xi)) + K(\phi_\xi, \psi_\xi, \psi_t - s\psi_\xi) + H.\end{aligned} \tag{4.3}$$

The initial data for (4.1), (4.2) is

$$(\phi,\psi)(\xi,0) = (\phi_0,\psi_0)(\xi), \qquad \psi_t(\xi,0) = s\psi_{0\xi} - z_*(\xi). \tag{4.4}$$

Define the following three norms for ψ, ϕ:

$$N_1(t) = \sup_{0\le\tau\le t} (\|\phi,\psi\|_2(\tau) + \|\psi_t\|_1(\tau)), \tag{4.5}$$

$$N_2(t) = \left\{\int_0^t \|\phi_\xi, \psi_\xi, \psi_t\|_1^2 d\tau\right\}^{1/2}, \tag{4.6}$$

$$N_3(t) = \left\{\int_0^t \int_{-\infty}^{\infty} (|\lambda_{1\xi}| + |\lambda_{2\xi}|)(\phi^2 + \psi^2)d\xi d\tau\right\}^{1/2}, \tag{4.7}$$

in which $\| \; \| = \| \; \|_0$ and $\| \; \|_k$ is the k^{th} Sobolev norm in space, e.g.

$$\|\phi\|_k = \left(\int_{-\infty}^{\infty} \phi^2 + \ldots + (\partial_\xi^k \phi)^2 d\xi \right)^{1/2}. \tag{4.8}$$

In these definitions we could replace ξ by x everywhere (without any changes) and correspondingly change $\left.\frac{\partial}{\partial t}\right|_\xi$ to $\left.\frac{\partial}{\partial t}\right|_x$.

Before proving the energy estimates, we derive two lemmas concerning the shock wave f_1, diffusion wave f_2 and linear hyperbolic wave f_3. Define $A_\pm = \lim_{\xi \to \pm\infty} A(\xi, t)$, $B_\pm = \lim_{\xi \to \pm\infty} B(\xi, t)$. Note that $A_\pm$, $B_\pm$ are positive and do not depend on t.

Lemma 4.1. *Let f_1 be a forward shock wave solution of* (1.1) *with strength δ, i.e. satisfying* (1.6). *Let f_2 be a corresponding diffusive wave solution of* (2.19), (2.20), *for initial data $f_I - f_1$ of magnitude δ i.e. satisfying* (1.7). *If δ is sufficiently small then*

(i) *There are positive constants c, C, and κ, independent of x, t, and δ, such that*

$$\begin{aligned} c < A_+ - \kappa\delta < A(\xi, t) < A_- + \kappa\delta < C, \\ c < B_+ - \kappa\delta < B(\xi, t) < B_- + \kappa\delta < C, \quad sA_\xi < \kappa\delta. \end{aligned} \tag{4.9}$$

(ii) *There is $\lambda > 0$ such that $\sup_\xi D_i < 0$ for $i = 1, 2, 3$ in which*

$$D_1 = A(A - \lambda), \quad D_2 = \lambda(\lambda - B),$$
$$D_3 = M^2 - 4(A - \lambda)(\lambda - B).$$

(iii) *Let λ, β be constants. If β is small enough then*

$$-\frac{\lambda}{2}(A^{-1}(M - sB))_\xi - |(A^{-1})_t| - \frac{\lambda}{2}[(A^{-1}B)_t + s(A^{-1})_{\xi t}] - \beta|(A^{-1})_\xi|$$
$$\geqq \begin{cases} -\kappa\delta e^{-\gamma t} & \text{for} \quad \xi > 0 \\ -\kappa|\lambda_{1x}| & \text{for} \quad \xi < 0 \end{cases} \tag{4.10}$$

for some constant κ.

Proof. This lemma was proved in [6] for the case of a shock, i.e. $f_2 \equiv 0$, with $\kappa = 0$ in (i), (iii). In that case $\partial_t \equiv 0$. Since $|f_2| < c\delta$, the inequalities (4.9), (ii) are only slightly perturbed. Since $|f_{2\xi}| + |f_{2t}| \leqq c\delta e^{-\gamma t}$ for $\xi > 0$ and $|f_{2\xi}| + |f_{2t}| \leqq c|\lambda_{1x}|$ for $\xi < 0$ the inequality (4.10) is derived.

The second lemma describes bounds on H, K, and Γ.

Lemma 4.2. *Let $H = H_1 + H_2 + H_3$, K, Γ be defined by* (3.12)–(3.14), (3.19), (3.10) *Denote*

$$F = (|\phi| + |\phi_\xi| + |\phi_{\xi\xi}| + |\psi| + |\psi_\xi| + |\psi_{\xi\xi}| + |\psi_t| + |\psi_{t\xi}|). \tag{4.11}$$

Then

$$\int_0^\infty \int (|H| + |H_\xi|) F \, d\xi dt \leqq c\delta N_1(t), \tag{4.12}$$

$$\int_0^\infty \int (|K| + |K_\xi|) F \, d\xi dt \leqq c\delta N_1(t) N_2(t), \tag{4.13}$$

$$\int\int (|\Gamma| + |\Gamma_\xi|) F \, d\xi dt \leqq c N_1(t) N_2(t)^2. \tag{4.14}$$

Proof of Lemma 4.2. First derive pointwise bounds on H_i, K, and Γ. For H_1 use (3.15), (2.32–2.35), (2.38) to obtain

$$\begin{aligned}|H_1|+|H_{1\xi}|&\leqq c(|e_1|+|e_2|+|u_{2xx}|+|u_{2xt}|+|u_{2x}|^2+|u_{2x}u_{2t}|)\\&\leqq c\delta(1+t)^{-3/2}(|y|^3+1)e^{-y^2}\end{aligned}\tag{4.15}$$

in which $y=(x-\lambda t)/2\sqrt{\alpha t}$. In estimating H_2, use (3.13) and the fact that $f_1-f_1^{-\infty}$ and f_2 have nearly disjoint supports as seen from (2.8) and (2.31), so that

$$\begin{aligned}|H_2|+|H_{2\xi}|&\leqq c|f_2|\,|f_1-f_1^{-\infty}|\leqq c\delta^2(t+1)^{-1/2}e^{-y^2}(\tanh\kappa(x-st)+1)\\&\leqq c\delta^2 e^{-c(|x|+t)}.\end{aligned}\tag{4.16}$$

For H_3 use (3.14), (2.44), (2.45) to estimate

$$\begin{aligned}|H_3|+|H_{3\xi}|&\leqq|f_3|^2+|f_{3x}|+|f_{3t}|\\&\leqq c\delta\{[1+t+ty^2]^{-3/2}+[1+t+t^{1/2}|y|]^{-2}\\&\quad+|t+1|^{-3/2}(|f_{1x}|+|f_{2x}|)\}.\end{aligned}\tag{4.17}$$

Estimate K using (3.19), (2.44), (2.45) to obtain

$$|K|+|K_\xi|<(|f_3|+|f_{3x}|)(|\phi_x|+|\psi_x|+|\psi_t|+|\phi_{xx}|+|\psi_{xx}|+|\psi_{xt}|)\tag{4.18}$$

Finally estimate $\Gamma=\Gamma(\phi_\xi,\psi_\xi,-\psi_t+s\psi_\xi)$ from its definition (3.10) as

$$|\Gamma|+|\Gamma_\xi|<(|\phi_\xi|+|\psi_\xi|+|\psi_t|)(|\phi_\xi|+|\phi_{\xi\xi}|+|\psi_\xi|+|\psi_{\xi\xi}|+|\psi_t|+|\psi_{t\xi}|).\tag{4.19}$$

It follows that

$$\int_0^t\int(|H|+|H_\xi|)Fd\xi d\tau\leqq\left(\sup_{0\leqq\tau\leqq t}(\int F^2d\xi)^{1/2}\right)\int_0^\infty(\int H^2+H_\xi^2d\xi)^{1/2}d\tau\leqq c\delta N_1(t),\tag{4.20}$$

$$\begin{aligned}&\int_0^t\int(|K|+|K_\xi|)Fd\xi d\tau\\&\leqq\int_0^t\int F(|f_3|+|f_{3\xi}|)(|\phi_\xi|+|\phi_{\xi\xi}|+|\psi_\xi|+|\psi_{\xi\xi}|+|\psi_t|+|\psi_{t\xi}|)d\xi d\tau\\&\leqq\int_0^t\sup_\xi(|f_3|+|f_{3\xi}|)\|\phi_\xi,\psi_\xi,\psi_t\|_1\|\phi,\psi,\psi_t,\phi_\xi,\psi_\xi\|_1d\tau\\&\leqq\left(\int_0^t\sup_\xi(|f_3|+|f_{3\xi}|)^2d\tau\right)^{1/2}N_1(t)\left(\sup_{0\leqq\tau\leqq t}\|\phi,\psi,\psi_t,\phi_\xi,\psi_\xi\|_1\right)\\&\leqq c\delta N_1(t)N_2(t),\end{aligned}\tag{4.21}$$

$$\begin{aligned}\int_0^t\int(|\Gamma|+|\Gamma_\xi|)Fd\xi d\tau&\leqq\int_0^t\int(|\phi|+|\phi_\xi|+|\psi|+|\psi_\xi|+|\psi_t|)\\&\quad\times(|\phi_\xi|+|\phi_{\xi\xi}|+|\psi_\xi|+|\psi_{\xi\xi}|+|\psi_t|+|\psi_{t\xi}|)^2d\xi d\tau\\&\leqq\sup_{\xi,\,0\leqq\tau\leqq t}(|\phi|+|\phi_\xi|+|\psi|+|\psi_\xi|+|\psi_t|)N_2(t)^2\leqq cN_1(t)N_2(t)^2,\end{aligned}\tag{4.22}$$

which concludes the proof of the lemma.

