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Classical and Quantum Mechanical Systems of Toda-Lattice Type

III. Joint Eigenfunctions of the Quantized Systems

Roe Goodman[1] and Nolan R. Wallach[2]

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. In a previous paper it was shown that certain Schrödinger operators $H=\Delta-V$ on $\mathbb{R}^{\ell}$ such as the Hamiltonians for the quantized one-dimensional lattice systems of Toda type (either non-periodic or periodic) are part of a family of mutually commuting differential operators $H=L_1,\dots,L_{\ell}$ on $\mathbb{R}^{\ell}$. The potential V in these cases is associated with a finite root system of rank ℓ, and the top-order symbols of the operators L_i are a set of functionally independent polynomials that generate the polynomial invariants for the Weyl group W of the root system. In this paper it is proved that the spaces of joint eigenfunctions for the family of operators L_i have dimension $|W|$. In the case of the periodic Toda lattices it is shown that the Hamiltonian has only bound states. An integrable holomorphic connection with periodic coefficients is constructed on a trivial $|W|$-dimensional vector bundle over $\mathbb{C}^{\ell}$, and it is shown that the joint eigenfunctions correspond exactly to the covariant constant sections of this bundle. Hence the eigenfunctions can be calculated (in principle) by integrating a system of ordinary differential equations. These eigenfunctions are holomorphic functions on $\mathbb{C}^{\ell}$, and are multivariable generalizations of the classical Whittaker functions and Mathieu functions. A generalization of Hill's determinant method is used to analyze the monodromy of the connection.

Contents

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0. Introduction

This paper is the third in a series of papers on classical and quantum mechanical systems of Toda lattice type (cf. [G-W2, G-W4]). The quantized Hamiltonians for these systems are Schrödinger operators $H=\Delta-V$ on $C^{\infty}(\mathbb{R}^{\ell})$ with the potential V determined by the following data: Let Φ be a finite set of linear functions on $\mathbb{R}^{\ell}$ that define either an irreducible Dynkin diagram of arbitrary type or a completed Dynkin diagram of type A, B, C, D, or E_6, and let c_α, for $\alpha\in\Phi$, be a constant. Then

$$V=\sum_{\alpha\in\Phi} c_\alpha^2 e^{-2\alpha}.$$

In [G-W2] it was shown that H is part of a family $L_1, L_2, \ldots, L_\ell$ of commuting differential operators whose top-order symbols are a set of functionally independent polynomials, and that these polynomials generate the polynomial invariants for the Weyl group of the root system associated with Φ.

For the completed Dynkin diagram of type A_ℓ, H is the quantized periodic Toda lattice Hamiltonian. We can write down an explicit "generating function" for the operators L_i in this case as follows:

Take Φ in $\mathbb{R}^n$, $n=\ell+1$, to consist of the linear functions $\alpha_i(x)=x_i-x_{i+1}$ [$i=1,2,\ldots,n$ with the indices read cyclically mod(n)]. Let $\mathscr{S}$ be the collection of all subsets $Q\subset\Phi$ such that $\alpha\perp\beta$ for $\alpha,\beta\in Q$. Given $Q\in\mathscr{S}$, set

$$e_Q=\exp\left\{-2\sum_{\alpha\in Q}\alpha\right\} \quad\text{and}\quad c_Q=\prod_{\alpha\in Q} c_\alpha.$$

Also let $I(Q)$ be the set of indices between 1 and n such that $(\partial/\partial x_i)e^{\alpha(x)}=0$ for all $\alpha\in Q$. In particular, when Q is empty, then $I(Q)=\{1,2,\ldots,n)$. Define a constant coefficient differential operator

$$P_Q(t)=\prod_{i\in I(Q)}\left\{\frac{\partial}{\partial x_i}+t\right\},$$

where t is an indeterminant. With this notation in place, define

$$L(t)=\sum_{Q\in\mathscr{S}}\left(-\frac{1}{2}\right)^{|Q|} c_Q e_Q P_Q(t). \tag{1}$$

One can prove (cf. Appendix II) that $[L(t), H]=0$ for all t. It follows by [G-W2, Theorem 5.2] that the coefficients D_j of t^{n-j} in $L(t)$ are mutually commuting differential operators for $j=1,2,\dots,n$. One has $D_1=(\partial/\partial x_1)+\dots+(\partial/\partial x_n)$ and $H=D_1^2-2D_2$. Setting $L_1=H$, $L_2=D_3,\dots,L_{n-1}=D_n$, we obtain a complete set of quantum invariants for the periodic Toda lattice system (cf. [Gu2] for the cases $n=3,4$, and 5; note that the linear momentum D_1 is an extra conserved quantity for this system).

We now return to the general case. Let the operators $L_1,\dots,L_\ell$ be as above. In Sect. 1 it is proved that if the potential V is determined by a completed Dynkin diagram and the coefficients c_α are all real and non-zero, then $L^2(\mathbb{R}^\ell)$ decomposes into a direct sum of finite-dimensional joint eigenspaces for $L_1,\dots,L_\ell$. Thus the problem of determining the joint eigenfunctions for these operators is especially important. It is natural to allow the coefficients $c=\{c_\alpha\}$ in the potential V to be complex, and to consider all the joint eigenfunctions, without any a priori condition of square-integrability. For $v=(v_1,\dots,v_\ell)\in\mathbb{C}^\ell$ we set

$$\mathscr{W}_{c,v}=\{f\in C^\infty(\mathbb{R}^\ell): L_i f=v_i f, \quad \text{for} \quad i=1,\dots,\ell\}.$$

The main purpose of the present paper is to analyze the joint eigenspaces $\mathscr{W}_{c,v}$. Let W be the finite Weyl group of the root system associated with Φ, and let w be its order. We prove that for all values of c and v, the space $\mathscr{W}_{c,v}$ has constant dimension w. For example, in the case of the periodic Toda lattice with n particles, this dimension is $n!$ (cf. Theorem 3.10; for invariance purposes the parameters c and v are replaced by an equivalent set in remainder of this paper). This result is proved by first constructing holomorphic $w\times w$ matrix-valued functions $\Gamma^i(c,v:z)$, for $z\in\mathbb{C}^\ell$ and $i=1,\dots,\ell$, such that:

(a) the operators $V_i=\partial/\partial z_i-\Gamma^i(c,v:z)$ mutually commute (Lemma 3.8). From the integrability condition (a) one knows that the space of solutions $\mathscr{E}_{c,v}$ of the first-order system

$$\partial F/\partial z_i=\Gamma^i(c,v)F, \quad \text{for} \quad i=1,\dots,\ell, \tag{2}$$

has dimension w. We then prove the key result (Lemma 3.9):

(b) the map $(F_1,\dots,F_w)\mapsto F_1$ is a bijection between $\mathscr{E}_{c,v}$ and $\mathscr{W}_{c,v}$.

Combining (a) and (b) we obtain the stated dimension result; furthermore, the integrability condition means that the joint eigenfunctions for the operators $L_1,\dots,L_\ell$ can be obtained by integrating a system of ordinary differential equations. Taking into account our earlier results [G-W4] on the solutions of the classical Hamiltonian flows for the generalized periodic Toda lattices, we see that these systems are "integrable" in all possible senses.

In the last part of the paper we turn to the problem of finding explicit expansions for the joint eigenfunctions. We know by properties (a) and (b) above that the eigenfunctions are holomorphic functions on $\mathbb{C}^\ell$. The coefficients of the operators L_j and the matrices Γ^i are multiply-periodic with purely imaginary independent periods τ_j, $j=1,\dots,\ell$. By Floquet theory there are mutually commuting $w \times w$ matrices $S_1, \ldots, S_\ell$ (the monodromy matrices), all depending holomorphically on the parameters c, v, such that

$$F(z+\tau_j)=S_jF(z),$$

for all $F \in \mathscr{E}_{c,v}$. We prove that the matrices S_j are diagonalizable for generic values of the parameters (c, v) (cf. Proposition 4.1). As a corollary, we show that for (c, v) in general position, there is a set $M(c, v)$ of w linear functionals on $\mathbb{C}^\ell$ which are all distinct mod $2L$ (L being the root lattice associated with Φ), and a basis $\{f_\Lambda | \Lambda \in M(c, v)\}$ for $\mathscr{W}_{c,v}$, such that the functions f_Λ have expansions

$$f_\Lambda(z)=e^{\langle \Lambda, z\rangle} \sum_{\mu \in L} a_\mu(\Lambda) e^{-2\langle \mu, z\rangle}. \tag{3}$$

The convergence in (3) is uniform on tube domains. We emphasize that in our treatment of these systems, the existence of a convergent expansion (3) is a consequence of the integrability of the system, and is not merely an *ansatz*.

The elements $\Lambda \in M(c, v)$ are the *monodromy exponents*. When Φ is an ordinary Dynkin diagram (the "generalized non-periodic Toda systems"), then we prove that $M(c, v)$ is an orbit of the Weyl group (Corollary 4.4). When Φ is a completed Dynkin diagram, then the explicit determination of $M(c, v)$ from c and v is quite difficult, the case $\ell=1$ being the problem of the Floquet exponents for the Mathieu equation (cf. [W-W; Chap. XIX Mathieu Functions]). We attack this problem in Sect. 4.3 using the following technique:

Write the Schrödinger equation

$$Hf_\Lambda(x)=v_1 f_\Lambda(x) \tag{4}$$

in terms of the expansion coefficients in (3); this gives an infinite set of homogeneous partial difference equations for the coefficients $a_v(\Lambda)$. We use a generalization of Hill's original method (as it applies to the Mathieu equation) and some results of von Koch ([vK1, vK2]; cf. the appendix to this paper) to show that the coefficient matrix of this set of difference equations has an absolutely convergent determinant $\Delta_{c,v}(\Lambda)$. Furthermore, the monodromy exponents Λ must satisfy the equation

$$\Delta_{c,v}(\Lambda)=0. \tag{5}$$

Conversely, for any $\Lambda \in \mathbb{C}^\ell$ lying on the variety (5), let $\mathscr{W}_{c,v}(\Lambda)$ be the space of functions that have convergent expansions of the form (3) and that satisfy (4). This space has finite non-zero dimension, and is invariant under the other operators $L_2, \ldots, L_\ell$. This yields $\ell-1$ more equations that suffice to determine the monodromy exponents (Theorem 4.6).

In the case of the quantized systems associated with an ordinary Dynkin diagram Φ, such as the non-periodic Toda lattice, the eigenfunctions have been previously constructed using representation theory ([G-W1, Go2, Ko]; cf. [O-P] for further references). The approach in the current paper via the system of first-order differential equations (2) was outlined in [Go2], and the result $\dim \mathscr{W}_{c,v} \leqq w$ (with equality for generic values of c and v) was proved there. Subsequently, Hashizume [Ha] carried out a direct construction of the eigenfunctions via the expansion (3) and reproved the generic dimension result. The fact that $\dim \mathscr{W}_{c,v}=w$ for *all* values of the parameters c and v was not previously known.

The most extensive previous study of the eigenfunctions for the quantized periodic Toda lattice was done by Gutzwiller [Gu1, Gu2], who uses some (partially heuristic) arguments involving matched asymptotic expansions to reduce the calculation of the monodromy to the case of one-variable infinite determinants of Hill's type. In [Gu2] the eigenfunctions are expanded in terms of Whittaker functions, rather than exponentials as in (3). This is a quantized version of a technique used by Kac and van Moerbeke to treat the N-particle periodic Toda lattice in terms of the $(N-1)$-particle non-periodic lattice. There seem to be several analytic points that are left open in this work. A WKB approximation for the eigenfunctions has been studied in [D-M], and general surveys of quantum integrable systems are found in [Fa, and O-P].

1. Hamiltonians with Long-Range Exponential Potentials

1.1. Solvable Lie Algebras and Exponential Potentials

In this section we study quantum Hamiltonians $L=\Delta-V$ on $\mathbb{R}^{\ell}$, where the potential V grows exponentially at infinity. By relating L to the unitary representations of certain exponential solvable Lie groups, we prove that the resolvent of L is compact, and we obtain exponential decay estimates for the eigenfunctions.

We begin by recalling some results from [G-W2], where the following class of Lie algebras was studied: Let $\mathfrak{b}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ such that $\mathfrak{b}=\mathfrak{a}\oplus\mathfrak{u}$, where $\mathfrak{a}$ is a commutative subalgebra of dimension ℓ, and $\mathfrak{u}$ is a commutative ideal. We assume that a positive-definite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{b}$ is given such that $\mathfrak{a}\perp\mathfrak{b}$ and the linear transformations ad H, for $H\in\mathfrak{a}$, are self-adjoint relative to this form.

By our assumptions, it is clear that $\mathfrak{u}$ admits an orthogonal direct sum decomposition into eigenspaces relative to $\operatorname{ad}\mathfrak{a}$:

$$\mathfrak{u}=\bigoplus_{\lambda\in\Phi}\mathfrak{u}_{\lambda}.$$

Here $\Phi\subset\mathfrak{a}^*$ and

$$\mathfrak{u}_{\lambda}=\{X\in\mathfrak{u}\,|\,[H,X]=\lambda(H)X,\,H\in\mathfrak{a}\}.$$

For the purposes of this paper we may assume that

$$\dim\mathfrak{u}_{\lambda}=1\quad\text{for all}\quad\lambda\in\Phi. \tag{1}$$

Let A and U denote the simply-connected Lie groups with Lie algebras $\mathfrak{a}$ and $\mathfrak{u}$ respectively. Denote by B the simply connected Lie group with Lie algebra $\mathfrak{b}$. From the structure of $\mathfrak{b}$ it follows directly that the exponential map exp: $\mathfrak{b}\to B$ and the map from $A\times U$ to B given by group multiplication are both analytic manifold isomorphisms.

Let $c\in\mathfrak{u}^*$, and set

$$\gamma_c(\exp X)=e^{ic(X)}, \tag{2}$$

for $X\in\mathfrak{u}$. Then γ_c is a unitary character of the group U. We form the induced unitary representation $T_c=\operatorname{Ind}_{U\to B}(\gamma_c)$ of B as usual. Using the splitting $B=A\cdot U$ and exponential coordinates on A, we may take $\mathscr{H}(T_c)=L^2(\mathfrak{a})$ (relative to

Lebesgue measure on $\mathfrak{a}$) as the Hilbert space for T_c. The action of B on a function $f \in L^2(\mathfrak{a})$ is given as

$$T_c(\exp H) f(a) = f(a-H), \tag{3}$$

for a and H in $\mathfrak{a}$, and

$$T_c(\exp X) f(a) = \exp\{ic(X) e^{-\lambda(a)}\} f(a), \tag{4}$$

for $X \in \mathfrak{u}_\lambda$.

There is a corresponding representation of the universal enveloping algebra $U(\mathfrak{b})$ on $C^\infty(\mathfrak{a})$ which we obtain by differentiating formulas (3) and (4) along one-parameter subgroups of B:

$$T_c(H) f = \partial(H) f \tag{5}$$

for $H \in \mathfrak{a}$, where $\partial(H) f(a) = (d/dt) f(a-tH)|_{t=0}$ and

$$T_c(X) f = ic(X) e^{-\lambda} f \tag{6}$$

for $X \in \mathfrak{u}_\lambda$.

Let $Z_1, \ldots, Z_m$ be any orthonormal basis of $\mathfrak{b}$. Set

$$\Omega = \sum_{i=1}^{m} Z_i^2$$

as an element of $U(\mathfrak{b})$. Then Ω does not depend on the particular choice of the orthonormal basis, and will be called the *Laplacian* for $\mathfrak{b}$, $\langle \cdot, \cdot \rangle$. We can choose the basis elements Z_i to be either in $\mathfrak{a}$ or in $\mathfrak{u}_\lambda$, $\lambda \in \Phi$. It then follows from (5) and (6) that the operator $L = T_c(\Omega)$ is given by

$$L = \Delta - \sum_{\lambda \in \Phi} |c_\lambda|^2 e^{-2\lambda}, \tag{7}$$

where Δ is the constant-coefficient Laplace operator on $\mathfrak{a}$ corresponding to $\langle \cdot, \cdot \rangle$ and c_λ is the restriction of the linear functional c to the subspace $\mathfrak{u}_\lambda$.

