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Asymptotic Completeness for a Quantum Particle in a Markovian Short Range Potential

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Abstract. Absence of bound states and asymptotic completeness are proven for a quantum particle in a time dependent random (Markovian) short range potential. Systems with confining potentials are also considered and unboundedness of the energy in time is shown.

1. Introduction and Results

In a previous paper ([1]) we started studying the quantum dynamics generated by random time dependent Hamiltonians of the form

$$H(t) = H_0 + V(\xi(t)), \tag{1.1}$$

where H_0 is a self adjoint operator on some Hilbert space $\mathscr{H}$ (typically $\mathscr{H} = L^2(\mathbb{R}^\nu)$ or $l^2(\mathbb{Z}^\nu)$ with $H_0 = -\Delta$), $\{\xi(t)|t\in\mathbb{R}\} = \underset{\sim}{\xi}$ a path of a stationary Markov process on some state space E with a unique invariant measure μ and $V(\cdot)$ a function on E with values in the self adjoint operators on $\mathscr{H}$.

In this paper we continue the analysis of such systems. The first and main part of our work is devoted to the case $\mathscr{H} = L^2(\mathbb{R}^\nu)$ for $\nu \geqq 3$ and $H_0 = -\Delta$, $V(\xi)$ multiplication by a short range potential $V(\xi, x)$ (i.e. sufficiently rapidly decaying at spatial infinity). From [1] we learn modulo some non-triviality condition assuring (1.1) to be "sufficiently time dependent" that such a system leaves any bounded region of its phase space in time mean (this is the "RAGE-theorem" 4.2 in [1]); however we don't know how. It may tend to spatial infinity, or have unbounded kinetic energy, or both. We only know states with bounded energy to approach spatial infinity like a free particle (from Corollary 4.4 in dimensions $\nu \geqq 5$). We prove this to be the right behaviour in general. More technically we show the dynamics generated by (1.1) to be asymptotically complete (with respect to the free one). Let us formulate our result as a

Theorem. *Let $H_0 = -\Delta$ be the ordinary kinetic energy on $L^2(\mathbb{R}^\nu)$ with $\nu \geqq 3$. Further assume the short range potential $V(\xi, x)$ and the process $\xi(\cdot)$ to satisfy the conditions 2.1–2.5 of Sect. 2. Then if $U(\underset{\sim}{\xi}|t, s)$ denotes the unitary propagator associated to (1.1),* *the wave operators*

$$\underset{\sim}{\Omega}^{\mp}(\underset{\sim}{\xi}|s) = \underset{t\to\pm\infty}{s\text{-lim}}\, U(\underset{\sim}{\xi}|t,s)^* e^{-iH_0(t-s)}$$

exist and are unitary with probability one. In particular all states $f \in L^2(\mathbb{R}^\nu)$ have time bounded energy (in the sense of [1]) and a free asymptotic as $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \| U(\underset{\sim}{\xi}|t,s)f - e^{-iH_0(t-s)}(\Omega^{\mp}(s))^* f \| = 0;$$

the scattering matrix

$$S = (\Omega^-)^* \Omega^+$$

being unitary.

Note the condition $\nu \geqq 3$ on space dimension. We think some additional work should relax this restriction; for a discussion of the (mainly technical) assumptions 2.1–2.5 we refer to Sect. 2. In the second part of this paper we consider the opposite situation of a confining potential (i.e. such that $\lim_{|x|\to\infty} V(\xi, x) = +\infty$). A simple application of the results in [1] shows that such systems always have time unbounded energy. Thus, as time goes on, states with higher and higher energy are excited (think of a harmonic oscillator, weakly and locally perturbed by a random force). For a precise formulation of the results see Sect. 6, which can be read independently of the rest of the paper.

The paper is organised as follows: Sects. 2 to 5 concern short range systems. In Sect. 2 the necessary assumptions on the potential V and the process ξ are listed and discussed, we also give some of their immediate consequences. The completeness proof then follows the beautiful time dependent approach of Enss, more precisely its Jafaev version (see [2, 3 and 4]). However the lack of energy conservation make useless the usual estimates on the cut-off free propagator. Instead we use weaker results on the full free propagator proved in an appendix; similar estimates have already been used by Jafaev in [5]. In order to compensate the weakness of these results, we will need some extra information on the interacting propagator $U(t,s)$, in the form of a local decay estimate proved in Sect. 3, using a stationary bound on kinetic energy. Equipped with this propagation estimates we proceed in Sect. 4 to the asymptotic completeness proof.

The main analytical work, now concentrated in the stationary energy estimate used in Sect. 3, is done in Sect. 5 and consists in controling boundary values of some resolvent $(L-z)^{-1}$ as z becomes real from the upper half plane. We use fairly standard methods somewhat reminiscent of the three body problem: weighted L^2-spaces, Birman–Schwinger kernels and Fredholm theory; however the setting is unusual since the operator L is neither self adjoint, nor elliptic and perhaps not even spectral. Finally Sect. 6 is devoted to the study of confining systems.