Now we proceed with the energy estimates, following the analysis of Kawashima and Matsumura [6]. There are four such estimates presented in Lemma 4.3; they are analogous to Lemmas 3.5–3.7 of [6].

Lemma 4.3 (Preliminary energy estimates). *Let ϕ, ψ solve* (3.17), (3.18) *with δ sufficiently small. Then for some constant c*

$$\text{i)}\ \|\phi(t)\|^2+\|\psi(t)\|_1^2+\|\psi_t(t)\|^2+\int_0^t \|\psi_\xi,\psi_t\|^2 d\tau$$
$$\leqq c(\|\phi(0)\|^2+\|\psi(0)\|_1^2+\|\psi_t(0)\|^2+\iint_{\Omega_+(t)} |\lambda_{1x}|\psi^2 dx d\tau$$
$$+\delta N_1(t)^2+\delta N_1(t)+\delta N_1(t)N_2(t)+N_1(t)N_2(t)^2), \tag{4.23}$$

$$\text{ii)}\ \|\phi_\xi(t)\|^2+\int_0^t \|\phi_\xi\|^2 d\tau - c\left(\|\psi_\xi,\psi_t(t)\|^2+\int_0^t \|\psi_\xi,\psi_t\|^2 d\tau\right)$$
$$\leqq c(\|\phi_\xi(0)\|^2+\|\psi_t(0)\|^2+\delta N_1(t)+\delta N_1(t)N_2(t)+N_1(t)N_2(t)^2), \tag{4.24}$$

$$\text{iii)}\ \|\phi_\xi(t)\|^2+\|\psi_\xi(t)\|_1^2+\|\psi_{t\xi}(t)\|^2+\int_0^t \|\psi_{t\xi},\psi_{\xi\xi}\|^2 d\tau - c\int_0^t \|\phi_\xi,\psi_\xi,\psi_t\|^2 d\tau$$
$$\leqq c(\|\phi_\xi(0)\|^2+\|\psi_\xi(0)\|_1^2+\|\psi_{t\xi}(0)\|^2$$
$$+\delta N_1(t)+\delta N_1(t)N_2(t)+N_1(t)N_2(t)^2), \tag{4.25}$$

$$\text{iv)}\ \|\phi_{\xi\xi}(t)\|^2+\int_0^t \|\phi_{\xi\xi}\|^2 d\tau - c(\|\psi_{\xi\xi},\psi_{t\xi}(t)\|^2+\int_0^t (\|\phi_\xi\|^2+\|\psi_\xi,\psi_t\|_1^2) d\tau)$$
$$\leqq c(\|\phi_{\xi\xi}(0)\|^2+\|\psi_{t\xi}(0)\|^2+\delta N_1(t)+\delta N_1(t)N_2(t)+N_1(t)N_2(t)^2). \tag{4.26}$$

Note that $\psi_t(0)=-z_*(0)$. A suitable linear combination of the four estimates (4.23)–(4.26) results in the main estimate of this section.

Lemma 4.4 (Principal energy estimate). *Under the assumptions of Lemma 4.3,*

$$N_1(t)(N_1(t)-c\delta-c\delta N_2(t)-N_2(t)^2)+N_2(t)^2 \leqq cN_1(0)^2 + \int_0^t \int_{\Omega_+(t)} |\lambda_{1x}|\psi^2 dx d\tau. \tag{4.27}$$

The norms N_1, N_2 are defined in (4.5), (4.6). The proof of Lemma 4.4 is immediate. Bounding the integral on the right of (4.27) is the object of Sect. 5.

Proof of Lemma 4.3.

(i) *Proof of* (4.23). Following [6] define

$$\begin{aligned} \mathrm{LHS}_1 &\equiv -\psi_\xi L_1 + A^{-1}(\psi_t - s\psi_\xi)L_2 = -A^{-1}(\psi_t - s\psi_\xi)\Gamma_1, \\ \mathrm{LHS}_2 &\equiv \phi L_1 + A^{-1}\psi L_2 = -A^{-1}\psi\Gamma_1. \end{aligned} \tag{4.28}$$

Integrate over $-\infty<\xi<\infty$, dropping some terms through integration by parts. After some rearrangements, one finds that

$$\begin{aligned} \mathrm{LHS}_1 = {} & [A^{-1}\{\tfrac{1}{2}(\psi_t-s\psi_\xi)^2+\tfrac{1}{2}\psi_\xi^2\}-\phi\psi_\xi]_t \\ & + A^{-1}[B(\psi_t-s\psi_\xi)^2+M(\psi_t-s\psi_\xi)\psi_\xi - A\psi_\xi^2] \\ & + \tfrac{1}{2}(A^{-1})_\xi[s(\psi_t-s\psi_\xi)^2+2\underline{\psi_t\psi_\xi}-s\psi_\xi^2]-(A^{-1})_t[\tfrac{1}{2}(\psi_t-s\psi_\xi)^2+\tfrac{1}{2}\psi_\xi^2]+[\;]_\xi, \end{aligned} \tag{4.29}$$

$$\begin{aligned}\mathrm{LHS}_2 = {} & [\tfrac{1}{2}\phi^2 + \tfrac{1}{2}A^{-1}B\psi^2 + A^{-1}\psi(\psi_t - s\psi_\xi) + \tfrac{1}{2}s(A^{-1})_\xi\psi^2]_t \\ & + A^{-1}[-(\psi_t - s\psi_\xi)^2 + \psi_\xi^2] + (1-s^2)(A^{-1})_\xi\psi\psi_\xi - \tfrac{1}{2}(A^{-1}(M-sB))_\xi\psi^2 \\ & - [\tfrac{1}{2}(A^{-1}B)_t\psi^2 + (A^{-1})_t\psi(\psi_t - s\psi_\xi) + \tfrac{1}{2}s(A^{-1})_{\xi t}\psi^2] + [\ ]_\xi . \end{aligned} \tag{4.30}$$

Add $\mathrm{LHS}_1 + \lambda \mathrm{LHS}_2$ (here λ is unrelated to the eigenvalues λ_i) to obtain

$$\begin{aligned}(E_1 + E_2 + \tilde{E}_2)_t + E_3 + E_4 + E_5 + E_6 + G + [\ ]_\xi \\ = -A^{-1}(\lambda\psi + (\psi_t - s\psi_\xi))\Gamma_1 ,\end{aligned} \tag{4.31}$$

in which

$$\begin{aligned} E_1 &= A^{-1}\left(\frac{\lambda}{2}A\phi^2 - A\phi\psi_\xi + \tfrac{1}{2}\psi_\xi^2\right), \\ E_2 &= A^{-1}\left(\frac{\lambda}{2}B\psi^2 + \lambda\psi(\psi_t - s\psi_\xi) + \tfrac{1}{2}(\psi_t - s\psi_\xi)^2\right), \\ \tilde{E}_2 &= \frac{\lambda s}{2}(A^{-1})_\xi\psi^2 , \\ E_3 &= A^{-1}((B-\lambda)(\psi_t - s\psi_\xi)^2 + M(\psi_t - s\psi_\xi)\psi_\xi + (\lambda - A)\psi_\xi^2), \\ E_4 &= -\frac{\lambda}{2}(A^{-1}(M-sB))_\xi\psi^2 , \\ E_5 &= -(A^{-1})_t[\tfrac{1}{2}(\psi_t - s\psi_\xi)^2 + \tfrac{1}{2}\psi_\xi^2 - \lambda\psi(\psi_t - s\psi_\xi)], \\ E_6 &= -\frac{\lambda}{2}[(A^{-1}B)_t + s(A^{-1})_{\xi t}]\psi^2 , \\ G &= \lambda(1-s^2)(A^{-1})_\xi\psi\psi_\xi + \tfrac{1}{2}(A^{-1})_\xi\{s(\psi_t - s\psi_\xi)^2 + 2(\psi_t - s\psi_\xi)\psi_\xi + s\psi_\xi^2\} . \end{aligned} \tag{4.32}$$

The quantities E_1, E_2, $\tilde{E}_2$, E_3, E_4, G are the same as in [6] except that A, B, M depend on f_2 as well as f_1, and hence on t as well as ξ.