The elliptic operator L satisfies global estimates relative to the unitary representation T_c. To describe these estimates, one introduces the Hilbert spaces $\mathscr{H}^k(T_c)$ of k-times differentiable vectors for the representation T_c, with norm $\|f\|_k$, and the Fréchet space

$$\mathscr{H}^\infty(T_c) = \bigcap_{k=0}^{\infty} \mathscr{H}^k(T_c)$$

of C^∞ vectors (cf. [Go1]). By standard results for Schrödinger operators, one knows that L is essentially self-adjoint on $C_c^\infty(\mathfrak{a})$. By results of Nelson and Goodman (loc. cit.) one has

$$\text{Domain}\,(L^k) = \mathscr{H}^k(T_c) \quad \text{for all positive } k. \tag{8}$$

Here L is considered as an unbounded self-adjoint operator on $L^2(\mathfrak{a})$, and the equality in (8) means that the norms $\|f\| + \|L^k f\|$ and $\|f\|_k$ are equivalent.

To explicate (8) in the present situation, we need a more direct description of the spaces $\mathscr{H}^k$ and $\mathscr{H}^\infty$. We shall say that the linear functional c is *non-degenerate* if for all $\lambda \in \Phi$, $c_\lambda \neq 0$. Pick a basis $\{H_i\}$ for $\mathfrak{a}$, and set $\partial_i = \partial(H_i)$.

Lemma 1.1. *Suppose that $c \in \mathfrak{u}^*$ is non-degenerate. Then for any non-negative integer k, the space $\mathscr{H}^k(T_c)$ consists of all functions $f \in L^2(\mathfrak{a})$ such that*

$$e^{-p\lambda}\partial^Q f \in L^2(\mathfrak{a}) \tag{9}$$

for all $\lambda \in \Phi$ and all integers p and multi-indices Q with $p+|Q| \leqq k$. Here the derivatives are taken in the sense of distributions. The norm $\|f\|_k$ of f in $\mathscr{H}^k(T_c)$ is equivalent to the sum of the L^2 norms of the functions in (9) *over all $\lambda \in \Phi$ and $p+|Q| \leqq k$.*

Proof. This is immediate from formulas (5) and (6) and the definition of $\mathscr{H}^k(T_c)$. □

Gel'fand and Shilov [Ge-Sh, Chap. II] have defined test function spaces $K\{M_p\}$ on $\mathbb{R}^n$ using families $\{M_p : p=0,1,\ldots\}$ of weight functions to control rate of decay at infinity. The weight functions that fit in the present context are

$$M_p(a) = e^{p|a|}, \tag{10}$$

where $|a|$ is any convenient norm on the vector space $\mathfrak{a}$. Define a family of norms $\{v_p : p=0,1,\ldots\}$ on $C^\infty(\mathfrak{a})$ by

$$v_p(f) = \sup M_p(a)\,|\partial^Q f(a)|$$

(sup over $a \in \mathfrak{a}$ and multi-indices Q with $|Q| \leqq p$). Then the space $K\{M_p\}$ consists of all C^∞ functions f such that $v_p(f) < \infty$ for all $p \geqq 0$, with topology defined by the countable family of norms $\{v_p\}$. The functions f in this space are obviously characterized by the property that f and all its derivatives decrease faster than any exponential at infinity.

Lemma 1.2. *Assume that the convex hull of the set of roots Φ contains a neighborhood of 0 in $\mathfrak{a}^*$. Suppose that c is non-degenerate. Then the space $\mathscr{H}^\infty(T_c) = K\{M_p\}$. More precisely, there are positive constants $C = C(\Phi, c)$ and $b = b(\Phi)$ so that for all positive integers p*

$$v_p(f) \leqq C\,\|f\|_{d+bp}, \tag{11}$$

where $d = [\ell/4]+1$, and

$$\|f\|_p \leqq C v(f)_{1+bp}. \tag{12}$$

Remark. Suppose that Φ spans $\mathfrak{a}^*$ and that there is a relation of the form

$$\sum_{\lambda \in \Phi} n_\lambda \lambda = 0, \tag{13}$$

where the coefficients $n_\lambda > 0$ for all λ. Then it is easy to check that Φ satisfies the conditions of Lemma 1.2.

Proof. It suffices to prove (11) and (12) when f is a C^∞ function with compact support on $\mathfrak{a}$, by standard density theorems. Define the *gauge function* of the set Φ by $w_\Phi(a) = \max_{\lambda \in \Phi} \lambda(a)$. Then the hypothesis on Φ implies that there is a constant $r > 0$ (which we may take to be an integer), such that $|a| \leqq r w_\Phi(a)$ for all $a \in \mathfrak{a}$. It follows that for any positive p,

$$\sup_{a\in\mathfrak{a}} e^{p|a|}|f(a)| \leqq \sum_{\lambda\in\Phi} \sup_{a\in\mathfrak{a}} e^{rp\lambda(a)}|f(a)|. \tag{14}$$

By standard Sobolev estimates one also has

$$\sup_{a\in\mathfrak{a}}|f(a)| \leqq C\|(1-\Delta)^d f\|, \tag{15}$$

where the norm on the right is the L^2 norm on $\mathfrak{a}$, and Δ is the constant-coefficient Laplacian on $\mathfrak{a}$. Estimate (11) now follows readily from (14), (15) and Lemma 1.1.

To obtain estimate (12), observe that if $\lambda\in\Phi$, then

$$\|e^{p\lambda}f\| \leqq C\sup_{a\in\mathfrak{a}} e^{(ps+1)|a|}|f(a)|, \tag{16}$$

where $s=\max_{\lambda\in\Phi}|\lambda|$ and C is the L^2 norm of $e^{-|a|}$. Estimate (12) now follows from (15), (16) and Lemma 1.1. □

Corollary 1.3. *There exists a positive integer t so that the inclusion map $\mathscr{H}^t\subset L^2(\mathfrak{a})$ is compact.*

Proof. For any positive integer p, denote by $\mathscr{E}_p$ the space of C^p functions f on $\mathfrak{a}$ such that the norm $v_p(f)$ is finite. This is a Banach space in this norm, and the inclusion map $\mathscr{E}_p\subset\mathscr{E}_q$ is compact, if $p>q$ (cf. [Ge-Sh; Chap. II, Sect. 2.3]). By Lemma 1.2, one has continuous inclusions

$$\mathscr{H}^{bp+d}\to\mathscr{E}_p\to\mathscr{E}_q\to\mathscr{H}^{(q-1)/b}.$$

If we take $p=2$, $q=1$, for example, then we see that $t=2b+d$ will suffice. □

Theorem 1.4. *Assume that the convex hull of the set of roots Φ contains a neighborhood of 0 in $\mathfrak{a}^*$, and suppose that $c\in\mathfrak{u}^*$ is non-degenerate. Then the Schrödinger operator L in (6) has compact resolvent on $L^2(\mathfrak{a})$, and the square-integrable eigenfunctions of L are in the space $K\{M_p\}$, where M_p is given by (10). Thus these eigenfunctions and their derivatives decrease faster than exponential at ∞, and the eigenfunctions form a complete orthonormal set in $L^2(\mathfrak{a})$.*

Proof. By (8) and Lemma 1.2, we know that the space of C^∞ vectors for the self-adjoint operator L coincides with the space $K\{M_p\}$. Furthermore, the resolvent of L raised to a suitably high power is compact, by (8) and the Corollary to Lemma 1.2. Hence the resolvent of L is compact, by the functional calculus for self-adjoint operators. □

1.2. Operators Commuting with the Laplacian

We continue with the notation of the previous section. To find quantum invariants for the system with Schrödinger operator L, we look in the enveloping algebra of $\mathfrak{b}$ for elements commuting with the Laplacian Ω. We write

$$U(\mathfrak{b})^\Omega=\{R\in U(\mathfrak{b}): R\Omega=\Omega R\}. \tag{1}$$

Let $T\to T^*$ be the canonical conjugate-linear involution on $U(\mathfrak{b})$ ($X^*=-X$ for X in $\mathfrak{b}$). Since $\Omega^*=\Omega$, the subalgebra $U(\mathfrak{b})^\Omega$ is self-adjoint. Furthermore, if $R=R^*$ is in $U(\mathfrak{b})^\Omega$, then $T_c(R)$ is essentially self-adjoint on $C_c^\infty(\mathfrak{a})$, by a theorem of Nelson and Stinespring [N-S]. The following result will be used later in the paper to establish quantum complete integrability for a family of systems. (Recall that a Lie algebra $\mathfrak{h}$, not assumed to be finite-dimensional, is called *solvable* if its derived series of ideals $D^{(1)}(\mathfrak{h})=\mathfrak{h}$, $D^{(k+1)}(\mathfrak{h})=[D^{(k)}(\mathfrak{h}), D^{(k)}(\mathfrak{h})]$ vanishes for some finite k.):

Proposition 1.5. *Assume that the hypotheses of Theorem 1.4 hold for Φ and c. Suppose that $R_1, \ldots, R_n \in U(\mathfrak{b})^{\Omega}$ are self-adjoint elements. Let $\mathfrak{h}$ be the complex Lie algebra of operators on $\mathscr{H}^{\infty}(T_c)$ generated by $T_c(R_1), \ldots, T_c(R_n)$. Assume that $\mathfrak{h}$ is a solvable Lie algebra. Then $\mathfrak{h}$ is commutative, i.e. the operators $T_c(R_j)$ mutually commute.*

Proof. For any real number μ, let

$$\mathscr{H}_{\mu} = \{f \in C^{\infty}(\mathfrak{a}) \cap L^2(\mathfrak{a}) : Lf = \mu f\}. \tag{2}$$

Then by Theorem 1.4, $\dim \mathscr{H}_{\mu} < \infty$ for every μ. Furthermore, $L^2(\mathfrak{a})$ is the orthogonal direct sum of the spaces $\mathscr{H}_{\mu}$, as μ ranges over the spectrum of L. Each subspace $\mathscr{H}_{\mu}$ is invariant under the action of $\mathfrak{h}$. Since $\mathfrak{h}$ is solvable, Lie's theorem implies that the derived algebra $[\mathfrak{h}, \mathfrak{h}]$ acts by nilpotent transformations on $\mathscr{H}_{\mu}$. But by the self-adjointness of the generators R_j, the subalgebra $[\mathfrak{h}, \mathfrak{h}]$ has a basis consisting of self-adjoint operators. This is only possible if $[\mathfrak{h}, \mathfrak{h}]$ acts by zero on each space $\mathscr{H}_{\mu}$. By the completeness of the eigenspaces of L, this in turn implies that $[\mathfrak{h}, \mathfrak{h}] = 0$. □

2. Some Generalities on Certain Rings of Differential Operators

2.1. Certain Rings of Differential Operators

Let V be a finite-dimensional real vector space. We identify the symmetric tensor algebra over V with the algebra $\mathscr{P}$ of all complex-valued polynomial functions on the dual space V^* as usual. We grade $\mathscr{P}$ by degree and we denote by $\mathscr{P}^j$ the space of all homogeneous elements of $\mathscr{P}$ of degree j. If $X \in V$, then $\partial(X)$ denotes the differential operator $(\partial(X)f)(v) = (d/dt) f(v - tX)|_{t=0}$. By the identification of $\mathscr{P}$ with the symmetric tensor algebra over V, we can extend the map $X \mapsto \partial(X)$ to an isomorphism from $\mathscr{P}$ onto the algebra of constant coefficient differential operators on V.

Let $U \subset V$ be an open subset, and suppose that $\mathscr{R} \subset C^{\infty}(U)$ is a subalgebra (under pointwise multiplication) containing the constants, such that $\partial(X)\mathscr{R} \subset \mathscr{R}$ for all $X \in V$. Denote by $\mathscr{Q}$ the algebra of differential operators on U generated by multiplication by $\mathscr{R}$ and $\partial(V)$. We give $\mathscr{Q}$ the usual filtration by maximum order of differentiation. If $D \in \mathscr{Q}$ is of order j, then we denote the top-order symbol of D by $\sigma_j(D)$. This is an element of $\mathscr{P}^j \otimes \mathscr{R}$.

In many cases the differential equations satisfied by the joint eigenfunctions of a set of differential operators (including the Toda lattice systems) can be put into the following framework: One has an algebra $\mathscr{Q}$ of (variable-coefficient) differential operators as above, and a subalgebra $\mathscr{B} \subset \mathscr{Q}$ which satisfies these conditions:

(DO_1) If $D \in \mathscr{B}$ is of order j, then $\sigma_j(D) \in \mathscr{P}^j$ (i.e. the top-order term of D is a constant-coefficient operator).

(DO_2) If $\mathscr{A}$ is the subalgebra of $\mathscr{P}$ generated by all the top-order symbols of elements of $\mathscr{B}$, then there are homogeneous elements $1 = e_1, e_2, \ldots, e_d \in \mathscr{P}$ (where $d < \infty$), such that

$$\mathscr{P} = \sum_{i=1}^{d} \mathscr{A} e_i$$

(i.e. $\mathscr{P}$ is integral over $\mathscr{A}$).

Given this information, we then obtain the following algebraic analogue of "separation of variables" for operators in $\mathcal{D}$ (this technique was first used by Harish-Chandra in connection with spherical functions [H-C]):

Lemma 2.1. *Suppose* (DO_1) *and* (DO_2) *are satisfied. Set* $E_i = \partial(e_i)$. *Given any differential operator* $D \in \mathcal{D}$, *there exist operators* $u_{ij} \in \mathcal{B}$ *and functions* $f_i \in \mathcal{R}$ *such that*

$$D = \sum_{i,j} f_i E_j u_{ij}. \tag{1}$$

Proof. We proceed by induction on the order of D. The operators in $\mathcal{D}$ of order zero are multiplications by functions in $\mathcal{R}$, so the result is obvious in this case. Suppose we have proved the result for operators of order $\leqq k$. Let $D \in \mathcal{D}$ have order $k+1$. Let $x_1, \ldots, x_n$ be linear coordinates on V'. For $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, write $x^I = x_1^{i_1} \ldots x_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. Since $D \in \mathcal{D}$, there are functions $f_I \in \mathcal{R}$, for $|I| = k+1$, such that $\sigma_{k+1}(D) = \sum f_I x^I$. Let k_i be the degrees of the elements e_i in condition (2) above. By condition (DO_2) there are elements $v_{I,j} \in \mathcal{A}$, homogeneous of degree $|I| - k_j$, such that each monomial x^I can be expressed in the form $x^I = \sum v_{I,j} e_j$. Let $u_{I,j} \in \mathcal{B}$ be of order $|I| - k_j$ and have top-order symbol $v_{I,j}$. The operator $T = \sum f_I E_j u_{I,j}$ is then of order $k+1$, with top order symbol $\sigma_{k+1}(D)$. Thus $D - T$ has order at most k, and is an element of $\mathcal{D}$. The result now follows from the inductive hypothesis. $\square$

Corollary 2.2. *Let the notation and assumptions be as in Lemma 2.1. Assume that* U *is convex. Let* $\chi : \mathcal{B} \to \mathbb{C}$ *be a homomorphism and set*

$$\mathcal{W}_\chi = \{f \in C^\infty(U) | u \cdot f = \chi(u) f \quad \textit{for all} \quad u \in \mathcal{B}\}.$$

Then $\dim \mathcal{W}_\chi \leqq d$.