2. Hypotheses and Preliminary Results

The conditions to be fulfilled by the potential $V(\xi, x)$ are of three distinct types:

1) *Decay conditions* are well known to be critical in scattering theory. The borderline for the existence of ordinary (i.e. short range) wave operators

$$\underset{t\to\pm\infty}{s\text{-lim}}\, e^{i(-\Delta+V)t}e^{i\Delta t}$$

being at

$$V(x)\sim|x|^{-1} \quad \text{as} \quad x\to\infty.$$

But, as already stressed by Jafaev [5], completeness of the scattering by time dependent potentials will be very hard to obtain, at least technically, for potentials decaying slower than $|x|^{-2}$ at infinity due to the infrared singularity of free propagation. In fact our method seems to break down at exactly this point for two reasons: first in the resolvent estimate of Sect. 5 where we are unable to use Agmon type arguments, and instead have to use the less effective Kato estimate (Lemma 5.2); second in the estimate on free propagation (last statement of Lemma 4.1). Note that in both cases the condition $\nu\geqq 3$ is also essential.
Since we don't want to worry about local singularities, we assume:

$$\begin{cases} W(\xi,x)=(1+|x|^2)^{s/2}V(\xi,x)\in L^\infty(E\times\mathbb{R}^\nu) \\ \text{for some } s>2. \end{cases} \tag{2.1}$$

2) *Smoothness conditions* are not usually needed if the potential is time independent, but as seen for example in Kato [6], any loss of regularity in the time behaviour of the potential have to be compensated by some smoothness in space in order to get good control of the evolution. The following will suffice:

$$\begin{aligned} &W(\xi,\cdot)\in C^n(\mathbb{R}^\nu) \\ &\text{for some } n>\operatorname{Max}\left(2,\frac{\nu}{4}\right), \text{ and} \\ &\partial_x^\alpha W\in C(E\times\mathbb{R}^\nu)\cap L^\infty(E\times\mathbb{R}^\nu)\ \forall|\alpha|\leqq n. \end{aligned} \tag{2.2}$$

We stress however not to know any physical motivation for such a restriction.

3) *Non-triviality conditions* clearly are necessary, for the system has to feel the time dependence of the potential which, destroying quantum coherence, is responsible for the unitarity of the wave operators. We assume

$$\begin{cases} \operatorname{Var}(V(\cdot,x)-V(\cdot,y))>0 \text{ for a.a. } (x,y)\in\mathcal{O}\times\mathbb{R}^\nu, \\ \text{where } \mathcal{O} \text{ is some non-empty open set in } \mathbb{R}^\nu. \end{cases} \tag{2.3}$$

We used the notation

$$\operatorname{Var}(f)=\int|f(\xi)|^2d\mu(\xi)-\left|\int f(\xi)d\mu(\xi)\right|^2\geqq 0$$

for any $f\in L^2(E,d\mu)$. This seems somewhat stronger than the corresponding assumption of our first paper, where (2.3) is only required to hold on $\mathcal{O}_1\times\mathcal{O}_2$ for some open sets $\mathcal{O}_1,\mathcal{O}_2\subset\mathbb{R}^\nu$. However a moment of reflection shows that under the smoothness condition (2.2) this is not a real restriction. Note also that (2.3) imply $\operatorname{Var}(V(\cdot,x))\not\equiv 0$ on $\mathbb{R}^\nu$, and since by (2.2) this is a continuous function of x there is an open set $\mathcal{O}' \subset \mathbb{R}^\nu$ such that

$$\mathrm{Var}(V(\cdot, x)) > 0 \quad \text{for} \quad x \in \mathcal{O}'. \tag{2.3}$$

We turn to the assumptions on the generator A of the Markov process $\{\xi(t)\}$. They are of two types (recall $(\Xi, \mathscr{X}, P)$ denotes the underlying probability structure, and $E(\cdot)$ is expectation with respect to P):

4) *Symmetry.* Doing scattering theory, we want the potential to be defined on the whole time axis, i.e. the process $\{\xi(t)\}$ to be indexed by $t \in \mathbb{R}$. A convenient way of doing this is to assume it to be symmetric:

$$(e^{-At}f)(\xi) = E(f(\xi(t)) \,|\, \xi(0) = \xi) = E(f(\xi(0)) \,|\, \xi(t) = \xi),$$

or equivalently its infinitesimal generator A to be self adjoint. Together with the assumptions already made in our first paper we obtain:

$$\begin{cases} A \text{ is a positive self-adjoint operator on } \mathfrak{h} = L^2(E, d\mu) \\ \text{with the non-degenerate ground state } 1\text{: } A1 = 0. \end{cases} \tag{2.4}$$

5) *Compactness.* Although not strictly necessary it will be technically very convenient to assume:

$$A \text{ has compact resolvent.} \tag{2.5}$$

This condition is fulfilled by any jump process on a finite state space, any diffusion on a compact manifold or even any $P(\phi)_1$-process, in particular the oscillator process (see [7]). (From the proof it will be clear than in fact only compactness of $V(\xi, x) \times (-\Delta - iA + 1)^{-1}$ really matters.)