It follows as in [6] from Lemma 4.1 that

$$\begin{aligned} E_1 + E_2 + \tilde{E}_2 &\leqq C(\phi^2 + \psi^2 + \psi_\xi^2 + (\psi_t - s\psi_\xi^2)), \\ E_1 + E_2 + \tilde{E}_2 &\geqq c(\phi^2 + \psi^2 + \psi_\xi^2 + (\psi_t - s\psi_\xi)^2), \\ E_3 &\geqq c(\psi_\xi^2 + (\psi_t - s\psi_\xi)^2) . \end{aligned} \tag{4.33}$$

Since $|(A^{-1})_\xi| + |(A^{-1})_t| < (|\lambda_{1x}| + |\lambda_{2x}|) < \delta$, then

$$\begin{aligned} |G| &\leqq \beta|(A^{-1})_\xi|\psi^2 + c\beta^{-1}\delta(\psi_\xi^2 + (\psi_t - s\psi_\xi)^2), \\ |E_5| &\leqq c\delta((\psi_t - s\psi_\xi)^2 + \psi_\xi^2) + |(A^{-1})_t|\psi^2 , \end{aligned} \tag{4.34}$$

for any small $\beta > 0$. By choosing δ and β small enough, it follows using (iii) in Lemma 4.1 that

$$E_3 + E_4 + E_5 + E_6 + G \geqq c((\psi_t - s\psi_\xi)^2 + \psi_\xi^2) - c(\chi_+|\lambda_{1x}| + \delta e^{-\gamma t})\psi^2 , \tag{4.35}$$

in which $\chi_+ = 0$ for $x \in \Omega_- \cup \Omega_2$, $\chi_+ = 1$ for $x \in \Omega_+$.

Now integrate the various terms in (4.31) over $-\infty<\xi<\infty$ and $0<\tau<t$ to obtain

$$\begin{aligned}
&\int_0^t\int (E_1+E_2+\tilde{E}_2)_t\,d\xi dt \geqq c\left[\int \phi^2+\psi^2+\psi_\xi^2+(\psi_t-s\psi_\xi)^2\int\right]_0^t,\\
&\int_0^t\int (E_3+E_4+E_5+E_6+G)\,d\xi dt \geqq c\int_0^t\int ((\psi_t-s\psi_\xi)^2+\psi_\xi^2)\,d\xi dt\\
&\qquad -c\delta N_1(t)^2-c\int_0^t\int_{\Omega_+(t)}|\lambda_{1x}|\psi^2\,dxdt, \qquad (4.36)\\
&\int_0^t\int |A^{-1}(\lambda\psi+(\psi_t-s\psi_\xi))\Gamma_1|\,d\xi dt \leqq c\int_0^t\int (|\Gamma|+|K|+|H|)F\,d\xi dt\\
&\qquad \leqq c(\delta N_1(t)+\delta N_1(t)N_2(t)+N_1(t)N_2(t)^2)\,.
\end{aligned}$$

Combine these together to obtain (4.23).

(ii) *Proof of* (4.24)–(4.26). These estimates are proved as in [6] and (i) with very little change.

5. Characteristic Energy Method

The object here is to estimate $N_3(t)^2=\int_0^t\int(|\lambda_{1x}|+|\lambda_{2x}|)(\phi^2+\psi^2)dxdt$ on the right-hand side of the energy estimate (4.27) in Lemma 4.4. Rewrite the system (3.17), (3.18) for ϕ, ψ by diagonalizing the left-hand side. As in (2.25), (2.26) the convection matrix

$$\begin{bmatrix} 0 & 1\\ A/B & M/B\end{bmatrix} \qquad (5.1)$$

has right eigenvectors r_1, r_2, left eigenvectors l_1, l_2 and eigenvalues λ_1, λ_2 given by

$$\begin{aligned}
r_1&=(-B/D,-(M-D)/2D)^\dagger\,, & r_2&=(B/D,(M+D)/2D)^\dagger\,,\\
l_1&=(-(M+D)/2B,1)\,, & l_2&=((-M+D)/2B,1)\,,\\
\lambda_1&=(M-D)/2B\,, & \lambda_2&=(M+D)/2B\,,
\end{aligned} \qquad (5.2)$$

in which $D=(M^2+4AB)^{1/2}$ and M, A, B are defined in (2.41) and depend only on f_1 and f_2. Define characteristic variables θ_1, θ_2 by

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\theta_1 r_1+\theta_2 r_2\,, \qquad (5.3)$$

which satisfy

$$\begin{aligned}
&c(\phi^2+\psi^2)<\theta_1^2+\theta_2^2<C(\phi^2+\psi^2)\,,\\
&c(\phi_x^2+\psi_x^2)<\theta_{1x}^2+\theta_{2x}^2+(|\lambda_{1x}|+|\lambda_{2x}|)(\theta_1^2+\theta_2^2)\\
&\qquad <C(\phi_x^2+\psi_x^2)+C(|\lambda_{1x}|+|\lambda_{2x}|)(\phi^2+\psi^2)\,,
\end{aligned} \qquad (5.4)$$

etc. for some constants c, C independent of x, t, δ. The characteristic form for (3.17), (3.18) is

$$\theta_{it}+\lambda_i\theta_{ix}=\sum_{k=1}^{2}\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k+B^{-1}(\psi_{xx}-\psi_{tt}-\Gamma-K-H)\,. \qquad (5.5)$$

We first obtain an energy estimate from the characteristic form:

Lemma 5.1. *For δ small enough,*

$$\int |\theta(x,t)|^2 dx + \tfrac{1}{2} \iint_{\Omega_2(t)} |\lambda_{2x}| \theta_2^2 dx d\tau + \tfrac{1}{2} \iint_{\Omega_-(t)} |\lambda_{1x}| \theta_1^2 dx d\tau$$
$$\leqq C \left\{ \sum_{i \neq j} \iint_{\Omega_j(t)} |\lambda_{jx}| \theta_i^2 dx d\tau + \iint_{\Omega_+(t)} |\lambda_{1x}| \theta_1^2 dx d\tau \right.$$
$$\left. + N_1(t)\{\delta + N_1(t) + \delta N_2(t) + N_2(t)^2\} + N_2(t)^2 \right\}. \tag{5.6}$$

Proof of Lemma 5.1. Multiply the characteristic form equation (5.5) by θ_i and integrate over $-\infty < x < \infty$, $0 < \tau < t$,

$$\int \theta_i^2(x,t) dx - \int_0^t \int \lambda_{ix} \theta_i^2 dx d\tau = \int \theta_i^2(x,0) dx$$
$$+ 2 \int_0^t \int \left[\theta_i \sum_{k=1}^{2} \theta_k (l_{it} + \lambda_i l_{ix}) \cdot r_k + \theta_i B^{-1} (\psi_{xx} - \psi_{tt} - \Gamma - K - H) \right] dx d\tau. \tag{5.7}$$

Throughout this analysis we use the fact that λ_i depends primarily on f_1 in Ω_2 (i.e. $x > 0$) and on f_2 in Ω_1 (i.e. $x < 0$).

First we partly handle the second term on the left by noting that, from (2.17) for Ω_2 and (2.29) for Ω_- and for some γ,

$$-\lambda_{1x} = -\lambda_{1x}(f_2) + O(f_{1x}) = |\lambda_{1x}| + O(\delta e^{-\gamma(|x|+t)}) \quad \text{for} \quad (x,t) \in \Omega_-, \tag{5.8}$$
$$-\lambda_{2x} = -\lambda_{2x}(f_1) + O(f_{2x}) = |\lambda_{2x}| + O(\delta e^{-\gamma(|x|+t)}) \quad \text{for} \quad (x,t) \in \Omega_2. \tag{5.9}$$

Thus

$$\int_0^t \int \lambda_{1x} \theta_1^2 + \lambda_{2x} \theta_2^2 dx d\tau \leqq - \iint_{\Omega_-(t)} |\lambda_{1x}| \theta_1^2 dx d\tau - \int \int_{\Omega_2} |\lambda_{2x}| \theta_2^2 dx d\tau$$
$$+ \iint_{\Omega_+(t)} |\lambda_{1x}| \theta_1^2 dx d\tau + c\delta N_1(t)^2. \tag{5.10}$$

Next use the bounds

$$\begin{aligned} & l_{it} + \lambda_i l_{ix} = O(1) \lambda_{ixx} + O(\delta e^{-\gamma(|x|+t)}) \quad \text{on} \quad \Omega_i, \\ & |l_{it} + \lambda_i l_{ix}| \leqq c |\lambda_{jx}| + O(\delta e^{-\gamma(|x|+t)}) \quad \text{on} \quad \Omega_j (i \neq j), \end{aligned} \tag{5.11}$$

which follow from (2.18), (2.36), (2.37), to estimate (integrating by parts)

$$\left| \iint_{\Omega_i(t)} \theta_i \sum_{k=1}^{2} \theta_k (l_{it} + \lambda_i l_{ix}) \cdot r_k dx dt \right| \leqq c\delta \left(\iint_{\Omega_i(t)} \theta^2 |\lambda_{ix}| dx dt + N_1(t)^2 \right), \tag{5.12}$$

$$\left| \iint_{\Omega_j(t)} \theta_i \sum_{k=1}^{2} \theta_k (l_{it} + \lambda_i l_{ix}) \cdot r_k dx dt \right|$$
$$\leqq \tfrac{1}{4} \iint_{\Omega_j(t)} \theta_j^2 |\lambda_{jx}| dx dt + c \iint_{\Omega_j(t)} \theta_i^2 |\lambda_{jx}| dx dt + c\delta N_1(t)^2, \tag{5.13}$$

for some constant c.