Proof. Let $X_1, \ldots, X_n$ be a basis for V, and write $\partial_i = \partial(X_i)$. By Lemma 2.1 there are functions $f_{ijk} \in \mathcal{R}$ and operators $u_{k\ell} \in \mathcal{B}$ such that

$$\partial_i E_j = \sum_{k,\ell} f_{ijk} E_\ell u_{k\ell}.$$

Thus if $f \in \mathcal{W}_\chi$, then

$$\partial_i E_j f = \sum_\ell \Gamma^i_{j\ell} E_\ell f, \tag{2}$$

where we have set

$$\Gamma^i_{j\ell} = \sum_k f_{ijk} \chi(u_{k\ell}).$$

We denote by Γ^i the $d \times d$ matrix whose j, ℓ entry is the function $\Gamma^i_{j\ell} \in \mathcal{R}$, and we let F be the column vector whose entries are the functions $E_\ell f$. Then the system (2) can be written as

$$\partial_i F = \Gamma^i F, \qquad i = 1, \ldots, n. \tag{3}$$

Let $\mathcal{E}$ be the space of all solutions F to the linear system (3). We have just seen that there is a map $T: \mathcal{W}_\chi \to \mathcal{E}$ given by $f \mapsto {}^t[E_1 f, \ldots, E_d f]$. Since $E_1 f = f$, this map is injective. The dimension estimate in the Corollary now follows from

Fix $x^0 \in U$. Then the map $\mathcal{E} \to \mathbb{C}^d$ given by $F \mapsto F(x^0)$ is injective. (4)

This is a standard result, but we give the proof for the sake of completeness. Given $x^1 \in U$, we set $X = x^0 - x^1$. Let X have components v_i relative to the basis $\{X_i\}$. Then by (3) one has

$$\partial(X)F = \sum_i v_i \Gamma^i F \tag{5}$$

for all $F \in \mathscr{E}$. Thus if we define $\Phi(t) = F(x^0 + tX)$, $A(t) = \sum_i v_i \Gamma^i(x^0 + tX)$, then by the convexity of U, $\Phi(t)$ and $A(t)$ are smooth functions of t on some interval $-\varepsilon < t < 1 + \varepsilon$, and by (5) we have $\Phi'(t) = A(t)\Phi(t)$. If we assume that $F(x^0) = 0$, then $\Phi(0) = 0$. It follows from the differential equation that $\Phi(t) = 0$ for $0 \leqq t \leqq 1$. This proves (4). $\square$

Note. It is clear that the dimension estimate in the corollary is true if U is only assumed to be connected.

2.2. Connections

We now recall some well-known facts about connections in several complex variables that will be used later in the paper. Let $U \subset \mathbb{C}^n$ be open and (for simplicity) real convex. Let $\Gamma^i : U \to \mathrm{End}(\mathbb{C}^d)$ be holomorphic for $i = 1, \ldots, n$. We define a connection ∇ on the trivial vector bundle $U \times \mathbb{C}^d$ over U by setting

$$\nabla_v F = \sum_{i=1}^{n} v_i(\partial_i F - \Gamma^i F),$$

for $v = \sum v_i \partial_i$ a holomorphic vector field on U and $F : U \to \mathbb{C}^d$ a holomorphic section. Here $\partial_i = \partial/\partial z_i$.

One says that ∇ is *integrable* (or *flat*) on U if for all holomorphic vector fields X, Y and all holomorphic sections F on U, one has $\nabla_X \nabla_Y F - \nabla_Y \nabla_X F - \nabla_{[X,Y]} F = 0$ on U. Let $\mathscr{E}$ be the space of *covariant constant* sections of the connection, i.e. the space of all holomorphic sections F such that $\nabla_X F = 0$ on U for all holomorphic vector fields X on U. The following result is a restatement of the Frobenius theorem on total differential systems in this situation:

Lemma 2.3. *The following properties of the connection ∇ are equivalent:*

(1) *For every $z \in U$, the map $\mathscr{E} \to \mathbb{C}^d$ given by $F \mapsto F(z)$ is surjective.*
(2) *For some $z^0 \in U$, the map $\mathscr{E} \to \mathbb{C}^d$ given by $F \mapsto F(z^0)$ is surjective.*
(3) *The connection ∇ is integrable.*

3. Joint Eigenfunctions for the Quantized Systems

3.1. Laplacians

We resume our study of the Laplace operator for the class of solvable Lie algebras introduced in Sect. 1. Given such an algebra $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{u}$, and a choice of inner product, we define a map $\lambda \mapsto \lambda^\flat$ from $\mathfrak{a}^*$ to $\mathfrak{a}$ by

$$\langle \lambda^\flat, H \rangle = \lambda(H) \quad \text{for} \quad H \in \mathfrak{a}.$$

Given $\lambda, \mu \in \mathfrak{a}^*$, define $\langle \lambda, \mu \rangle = \langle \lambda^\flat, \mu^\flat \rangle$. If $\lambda \in \mathfrak{a}^*$, let s_λ be the orthogonal reflection of $\mathfrak{a}$ about the hyperplane $\lambda = 0$. That is,

$$s_\lambda(H) = H - \frac{2\lambda(H)}{\langle \lambda, \lambda \rangle} \cdot \lambda^\flat .$$

Let W be the subgroup of the orthogonal group of $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ generated by the reflections s_λ, for $\lambda \in \Phi$. The action of W on $\mathfrak{a}$ extends naturally to an action on the symmetric tensor algebra $S(\mathfrak{a})$. Since $\mathfrak{a}$ is abelian, we may identify $S(\mathfrak{a})$ with the universal enveloping algebra $U(\mathfrak{a})$. We denote by $U(\mathfrak{a})^W$ the invariants in $U(\mathfrak{a})$ under W.

Let Ω be the *Laplacian* for $\mathfrak{b}$, $\langle \cdot, \cdot \rangle$, as in Sect. 1. Recall that

$$U(\mathfrak{b})^\Omega = \{ T \in U(\mathfrak{b}) | T\Omega = \Omega T \} .$$

The algebra $U(\mathfrak{b})^\Omega$ was extensively studied in [G-W2]. Identify $\mathfrak{a}$ with the abelian Lie algebra $\mathfrak{b}/\mathfrak{u}$. Then the natural homomorphism $\mathfrak{b} \to \mathfrak{b}/\mathfrak{u}$ extends to an algebra homomorphism $\mu: U(\mathfrak{b}) \to U(\mathfrak{a})$. The operators commuting with Ω then have the following symmetry property:

Theorem 3.1 [G-W2, Theorem 2.6]. *One has* $\mu(U(\mathfrak{b})^\Omega) \subset U(\mathfrak{a})^W$.

3.2. Root Systems and Invariants

To obtain further information about $U(\mathfrak{b})^\Omega$, we shall require that Φ be suitably related to a root system. Recall that a subset $\pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{a}^*$ defines a *connected Dynkin diagram* if π is a set of simple positive roots for an irreducible reduced root system $R_\pi \subset \mathfrak{a}^*$ (see [Bo]). This is equivalent to the following conditions:

(D_1) π is a basis of $\mathfrak{a}^*$.

(D_2) The group W_π generated by the reflections s_α, for $\alpha \in \pi$, is finite.

(D_3) The numbers $a_{ij} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$ are non-positive integers, for all $i \neq j$.

(D_4) If $\pi = \pi_1 \cup \pi_2$ with $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \pi_1$ and $\beta \in \pi_2$, then either π_1 or π_2 is empty.

We now assume that $\pi \subset \mathfrak{a}$ is a connected Dynkin diagram, and that Φ is given in terms of π as follows: Either

$(NP)\, \Phi = \pi$ ("generalized non-periodic Toda lattice systems"), or

$(P)\, \Phi = \tilde{\pi}$ ("generalized periodic Toda lattice systems").

In case (P), $\tilde{\pi} = \pi \cup \{\alpha_{\ell+1}\}$ is an *extended* Dynkin diagram, i.e. $\alpha_{\ell+1} = -\beta$, with $\beta \in R_\pi$ and $\langle \beta, \alpha_i \rangle \geqq 0$ for $i = 1, \dots, \ell$. The possible choices for β were determined in [G-W2, Sect. 5]. Note that the group $W = W_\pi$ in this case (see D_2). We thus have the following fundamental result of Chevalley and Harish-Chandra [He, Chap. III, Sect. 3]:

Theorem 3.2. (1) *$U(\mathfrak{a})^W$ is a polynomial ring in ℓ homogeneous generators $u_1, \dots, u_\ell$.*

(2) *Let w be the order of the group W. Then there exist homogeneous elements $1 = e_1, \dots, e_w$ in $U(\mathfrak{a})$ such that $U(\mathfrak{a})$ is free as a $U(\mathfrak{a})^W$ module with basis $e_1, \dots, e_w$. (Note that $U(\mathfrak{a})$ is graded by degree, since $\mathfrak{a}$ is abelian.)*

(3) *The degrees d_i of the elements u_i and the degrees k_j of the elements e_j are uniquely determined by W.*

3.3. Structure of $U(\mathfrak{b})^{\Omega}$

Let Φ be defined as in Sect. 3.2 in terms of a Dynkin diagram π. Choose a unit vector $X_i \in \mathfrak{u}_{\alpha_i}$ for each i. When $\Phi = \tilde{\pi}$, there is a linear relation

$$\sum_{i=1}^{\ell+1} n_i \alpha_i = 0\,, \tag{1}$$

where each n_i is a positive integer and $n_{\ell+1} = 1$. We set $\sum n_i = h$. (When $-\alpha_{\ell+1}$ is the highest root of the root system R_π relative to π, then h is the *Coxeter number* of R_π.) Set

$$\xi = \prod_{i=1}^{\ell+1} X_i^{n_i}.$$

Thus ξ has degree h, and relation (1) implies that ξ is in the center of $U(\mathfrak{b})$. (In fact, it is easy to verify that ξ generates the center.) In particular, $\xi \in U(\mathfrak{b})^{\Omega}$. Note also that $\mu(\xi) = 0$.

Let $U_j(\mathfrak{b})$ be the standard filtration on $U(\mathfrak{b})$, generated by $\mathfrak{b}$. Let $U_+(\mathfrak{b})$ denote the ideal of elements with zero constant term.

Theorem 3.3. *Assume that Φ is one of the following:*

Case 1: *$\Phi = \pi$, an arbitrary connected Dynkin diagram*

Case 2: *$\Phi = \tilde{\pi}$, where π is of type A, B, C, D or E_6 and $-\alpha_{\ell+1}$ is the largest positive root of R_π*

Case 3: *$\Phi = \tilde{\pi}$, where π is of type B_ℓ or C_ℓ and $-\alpha_{\ell+1}$ is the short dominant root.*

Fix homogeneous generators $u_1, \ldots, u_\ell$ of $U(\mathfrak{a})^W$, and in Case 1 set $\xi = 0$, $X_{\ell+1} = 0$. Then the following holds:

(i) *There exist elements $\Omega_1, \ldots, \Omega_\ell$ in $U_+(\mathfrak{b})^{\Omega}$ such that*

(a) *$\mu(\Omega_i) = u_i$ and $\deg \Omega_i = \deg u_i$ relative to the standard filtration on $U(\mathfrak{b})$, for $i = 1, \ldots, \ell$;*

(b) *Ω_i is in the subalgebra of $U(\mathfrak{b})$ generated by $\mathfrak{a}$ and $X_1^2, \ldots, X_\ell^2, X_{\ell+1}^2$;*

(c) *$U(\mathfrak{b})^{\Omega}$ is generated as an algebra by $\Omega_1, \ldots, \Omega_\ell$ and ξ.*

(ii) *The elements $\Omega_1, \ldots, \Omega_\ell$ mutually commute and are algebraically independent.*

Proof. In Case 1, this follows from [G-W2, Theorems 2.5 and 4.1]. In Cases 2 and 3, let Ω_j be the element $w'(u_j)$ constructed inductively in Sect. 3 of [G-W2] for $j = 1, \ldots, \ell$. These operators are alternating sums of elements for which conditions ($\mathrm{i_a}$) and ($\mathrm{i_b}$) hold, by [G-W2, Sect. 2, Formula (2.7)]. Property ($\mathrm{i_c}$) now follows from [G-W2, Theorem 5.2] in both Cases 2 and 3. By [loc. cit.], property (ii) is known to hold for Case 2 and the diagram of type C_ℓ in Case 3.

It remains to prove that the operators Ω_j mutually commute in the case of the extended diagram B_ℓ with $-\alpha_{\ell+1}$ the short dominant root. To establish this algebraic property, we use the following analytic argument: Consider the representation T_c of Sect. 1.1 (c non-degenerate). It is easily verified that the kernel of this representation of $U(\mathfrak{b})$ is the ideal generated by the central element $\xi - \gamma(c)$, where $\gamma(c)$ is the scalar $T_c(\xi)$. To prove that the operators Ω_j mutually commute, it will suffice by Proposition 1.5 to show that the Lie algebra $\mathfrak{h}$ generated by $T_c(\Omega_j)$, $j = 1, \ldots, \ell$, is solvable.

Letting $x_1, \dots, x_\ell$ be orthogonal coordinates on $\mathfrak{a}$, we take the elementary symmetric functions of $x_1^2, \dots, x_\ell^2$ as the basic Weyl group invariants $u_1, \dots, u_\ell$. (W consists of permutations and sign changes of the coordinates in this case.) Their degrees are thus $2, 4, \dots, 2\ell$, and hence $\Omega_i \in U^{2i}(\mathfrak{b})$ for $i = 1, 2, \dots, \ell$. (Here $U^k(\mathfrak{b})$ denotes the elements of degree $\leqq k$, relative to the standard filtration on the enveloping algebra.)

Take $A, B \in U(\mathfrak{b})^\Omega$ of degrees $2j$ and $2k$, respectively. By [G-W2, Lemma 3.6],

$$[A, B] = \xi^2 C(A, B), \tag{2}$$

where $C(A, B) \in U^{2j+2k-2\ell-2}(\mathfrak{b})$ and commutes with Ω. In particular, if $\max\{j, k\} \leqq 2\ell$, then $C(A, B)$ has degree at most $2\ell - 2$. Now $T_c(\xi) = \gamma(c) I$. Thus

$$[T_c(A), T_c(B)] = \gamma(c)^2 T_c(C(A, B)). \tag{3}$$

We conclude from (2) and (3) that $\mathfrak{h} = T_c(U^{2\ell}(\mathfrak{b})^\Omega)$ is a Lie algebra, and if $Z \in [\mathfrak{h}, \mathfrak{h}]$, then the order of Z (as a differential operator) is at most $2\ell - 2$. Let $\mathfrak{h}_0 = \mathfrak{h}$ and $\mathfrak{h}_{i+1} = [\mathfrak{h}, \mathfrak{h}_i]$. Given $Y \in \mathfrak{h}_i$, one verifies inductively from (2) that $\operatorname{ord} Y \leqq 2\ell - 2i$. Thus $\mathfrak{h}_{\ell+1} = 0$. This implies that $\mathfrak{h}$ is nilpotent. □

Remark. The proofs of the cited results in [G-W2] exploit the empirical fact that the degrees of the basic invariants are at most twice the rank for the classical root systems and E_6. It is possible that the theorem is true for all the exceptional extended diagrams.

3.4. Representations of $\mathfrak{b}$

Throughout the remainder of this section we shall assume that Φ satisfies Cases 1, 2, or 3 of Theorem 3.3. Let $d = \dim \mathfrak{u} = \operatorname{Card} \Phi$. Given $c \in \mathfrak{u}^*$, set $c_i = c(X_i)$, where $X_i \in \mathfrak{u}_{\alpha_i}$ as in Sect. 3.3. Identify c with the point $(c_1, \dots, c_d) \in \mathbb{C}^d$. Let T_c be the representation of $U(\mathfrak{b})$ on $C^\infty(\mathfrak{a})$ as in Sect. 1.1. When $\Phi = \tilde{\pi}$, then the central element ξ acts by

$$T_c(\xi) = \prod_{j=1}^{d} (ic_j)^{n_j} I. \tag{1}$$

We now relate this representation of $U(\mathfrak{b})$ to the rings of differential operators in Sect. 2.1, taking $\mathfrak{a}$ as the vector space V. We may identify $U(\mathfrak{a})$ with $\mathscr{P}$. Let $\mathscr{R}$ be the algebra of C^∞ functions on $\mathfrak{a}$ generated over $\mathbb{C}$ by 1 and $e^{-\alpha_1}, \dots, e^{-\alpha_d}$, and let $\mathscr{Q}$ be the algebra of differential operators on $\mathfrak{a}$ with coefficients in $\mathscr{R}$. Clearly $T_c(U(\mathfrak{b})) \subset \mathscr{Q}$, with equality if all $c_j \neq 0$. We set $\mathscr{B} = T_c(U(\mathfrak{b})^\Omega)$ and $\mathscr{A} = U(\mathfrak{a})^W$. We claim that conditions (DO_1) and (DO_2) in Sect. 2.1 are satisfied in this case.

Indeed, by Eq. (1) and Theorem 3.3 $(\mathrm{i_c})$, we see that

$$\mathscr{B} = T_c(\mathscr{J}), \tag{2}$$

where $\mathscr{J}$ is the subalgebra of $U(\mathfrak{b})$ generated over $\mathbb{C}$ by $\Omega_1, \dots, \Omega_\ell$. By $(\mathrm{i_a})$ of Theorem 3.3, we have

$$\Omega_j - u_j \in \mathfrak{u}^2 \cdot U_{d_j - 2}(\mathfrak{b}), \tag{3}$$

where $d_j = \deg u_j$. From (2) and (3) it follows that $T_c(\Omega_j)$ is a differential operator of order d_j, and

$$\sigma_{d_j}(\Omega_j) = u_j. \tag{4}$$

Thus conditions (DO_1) and (DO_2) follow from (2), (4), and Theorem 3.2. Furthermore, in (DO_2) we may take $d=w$, the order of W.

Corollary 2.2 combined with these observations implies the following dimension estimate:

Theorem 3.4. *Assume that Φ is of type Case 1, 2 or 3 in Theorem 3.3. Let $\mathscr{J}$ be the subalgebra of $U(\mathfrak{b})$ generated over $\mathbb{C}$ by $\Omega_1, \ldots, \Omega_\ell$, and let $\chi: \mathscr{J} \to \mathbb{C}$ be a homomorphism. Let $U \subset \mathfrak{a}$ be an open convex subset. Put*

$$\mathscr{W}_{c,\chi}(U) = \{f \in C^\infty(U) | T_c(\omega) f = \chi(\omega) f \quad for \quad \omega \in \mathscr{J}\}. \tag{5}$$

Then $\dim \mathscr{W}_{c,\chi}(U) \leqq w$.

3.5. Factorization of $U(\mathfrak{b})$

We next show that the inequality in the dimension estimate of Theorem 3.4 can be replaced by equality. For this, we shall continue to assume Φ is of type Case 1, 2, or 3 in Theorem 3.3. In these cases one knows that the algebra $\mathscr{J}$ in Sect. 3.4 is commutative and algebraically isomorphic to $U(\mathfrak{a})^W$ under the homomorphism μ. Let $1 = e_1, \ldots, e_w$ be the elements in Theorem 3.2, indexed by increasing degree, and set

$$\mathscr{H} = \bigoplus_{i=1}^{w} \mathbb{C} e_i$$

[the space of W-harmonics in $U(\mathfrak{a})$]. We denote by $U_j(\mathfrak{b})$ the standard filtration on $U(\mathfrak{b})$. If $L \subset U(\mathfrak{b})$ is a linear subspace, we set $L_j = L \cap U_j(\mathfrak{b})$.

Lemma 3.5. *The map $U(\mathfrak{u}) \otimes \mathscr{H} \otimes \mathscr{J} \to U(\mathfrak{b})$ given by $z \otimes e \otimes \omega \mapsto ze\omega$ is a linear isomorphism. Furthermore, for every $j \geqq 0$,*

$$U_j(\mathfrak{b}) = \sum_{r+s+t=j} U_r(\mathfrak{u}) \cdot \mathscr{H}_s \cdot \mathscr{J}_t. \tag{1}$$

Proof. From Theorem 3.2 and Theorem 3.3 it is clear that the spaces $U_j(\mathfrak{b})$ and $\sum U_r(\mathfrak{u}) \otimes \mathscr{H}_s \otimes \mathscr{J}_t$ (sum over $r+s+t=j$) have the same dimension. So it is enough to prove Eq. (1). It is trivial for $j=0$. We assume the result is true for $j \leqq k-1$, and look at the case $j=k$. By Theorem 3.2 it suffices to consider elements $x \in U(\mathfrak{b})$ of the form zeu, where $z \in U(\mathfrak{u})$, $e \in \mathscr{H}$, $u \in U(\mathfrak{a})^W$, and the sum of the degrees of z, e, and u is at most k. By Theorem 3.3, there exists an element $\omega \in \mathscr{J}$ such that $\deg \omega = \deg u$ and $\omega - u \in \mathfrak{u}^2 \cdot U(\mathfrak{b})$. It follows that $x - ze\omega \in \mathfrak{u}^2 \cdot U_{k-2}(\mathfrak{b})$. Now apply the induction hypothesis. □

Let $c \in \mathbb{C}^d$. Recall that in the representation T_c of $\mathfrak{b}$ in Sect. 3.4, an element $z \in U(\mathfrak{u})$ acts by multiplication by the function $\tau_c(z) = T_c(z) \cdot 1 \in \mathscr{R}$. This defines a homomorphism $\tau_c: U(\mathfrak{u}) \to \mathscr{R}$. Let $\mathscr{Z}$ be the vector space of all holomorphic maps from $\mathfrak{a}_C$ to $\mathscr{H}$ whose coefficients are in the algebra $\mathscr{R}$.

Corollary 3.6. *Let $\chi: \mathscr{J} \to \mathbb{C}$ be a homomorphism. There is a unique linear map $S_{c,\chi}: U(\mathfrak{b}) \to \mathscr{Z}$ such that*

$$S_{c,\chi}(ze\omega) = \chi(\omega) \tau_c(z) e$$

for $z \in U(\mathfrak{u})$, $e \in \mathscr{H}$, and $\omega \in \mathscr{J}$.

Proof. There is a linear map $U(\mathfrak{u})\otimes\mathscr{H}\otimes\mathscr{J}\to\mathscr{Z}$ which carries $z\otimes e\otimes\omega$ to $\chi(\omega)\tau_c(z)e$, for $z\in U(\mathfrak{u})$, $e\in\mathscr{H}$, and $\omega\in\mathscr{J}$. Using the isomorphism of Lemma 3.5, we obtain the map $S_{c,\chi}$. □

From the proof of Lemma 3.5 we can extract some explicit information about the factorization (1) when applied to elements of $U(\mathfrak{a})$. Denote by $U_+(\mathfrak{a})$ the elements with zero constant term in $U(\mathfrak{a})$, and by $U_+(\mathfrak{a})^W$ the subring of W-invariants. Since there are no linear invariants, we note that if $0\neq u\in U_+(\mathfrak{a})^W$, then $\deg u\geqq 2$. Let $P:U(\mathfrak{a})\to\mathscr{H}$ be the projection onto the harmonics along the ideal in $U(\mathfrak{a})$ generated by $U_+(\mathfrak{a})^W$. Let $H_1,\ldots,H_\ell$ be the basis for $\mathfrak{a}$ dual to π. Let a basis $e_1,\ldots,e_w$ for $\mathscr{H}$ be fixed as above, and define the integers m_r by the property that $\deg e_k\leqq r$ if and only if $k\leqq m_r$. We know from Theorem 3.2 that there are unique linear maps $I_{ip}:U(\mathfrak{a})\to U_+(\mathfrak{a})^W$ such that if ϕ is homogeneous of degree r, then

$$H_i\phi=P(H_i\phi)+\sum_{p=1}^{m_r-1}e_pI_{ip}(\phi)\,. \tag{2}$$

By Theorem 3.3 we can define unique linear maps $\Omega_{ip}:U(\mathfrak{a})\to\mathscr{J}_+$ such that

$$\Omega_{ip}(\phi)=I_{ip}(\phi)\bmod \mathfrak{u}^2U(\mathfrak{b})\,, \tag{3}$$

and $\deg\Omega_{ip}(\phi)=deg\,I_{ip}(\phi)$ for any $\phi\in U(\mathfrak{a})$. Combining (2) and (3), we obtain the identity

$$H_i\phi=P(H_i\phi)+\sum_{p=1}^{m_r-1}e_p\Omega_{ip}(\phi)\bmod \mathfrak{u}^2U(\mathfrak{b})\,, \tag{4}$$

for all $\phi\in U(\mathfrak{a})$ which are homogeneous of degree r.

Lemma 3.7. *Suppose $\omega\in\mathscr{J}$. Assume that $\omega_1,\omega_2,\ldots$ are elements in $\mathscr{J}$ such that*

$$\mu(\omega)=\sum_k e_k\omega_k\bmod \mathfrak{u}U(\mathfrak{b})\,. \tag{5}$$

Then $\omega_1=\omega$ and $\omega_k=0$ for $k>1$.

Proof. Apply the homomorphism μ to Eq. (5). By Theorem 3.1 we have $\mu(\omega_k)\in U(\mathfrak{a})^W$, so by Theorem 3.2(2), $\mu(\omega_1)=\mu(\omega)$ and $\mu(\omega_k)=0$ for $k>1$. Now apply Theorem 3.3. □

3.6. Integrability of the Connection

Continuing with the notation and assumptions of the previous section, we now choose a basis $1=z_1,z_2,\ldots$ for $U(\mathfrak{u})$ such that $[H,z_p]=\lambda_p(H)z_p$, for $H\in\mathfrak{a}$, where $\lambda_p\in\mathfrak{a}^*$. By Lemma 3.5, there are unique elements $u_{ijpq}\in\mathscr{J}$ such that

$$H_ie_j=\sum_{p,q}z_pe_qu_{ijpq} \tag{1}$$

for $1\leqq i\leqq\ell$ and $1\leqq j\leqq w$. Fixing a homomorphism $\chi:\mathscr{J}\to\mathbb{C}$ and $c\in\mathbb{C}^{\ell+1}$, we define functions Γ^i_{jq} on $\mathfrak{a}$ by

$$\Gamma^i{}_{jq}=\sum_p\tau_c(z_p)\chi(u_{ijpq})\,. \tag{2}$$

[Recall from Sect. 3.5 that $\tau_c(z)=T_c(z)\cdot 1$.] These functions are in the algebra $\mathscr{R}$ generated by the exponentials $\{e^\alpha|\alpha\in\Phi\}$, and thus extend holomorphically to $\mathfrak{a}_{\mathbb{C}}$. Let $\Gamma^i:\mathfrak{a}_{\mathbb{C}}\to\operatorname{End}\mathscr{H}$ be defined by

$$\Gamma^i(x)e_j=\sum_{q=1}^{w}\Gamma^i{}_{jq}(x)e_q\,, \tag{3}$$

for $x\in\mathfrak{a}_{\mathbb{C}}$. Define a connection ∇ on the trivial vector bundle $\mathfrak{a}_{\mathbb{C}}\times\mathscr{H}$ over $\mathfrak{a}_{\mathbb{C}}$ by

$$\nabla_{H_i}=\partial(H_i)-\Gamma^i\,,\quad\text{for}\quad 1\leqq i\leqq\ell\,. \tag{4}$$

Lemma 3.8. *The connection ∇ is integrable.*

Proof. Using the map $S_{c,\chi}$ from Corollary 3.6, we see that $\Gamma^i(x)$ is simply the linear map

$$e\mapsto S_{c,\chi}(H_ie)(x)$$

on $\mathscr{H}$. By a direct calculation using (1) and (2) one finds that

$$S_{c,\chi}(H_iH_je)=(\partial(H_i)\Gamma^j)e+\Gamma^j\Gamma^ie\,, \tag{5}$$

for $e\in\mathscr{H}$ and $1\leqq i,j\leqq\ell$. The left side of (5) is symmetric in i and j, so (5) implies that

$$\partial(H_i)\Gamma^j+\Gamma^j\Gamma^i=\partial(H_j)\Gamma^i+\Gamma^i\Gamma^j\,. \tag{6}$$

It follows immediately from (6) that $[\nabla_i,\nabla_j]=0$, which proves that ∇ is integrable. $\square$

3.7. Covariant Constant Sections

Let ∇ be the connection defined in Sect. 3.6, and suppose F is a C^∞ map from $\mathfrak{a}$ to $\mathscr{H}^*$ that is *covariant constant* relative to the dual connection:

$$\partial(H_j)F={}^t\Gamma^jF\,,\qquad j=1,\dots,\ell\,. \tag{1}$$

Define the functions $f_j(x)=\langle F(x),e_j\rangle$, $x\in\mathfrak{a}$, using the basis $e_1,\dots,e_w$ of $\mathscr{H}$. (Recall that $e_1=1$.) In terms of the components f_j, the system (1) becomes

$$\partial(H_j)f_k=\sum_{q=1}^{w}\Gamma^j{}_{kq}f_q \tag{1'}$$

for $1\leqq j\leqq\ell$ and $1\leqq k\leqq w$. The following result is the key step for determining the dimension of the space of joint eigenfunctions for the operators $T_c(\Omega_j)$:

Lemma 3.9. *If F satisfies* (1) *then*

$$f_j=\partial(e_j)f_1 \tag{2}$$

for $j=1,\dots,w$, and

$$T_c(\omega)f_1=\chi(\omega)f_1 \tag{3}$$

for all $\omega\in\mathscr{J}$.

Proof. We shall prove (2) and (3) by induction on the degrees of e_j and ω. They are trivially true if both degrees are 0, and we assume inductively that (2) and (3) are true whenever the degrees of e_j and ω are at most n.

We shall first prove that (3) is true for any $\omega \in \mathscr{J}$ of degree $n+1$. Set $u=\mu(\omega)$. We may assume by the induction hypothesis that u is homogeneous of degree $n+1$. There exist $\phi_j \in U(\mathfrak{a})$, homogeneous of degree n, such that

$$u = \sum_{j=1}^{\ell} H_j \phi_j . \tag{4}$$

Applying the factorization in Lemma 3.5 to ϕ_j, we obtain elements $\omega_{jpq} \in \mathscr{J}$ of degree at most n, such that

$$u = \sum_{j=1}^{\ell} \sum_{q=1}^{m_n} \sum_{p} H_j z_p e_q \omega_{jpq} . \tag{5}$$

Define

$$S = \sum_{j=1}^{\ell} \sum_{q=1}^{m_n} \sum_{p} z_p H_j e_q \omega_{jpq} . \tag{6}$$

Then $S = u \bmod \mathfrak{u} \cdot U(\mathfrak{b})$. Factoring $H_j e_q$ according to Sect. 3.6(1), we can write S as

$$S = \sum_{j=1}^{\ell} \sum_{q=1}^{m_n} \sum_{p,r} \sum_{k=1}^{w} z_p z_r e_k u_{jqrk} \omega_{jpq} . \tag{7}$$

Although this formula for S looks quite unpromising, it actually has the original operator ω hidden in it. To make this explicit, we recall that $z_1 = 1$, so if we isolate the terms with $p=r=1$ in (7), we see that

$$u = \sum_{j=1}^{\ell} \sum_{q} \sum_{k=1}^{w} e_k u_{jq1k} \omega_{j1q} , \tag{8}$$

$\bmod \mathfrak{u} \cdot U(\mathfrak{b})$. Hence by Lemma 3.7,

$$\omega = \sum_{j=1}^{\ell} \sum_{q} u_{jq11} \omega_{j1q} \tag{9}$$

and

$$0 = \sum_{j=1}^{\ell} \sum_{q} u_{jq1k} \omega_{j1q} \tag{10}$$

for $k>1$. Thus the formula for S becomes

$$S = \omega + \sum_{j=1}^{\ell} \sum_{p+r>1} \sum_{q} \sum_{k=2}^{w} z_p z_r e_k u_{jqrk} \omega_{jpq} . \tag{11}$$

Observe that ω_{jpq} and u_{jqrk} each have degrees at most n, and if $u_{jqrk} \neq 0$ with $p+r>1$, then $\deg e_k \leqq n$. Thus we may use the induction hypothesis to calculate

$$T_c(S) f_1 = T_c(\omega) f_1 + \sum_{j=1}^{\ell} \sum_{p+r>1} \sum_{q} \sum_{k=2}^{w} T_c(z_p z_r) \chi(u_{jqrk} \omega_{jpq}) f_k . \tag{12}$$

By (10), the right side of (12) is unchanged if we include the terms with $p=r=1$. Thus by (9) the terms with $k=1$ that are omitted on the right side of (12) sum to $\chi(\omega) f_1$. By adding and subtracting these terms, we may write (12) as

$$T_c(S) f_1 = T_c(\omega) f_1 - \chi(\omega) f_1 + \sum_{i,p,q} \sum_{k=1}^{w} T_c(z_p) \chi(\omega_{jpq}) \left\{ \sum_{r} T_c(z_r) \chi(u_{jqrk}) \right\} f_k . \tag{13}$$

We may now simplify (13) by first noting that the summation over r in the braces gives $\Gamma^j{}_{qk}$. By Eq. (1)′, the summation over r and k in (13) thus gives $\partial(H_j)f_q$. But $\deg e_q \leqq n$ for all q such that $\omega_{jpq} \neq 0$. Hence by the induction hypothesis $\partial(H_j)f_q = \partial(H_j e_q)f_1$ for q in this range. With these observations, we may write (13) as

$$T_c(S)f_1 = T_c(\omega)f_1 - \chi(\omega)f_1 + \sum_{j=1}^{\ell} \sum_{p,q} T_c(z_p)\chi(\omega_{jpq})\partial(H_j e_q)f_1 . \tag{14}$$

Again by the induction hypothesis we can write

$$\chi(\omega_{jpq})\partial(H_j e_q)f_1 = \partial(H_j e_q)T_c(\omega_{jpq})f_1 = T_c(H_j e_q \omega_{jpq})f_1 .$$

Substituting this into (14), we finally get

$$T_c(S)f_1 = T_c(\omega)f_1 - \chi(\omega)f_1 + T_c(S)f_1 . \tag{15}$$

Hence $T_c(\omega)f_1 = \chi(\omega)f_1$, completing the induction step for Eq. (3).