Next since B^{-1} is bounded,

$$\int_0^t \int \theta_i B^{-1} (\Gamma + K + H) dx dt \leqq c(\delta + \delta N_2(t) + N_2(t)^2) N_1(t) \tag{5.14}$$

from Lemma 4.2. Finally use $|B_x|+|B_t|<c(|\lambda_{1x}|+|\lambda_{2x}|)<c\delta$ and $\phi_x=-\psi_t$ to estimate

$$\left|\int_0^t\int \theta_i B^{-1}(\psi_{xx}-\psi_{tt})dxdt\right| \leqq c\left(\int_0^t\int \theta_x^2+\theta_t^2+(|\lambda_{1x}|+|\lambda_{2x}|)(|\theta\theta_x|+|\theta\theta_t|dxdt\right.$$
$$\left.+\int|\theta\theta_t(t)|+|\theta\theta_t|(0)dx\right)$$
$$\leqq c(N_2(t)^2+\delta\iint(|\lambda_{1x}|+|\lambda_{2x}|)\theta^2dxdt+N_1(t)^2) \quad (5.15)$$

Combine (5.7), (5.10), (5.12)–(5.15) to obtain the desired estimate (5.6) and complete the proof of Lemma 5.1.

Since N_1 and N_2 can be bounded through the energy estimate (4.27), we now only need to find bounds on the three integrals on the right of (5.6):

$$I_1=\iint_{\Omega_1(t)}|\lambda_{1x}|\theta_2^2dxd\tau,$$
$$I_2=\iint_{\Omega_2(t)}|\lambda_{2x}|\theta_1^2dxd\tau \quad\text{and}\quad I_3=\iint_{\Omega_+(t)}|\lambda_{1x}|\theta_1^2dxd\tau.$$

The estimates are derived using the characteristic energy method developed by Liu [7]. Actually we use a somewhat simplified version in which integration is performed along piecewise linear approximations of the characteristics. Although simpler, this version is less robust since its validity depends in an additional way on the shock being weak.

Write the eigenvalue λ_i as piecewise constant part $\bar{\lambda}_i$ plus an error $\tilde{\lambda}_i$, i.e.

$$\lambda_i=\bar{\lambda}_i+\tilde{\lambda}_i, \quad (5.16)$$

in which

$$\bar{\lambda}_i(x,t)=\begin{cases}\lambda_i(f_1^{\infty})=\lambda_i^+, & x>st\\ \lambda_i(f_1^{-\infty})=\lambda_i^-, & x<st.\end{cases} \quad (5.17)$$

Define approximate, piecewise linear characteristics $X_1(\tau,t,x)$, $X_2(\tau,t,x)$ (cf. Fig. 2) satisfying

$$\frac{\partial}{\partial\tau}X_i(\tau,t,x)=\bar{\lambda}_i(X_i,\tau), \qquad X_i(t,t,x)=x, \quad (5.18)$$

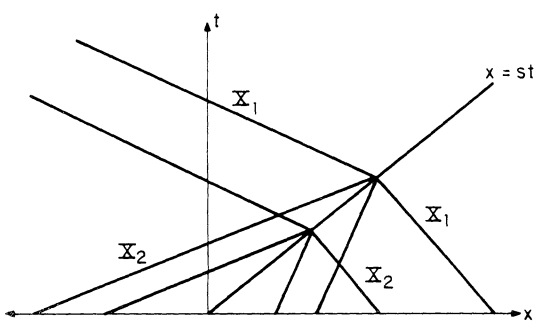

Fig. 2. Piecewise linear characteristics X_1 and X_2

i.e.

$$X_1(\tau,t,x)=\begin{cases} x+\lambda_1^+(\tau-t) & \text{if} \quad x>st, X_1>s\tau \\ x+\lambda_1^-(\tau-t) & \text{if} \quad x<st, X_1<s\tau \\ \dfrac{(s-\lambda_1^-)}{(s-\lambda_1^+)}(x-\lambda_1^+t)+\lambda_1^-\tau & \text{if} \quad x>st, X_1<s\tau \end{cases}$$

$$X_2(\tau,t,x)=\begin{cases} x+\lambda_2^+(\tau-t) & \text{if} \quad x>st \\ x+\lambda_2^-(\tau-t) & \text{if} \quad x<st\,. \end{cases}$$

Note that X_2 is not defined past the time that it intersects $X_2=s\tau$. Equation (5.5) can be rewritten as

$$\theta_{it}+\bar{\lambda}_i\theta_{ix}=\sum_{k=1}^{2}\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k+B^{-1}(\psi_{xx}-\psi_{tt}-\Gamma-K-H)-\tilde{\lambda}_i\theta_{ix}\,. \tag{5.19}$$

Multiply by θ_i and integrate along X_i to obtain

$$\begin{aligned}\theta_i(x,t)^2=\tfrac{1}{2}\int_0^t\theta_i&\left\{\sum_{k=1}^{2}\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k\right.\\ &\left.+B^{-1}(\psi_{xx}-\psi_{tt}-\Gamma-K-H)-\tilde{\lambda}_i\theta_{ix}\right\}(X_i(\tau,t,x),\tau)d\tau+\theta_i(X_i(0,t,x),0)^2\,.\end{aligned}$$

For any non-negative function $g(x,t)$ we integrate to get

$$\begin{aligned}\int_0^t\int g\theta_i^2\,dx d\tau&=\tfrac{1}{2}\int_0^t\int\tilde{G}\theta_i\left\{\sum_{k=1}^{2}\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k\right.\\ &\quad\left.+B^{-1}(\psi_{xx}-\psi_{tt}-\Gamma-K-H)-\tilde{\lambda}_i\theta_{ix}\right\}(x,\tau)dxd\tau+\int_{-\infty}^{\infty}(\tilde{G}\theta_i^2)(x,0)dx\\ &\leqq\tfrac{1}{2}\int_0^t\int G\theta_i\left\{\sum_{k=1}^{2}\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k+B^{-1}(\psi_{xx}-\psi_{tt}-\Gamma-K-H)-\tilde{\lambda}_i\theta_{ix}\right\}\\ &\quad+\int_{2-\infty}^{\infty}(G\theta_i^2)(x,0)dx\,,\end{aligned} \tag{5.20}$$

in which

$$\begin{aligned}\tilde{G}(x,\tau)&=\int_\tau^t g(X_i(\sigma,\tau,x),\sigma)\frac{\partial X_i}{\partial x}(\sigma,\tau,x)d\sigma\,,\\ G(x,\tau)&=\int_\tau^\infty g(X_i(\sigma,\tau,x),\sigma)\frac{\partial X_i}{\partial x}(\sigma,\tau,x)d\sigma\,.\end{aligned} \tag{5.21}$$

The inequality in (5.20) follows from the non-negativity of g and $(\partial X_i/\partial x)$.

Since X_2 can only be extended forward for a finite time, until it hits the line $x'=st'$, define $\partial X_2/\partial x=0$ for σ past that time. Note that $\left(\frac{\partial X_i}{\partial x}\right)^{-1}$ is piecewise constant and bounded, i.e. for $\sigma>\tau$, $X_i(\sigma,\tau,x)$ satisfies

$$\frac{\partial X_1}{\partial x}=\begin{cases}1 & \text{for} \quad (x>s\tau, X_1>s\sigma) \quad \text{or} \quad (x<s\tau, X_1<s\sigma)\\ (s-\lambda_1^-)(s-\lambda_1^+)^{-1} & \text{for} \quad x>s\tau, X_1<s\sigma\,,\end{cases}$$

$$\frac{\partial X_2}{\partial x}(\sigma,\tau,x)=\begin{cases}1 \quad \text{for} \quad (x>s\tau, \sigma<(s-\lambda_2^+)^{-1}(x-\lambda_2^+\tau)) \quad \text{or}\\ \qquad\qquad (x<s\tau, \sigma<(s-\lambda_2^-)^{-1}(x-\lambda_2^-\tau))\\ 0 \quad \text{otherwise}.\end{cases} \tag{5.22}$$