With (3) now established for all ω of degree $\leqq n+1$, we turn to (2). Let $\deg e_p = n$. Taking $\phi = e_p$ in Sect. 3.5(4) and using the definition of the connection coefficients $\Gamma^j{}_{pq}$, we see that

$$\Gamma^j{}_{pq} = 0 \quad \text{if} \quad \deg e_q > n+1 \quad \text{or if} \quad \deg e_q = n , \tag{16}$$

$$\Gamma^j{}_{pq} \in \mathbb{R} \quad \text{if} \quad \deg e_q = n+1 , \tag{17}$$

and

$$P(H_j e_p) = \sum_{q > m_n} \Gamma^j{}_{pq} e_q . \tag{18}$$

Thus from (18) we can write formula Sect. 3.6(1) as

$$H_j e_p = P(H_j e_p) + \sum_{q < m_n} \sum_r z_r e_q u_{jprq} . \tag{19}$$

Since $\deg e_q < n$ and $\deg u_{jprq} \leqq n+1$ on the right side of (19), we may use the induction hypothesis as extended in the first part of this proof to calculate

$$\partial(H_j e_p)f_1 = \partial(P(H_j e_p))f_1 + \sum_{q < m_n} \sum_r T_c(z_r)\chi(u_{jprq})f_q = \partial(P(H_j e_p))f_1 + \sum_{q < m_n} \Gamma^j{}_{pq} f_q .$$

On the other hand, we can use Eq. (1)′ and the induction hypothesis to write

$$\partial(H_j e_p)f_1 = \sum_q \Gamma^j{}_{pq} f_q .$$

Comparing these two formulas, we find that

$$\partial(P(H_j e_p))f_1 = \sum_{q > m_n} \Gamma^j{}_{pq} f_q . \tag{20}$$

The final step in the induction is now an easy consequence of (20). Indeed, it is a standard property of harmonic polynomials that $\mathscr{H}^{n+1} = P(\mathfrak{a} \cdot \mathscr{H}^n)$. Given $e_k \in \mathscr{H}^{n+1}$, we can thus find coefficients $a_{kjp} \in \mathbb{R}$ such that

$$e_k = \sum_{j,p} a_{kjp} P(H_j e_p) .$$

From (17) and (18) it then follows that

$$\delta_{kq} = \sum_{q > m_n} \sum_{j=1}^{\ell} a_{jkp} \Gamma^j{}_{pq} \, .$$

Using this in (20), we obtain $\partial(e_k) f_1 = f_k$, completing the induction. □

Combining Lemmas 2.3, 3.8, and 3.9, we obtain our main result (recall that we are assuming that Φ satisfies Cases 1, 2, or 3 of Theorem 3.3):

Theorem 3.10. *Let $\mathscr{W}_{c,\chi}$ denote the space of all holomorphic functions f on $\mathfrak{a}_{\mathbb{C}}$ such that*

$$T_c(\omega) f = \chi(\omega) f \quad \text{for} \quad \omega \in \mathscr{J} \, . \tag{21}$$

Let $\mathscr{E}_{c,\chi}$ denote the space of all holomorphic maps F from $\mathfrak{a}_{\mathbb{C}}$ to $\mathscr{H}^*_{\mathbb{C}}$ such that

$$\partial(H_j) F = {}^t\Gamma^j F \quad \text{for} \quad j = 1, \ldots, \ell \, . \tag{22}$$

Then

(i) $\dim \mathscr{W}_{c,\chi} = w$.

(ii) *The map $F \mapsto \langle F, e_1 \rangle$ defines a linear isomorphism from $\mathscr{E}_{c,\chi}$ onto $\mathscr{W}_{c,\chi}$.*

4. The Monodromy of the Systems

4.1. The Space $\mathscr{W}_{c,\nu}$

We retain the notation and hypotheses of Sect. 3, and assume that the set Φ is as in Cases 1, 2, or 3 of Theorem 3.3. Thus we know that $U(\mathfrak{b})^{\Omega}$ is abelian and is the polynomial algebra in $\xi, \Omega_1, \ldots, \Omega_\ell$, where the top-order symbols $\mu(\Omega_1), \ldots, \mu(\Omega_\ell)$ are basic homogeneous invariants for W in $U(\mathfrak{a})$ and ξ generates the center of $U(\mathfrak{b})$.

Recall that we are denoting by $\mathscr{J}$ the algebra generated by $\Omega_1, \ldots, \Omega_\ell$. Let $\chi : \mathscr{J} \to \mathbb{C}$ be a homomorphism. Then χ can be parametrized by $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ via

$$\chi(\omega) = \mu(\omega)(\nu) \, , \qquad \omega \in \mathscr{J} \, . \tag{1}$$

(Here we view $\mu(\omega)$ as a polynomial function on $\mathfrak{a}^*$ as usual.) We write $\chi = \chi_\nu$; under the assumptions above, we then have $\chi_\nu = \chi_\mu$ if and only if $\mu = s\nu$ for some element $s \in W$. For $c \in \mathbb{C}^d$, let the representation T_c of $\mathfrak{b}$ be as in Sect. 3.4. In this section we shall study in greater detail the $|W|$-dimensional space $\mathscr{W}_{c,\chi}$ of solutions to the system

$$T_c(\Omega_j) f = \chi(\Omega_j) f, \quad \text{for} \quad j = 1, \ldots, \ell \tag{2}$$

(cf. Theorem 3.10). Taking $\chi = \chi_\nu$, we denote this space as $\mathscr{W}_{c,\nu}$.

Since the functions in $\mathscr{W}_{c,\nu}$ are holomorphic on $\mathfrak{a}_{\mathbb{C}}$, we can exploit the periodicity of the coefficients of the operators $T_c(\Omega_i)$ under complex translations, as follows: Recall that $H_j \in \mathfrak{a}$ is defined by $\alpha_i(H_j) = \delta_{ij}$, for $i = 1, \ldots, \ell$. Define the translation operator T_j on $C^\infty(\mathfrak{a}_{\mathbb{C}})$ by $(T_j f)(y) = f(y + i\pi H_j)$ for $y \in \mathfrak{a}_{\mathbb{C}}$ (where $i = (-1)^{1/2}$). The operator T_j obviously commutes with the operators $T_c(H)$, $H \in \mathfrak{a}$, and with the operators $T_c(X_k^2)$, $1 \leqq k \leqq d$. Hence by Theorem 3.3(b) it follows that T_j commutes with $T_c(\omega)$ for all $\omega \in U(\mathfrak{b})^{\Omega}$. Thus

$$T_j : \mathscr{W}_{c,\nu} \to \mathscr{W}_{c,\nu} \tag{3}$$

for $j = 1, \ldots, \ell$. Obviously $T_i T_j = T_j T_i$.

Let $f \in \mathscr{W}_{c,v}$ be a joint eigenfunction of the operators T_i, $i=1,\dots,\ell$. (By the finite-dimensionality of $\mathscr{W}_{c,v}$, there always exists at least one such eigenfunction.) Thus

$$T_i f = \mu_i f \quad \text{for} \quad i=1,\dots,\ell,$$

where $\mu_i \in \mathbb{C}$. Choose $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, so that

$$e^{i\pi\Lambda(H_j)} = \mu_j, \quad \text{for} \quad j=1,\dots,\ell. \tag{4}$$

Of course, Λ is not uniquely defined by (4), but the *coset* $\Lambda + 2L$ is well-defined as an element of $\mathfrak{a}_{\mathbb{C}}^*/2L$, where

$$L = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$$

is the root lattice. If we now set $\phi(h) = e^{-\Lambda(h)} f(h)$, for $h \in \mathfrak{a}_{\mathbb{C}}^*$, then ϕ is a holomorphic function which satisfies the periodicity conditions

$$\phi(h + i\pi H_j) = \phi(h), \quad \text{for} \quad j=1,\dots,\ell. \tag{5}$$

Thus ϕ has a Fourier series (multiple Laurent series) expansion

$$\phi(h) = \sum_{\mu \in L} a_\mu e^{2\mu(h)},$$

which converges uniformly and absolutely on all strips

$$\{h \in \mathfrak{a}_{\mathbb{C}} \mid |\mathrm{Re}\,\alpha_i(h)| \leqq C \quad \text{for} \quad i=1,\dots,\ell\}.$$

For the eigenfunction f, we thus have an expansion

$$f = e^{\Lambda} \sum_{\mu \in L} a_\mu e^{2\mu}, \tag{6}$$

with the same convergence properties.

Definition. Let $M(c,v) \subset \mathfrak{a}_{\mathbb{C}}^*/2L$ consist of all cosets $\Lambda + 2L$ corresponding to joint eigenvalues of the operators $T_1, \dots, T_\ell$ as in (4). Call $M(c,v)$ the (semi-simple part of the) *monodromy* of the system (2).

From the results just cited, we see that $M(c,v)$ is non-empty, and has at most $|W|$ elements. The rest of this paper will be largely devoted to the study of the monodromy.

Example. Take $c=0$. Then the representation T_0 is trivial on the subalgebra $\mathfrak{u}$, and is just the regular representation on $\mathfrak{a}$. The operators $T_0(\Omega_i) = \partial(u_i)$ for $i=1,\dots,\ell$ in this case. In the notation of [He, Chap. III, Sect. 3.4], the space $\mathscr{W}_{0,v} = \mathscr{E}_v(\mathfrak{a})$. Let W_v be the subgroup of W that fixes v, and let $\mathscr{H}^v$ be the space of W_v harmonic polynomials. Then by [loc. cit., Theorem 3.13] we have

$$\mathscr{W}_{0,v} = \bigoplus_{s \in W} \mathscr{H}^{sv} e^{sv}.$$

Thus $M(0,v)$ consists of all the cosets $sv + 2L$, $s \in W$. If we make the stronger assumption that

$$\text{for all } \alpha \in R_\pi, \frac{(v,\alpha)}{(\alpha,\alpha)} \notin \mathbb{Z} \tag{7}$$

(cf. Sect. 3.2 for notation), then the cosets $sv+2L$ are all distinct. This follows easily from the fact that for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the subgroup $\{s \in W | s\lambda - \lambda \in L\}$ is generated by reflections (cf. [Bo, p. 227, ex. 1 and p. 75, Proposition 2]). Thus $W_{sv}=\{1\}$ and $\mathscr{H}^{sv}=\mathbb{C}$ for all $s \in W$ in this case. It follows that *when v satisfies* (7), *then $M(0, v)$ has cardinality $|W|$, and the elements of $M(0, v)$ parametrize a basis for $W_{0,v}$*. We shall show in the next section that this property continues to hold for generic values of the pair c, v.

4.2. *General Properties of the Monodromy*

Because the system 4.1 (2) is equivalent to the system 3.7(1)′ by Theorem 3.10, we can also obtain the monodromy of these systems via Floquet's theorem, as follows:

Fix the basis for $\mathscr{H}$ as in Theorem 3.2 and identify $\mathscr{H}$ with $\mathbb{C}^w$ via this basis ($w=|W|$). For $c \in \mathbb{C}^d$ (or in intrinsic terms, $c \in \mathfrak{u}_{\mathbb{C}}^*$) and $v \in \mathfrak{a}_{\mathbb{C}}^*$, let $\Gamma^i(c, v: h)$ be the $w \times w$ connection matrix evaluated at the point $h \in \mathfrak{a}_{\mathbb{C}}$, as defined in Sect. 3.6(2). From the theory of completely integrable systems of holomorphic differential equations, there exists a unique holomorphic map

$$\Psi: \mathbb{C}^d \times \mathfrak{a}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}} \to \mathrm{GL}(w, \mathbb{C}),$$

such that

$$\partial(H_i)\Psi(c, v: h) = \Gamma^i(c, v: h)\Psi(c, v: h), \quad \Psi(c, v: 0) = I. \tag{1}$$

By Theorem 3.10, if $f_1, \ldots, f_w$ are the entries in the first row of Ψ, then this set of functions comprises a basis for the space $\mathscr{W}_{c,v}$, and the remaining entries of Ψ are given by

$$\Psi_{ij} = \partial(e_i) f_j. \tag{2}$$

Let $S_j(c, v) \in \mathrm{GL}(w, \mathbb{C})$ be the matrix of the translation operator T_j on $\mathscr{W}_{c,v}$ relative to this basis. From (2) it follows that

$$\Psi(c, v: h + i\pi H_j) = \Psi(c, v: h) S_j(c, v) \tag{3}$$

for $j=1, \ldots, \ell$. Furthermore, the matrices S_j mutually commute and the eigenvalues of S_j are the numbers $e^{i\pi\Lambda(H_j)}$, as Λ ranges over $M(c, v)$.

With this notation in place we can now prove the following general properties of the monodromy:

Proposition 4.1. *There exists a non-zero holomorphic function ϕ on $\mathbb{C}^d \times \mathfrak{a}_{\mathbb{C}}^*$ such that if $\phi(c, v) \neq 0$, then $M(c, v)$ has cardinality w.*

Proof. Since the matrices $S_j(c, v)$ mutually commute and have eigenvalues $e^{i\pi\Lambda(H_j)}$, where $\Lambda \in M(c, v)$, it follows that any linear combination

$$A(c, v) = \sum_{j=1}^{\ell} a_j S_j(c, v) \tag{4}$$

($a_j \in \mathbb{C}$) will have eigenvalues

$$\sum_{j=1}^{\ell} a_j e^{i\pi\Lambda(H_j)}, \tag{5}$$

with Λ ranging over $M(c, v)$. In particular, the cardinality of $M(c, v)$ is at least as large as the number of distinct eigenvalues of the matrix $A(c, v)$.

Now fix $v_0 \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying condition 4.1 (7), set $c=0$, and pick the constants a_i in (4) so that the numbers (5) are all distinct, as Λ ranges over $M(0, v_0)$. (This can always be done, since the row vectors

$$\mu_\Lambda = [e^{i\pi\Lambda(H_1)}, \ldots, e^{i\pi\Lambda(H_\ell)}]$$

are all distinct when the cosets $\Lambda + 2L$ are disjoint.) The matrix $A(0, v_0)$ will then have w distinct eigenvalues. To see that this is a generic property, recall that if $Z \in \mathrm{End}(\mathbb{C}^w)$ and $\mathrm{ad}\, Z : \mathrm{End}(\mathbb{C}^w) \to \mathrm{End}(\mathbb{C}^w)$ is defined by $\mathrm{ad}\, Z(X) = [Z, X]$ as usual, then

$$\det(\mathrm{ad}\, Z - tI) = \sum_{j=w}^{w^2} (-1)^j t^j D_j(Z).$$

If Z has eigenvalues $\xi_1, \ldots, \xi_w$, counted according to algebraic multiplicity, then

$$D_w(Z) = (-1)^{w(w-1)/2} \prod_{i<j} (\xi_i - \xi_j)^2 .$$

Thus if we set $\phi(c, v) = D_w(A(c, v))$, then ϕ is clearly holomorphic on $\mathbb{C}^d \times \mathfrak{a}_{\mathbb{C}}^*$, and $\phi(0, v_0) \neq 0$. As observed above, $\mathrm{Card}\, M(c, v) = w$ on the set where $\phi(c, v) \neq 0$. $\square$

By the *multiplicity* of an element $\Lambda \in M(c, v)$ we shall mean the dimension of the corresponding joint generalized eigenspace

$$\mathscr{W}_{c,v}(\Lambda) = \{f \in \mathscr{W}_{c,v} : (T_j - \mu_j)^k f = 0 \text{ for some } k \text{ and all } j\},$$

where $\mu_j = e^{i\pi\Lambda(H_j)}$. Of course, if $\mathrm{Card}\, M(c, v) = w$, which we now know to be the case generically, then all multiplicities are one.

Proposition 4.2. *For all c, v, the sum (counting multiplicity) of the elements of $M(c, v)$ is in $2L$.*

Proof. Let Ψ be the fundamental solution matrix, as in (1). By Abel's formula for homogeneous systems of ordinary differential equations, we have $\partial(H_i)\det\Psi = (\mathrm{tr}\, \Gamma^i)\det\Psi$. But from Sect. 3.7(16) we know that $\mathrm{tr}(\Gamma^i) = 0$. Hence $\det\Psi$ is constant on $\mathfrak{a}_{\mathbb{C}}$. Evaluating it at 0, we see that $\det\Psi = 1$, and thus by formula (3) we conclude that

$$\det S_j = 1, \quad \text{for} \quad j = 1, \ldots, \ell. \tag{6}$$

If m_Λ is the multiplicity of Λ, then Eq. (6) implies that

$$\sum_{\Lambda \in M(c,v)} m_\Lambda \Lambda(H_j) \in 2\mathbb{Z}$$

for $j = 1, \ldots, \ell$. Hence $\sum m_\Lambda \Lambda \in 2L$, as claimed. $\square$

Recall from Sect. 3.3 the central element ξ (which equals 0 for the generalized non-periodic systems, and generates the center of $U(\mathfrak{b})$ for the generalized periodic systems). We now show that the monodromy only depends on the parameter c via the value of the scalar $T_c(\xi)$ [cf. Sect. 3.4(2)]:

Proposition 4.3. *Suppose $c, c' \in \mathbb{C}^d$ and $T_c(\xi) = T_{c'}(\xi)$. Then $M(v, c) = M(v, c')$ for any $v \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof. If $h \in \mathfrak{a}_{\mathbb{C}}$ and $f \in C^{\infty}(\mathfrak{a}_{\mathbb{C}})$, then we set $(R(h)f)(x) = f(x-h)$, for $x \in \mathfrak{a}_{\mathbb{C}}$. Then $R(h)$ commutes with the operators T_j, and a simple calculation shows that

$$R(h)^{-1} T_c(z) R(h) = T_{h \cdot c}(z) \quad \text{for} \quad z \in \mathfrak{b}. \tag{7}$$

Here we let $h \in \mathfrak{a}_{\mathbb{C}}$ act on $c \in \mathbb{C}^d$ by

$$(h \cdot c)_j = e^{-\alpha_j(h)} c_j, \quad \text{for} \quad j = 1, \ldots, d. \tag{8}$$

[Recall that $d = \dim(\mathfrak{u})$; the action in (8) is equivalent to the coadjoint action of $\exp \mathfrak{a}_{\mathbb{C}}$ on $\mathfrak{u}_{\mathbb{C}}^*$.] From (7) we see that

$$R(h) : \mathscr{W}_{c,v} \to \mathscr{W}_{h \cdot c, v}$$

bijectively. It then follows from formula (2) that the fundamental solution matrix satisfies $\Psi(c, v : x) = \Psi(h \cdot c, v : x - h) A$, for some matrix A depending on c, v and h. Setting $x = h$, we find that $A = \Psi(c, v : h)$ and thus obtain the functional equation

$$\Psi(c, v : x) = \Psi(h \cdot c, v : x - h) \Psi(c, v : h) \tag{9}$$

for all $c \in \mathbb{C}^d$, $v \in \mathfrak{a}_{\mathbb{C}}^*$, and $x, h \in \mathfrak{a}_{\mathbb{C}}$. In particular, taking $x = 0$ and then $x = i\pi H_j$ in (9) yields the relation

$$S_j(c, v) = \Psi(c, v : h)^{-1} S_j(h \cdot c, v) \Psi(c, v : h). \tag{10}$$

Given (10), it is now easy to complete the proof. Suppose first that for all j, we have $c_j \neq 0$ and $c'_j \neq 0$. Then there exists $h \in \mathfrak{a}_{\mathbb{C}}$ such that

$$c'_j = e^{-\alpha_j(h)} c_j \quad \text{for} \quad j = 1, \ldots, \ell \tag{11}$$

(recall that $\alpha_1, \ldots, \alpha_\ell$ are linearly independent). If we also assume that

$$T_{c'}(\xi) = T_c(\xi) \tag{12}$$

in the extended diagram case where $d = \ell + 1$, then condition (12) implies that (11) also holds for $j = d$, and hence $h \cdot c = c'$. [In the non-periodic case, $d = \ell$ and condition (12) is vacuous.] By (10) the matrices $S_j(c, v)$ and $S_j(c', v)$ are similar for $j = 1, \ldots, \ell$; in particular, the joint spectra are the same, which proves the proposition in this case.

Now suppose some $c_j = 0$. By the linear independence of the set of roots $\{\alpha_i : i \neq j\}$, there exists $h \in \mathfrak{a}$ such that

$$\alpha_i(h) > 0 \tag{13}$$

for all $i \neq j$, with $1 \leq i \leq d$. From (11) we thus have

$$\lim_{t \to +\infty} S_i((th) \cdot c, v) = S_i(0, v), \tag{14}$$

for $i = 1, \ldots, \ell$. But the characteristic polynomial of $S_i((th) \cdot c, v)$ is independent of t by (10), so by (14) we see that $S_i(c, v)$ and $S_i(0, v)$ have the same eigenvalues. Thus

$$M(c, v) = M(0, v). \tag{15}$$

This completes the proof. □

Since Eq. (13) can always be satisfied for $i = 1, \ldots, \ell$, the proof just given shows that (15) holds for all the generalized non-periodic systems. Taking into account the example at the end of Sect. 4.1, we thus have completely determined the monodromy in the following cases:

Corollary 4.4. (a) *For the generalized non-periodic Toda lattice systems,* $M(c,v) = \{sv+2L | s\in W\}$ *for all* $c\in\mathbb{C}^\ell$ *and all* $v\in\mathfrak{a}_{\mathbb{C}}^*$. *Furthermore, when* v *satisfies Sect.* 4.1(7), *then* $|M(c,v)|=|W|$, *and the monodromy matrices* $S_j(c,v)$ *are semi-simple.*

(b) *For the generalized periodic Toda lattice systems, the same conclusions hold whenever at least one* $c_j=0$.

In the case of an extended Dynkin diagram $\Phi=\{\alpha_1,\dots,\alpha_{\ell+1}\}$ as in Theorem 3.3, a subgroup of the Weyl group acts on the space of eigenfunctions and on the monodromy. Define

$$W_\Phi=\{s\in W | s\Phi\subset\Phi\}.$$

An element s of W_Φ acts on $\mathfrak{a}$ by definition. We let s act on $\mathfrak{u}$ by $s\cdot X_i=X_j$ if $\alpha_i=s\cdot\alpha_j$, where $\{X_i\}$ is the basis for $\mathfrak{u}$ as in Sect. 3.3. This action preserves the commutation relations in $\mathfrak{b}$, and hence defines an action of W_Φ as automorphisms of $U(\mathfrak{b})$. These actions commute with the projection $\mu: U(\mathfrak{b})\to U(\mathfrak{a})$. Let $c\in\mathbb{C}^{\ell+1}$ have all components equal. Then for $s\in W_\Phi$ and $z\in U(\mathfrak{b})$, we have

$$T_c(s\cdot z)=\varrho(s)\,T_c(z)\,\varrho(s)^{-1}, \tag{16}$$

where ϱ is the natural action of W on $C^\infty(\mathfrak{a})$: $\varrho(s)f(h)=f(s^{-1}\cdot h)$ for $h\in\mathfrak{a}$.

Proposition 4.5. *For any* $v\in\mathfrak{a}_{\mathbb{C}}^*$, *the space* $\mathscr{W}_{c,v}$ *and the monodromy* $M(c,v)$ *are invariant under* W_Φ.

Proof. Recall that $\chi_v=\chi_{s\cdot v}$ for any $s\in W$. By definition of the action of W_Φ, it is immediate that every $s\in W_\Phi$ fixes Ω. Furthermore, it follows from Theorem 3.3(1a) that the elements Ω_i are all left fixed by s, since this is true of their top-order terms u_i. Thus by (16) we find that $\mathscr{W}_{c,v}$ is invariant under $\varrho(s)$. Since the lattice L is invariant under W, s induces a transformation of $M(c,v)$. □

Example. The most extensive group of symmetries W_Φ occurs in the case of the A_ℓ completed Dynkin diagram (the original "periodic Toda lattice"), for which the subgroup W_Φ is the group $\mathbb{Z}/(\ell+1)\mathbb{Z}$, acting by cyclic permutations of coordinates. To verify this, we realize the A_ℓ root system in $\mathbb{R}^n$, $n=\ell+1$, as usual [Bo, Chap. VI, Sect. 4.7]: Let e_i, $1\leqq i\leqq n$, be the standard basis for $\mathbb{R}^n$, and take $\mathfrak{a}\subset\mathbb{R}^n$ to be the subspace $\sum x_i=0$. Define linear functionals α_i on $\mathfrak{a}$ by

$$\alpha_i=(e_i-e_{i+1})|_{\mathfrak{a}}, \quad \text{for} \quad i=1,\dots,n.$$

(Here and in the following the indices are to be read cyclically mod n: $e_{n+1}=e_1$.) The functionals $\alpha_1,\dots,\alpha_\ell$ comprise a base for the A_ℓ root system, and α_n is the negative of the highest root relative to this base. The Weyl group W is the permutation group on n letters, acting on $\mathbb{R}^n$ by permutations of the coordinates. This action leaves $\mathfrak{a}$ invariant, and it is clear that the only permutations that preserve the set Φ are the iterates of the shift $e_i\to e_{i+1}$. □

Remark. In general, one can show that the group W_Φ is isomorphic to the quotient of the weight lattice of the root system modulo the root lattice, hence its order is the "connection index" of the root system (cf. [Bo; Chap. VI, Sect. 4.3 and Planches II–V]). This fact has also been observed by Olive and Turok [O–T].

4.3. *Equations for the Monodromy*

Since Corollary 4.4 gives complete information about the monodromy for generalized non-periodic systems, we shall henceforth consider only the case of an extended Dynkin diagram $\Phi=\{\alpha_1,\ldots,\alpha_\ell,\alpha_{\ell+1}\}$. We show, using a multidimensional version of the celebrated technique introduced by G.W. Hill [Hi] and developed by von Koch [vK1] (see the appendix to this paper), that the monodromy exponents are restricted to lie on a complex variety in $\mathfrak{a}^*_{\mathbb{C}}/2L$ defined by the vanishing of an infinite determinant associated with the operator Ω. The complete set of equations for these exponents is then furnished by finite-dimensional eigenvalue conditions for the remaining operators in $U(\mathfrak{b})^{\Omega}$.

We fix the parameter $c\in\mathbb{C}^\ell$ with $T_c(\xi)\neq 0$. As shown in the proof of Proposition 4.3, we may assume for calculating the monodromy that the components of c are all equal, and we set $c_i^2=-\kappa$. Let $v\in\mathfrak{a}^*_{\mathbb{C}}$, and set $\sigma=\chi_v(\Omega)=\langle v,v\rangle$ (where $\langle\cdot,\cdot\rangle$ denotes the complex-bilinear extension to $\mathfrak{a}^*_{\mathbb{C}}$ of the inner product on $\mathfrak{a}^*$).

Let $\Lambda+2L\in M(c,v)$. Thus there is a non-zero function $f\in\mathscr{W}_{c,v}$ which has an expansion Sect. 4.1(6). We also know (since f is holomorphic on $\mathfrak{a}_{\mathbb{C}}$) that the coefficients a_μ in the expansion are exponentially decreasing. Set $\phi(y)=f(iy)$, for $y\in\mathfrak{a}(i=(-1)^{1/2})$, and let S be the operator $T_c(\Omega)$ acting on $C^\infty(i\mathfrak{a})$. Identify $i\mathfrak{a}$ with $\mathbb{R}^\ell$ using an orthonomal basis relative to the inner product on $\mathfrak{a}$. Then ϕ satisfies Eqs. Sect. A.3(3)–(5), so by Theorem A.5 the monodromy exponent Λ satisfies the equation

$$\Delta_{\sigma,\kappa}(\Lambda)=0, \tag{1}$$

where Δ is the infinite determinant associated with S defined in Sect. A.3.

We can obtain a complete set of equations for the monodromy by using the remaining operators $\Omega_2,\ldots,\Omega_\ell$. Let L_j be the operator $L_c(\Omega_j)$ acting on $C^\infty(i\mathfrak{a})$, for $j=1,2,\ldots,\ell$ (we set $\Omega=\Omega_1$). For any $\Lambda\in\mathfrak{a}^*_{\mathbb{C}}$, let $\mathscr{E}_{\kappa,\sigma}(\Lambda)$ denote the space of functions ϕ on $i\mathfrak{a}$ satisfying Sect. A.3(3)–(4) [for some choice of coefficients a_μ satisfying Sect. A.3(5)]. Since the operators L_j mutually commute, they leave this space invariant; we write $L_j(\kappa,\sigma,\Lambda)$ for the restriction of L_j to this space. By Theorem A.5, $\dim\mathscr{E}_{\kappa,\sigma}(\Lambda)<\infty$, and Eq. (1) is the necessary and sufficient condition for $\mathscr{E}_{\kappa,\sigma}(\Lambda)$ to be non-zero. For $t\in\mathbb{C}$, let $p_j(\kappa,\sigma,\Lambda;t)=\det(tI-L_j(\kappa,\sigma,\Lambda))$ be the characteristic polynomial of $L_j(\kappa,\sigma,\Lambda)$.

Theorem 4.6. *Let $\Lambda, v\in\mathfrak{a}^*_{\mathbb{C}}$. Set $\sigma_j=\chi_v(\omega_j)$ for $j=1,2,\ldots,\ell$. Then $\Lambda+2L\in M(c,v)$ if and only if* (1) *holds and*

$$p_j(\kappa,\sigma,\Lambda;\sigma_j)=0\,,\quad \text{for}\quad j=2,3,\ldots,\ell\,. \tag{2}$$

Proof. Conditions (1) and (2) are obviously necessary for $\Lambda+2L\in M(c,v)$, by the remarks preceding the theorem, since L_j acts by the scalar σ_j on the space $\mathscr{W}_{c,v}$. (Recall that the functions in this space are holomorphic on $\mathfrak{a}_{\mathbb{C}}$, so we can consider their restrictions to $i\mathfrak{a}$.)