For three different choices of g we shall utilize (5.20). For each choice the various terms on the right-hand side of (5.20) will be estimated as:

$$\left|\int_0^t\int_{-\infty}^{\infty}\theta_i\theta_k(l_{it}+\lambda_i l_{ix})\cdot r_k G\,dx\,d\tau\right| \leqq (\sup|G|)\int_0^t\int_{-\infty}^{\infty}(|\lambda_{1x}|+|\lambda_{2x}|)\theta^2\,dx\,d\tau$$
$$\leqq (\sup|G|)N_3(t)^2\,, \tag{5.23}$$

$$\left|\int_0^t\int_{-\infty}^{\infty}\tilde{\lambda}_i\theta_i\theta_{ix}G\,dx\,d\tau\right| \leqq \left(\int_0^t\int_{-\infty}^{\infty}\theta_{ix}^2\,dx\,dt\right)^{1/2}\left(\int_0^t\int_{-\infty}^{\infty}(\tilde{\lambda}_i G)^2\theta_i^2\,dx\,d\tau\right)^{1/2}$$
$$\leqq cN_2(t)\left(\int_0^t\int_{-\infty}^{\infty}(\tilde{\lambda}_i G)^2\theta_i^2\,dx\,d\tau\right)^{1/2}, \tag{5.24}$$

$$\left|\int_0^t\int_{-\infty}^{\infty}G\theta_i B^{-1}(\Gamma+K+H)\,dx\,d\tau\right| \leqq c(\sup|G|)\int_0^t\int_{-\infty}^{\infty}|\theta_i|(|\Gamma|+|K|+|H|)\,dx\,d\tau$$
$$\leqq c(\sup|G|)N_1(t)(\delta+\delta N_2(t)+N_2(t)^2)\,, \tag{5.25}$$

$$\int_{-\infty}^{\infty}(G\theta_i^2)(x,0)\,dx=(\sup|G|)N_1(0)^2\,.$$

Since $G(x,t)$ has a jump discontinuity along the line $x=st$, write

$$\begin{aligned}\frac{\partial}{\partial x}G&=\bar{G}(x)\delta(x-st)+G_x(x,t)\,,\\ \frac{\partial}{\partial t}G&=s^{-1}\bar{G}(x)\delta(x-st)+G_t(x,t)\,,\end{aligned} \tag{5.26}$$

in which $\bar{G}(x)$ is the size of the jump in G at $(x,t=s^{-1}x)$ and G_x, G_t are regular functions. We need a bound on integrals along the line $x=st$. From Eq. (3.18), $\psi_{tt}^2<c(\phi_x^2+\psi_x^2+\psi_t^2+\psi_{xx}^2+H^2+\phi_x^4+\psi_x^4+\psi_t^4)$. Since $\int_0^{\infty}\int H^2\,dx\,d\tau<c\delta^2$ and $|\phi_x|+|\psi_x|+|\psi_t|<N_1$, then the Sobolev inequality implies that

$$\int_0^t(\psi_x^2+\psi_\tau^2)(s\tau,\tau)\,d\tau^{1/2} \leqq \int_0^t\int_{-\infty}^{\infty}(\psi_x^2+\psi_t^2+\psi_{xx}^2+\psi_{xt}^2+\psi_{tt}^2)\,dx\,d\tau^{1/2}$$
$$\leqq c(\delta+N_2(t)+N_1(t)N_2(t))\,. \tag{5.27}$$

Now we are ready [using (5.4)] to estimate the terms in (5.20) containing ψ_{xx}, ψ_{tt}:

$$\begin{aligned}
&\left|\int_0^t\int_{-\infty}^{\infty} G\theta_i B^{-1}(\psi_{xx}-\psi_{\tau\tau})\,dx\,d\tau\right| = \Bigg|-\int_0^t\int_{-\infty}^{\infty}\Big[GB^{-1}(\theta_{ix}\psi_x-\theta_{i\tau}\psi_\tau)\\
&\quad+G((B^{-1})_x\theta_i\psi_x-(B^{-1})_\tau\theta_i\psi_\tau)+B^{-1}(G_x\theta_i\psi_x-G_\tau\theta_i\psi_\tau)\Big]\,dx\,d\tau\\
&\quad-\left.\int_{-\infty}^{\infty} G\theta_i B^{-1}\psi_\tau\,dx\right]_0^t-\int_0^t \bar{G}\theta_i B^{-1}\psi_x(s\tau,\tau)\,d\tau+\int_0^{st}\bar{G}\theta_i B^{-1}\psi_t(x,s^{-1}x)\,dx\Bigg|\\
&\leqq c(\sup|G|)\int_0^t\int_{-\infty}^{\infty}\psi_x^2+\psi_\tau^2+\phi_x^2+\phi_\tau^2\,dx\,d\tau\\
&\quad+c(\sup|G|)\left(\int_0^t\int_{-\infty}^{\infty}\psi_x^2+\psi_\tau^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t\int_{-\infty}^{\infty}(|\lambda_{1x}|+|\lambda_{2x}|)\theta^2\,dx\,d\tau\right)^{1/2}\\
&\quad+c\left(\int_0^t\int_{-\infty}^{\infty}\psi_x^2+\psi_\tau^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t\int_{-\infty}^{\infty}(G_x^2+G_\tau^2)\theta^2\,dx\,d\tau\right)^{1/2}\\
&\quad+c(\sup|G|)\left(\int_{-\infty}^{\infty}\theta^2(0)\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}\psi_t^2(0)\,dx\right)^{1/2}\\
&\quad+c(\sup|G|)\left(\int_{-\infty}^{\infty}\theta^2(t)\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}\psi_t^2(t)\,dx\right)^{1/2}\\
&\quad+c(\sup|\bar{G}|)\left(\int_0^t(\psi_x^2+\psi_\tau^2)(s\tau,\tau)\,d\tau\right)^{1/2}\left(\int_0^t\theta^2(s\tau,\tau)\,d\tau\right)^{1/2}\\
&\leqq c(\sup|G|)(N_2(t)^2+N_2(t)N_3(t)+N_1(t)^2)\\
&\quad+cN_2(t)\left(\int_0^t\int_{-\infty}^{\infty}(G_x^2+G_\tau^2)\theta^2\,dx\,d\tau\right)^{1/2}\\
&\quad+c(\sup|G|)(\delta+N_2(t)+N_1(t)N_2(t))\left(\int_0^t\theta^2(s\tau,\tau)\,d\tau\right)^{1/2}. \qquad (5.28)
\end{aligned}$$

Combine (5.20), (5.23)–(5.25), (5.28) to obtain

Lemma 5.2. *Let g be non-negative and define G by* (5.21). *Then*

$$\begin{aligned}
&\int_0^t\int g\theta_i^2\,dx\,d\tau\leqq c(\sup|G|)(\delta N_1(t)+N_1(t)^2+N_1(t)N_2(t)^2+N_2(t)^2+N_3(t)^2)\\
&\quad+cN_2(t)\left(\int_0^t\int_{-\infty}^{\infty}((\tilde{\lambda}_iG)^2+G_x^2+G_\tau^2)\theta^2\,dx\,d\tau\right)^{1/2}\\
&\quad+c(\sup|G|)(\delta+N_2(t)+N_1(t)N_2(t))\left(\int_0^t\theta^2(s\tau,\tau)\,d\tau\right)^{1/2}. \qquad (5.29)
\end{aligned}$$

Since N_1, N_2 are bounded through Lemma 4.4 and we are in the process of bounding N_3, it will suffice to show that $(\sup|G|)<\delta$ and to bound the two integrals on the right of (5.29) for each case.

The estimate from the characteristic energy method is

Lemma 5.3. *For δ sufficiently small,*

$$\begin{aligned}&\iint_{\Omega_1(t)} |\lambda_{1x}|\theta_2^2 dx d\tau + \iint_{\Omega_2(t)} |\lambda_{2x}|\theta_1^2 dx d\tau + \iint_{\Omega_+(t)} |\lambda_{1x}|\theta_1^2 dx d\tau \\ &\quad \leqq c\delta(\delta N_1(t) + N_1(t)^2 + N_1(t)N_2(t)^2 + N_2(t)^2 + \delta^{-1/2}N_3(t)^2) \\ &\quad + c\delta(\delta + N_2(t) + N_1(t)N_2(t))\left(\int_0^t \theta^2(s\tau,\tau)d\tau\right)^{1/2}. \end{aligned} \tag{5.30}$$

Proof. i) *Estimate on* $|\lambda_{1x}|\theta_2^2$. Let $g=\chi_1|\lambda_{1x}|$ in which $\chi_1=1$ for $x<0$, $\chi_1=0$ for $x>0$. Then

$$g(x,\tau)=\begin{cases}\lambda_{1x}(x,\tau)+O(\delta e^{-\gamma(|x|+\tau)}), & (x,\tau)\in\Omega_+,\\ -\lambda_{1x}(x,\tau)+O(\delta e^{-\gamma(|x|+\tau)}), & (x,\tau)\in\Omega_-,\\ 0, & (x,\tau)\in\Omega_2.\end{cases} \tag{5.31}$$

The relevant characteristics in (3.52) are X_2 which are entirely contained within $\Omega_1=\Omega_+\cup\Omega_-$ (cf. Fig. 2) and in Ω_1 the characteristics are straight lines so that $\partial X_2/\partial x\equiv 1$. Fix $(x,\tau)\in\Omega_1$ and denote

$$\begin{aligned}[T_1,T_2]&=\{\sigma:\tau<\sigma,(X_2(\sigma,\tau,x),\sigma)\in\Omega_+\},\\ [T_3,T_4]&=\{\sigma:\tau<\sigma,(X_2(\sigma,\tau,x),\sigma)\in\Omega_-\}.\end{aligned} \tag{5.32}$$

Depending on the sign of κ and the location of (x,τ), the T_i's take on the values τ, ∞ (endpoints), $\tau-x/\lambda_2^-$ (intersection with $x=0$), or T satisfying

$$y_1=y(X_2,T)=(X_2(T,\tau,x)-\lambda_1^-T)/(\sqrt{\alpha T})=(x-\lambda_2^-\tau+(\lambda_2^--\lambda_1^-)T)/(\sqrt{\alpha T}) \tag{5.33}$$

[intersection with (Ω_+,Ω_- border)]. It follows that for (x,τ) large, $T=O(x)+O(\tau)$, and thus that for each i

$$|\partial_t T_i|+|\partial_x T_i|<c, \tag{5.34}$$

with c a constant independent of x, τ, δ.