To prove sufficiency of (1) and (2), we use the fact that the operators L_j mutually commute, and thus they have a joint eigenfunction in the finite-dimensional space $\mathscr{E}_{\kappa,\sigma}(\Lambda)$: For every Λ satisfying (1), there exist complex eigenvalues $\sigma=\gamma_1,\gamma_2,\ldots,\gamma_\ell$ and a non-zero function $\phi\in\mathscr{E}_{\kappa,\sigma}(\Lambda)$, such that

$$L_j\phi=\gamma_j\phi, \quad \text{for} \quad j=1,\ldots,\ell. \tag{3}$$

If Eqs. (2) hold, then we may take $\gamma_j=\sigma_j$ in (3). It follows from Lemma A.4 that ϕ extends holomorphically to a complex tubular neighborhood U of $i\mathfrak{a}$, and satisfies $L_c(\Omega_j)\phi=\chi_\nu(\Omega_j)\phi$ on U. By Lemma 2.3 and Theorem 3.10, we conclude that ϕ extends holomorphically to $\mathfrak{a}_{\mathbb{C}}$ and is in the space $\mathscr{W}_{c,\nu}$. Since ϕ has the expansion Sect. A.3(4), it follows that $\Lambda+2L\in M(c,\nu)$. $\square$

Appendix I: Infinite Determinants and Equations of Mathieu Type in Several Variables

AI.1. Infinite Determinants

We recall some results in the "classical" theory of absolutely convergent infinite determinants, as developed by H. Poincaré and H. von Koch (cf. [Ri, Chap. 2] and the references cited there; the "modern" operator-theoretic treatment of infinite determinants, as in [Si], does not seem to be sufficiently delicate for our needs). Let L be an index set. If $s:L\to L$ is a bijection, define

$$\operatorname{Supp}(s)=\{\mu\in L: s(\mu)\neq\mu\}.$$

The *restricted symmetric group* of L is the group S_∞ of all permutations s with $\operatorname{Supp}(s)$ finite. Let sgn: $S_\infty\to\{\pm1\}$ be the homomorphism that is -1 on transpositions.

Let $A=[A_{\mu\nu}]$ be a matrix of complex numbers indexed by $L\times L$. Assume that the diagonal elements $A_{\mu\mu}$ satisfy

$$\sum_{m\in L}|A_{\mu\mu}-1|<\infty. \tag{1}$$

In particular, at most a finite number of the $A_{\mu\mu}$ are zero. For $s\in S_\infty$, set

$$A(s)=\prod_{\mu\in L}A_{\mu,s\mu}. \tag{2}$$

Since s has finite support, it is clear from (1) that the infinite product (2) converges absolutely.

Definition. A has an *absolutely convergent* determinant if (1) holds and

$$\pi(A)=\sum_{s\in S_\infty}|A(s)|<\infty.$$

In this case, we define

$$\det A=\sum_{s\in S_\infty}\operatorname{sgn}(s)A(s).$$

Remarks. 1. Suppose A has an absolutely convergent determinant. Let (μ_0,ν_0) be a fixed pair of indices, and consider the matrix A' formed by multiplying the μ_0,ν_0 entry of A by the complex number λ. Since each entry of A occurs at most once in any term $A(s)$, one has $\pi(A')\leqq(1+|\lambda|)\pi(A)$, and hence A' has an absolutely convergent determinant. Thus, for example, to establish that a matrix A satisfying condition (1) has an absolutely convergent determinant, it suffices to show that the matrix A'' has an absolutely convergent determinant, where A'' is obtained from A by replacing any zero diagonal elements by 1.

2. If condition (1) holds, let D be the infinite product of the non-zero diagonal elements of A. Set $B_{mn}=A_{mn}/A_{mm}$ for all m such that $A_{mm}\neq 0$, and otherwise $B_{mn}=A_{mn}$. Then the diagonal elements of B are all 0 or 1, and it is straightforward to show that A has an absolutely convergent determinant if and only if B does. In this case, $\det(A)=D\det(B)$.

Lemma A.1. *Assume that A has an absolutely convergent determinant. Given $M\subset L$, set $A^{(M)}=[A_{mn}]_{m,n\in M}$. Then*

$$\det A=\lim_{M\to\infty}\det A^{(M)}. \tag{3}$$

Here $M\to\infty$ means taking the limit along the net of all finite subsets of L.

Proof. As noted in Remark 2 above, it suffices to consider the case in which a finite number of the diagonal entries are 0, and the rest are 1. For every finite subset $M\subset L$, we set

$$S(M)=\{s\in S_\infty : \operatorname{Supp}(s)\subset M\}.$$

For every $\varepsilon>0$, the convergence condition (3) implies that there is a finite set $M_\varepsilon\subset L$ such that

$$\sum_{s\notin S(M_\varepsilon)}|A(s)|<\varepsilon. \tag{4}$$

We may choose M_ε large enough to include all the indices μ such that $A_{\mu\mu}=0$. But if $M\supset M_\varepsilon$ is finite and $\operatorname{Supp}(s)\subset M$, then $A^{(M)}(s)=A(s)$. It follows from (4) that

$$|\det A-\det A^{(M)}|<\varepsilon,$$

which proves the lemma. □

We want to have conditions on A so that the existence of a non-trivial bounded sequence x satisfying $Ax=0$ is equivalent to the condition $\det A=0$. The first sufficient condition for this to hold was given by Poincaré. For our purposes we shall need the more refined conditions found by von Koch ([vK2] and the references cited there):

Lemma A.2. *Let $r\geqq 2$ be an integer, and define for $\mu,\nu\in L$*

$$Z^{(r)}_{\mu\nu}=\sum{}'|A_{\mu\mu_1}A_{\mu_1\mu_2}\cdots A_{\mu_{r-1}\nu}|, \tag{5}$$

where $\sum'$ denotes the sum over all $\mu_1,\dots,\mu_{r-1}\in L$ which are distinct among themselves and not equal to μ or ν. Assume that there exists an integer $n\geqq 2$ such that the series of "circular products" of length r

$$\sum_{\mu\in L}Z^{(r)}_{\mu\mu} \tag{6}$$

converges for $r=2,3,\dots,2n-1$, and the series of "semi-circular products" of length r

$$\sum_{\mu,\nu\in L}Z^{(r)}_{\mu\nu} \tag{7}$$

converges for $r=n, n+1, \ldots, 2n-1$. *Then*

(i) *A has an absolutely convergent determinant;*

(ii) *The system of equations:*

$$\text{for all } \mu \in L, \sum_{\lambda \in L} A_{\mu\lambda} x_\lambda = 0$$

has a non-zero solution x with $\sup_\mu |x_\mu| < \infty$ *if and only if* $\det A = 0$.

AI.2. Multidimensional Jacobi Matrices

Let $\Phi = \{\alpha_1, \ldots, \alpha_{\ell+1}\} \subset \mathbb{R}^\ell$ be such that

(a) the vectors $\alpha_1, \ldots, \alpha_\ell$ are a basis for $\mathbb{R}^\ell$

(b) there are integers $n_i \geqq 1$ such that

$$\alpha_{\ell+1} = -\sum_{i=1}^{\ell} n_i \alpha_i .$$

We set $h = 1 + n_1 + \ldots + n_\ell$, and denote by L the lattice in $\mathbb{R}^\ell$ generated by $\alpha_1, \ldots, \alpha_\ell$.

Let $A = [A_{\mu\nu}]$ be a matrix indexed by $L \times L$. We shall assume that the off-diagonal elements satisfy the following condition:

(c) If $\nu \notin \mu + \Phi$, then $A_{\mu\nu} = 0$.

For example, if $\ell = 1$ and $\alpha_1 = -\alpha_2$, then (c) says that A is a doubly-infinite tridiagonal (Jacobi) matrix. In this case the condition for the absolute convergence of the determinant of A was found by von Koch (cf. [W–W, p. 37]). For $\ell > 1$, we shall show that von Koch's general criterion (Lemma A.2) applies, given the following bounds on the elements of A:

Lemma A.3. *Assume that the product of the diagonal elements of* A *is absolutely convergent, and that there exists a number* $\varepsilon > \ell/h$ *and a constant* C *such that*

$$|A_{\mu\nu}| \leqq \frac{C}{(1 + \|\mu\|)^\varepsilon} \tag{1}$$

for all $\mu \in L$ *and* $\nu \in \mu + \Phi$. *Then the conditions of Lemma A.2 are satisfied with* $n = h$. *Furthermore, if the entries of A depend on a parameter in such a way that the series Sect. A.1(1) converges uniformly and estimate* (1) *holds uniformly in the parameter, then the limit Sect. A.1(3) giving* $\det A$ *holds uniformly in the parameter.*

Proof. We introduce the following notions: If $\mu \in L$ and $\nu \in \mu + \Phi$, we shall write $\mu \to \nu$. By a *Φ-path* in L of *length r* we shall mean a map $\gamma: \{1, 2, \ldots, r, r+1\} \to L$ which satisfies

$$\mu_1 \to \mu_2 \to \ldots \to \mu_r \to \mu_{r+1},$$

where $\mu_j = \gamma(j)$. We say the path is *closed* if $\gamma(r+1) = \gamma(1)$. Set

$$A_\gamma = \prod_{i=1}^{r} A_{\gamma(i), \gamma(i+1)} .$$

Consider now the conditions of Lemma A.2. By (c) above, the only non-zero terms in the summations Sect. A.1(5) for $Z_{\mu\nu}^{(r)}$ are those for which

$$\mu \to \mu_1 \to \mu_2 \to \ldots \to \mu_{r-1} \to \nu$$

is a Φ-path. Thus

$$Z_{\mu\nu}^{(r)} \leqq \sum_{\gamma \in P_r(\mu, \nu)} |A_\gamma|, \tag{2}$$

where $P_r(\mu, \nu)$ is the set of all Φ-paths of length r beginning at μ and ending at ν. But for fixed μ and fixed r, there are at most $(\ell+1)^r$ such paths, and every point on the path is at a bounded distance from μ. From estimate (1) and (2) we thus have

$$Z_{\mu\nu}^{(r)} \leqq \frac{C'}{(1+\|\mu\|)^{\varepsilon r}}, \tag{3}$$

where the constant C' depends only on r, the set Φ, and the constant C in (1). Also, $Z_{\mu\nu}^{(r)}=0$ if $\|\mu-\nu\|>M$, where M is a constant depending on r and Φ, but independent of μ and ν. By the condition $\varepsilon h>\ell$, it is thus clear from (3) that the series Sects. A.1(6) and A.1(7) converge as soon as $r \geqq h$.

It remains to consider the series Sect. A.1(6) when $r<h$. For this, we make the key observation that

$$Z_{\mu\mu}^{(r)}=0 \text{ unless } r \text{ is divisible by } h. \tag{*}$$

Thus the convergence condition on Sect. A.1(6) is vacuous when $r=1, 2, \ldots, h-1$, so the lemma will follow from (*). (The uniformity statements follow from the estimates above and von Koch's estimates for $\det A$ [loc. cit.].)

To prove (*), write $\mu=\mu_1$, and assume that there is a non-zero term $A_{\mu_1\mu_2}\ldots A_{\mu_r\mu_1}$ in $Z_{\mu\mu}^{(r)}$. Since $\mu_i \neq \mu_j$ for $i \neq j$, we have a closed Φ-path $\gamma: \mu_1 \to \mu_2 \to \ldots \to \mu_r \to \mu_1$, by condition (c). For $1 \leqq i \leqq \ell+1$, let a_i be the number of times that $\mu_{k+1}=\mu_k+\alpha_i$. (Here we set $\mu_{r+1}=\mu_1$.) Writing $a_{\ell+1}=b$, we find that because γ is closed, the numbers a_i must satisfy $a_i=bn_i$, $1 \leqq i \leqq \ell$. But $a_1+\ldots+a_\ell+b=r$, so it follows that $r=bh$, which proves (*). □

AI.3. Mathieu Equations in Several Variables

Let the set of vectors $\Phi \subset \mathbb{R}^\ell$ be as in Sect. A.2. Define the complex-valued potential

$$V(y)=\sum_{\alpha \in \Phi} e^{-2i\langle \alpha, y\rangle} \tag{1}$$

for $y \in \mathbb{R}^\ell$ $(i=(-1)^{1/2})$, and define the Schrödinger operator

$$S=-\sum_{j=1}^{\ell} (\partial/\partial y_j)^2+\kappa V(y), \tag{2}$$

where $\kappa \in \mathbb{C}$ is a constant. Notice that V is multiply-periodic with periods πH_j, where $H_1, \ldots, H_\ell$ is the dual basis to $\alpha_1, \ldots, \alpha_\ell$. Let

$$P=\sum_{j=1}^{\ell} \mathbb{Z} H_j$$

be the lattice generated by $H_1, \ldots, H_\ell$. As in Sect. A.2, we let L be the dual lattice generated by $\alpha_1, \ldots, \alpha_\ell$.

In this section we consider the eigenfunctions ϕ of S:

$$S\phi=\sigma\phi, \quad \sigma \in \mathbb{C}, \tag{3}$$

which admit convergent expansions

$$\phi(y)=e^{i\langle \Lambda, y\rangle} \sum_{\mu\in L} a_\mu e^{2i\langle \mu, y\rangle} \tag{4}$$

for some $\Lambda\in\mathbb{C}^\ell$. (Here $\langle \Lambda, y\rangle$ denotes the complex-bilinear extension to $\mathbb{C}^\ell$ of the inner product on $\mathbb{R}^\ell$.)

Example. If $\ell=1$ and $\Phi=\{\alpha, -\alpha\}$, then $V(y)=2\cos\alpha(y)$, and (3) is Mathieu's equation [W-W, Chap. XIX]. In this case (3) always admits solutions of the form (4), by Floquet's theorem.

Lemma A.4. *Suppose ϕ is a solution of* (3). *If ϕ admits an expansion* (4) *that converges weakly (in the sense of distributions, say), then ϕ is a real-analytic function and the coefficients a_μ in* (4) *are exponentially decreasing: There exists an $r>0$ such that*

$$\sup_{\mu\in L}|a_\mu|\, e^{r\|\mu\|}<\infty\,. \tag{5}$$

In particular, if $\{a_\mu : \mu\in L\}$ is a bounded sequence such that the distribution defined by (4) *is an eigendistribution for S, then this sequence necessarily satisfies* (5).

Proof. The distribution $e^{-i\langle\Lambda,y\rangle}\phi(y)$ is a periodic eigendistribution for the operator $S_\Lambda=M_\Lambda^{-1}SM_\Lambda$, where M_Λ is multiplication by $e^{i\langle \Lambda, y\rangle}$. But S_Λ is an elliptic operator with analytic, πP-periodic coefficients, so the lemma follows by the analytic hypoellipticity of this operator on the torus $\mathbb{R}^\ell/\pi P$. $\square$

We denote by $\mathscr{E}_{\kappa,\sigma}(\Lambda)$ the space of functions ϕ on $\mathbb{R}^\ell$ satisfying (3) and (4) [for some choice of coefficients a_μ satisfying (5)]. Since the eigenspaces of S_Λ on $C^\infty(\mathbb{R}^\ell/\pi P)$ are finite-dimensional, it follows as in the proof of Lemma A.4 that $\dim\mathscr{E}_{\kappa,\sigma}(\Lambda)<\infty$.

We can now prove, using a multi-dimensional version of the celebrated technique introduced by Hill [Hi] and developed by von Koch [vK1], that the space $\mathscr{E}_{\kappa,\sigma}(\Lambda)$ is non-zero if and only if the monodromy exponent Λ lies on a variety defined by the vanishing of an infinite determinant. To obtain this result, we calculate from Eq. (3) that the coefficients a_μ in the expansion (4) of ϕ satisfy the following infinite set of homogeneous partial difference equations:

$$[\langle \Lambda+2\mu, \Lambda+2\mu\rangle-\sigma]a_\mu=-\kappa\sum_{j=1}^{\ell+1} a_{\mu+\alpha_j} \tag*{$(6)_\mu$}$$

for all μ in the lattice L. We can express the condition on Λ so that the system (6) have a non-zero bounded solution $\{a_\mu\}$ as follows:

For $\mu\in L$, $\mu\neq 0$, we multiply Eq. $(6)_\mu$ by the non-zero scalar

$$\langle 2\mu, 2\mu\rangle^{-1} E\left\{-\frac{\langle \Lambda, \Lambda\rangle+4\langle \Lambda, \mu\rangle-\sigma}{4\langle \mu, \mu\rangle}\right\},$$

where $E\{z\}=\exp[z+(z^2/2)+\ldots+(z^\ell/\ell)]$ for $z\in\mathbb{C}$. This gives an equivalent homogeneous system of the form

$$\sum_{\lambda\in L} A_{\mu\lambda}a_\lambda=0\,, \tag*{$(7)_\mu$}$$

where the coefficient matrix $A=[A_{\mu\lambda}]$ has diagonal entries

$$A_{\mu\mu}=\left\{1+\frac{\langle\Lambda,\Lambda\rangle+4\langle\Lambda,\mu\rangle-\sigma}{4\langle\mu,\mu\rangle}\right\}E\left\{-\frac{\langle\Lambda,\Lambda\rangle+4\langle\Lambda,\mu\rangle-\sigma}{4\langle\mu,\mu\rangle}\right\} \tag{8}$$

if $\mu\neq 0$, and $A_{00}=\langle\Lambda,\Lambda\rangle-\sigma$. The only non-zero off-diagonal entries of A are $A_{0\lambda}=\kappa$ for $\mu=0$, and

$$A_{\mu\lambda}=\frac{\kappa}{\langle 2\mu,2\mu\rangle}E\left\{-\frac{\langle\Lambda,\Lambda\rangle+4\langle\Lambda,\mu\rangle-\sigma}{4\langle\mu,\mu\rangle}\right\} \tag{9}$$

for $\mu\neq 0$, where in both cases $\lambda=\mu+\alpha_j$, with $j=1,\ldots,\ell+1$.

Theorem A.5. *The matrix A has an absolutely convergent determinant $\Delta_{\sigma,\kappa}(\Lambda)$ which is a non-constant holomorphic function of $\Lambda\in\mathbb{C}^{\ell}$, $\kappa\in\mathbb{C}$, and $\nu\in\mathbb{C}$. A necessary and sufficient condition that Eq.* (3) *admit a non-trivial C^{∞} solution of the form* (4) *is that $\Delta_{\sigma,\kappa}(\Lambda)=0$.*

Proof. By the standard estimates for Weierstrass infinite products, we obtain from (8) that

$$|1-A_{\mu\mu}|\leqq\frac{C(\Lambda,\sigma)}{(1+\|\mu\|)^{\ell+1}}, \tag{10}$$

where $C(\Lambda,\sigma)$ is a locally bounded function of Λ,σ and is independent of μ. Thus Sect. A.1 (1) converges, uniformly on compacta in Λ, σ. For the off-diagonal entries we have an estimate

$$|A_{\mu\lambda}|\leqq\frac{|\kappa|C(\Lambda,\sigma)}{(1+\|\mu\|)^{2}} \tag{11}$$

from (9), with $C(\Lambda,\sigma)$ another locally bounded function of Λ,σ. Thus estimate Sect. A.2(1) holds with $\varepsilon=2$, uniformly on compacta in the parameters. Since the number $h\geqq\ell+1$ in Lemma A.3, we certainly have $\varepsilon h>\ell$, and the hypotheses of Lemma A.3 are satisfied by A. The theorem thus follows by Lemmas A.2 and A.4. □

Remarks. The infinite determinant $\Delta_{\sigma,\kappa}(\Lambda)$ can be factored as follows: Let

$$D_{\sigma}(\Lambda)=\prod_{\mu\in L}A_{\mu\mu}, \tag{12}$$

where the diagonal elements of A are given by (8). Then by estimate (10), we find that D_{σ} is a holomorphic function of $\Lambda\in\mathbb{C}^{\ell}$, and $D_{\sigma}(\Lambda)=0$ if and only there exists $\mu\in L$ such that

$$\langle\Lambda+2\mu,\Lambda+2\mu\rangle=\sigma. \tag*{$(13)_{\mu}$}$$

In particular, the zero set of D_{σ} is invariant under translations by elements of $2L$. Suppose $D_{\sigma}(\Lambda)\neq 0$. Then we can get a system of equations equivalent to $(6)_{\mu}$ by dividing by the coefficient of a_{μ}:

$$\sum_{\lambda\in L}B_{\mu\lambda}a_{\lambda}=0. \tag*{$(14)_{\mu}$}$$

Here the coefficient matrix $B=[B_{\mu\lambda}]$ has diagonal entries $B_{\mu\mu}=1$. The only non-zero off-diagonal entries of B are

$$B_{\mu\lambda}=\frac{\kappa}{\langle\Lambda+2\mu,\Lambda+2\mu\rangle-\sigma}, \tag{15}$$

where $\lambda=\mu+\alpha_j$, with $j=1,\ldots,\ell+1$.

By the same argument as for the matrix A, we see that the function $\Gamma_{\sigma,\kappa}=\det B$ is a holomorphic function of Λ on the complement of the unions of the varieties $(13)_\mu$, $\mu\in L$. It is clear that

$$\Gamma_{\sigma,\kappa}(\Lambda+2\mu)=\Gamma_{\sigma,\kappa}(\Lambda) \tag{16}$$

for all $\mu\in L$. By Remark 2 before Lemma A.1, we have the factorization

$$\Delta_{\sigma,\kappa}(\Lambda)=D_0(\Lambda)\Gamma_{\sigma,\kappa}(\Lambda)\,. \tag{17}$$

This shows that $\Gamma_{\sigma,\kappa}$ is a meromorphic function on $\mathbb{C}^\ell$, multiply-periodic with periods $2L$. We also conclude that the equation $\Delta_{\sigma,\kappa}(\Lambda)=0$ is invariant under translations of Λ by $2L$, and hence defines an analytic variety on the manifold $\mathbb{C}^\ell/2L$.

Appendix II: Invariant Operators for the Periodic Toda Lattice

In this appendix we derive formula (1) of the introduction. We realize the A_ℓ root system in $\mathbb{R}^n$, $n=\ell+1$, as usual [Bo, Chap. VI, Sect. 4.7]: Let e_i, $1\leqq i\leqq n$, be the standard basis for $\mathbb{R}^n$, and take $\mathfrak{a}\subset\mathbb{R}^n$ to be the subspace $\sum x_i=0$. Define $h_i=e_i-u$, for $i=1,\ldots,n$, where $u=(1/n,\ldots,1/n)$. Then $h_i\in\mathfrak{a}$, and $h_1+\ldots+h_n=0$. We define linear functionals α_i on $\mathfrak{a}$ by $\alpha_i(x)=x_i-x_{i+1}$, for $i=1,\ldots,n$. (Here and in the following the indices will always be read cyclically mod$n$: $x_{n+1}=x_1$.) The functionals $\alpha_1,\ldots,\alpha_\ell$ comprise a base for the A_ℓ root system, and $\alpha_{\ell+1}$ is the negative of the highest root relative to this base. One has the relation

$$\alpha_1+\ldots+\alpha_{\ell+1}=0\,. \tag{1}$$

The Weyl group W is the permutation group on n letters, and its action on $\mathfrak{a}$ is by permutations of $h_1,\ldots,h_n$.

We introduce the following notation: Set $\Phi=\{\alpha_1,\ldots,\alpha_{\ell+1}\}$ and $I=\{1,\ldots,\ell+1\}$. If $P\subset\Phi$, let

$$I(P)=\{i\in I|\text{there exists } \alpha\in P \text{ with } \alpha(h_i)\neq 0\}$$

(with the convention that $I(\text{void})=\text{void}$). It is clear that

(a) If $P\subset\Phi$ and $P=P'UP''$, then $I(P)=I(P')\cup I(P'')$.

Furthermore, a special property of the type A_ℓ completed Dynkin diagram is that

(b) If P', $P''\subset\Phi$ and $P'\perp P''$, then $I(P')\cap I(P'')=\text{void}$.

Indeed, if $i\in I(P')$, then either α_i or α_{i-1} is in P'. But $(\alpha_i,\alpha_{i-1})\neq 0$, so neither of these roots can be in P''. Hence $\alpha(h_i)=0$ for all $\alpha\in P''$, so $i\notin I(P'')$. This proves (b). Now consider the family of *disconnected* subsets of P, in the sense of Dynkin diagrams:

$$\mathscr{S}(P)=\{Q\subset P|\alpha\perp\beta \text{ for all } \alpha,\beta\in Q\}\,.$$

(In particular, note that $\mathscr{S}(P)$ always contains the empty set.) It is clear from the definition that

(c) If P' and P'' are subsets of Φ and $P' \perp P''$, then $\mathscr{S}(P' \cup P'')$ consists of all sets $Q = Q' \cup Q''$, where $Q' \in \mathscr{S}(P')$ and $Q'' \in \mathscr{S}(P'')$.

Let $t \in \mathbb{C}$. Given $Q \subset P \subset \Phi$, set

$$H_{P/Q}(t) = \prod_{i \in I(P) \sim I(Q)} (h_i + t) \tag{2}$$

as an element in the complexified enveloping algebra $U(\mathfrak{a})$. When Q is empty, we write $H_{P/Q}(t) = H_P(t)$. In particular, when $P = \Phi$, then we set $H_\Phi(t) = H(t)$. This element of degree $\ell + 1$ is W-invariant, and when we expand it as a polynomial in t, the coefficients of $1, t, \ldots, t^{\ell - 1}$ furnish a set of ℓ independent homogeneous generators for $S(\mathfrak{a})^W$ (cf. [Bo, loc. cit.]; note that the coefficient of t^ℓ is zero by (1)). From properties (a) and (b) above we obtain the following multiplicative property for these polynomials:

(d) Suppose $Q' \subset P'$ and $Q'' \subset P''$ are subsets of Φ such that $P' \perp P''$. Set $P = P' \cup P''$ and $Q = Q' \cup Q''$. Then $H_{P/Q}(t) = H_{P'/Q'}(t) H_{P''/Q''}(t)$.

Form the Lie algebra $\mathfrak{b}$ as in Sect. 3 with roots $\Phi = \{\alpha_1, \ldots, \alpha_{\ell+1}\}$, and let $X_i = X_{\alpha_i}$ be chosen as in Sect. 3.3. Then $h_1, \ldots, h_\ell, X_1, \ldots, X_{\ell+1}$ is a basis for $\mathfrak{b}$, and the commutation relations in $\mathfrak{b}$ are

$$[h_i, X_j] = \begin{cases} X_j & \text{if } i = j \\ -X_j & \text{if } i = j+1 \end{cases} \tag{3}$$

for $i, j = 1, \ldots, \ell + 1$.

For an inductive study of the Laplacian Ω for $\mathfrak{b}$ we recall the following constructions from [G-W2]: Let $P \subset \Phi$ and set

$$\mathfrak{a}(P) = \sum_{i \in I(P)} \mathbb{C} h_i, \qquad \mathfrak{u}(P) = \sum_{\alpha \in P} \mathbb{C} X_\alpha,$$

and $\mathfrak{b}(P) = \mathfrak{a}(P) + \mathfrak{u}(P)$. Let $\sigma_P : \mathfrak{b}(P) \to \mathfrak{a}(P)$ be the projection corresponding to this direct sum decomposition. We extend σ_P to a homomorphism from $U(\mathfrak{b}(P))$ to $U(\mathfrak{a}(P))$, and call $\sigma_P(T)$ the *symbol* of T, for $T \in U(\mathfrak{b}(P))$. For any subset $Q \subset P$ let

$$\mathfrak{b}(P)_Q = \mathfrak{a}(P) + \mathfrak{u}(P \sim Q).$$

This is a subalgebra of $\mathfrak{b}(P)$. The Laplacians for the algebras $\mathfrak{b}(P)_Q$ are

$$\Omega(P)_Q = \sum_{i \in I(P)} h_i^2 + \sum_{\alpha \in P \sim Q} X_\alpha^2.$$

When $Q =$ void, we simply write $\Omega(P)$. Note that

$$\Omega(P)_Q = \Omega(P \sim Q) + \Delta_{P,Q}, \tag{4}$$

where $\Delta_{P,Q} \in U(\mathfrak{a}(P))$ and commutes with $U(\mathfrak{b}(P)_Q)$. This is obvious, since $[h_i, X_\alpha] = 0$ if $\alpha \in P \sim Q$ and $i \notin I(P \sim Q)$.

If $Q \subset P$ set

$$X_Q = \prod_{\alpha \in Q} X_\alpha.$$

Note that X_Q commutes with $H_{P/Q}(t)$, since $\alpha(h_i) = 0$ for all $\alpha \in Q$ and $i \notin I(Q)$. With this notation in place, we can now define the following polynomial in $t \in \mathbb{C}$

with coefficients in $U(\mathfrak{b}(P))$:

$$L_P(t)=\sum_{Q\in\mathscr{S}(P)}(-\tfrac{1}{2})^{|Q|}X_Q^2H_{P/Q}(t)\,. \tag{5}$$

For example, if $P=\{\alpha_i\}$, then $\mathscr{S}(P)$ consists of the empty set and P, so

$$L_P(t)=(h_i+t)(h_{i+1}+t)-\tfrac{1}{2}X_{\alpha_i}^2$$

in this case. A simple calculation using (3) then shows that

$$[\Omega(P),L_P(t)]=0 \tag{6}$$

when $|P|=1$.

Before proving (6) in general, we observe that if $P=P'\cup P''$ with $P'\perp P''$, then

$$L_P(t)=L_{P'}(t)L_{P''}(t)\,. \tag{7}$$

Furthermore,

$$\Omega(P)=\Omega(P')+\Omega(P'')\,, \tag{8}$$

and we have

$$[\Omega(P'),\Omega(P'')]=0\,,\qquad[\Omega(P'),L_{P''}(t)]=0\,,\qquad[\Omega(P''),L_{P'}(t)]=0\,. \tag{9}$$

[These all follow easily from properties (a), (b), (c), and (d) above.]

Theorem. *For any subset $P\subset\Phi$, one has $[L_P(t),\,\Omega(P)]=0$ for all $t\in\mathbb{C}$.*

Proof. When $|P|=1$, then the theorem follows by direct calculation, as noted above. Assume that the theorem is true for all subsets of cardinality less than $|P|$.

Case 1. $P=P'\cup P''$, with $P'\perp P''$ and both non-empty. The induction hypothesis applies to P' and P''. By (7)–(9) we see immediately that the theorem is true for P.

Case 2. P is connected. We recall the following results from [G-W2]: Let $Q\subset P$ with $Q\neq P$. Set $Q'=P\sim Q$. Since Q is an ordinary Dynkin diagram, [G-W2, Theorem 4.1] implies that there is a unique element $w_{P,Q}(t)\in U(\mathfrak{b}(P)_{Q'})$ such that

(i) $w_{P,Q}(t)$ has symbol $H_P(t)$.

(ii) $w_{P,Q}(t)$ commutes with $\Omega(P)_{Q'}$.

We claim that

$$w_{P,Q}(t)=H_{P/Q}(t)L_Q(t)\,. \tag{10}$$

To prove (10), note that the right side of (10) has symbol $H_{P/Q}(t)\cdot H_Q(t)$, which is simply $H_P(t)$. Hence property (i) is satisfied. Furthermore, since $L_Q(t)$ commutes with $\Omega(Q)$ by the induction hypothesis, we see from (4) (with Q replaced by Q') that $L_Q(t)$ also commutes with $\Omega(P)_{Q'}$. Finally, $H_{P/Q}(t)$ commutes with $\Omega(P)_{Q'}$ because $[h_i,X_\alpha]=0$ for $\alpha\in Q$ and $i\notin I(Q)$. Thus the right side of (10) also satisfies property (ii). This proves that equality holds in (10).

To complete the induction, we use [G-W2, Lemma 3.2]: Since $\deg(H_P(t))=|P|+1$, the element

$$w'(H_P(t))=\sum_{Q\neq P}(-1)^{|P|-|Q|+1}w_{P,Q}(t)$$

(sum over all proper subsets Q of P, including the empty set) commutes with $\Omega(P)$ and has symbol $H_P(t)$. By (10) it thus only remains to check that

$$L_P(t) = \sum_{Q \neq P} (-1)^{|P|-|Q|+1} H_{P/Q}(t) L_Q(t). \tag{11}$$

The proof of (11) is a straightforward calculation starting from formula (4) and using the fact that every totally disconnected subset $Q \in \mathscr{S}(P)$ is contained in some *proper* subset of P, since P is connected. We leave the details to the reader.

The differential operator in Eq. (1) of the introduction is $T_c(L(t))$, with T_c the representation of $\mathfrak{b}$ defined in Sect. 1.1.

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