In Ω_1, $(\partial/\partial\sigma)\lambda_1(X_2(\sigma,\tau,x),\sigma)=\lambda_2^-\lambda_{1x}+\lambda_{1\sigma}$ and $|\lambda_{1\sigma}|<c\delta(1+\sigma)^{-3/2}$. Thus

$$\int_{T_1}^{T_2}|\lambda_{1\sigma}|d\sigma+\int_{T_3}^{T_4}|\lambda_{1\sigma}|d\sigma\leqq\int_\tau^\infty c\delta(1+\sigma)^{-3/2}d\sigma\leqq c\delta(1+\tau)^{-1/2}. \tag{5.35}$$

Using $\partial X_2/\partial x=1$, (5.35) and (5.31), we may write

$$\begin{aligned}G(x,\tau)&=\int_{T_1}^{T_2}\lambda_{1x}(X_2(\sigma,\tau,x),\sigma)d\sigma-\int_{T_3}^{T_4}\lambda_{1x}(X_2(\sigma,\tau,x),\sigma)d\sigma\\ &=\int_{T_1}^{T_2}(\lambda_{1x}+(\lambda_2^-)^{-1}\lambda_{1\sigma})d\sigma-\int_{T_3}^{T_4}(\lambda_{1x}+(\lambda_2^-)^{-1}\lambda_{1\sigma})d\sigma+O(\delta(1+\tau)^{-1/2})\\ &=(\lambda_2^-)^{-1}(\lambda_1]_{T_1}^{T_2}-\lambda_1]_{T_3}^{T_4})+O(\delta(1+\tau)^{-1/2})\leqq c\delta(1+\tau)^{-1/2},\end{aligned} \tag{5.36}$$

since (2.31) implies that $\lambda_1=\lambda_1^-+O(1+\tau)^{-1/2}$ for $x<0$. Next differentiate G to obtain

$$\begin{aligned}|G_x|=(\lambda_2^-)^{-1}\sum_{i=1}^4\beta_i\{(\partial_xT_i)\lambda_2^-\lambda_{1x}(X_2(T_i,\tau,x),T_i)\\ +\lambda_{1\sigma}(X_2(T_i,\tau,x),T_i)+\lambda_{1x}(X_2(T_i,\tau,x),T_i)\}+O(\delta(1+\tau)^{-1})\leqq c\delta(1+\tau)^{-1},\end{aligned} \tag{5.37}$$

in which $\beta_i = \pm 1$. Similarly,

$$|G_\tau| \leqq c\delta(1+\tau)^{-3/2}. \tag{5.38}$$

Moreover $G \equiv 0$ for $x>0$ and $|\tilde{\lambda}_i| < c\delta(1+\tau)^{-1/2}$ for $x<0$. Therefore

$$\sup|G| < c\delta,$$

$$\int_0^t \int_{-\infty}^{\infty} ((\tilde{\lambda}_i G)^2 + G_x^2 + G_\tau^2)\theta^2 \, dx d\tau \leqq c\delta^2 \int_0^t \int_{-\infty}^{\infty} (1+\tau)^{-2}\theta^2 \, dx d\tau \leqq c\delta^2 N_1(t)^2. \tag{5.39}$$

Using (5.39) in (5.29) establishes the bound (5.30) for the integral of $|\lambda_{1x}|\theta_2^2$.

ii) *Estimate of* $|\lambda_{1x}|\theta_1^2$. Let $g = \chi_1 |\lambda_{1x}|$ as in (i). Now the relevant characteristics are $X_1(\sigma, \tau, x)$ with $X_1 < 0$ which may enter Ω_1 from Ω_2 and may cross $X_1 = s\sigma$ as well as $y = y_1$. Thus $(\partial X_1/\partial x)(\sigma, \tau, x)$ has values 1 for $x < s\tau$ and $(s-\lambda_1^-)/(s-\lambda_1^+)$ for $x > s\tau$. Denote

$$\begin{aligned} [T_1, T_2] &= \{\sigma : \tau \leqq \sigma, (X_1(\sigma, \tau, x), \sigma) \in \Omega_+\}, \\ [T_3, T_4] &= \{\sigma : \tau \leqq \sigma, (X_1(\sigma, \tau, x), \sigma) \in \Omega_-\}. \end{aligned} \tag{5.40}$$

Depending on the sign of κ and the position of (x, τ), each T_i takes on one of the values $s_1 = \tau$, $s_2 = \infty$ (lower and upper limits), $s_3 = (x - \lambda_1^- \tau)/\lambda_1^-$, $s_4 = -(s-\lambda_1^-)(x - \lambda_1^+ \tau)/\lambda_1^-(s - \lambda_1^+)$ (intersections with $X_1 = 0$ for $0 < x < s\tau$ and $s\tau < x$ respectively), or s_5 satisfying $y = y_1$, i.e.

$$y_1 = \frac{X_1(s_5, \tau, x) - \lambda_1^- s_5}{\sqrt{\alpha s_5}}.$$

There are three possibilities depending on whether $x<0$, $0<x<s\tau$ or $x>s\tau$; however in each case $X_1(s_5, \tau, x) = d_1 x + d_2 \tau + \lambda_1^- s_5$ (with three different positive values of d_1, d_2). Thus

$$\begin{aligned} &s_5 = \alpha y_1^{-2}(d_1 x + d_2\tau)^2, \quad \partial_x s_5 = 2d_1 \alpha y_1^{-2}(d_1 x + d_2 \tau), \quad \partial_\tau s_5 = 2d_2\alpha y_1^{-2}(d_1 x + d_2\tau), \\ &|\lambda_{1x}(X_1(s_5, \tau, x), s_5)| \leqq c\delta(1+s_5)^{-1}. \end{aligned} \tag{5.41}$$

For $i<5$, $|\partial_x s_i| + |\partial_\tau s_i| < c$ and $|\lambda_{1x}(\sigma = s_i)| \leqq c\delta(1+\tau)^{-1}$. Thus for each $i \leqq 5$,

$$(|\partial_x s_i| + |\partial_\tau s_i|)|\lambda_{1x}|(X_1(s_i, \tau, x), s_i) \leqq c\delta(1+\tau)^{-1}. \tag{5.42}$$

Evaluate G as

$$G(x,\tau) = \kappa_1 \int_{T_1}^{T_2} \lambda_{1x} d\sigma - \kappa_2 \int_{T_3}^{T_4} \lambda_{1x} d\sigma + O(e^{-\gamma\tau}),$$

in which κ_1, κ_2 are piecewise constant, having a jump along $x = s\tau$ and taking on values either 1 or $(s-\lambda_1^-)/(s-\lambda_1^+)$. As before λ_{1x} may be replaced by $(\partial/\partial\sigma)\lambda_1$ by adding in terms proportional to $\lambda_{1\sigma}$ so that $G = G(x, \tau)$ satisfies

$$\begin{aligned} G &= (\lambda_2^-)^{-1}(\kappa_1 \lambda_1]_{T_1}^{T_2} - \kappa_2 \lambda_1]_{T_3}^{T_4}) + O(\delta(1+\tau)^{-1/2}) \leqq c\delta(1+\tau)^{-1/2}, \\ |G_x| &= |(\lambda_2^-)^{-1} \sum_{i=1}^{4} \beta_i \{(\partial_x T_i)(\lambda_2^- \lambda_{1x} + \lambda_{1\tau}) + \lambda_{1x}\partial_x X_1\} + O(\delta(1+\tau)^{-1})| \\ &\leqq c\delta(1+\tau)^{-1}, \\ |G_\tau| &\leqq c\delta(1+\tau)^{-1}, \end{aligned} \tag{5.43}$$

in which the δ-function part of G_x, G_t is ignored as in (5.26). That part is compensated for by the integral along $x'=st'$ in (5.30). Also for all x,

$$|\tilde{\lambda}_1| \leqq c\delta(1+\tau)^{-1/2} + c|\lambda_{2x}|. \tag{5.44}$$

Thus

$$\sup|G| < \delta,$$

$$\begin{aligned}\int_0^t \int ((\tilde{\lambda}_1 G)^2 + G_x^2 + G_\tau^2)\theta^2 \,dx d\tau &\leqq c\delta^2 \int_0^t \int ((1+\tau)^{-2} + \delta|\lambda_{2x}|)\theta^2 \,dx d\tau \\ &\leqq c\delta^2 N_1(t)^2 + c\delta^3 N_3(t)^2 .\end{aligned} \tag{5.45}$$

Substitute (5.45) into (5.29) to complete the estimate for $|\lambda_{1x}|\theta_1^2$ in (5.30).

iii) *Estimate of* $|\lambda_{2x}|\theta_1^2$. Let $g=\chi_2|\lambda_{2x}|$ with χ_2 the characteristic function of $\Omega_2(t)$, i.e.

$$\begin{aligned} g(x,\sigma) &= \begin{cases} |\lambda_{2x}(x,\sigma)|, & x>0 \\ 0, & x<0 \end{cases} \\ &= \begin{cases} -\lambda_{2x}(f_1(x-s\sigma)) + O(\delta e^{-\gamma(|x|+\sigma)}), & x>0 \\ 0, & x<0 \end{cases}. \end{aligned} \tag{5.46}$$

The relevant characteristics here are $X_1(\sigma,\tau,x)$ with $X_1>0$ (i.e. $X_1 \in \Omega_2$). In Ω_2 the X_1 characteristic has piecewise constant slope, so that for $\sigma>\tau$,

$$\frac{\partial X_1}{\partial x}(\sigma,\tau,x) = \begin{cases} 1 & \text{if} \quad (x>s\tau, X_1>s\sigma) \quad \text{or} \quad (x<s\tau, X_1<s\sigma) \\ (s-\lambda_1^-)/(s-\lambda_1^+) & \text{if} \quad x>s\tau, X_1<s\sigma \end{cases}. \tag{5.47}$$

Denote

$$\begin{aligned} [T_1, T_2] &= \{\sigma : \tau<\sigma, X_1(\sigma,\tau,x)>s\sigma\}, \\ [T_2, T_3] &= \{\sigma : \tau<\sigma, 0<X_1(\sigma,\tau,x)<s\sigma\}. \end{aligned} \tag{5.48}$$

If $x<s\tau$ so that $X_1<s\sigma$ for all $\sigma>\tau$, set $T_1=T_2=\tau$. We can identify T_i as $T_1=\tau$,

$$\begin{aligned} T_2 &= \begin{cases} \tau, & \text{if} \quad x<s\tau \\ (x-\lambda_1^+\tau)/(s-\lambda_1^+), & \text{if} \quad x>s\tau \end{cases}, \\ T_3 &= \begin{cases} \tau - x/\lambda_1^-, & \text{if} \quad x<s\tau. \\ -(x-\lambda_1^+\tau)(s-\lambda_1^-)/(s-\lambda_1^+)\lambda_1^-, x>s\tau & \quad x>s\tau. \end{cases} \end{aligned} \tag{5.49}$$

As before for each i, $|\partial_t T_i| + |\partial_x T_i| < c$. Note that $X_1(T_3,\tau,x)=0$. Now evaluate G as

$$\begin{aligned} G(x,\tau) &= -(s-\lambda_1^+)(s-\lambda_1^-)^{-1} \int_\tau^{T_2} \lambda_{2x}(x+\lambda_1^+(\sigma-\tau)-s\sigma)d\sigma \\ &\quad - \int_{T_2}^{T_3} \lambda_{2x}(X_1(T_2,\tau,x)+\lambda_1^-(\sigma-T_2)-s\sigma)d\sigma + O(\delta e^{-\gamma\tau}) \\ &= (s-\lambda_1^-)^{-1} \left\{ \int_\tau^{T_2} (\partial/\partial\sigma)\lambda_2 d\sigma + \int_{T_2}^{T_3} (\partial/\partial\sigma)\lambda_2 d\sigma \right\} + O(\delta e^{-\gamma\tau}) \\ &= (s-\lambda_1^-)^{-1}\{\lambda_2(0,T_3) - \lambda_2(x,\tau)\} + O(\delta e^{-\gamma\tau}) \leqq c\delta, \end{aligned} \tag{5.50}$$

since λ_2 has total variation of size δ. Next

$$\begin{aligned} |G_x(x,\tau)| &= (s-\lambda_1^-)^{-1}\{X_{1x}(T_3,\tau,x)\lambda_{2x}(0,T_3) - \lambda_{2x}(x,\tau)\} + O(\delta e^{-\gamma\tau}) \\ &\leqq c(|\lambda_{2x}(x,\tau)| + \delta e^{-\gamma\tau}), \end{aligned} \tag{5.51}$$

since $|\lambda_{2x}(0, T_3)| \leqq c\delta e^{-\gamma T_3} \leqq c\delta e^{-\gamma\tau}$. Similarly

$$|G_\tau(x,\tau)| \leqq c(|\lambda_{2x}(x,\tau)| + \delta e^{-\gamma\tau}). \tag{5.52}$$

Also in $x>0$, $|\tilde{\lambda}_2| \leqq c|\lambda_{2x}| \leqq c\delta$. Thus

$$\sup|G| < c\delta,$$

$$\int_0^t \int ((\tilde{\lambda}_1 G)^2 + G_x^2 + G_\tau^2)\theta^2 \, dx d\tau \leqq c \int_0^t \int (\delta|\lambda_{2x}| + \delta^2 e^{-2\gamma\tau})\theta^2 \, dx d\tau$$
$$\leqq c\delta^2 N_1(t)^2 + c\delta N_3(t)^2. \tag{5.53}$$

Substitute (5.53) into (5.29) to complete the estimate for $|\lambda_{2x}|\theta_1^2$ in (5.30). This completes the proof of Lemma 5.3.

Note. The same estimation method does not work for $\iint_{\Omega_2(t)} |\lambda_{2x}|\theta_2^2 \, dx d\tau$, because the X_2 characteristics have endpoints on the line $x' = st'$ on which $|\lambda_{2x}|$ is large. Fortunately this term was estimated in Lemma 5.1.

We finish the characteristic energy estimates by estimating $\int_0^t \theta^2(s\tau,\tau)\,d\tau$. For $x \geqq st$,

$$\theta_i(x,t)^2 = \tfrac{1}{2}\int_0^t \theta_i \left\{ \sum_{k=1}^{2} \theta_k(l_{i\tau} + \lambda_i l_{ix}) \cdot r_k \right.$$
$$\left. + B^{-1}(\psi_{xx} - \psi_{\tau\tau} - \Gamma - K - H) - \tilde{\lambda}_i \theta_{ix} \right\}(x - \bar{\lambda}_i^+(t-\tau), \tau)\,d\tau + \theta_i(x - \bar{\lambda}_i^+ t, 0)^2. \tag{5.54}$$

Thus

$$\int_0^t \theta_i(s\tau,\tau)^2 d\tau = s^{-1}\int_0^{st} \theta_i(x, s^{-1}x)^2 dx \leqq s^{-1}\int_0^{st} \theta_i(x, s^{-1}x)^2 dx + \int_{st}^{\infty} \theta_i(x,t)^2 dx$$
$$= \tfrac{1}{2}(s - \bar{\lambda}_i^+)^{-1} \int_0^t \int_{s\tau}^{\infty} \theta_i \left\{ \sum_{k=1}^{2} \theta_k(l_{i\tau} + \lambda_i l_{ix}) \cdot r_k \right.$$
$$\left. + B^{-1}(\psi_{xx} - \psi_{\tau\tau} - \Gamma - K - H) - \tilde{\lambda}_i \theta_{ix} \right\}(x,\tau)\,dx d\tau + \int_0^{\infty} \theta_i(x,0)^2 dx. \tag{5.55}$$

Estimate various terms separately as

$$\left| \int_0^t \int_{s\tau}^{\infty} \theta_i \sum_{k=1}^{2} \theta_k(l_{i\tau} + \lambda_i l_{ix}) \cdot r_k \, dx d\tau \right| \leqq \int_0^t \int_{-\infty}^{\infty} \theta^2(|\lambda_{1x}| + |\lambda_{2x}|)\,dx d\tau \leqq N_3(t)^2,$$
$$\int_0^{\infty} \theta_i^2(x,0)\,dx \leqq N_1(t)^2, \tag{5.56}$$

$$\left| \int_0^t \int_{s\tau}^{\infty} \theta_i B^{-1}(\Gamma + K + H)\,dx d\tau \right| \leqq cN_1(t)(\delta + \delta N_2(t) + N_2(t)^2), \tag{5.57}$$

by Lemma 4.2. Next since $|\tilde{\lambda}_i| \leqq c|\lambda_{2x}| < c\delta$ for $x>0$,

$$\left| \int_0^t \int_{s\tau}^{\infty} \tilde{\lambda}_i \theta_i \theta_{ix} \, dx d\tau \right| \leqq \left(\int_0^t \int_0^{\infty} \theta_{ix}^2 \, dx d\tau \right)^{1/2} \left(\int_0^t \int_0^{\infty} \tilde{\lambda}_i^2 \theta_i^2 \, dx d\tau \right)^{1/2}$$
$$\leqq c\delta^{1/2} N_2(t) N_3(t). \tag{5.58}$$

Finally, using (5.27) in the last step,

$$\left|\int_0^t\int_{s\tau}^{\infty} B^{-1}\theta_i(\psi_{xx}-\psi_{\tau\tau})dxd\tau\right| = \left|\int_0^t\int_{s\tau}^{\infty} B^{-1}(-\theta_{ix}\psi_x+\theta_{i\tau}\psi_\tau)-(B^{-1})_x\theta_i\psi_x\right.$$

$$+(B^{-1})_\tau\theta_i\psi_\tau dxd\tau+\int_0^{\infty} B^{-1}\theta_i\psi_\tau(x,0)dx-\int_0^{st} B^{-1}\theta_i\psi_\tau(x,s^{-1}x)dx$$

$$\left.-\int_{st}^{\infty} B^{-1}\theta_i\psi_\tau(x,t)dx-\int_0^t B^{-1}\theta_i\psi_x(s\tau,\tau)d\tau\right|$$

$$\leqq c\left[N_2(t)^2+\delta^{1/2}N_2(t)N_3(t)+N_1(t)^2\right.$$

$$\left.+\left(\int_0^t(\psi_x^2+\psi_\tau^2)(s\tau,\tau)d\tau\right)^{1/2}\left(\int_0^t\theta^2(s\tau,\tau)d\tau\right)^{1/2}\right]$$

$$\leqq c\left[N_1(t)^2+N_2(t)^2+\delta^{1/2}N_2(t)N_3(t)\right.$$

$$\left.+(\delta+N_2(t)+N_1(t)N_2(t))\left(\int_0^t\theta^2(s\tau,\tau)d\tau\right)^{1/2}\right]. \tag{5.59}$$

Combine (5.55)–(5.59) to obtain

$$\int_0^t\theta^2(s\tau,\tau)d\tau\leqq c[\delta N_1(t)+N_1(t)^2+N_2(t)^2+N_1(t)N_2(t)^2+N_3(t)^2]$$

$$+c(\delta+N_1(t)N_2(t)+N_2(t))\left(\int_0^t\theta^2(s\tau,\tau)d\tau\right)^{1/2}, \tag{5.60}$$

from which it follows that

Lemma 5.4. *For δ sufficiently small,*

$$\int_0^t\theta^2(s\tau,\tau)d\tau\leqq c[\delta^2+N_1^2+N_1^2N_2^2+N_2^2+N_3^2](t). \tag{5.61}$$

Now combine (5.30) of Lemma 5.3 with (5.61) of Lemma 5.4 to obtain, after some recombination.

$$\iint_{\Omega_1(t)}|\lambda_{1x}|\theta_2^2dxd\tau+\iint_{\Omega_2(t)}|\lambda_{2x}|\theta_1^2dxd\tau+\iint_{\Omega_+(t)}|\lambda_{1x}|\theta_1^2dxd\tau$$

$$\leqq c\delta(\delta^2+N_1^2+N_1^2N_2^2+N_2^2+\delta^{-1/2}N_3^2)(t). \tag{5.62}$$

Combine this with (4.27) of Lemma 4.4 to obtain

$$N_1(N_1-c\delta-c\delta N_2-cN_2^2-c\delta N_1N_2^2)+N_2^2\leqq cN_1(0)^2+c\delta^3+c\delta^{1/2}N_3^2. \tag{5.63}$$

Next combine (5.62) with (5.6) of Lemma 5.1 using the inequality

$$N_3(t)^2\leqq c\iint_{\Omega_1(t)}|\lambda_{1x}|\theta^2dxd\tau+c\iint_{\Omega_2(t)}|\lambda_{2x}|\theta^2dxd\tau$$

to obtain

$$N_3^2\leqq cN_1\{\delta+N_1+\delta N_1N_2^2+\delta N_2+N_2^2)\}+cN_2^2+c\delta^3. \tag{5.64}$$

Finally combine (5.64) with (5.63) to obtain

$$N_1(N_1 - c\delta - c\delta N_2 - cN_2^2 - c\delta N_1 N_2^2) + N_2^2 \leqq cN_1(0)^2 + c\delta^3 . \tag{5.65}$$

This inequality implies that either N_1 and N_2 are both small or that N_1, N_2 are of $O(1)$ size independent of δ. However since N_1, N_2 start off with size δ^2 and are continuous, they must remain small. It follows that, for $N_1(0)$ sufficiently small,

$$N_1^2 + N_2^2 + N_3^2 \leqq c\delta^2 + cN_1(0)^2 . \tag{5.66}$$

By assumption $\|\phi, \psi\|_2(0)^2 + \|\partial_t \psi\|_1(0)^2 \leqq c\delta^2$. This establishes the main result of this paper.

Theorem 5.1. *Let δ be sufficiently small, then for any $t \geqq 0$,*

$$\sup_{0 \leqq \tau \leqq t} \{\|\phi, \psi\|_2(\tau)^2 + \|\partial_t \psi\|_1(\tau)^2\} + \int_0^t \|\phi_x, \psi_x, \psi_t\|_1^2 d\tau \leqq c\delta^2 . \tag{5.67}$$

Finally we prove Theorem 1. Since $\phi_x = \varrho_*$, $\psi_x = m_*$, $\psi_t = -z_*$, then

$$\sup_{0 \leqq \tau \leqq t} \|\varrho_*, m_*, z_*\|_1(\tau)^2 + \int_0^t \|\varrho_*, m_*, z_*\|_1^2 d\tau \leqq c\delta^2 . \tag{5.68}$$

Since this inequality is also satisfied by (ϱ_2, m_2, z_2) and (ϱ_3, m_3, z_3), we obtain

$$\sup_{0 \leqq \tau \leqq t} \|(\varrho - \varrho_1), (m - m_1), (z - z_1)\|_1(\tau)^2 + \int_0^t \|(\varrho - \varrho_1), (m - m_1), (z - z_1)\|_1^2 d\tau \leqq c\delta^2 . \tag{5.69}$$

Use the formula for f in terms of ϱ, m, z and the equation for f_t in terms of f_x and f to obtain

$$\sup_{0 \leqq \tau \leqq t} \int_{-\infty}^{\infty} (f - f_1)^2 + (f_x - f_{1x})^2 + (f_t - f_{1t})^2 dx + \int_0^t \int_{-\infty}^{\infty} (f - f_1)^2 + (f_x - f_{1x})^2 + (f_t - f_{1t})^2 dx d\tau \leqq c\delta^2 , \tag{5.70}$$

which is the inequality (1.8).

We now prove (1.9). From (4.25), (4.26) and integration of ψ_t times (3.18), we have

$$\|\psi_t, \phi_x, \psi_x\|_1^2(t_2) \leqq c\|\psi_t, \phi_x, \psi_x\|_1^2(t_1) + c\int_{t_1}^{t_2} \|\phi_x, \psi_x, \psi_t\|_1^2 d\tau + c\int_{t_1}^{t_2} \int_{-\infty}^{\infty} |\lambda_{1x}| \psi^2 dx d\tau , \qquad 0 \leqq t_1 \leqq t_2 < \infty .$$

This implies (by integration of t_1) that

$$\|\phi_{xx}, \psi_{xx}\|^2(t) \leqq c \int_{t-1}^{t} \|\phi_x, \psi_x, \psi_t\|_1^2 d\tau + c\int_{t-1}^{t} \int_{-\infty}^{\infty} |\lambda_{1x}| \psi^2 dx d\tau . \tag{5.71}$$

The estimate (5.66) implies

$$\int_0^t \|\phi_x, \psi_x, \psi_t\|_1^2 d\tau + \int_0^t \int_{-\infty}^{\infty} |\lambda_{1x}| \psi^2 dx d\tau \leqq c\delta^2 ,$$

which is slightly stronger than (5.67). It follows that the right-hand side of (5.71) tends to zero as $t \to \infty$. This proves (1.9) and concludes the proof of Theorem 1.

Appendix A. The Chapman-Enskog Error for the Broadwell Equation

Suppose that ϱ, m and $z = z(\varrho, m)$ satisfy the model Navier-Stokes equations with errors, i.e.

$$\varrho_t + m_x = e_1, \qquad m_t + z(\varrho, m)_x = e_2, \tag{A.1}$$

with $u = m/\varrho$ and

$$z = \varrho F(u) - v(u) u_x,$$

in which F and v are given by (2.11), (2.16). The third Broadwell equation ($z_t + m_x = Q(f, f)$) then has the error

$$E = Q(f, f) - (z_t + m_x) = \tfrac{1}{8}\{(\varrho - z)^2 - 4(z^2 - m^2)\} - (z_t + m_x). \tag{A.2}$$

To calculate E, first eliminate $z = \varrho F - v u_x$ to get

$$E = \tfrac{1}{8}\{\varrho^2 - 2\varrho^2 F - 3\varrho^2 F^2 + 4m^2\} + \{\tfrac{1}{4}(\varrho + 3\varrho F) v u_x - \tfrac{3}{8}(v u_x)^2 - (\varrho F - v u_x)_t - m_x\}. \tag{A.3}$$

The undifferentiated terms vanish due to the definition of F. Next eliminate time derivatives $(\varrho F)_t$ using the Navier-Stokes equations to obtain

$$\begin{aligned} E = &\{\tfrac{1}{4}\varrho(1 + 3F) v u_x + (F - uF' - 1) m_x + F'(\varrho F)_x\} \\ &+ \{-F'(v u_x)_x + (v u_x)_t - \tfrac{3}{8}(v u_x)^2\} + \{(-F + uF')e_1 - F' e_2\}. \end{aligned} \tag{A.4}$$

The terms in the first bracket, which are linear in first derivatives, vanish due to the choice of viscosity v. Therefore

$$E = \{-F'(v u_x)_x + (v u_x)_t - \tfrac{3}{8}(v u_x)^2\} + (-F + uF')e_1 - F' e_2. \tag{A.5}$$

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