We will use freely the spectral representation

$$\mathfrak{h} = L^2(E, d\mu) \simeq l^2(\mathbb{N}), \quad f(\xi) \leftrightarrow \{f_n\},$$

with

$$\|f\|^2 = \int |f(\xi)|^2 d\mu(\xi) = \sum_n |f_n|^2, \quad (Af)_n = \lambda_n f_n,$$

$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ being the repeated eigenvalues of A. Either $\mathfrak{h}$ is finite dimensional, or $\lim_{n \to \infty} \lambda_n = +\infty$. Note also that the eigenspace to $\lambda_0 = 0$ contains the constant functions of ξ.

Conditions 2.1–2.5 are all the assumptions we will need in this paper. We now briefly discuss some immediate consequences.

Clearly all hypotheses of Sect. 4 in [1] are fulfilled, so there is a nice propagator $U(\underset{\sim}{\xi} | t, s)$ for the Schrödinger equation

$$i\partial_t \psi_t = (-\Delta + V(\xi(t), x))\psi_t. \tag{2.6}$$

Since the smoothness condition (2.2) imply (i)–(v) of [1] with the identifications:

$$X = L^2(\mathbb{R}^\nu), \quad Y = H^n(\mathbb{R}^\nu), \quad K(\xi) = i(-\Delta + V(\xi, x)), \quad S = (1 - \Delta)^{n/2},$$

they are two constants M', β' such that for $t \geqq s$:

$$\|(1 - \Delta)^{n/2} U(\underset{\sim}{\xi} | t, s)(1 - \Delta)^{-n/2}\| \leqq M' e^{\beta'(t-s)} \quad P-\text{a.s.}. \tag{2.7}$$

Furthermore, via Cook's estimate (see [8]), the decay condition (2.1) imply the existence of wave operators

$$\Omega^{\pm}(\underset{\sim}{\xi}|s) = \operatorname*{s\text{-}lim}_{t\to\mp\infty} U(\underset{\sim}{\xi}|t,s)^* e^{i\Delta(t-s)}$$

with the intertwinning property

$$\Omega^{\pm}(\underset{\sim}{\xi}|t) = U(\underset{\sim}{\xi}|t,s)\Omega^{\pm}(\underset{\sim}{\xi}|s)e^{-i\Delta(t-s)} \tag{2.8}$$

Finally on $L^2(\mathbb{R}^\nu \times \mathbb{R}^\nu \times E; d^\nu x d^\nu y d\mu(\xi))$, let

$$L_0 = p^2 - k^2 - iA, \tag{2.9}$$

where $\vec{p} = -i\vec{\nabla}_x$ and $\vec{k} = -i\vec{\nabla}_y$. L_0 clearly generates the contraction semigroup

$$e^{-iL_0 t} = e^{i\Delta t} \otimes e^{-i\Delta t} \otimes e^{-At}.$$

Multiplication with $(V(\xi,x) - V(\xi,y))$ being a bounded self-adjoint operator, it then follows from a standard perturbation theorem (see for example [9] theorem X.50) that

$$L = L_0 + V(\xi,x) - V(\xi,y) \quad D(L_0) = D(L) \tag{2.10}$$

also generate a contraction semigroup, easily identified with the expectation semigroup of [1]:

$$E[\psi_t \otimes \bar{\phi}_t | \xi(t)] = e^{-iLt}\psi_0 \otimes \bar{\phi}_0 \otimes 1 \tag{2.11}$$

for solutions ψ_t and ϕ_t of the Schrödinger equation (2.6). The map $\phi \to \bar{\phi}$ denoting the anti-unitary complex conjugation on $L^2(\mathbb{R}^\nu)$.

3. Propagator Estimate

As explained in the introduction our completeness proof relies on some estimate of the interacting propagation. In this section we derive it from a stationary estimate of the kinetic energy to be proved in Sect. 5.

Let C be Hilbert–Schmidt on $L^2(\mathbb{R}^\nu)$, and assume its integral kernel, which we also denote by C, satisfies

$$C \in H^n(\mathbb{R}^\nu) \otimes H^n(\mathbb{R}^\nu).$$

Then for $\psi \in \mathcal{S}(\mathbb{R}^\nu)$, the Schwartz space of test functions, and with $S = (1-\Delta)^{n/2}$ a simple computation shows

$$(SU(\underset{\sim}{\xi}|t,0)\psi, CSU(\underset{\sim}{\xi}|t,0)\psi) = (\psi_t \otimes \bar{\psi}_t, (S \otimes S)C), \tag{3.1}$$

where we have set $\psi_t = U(\underset{\sim}{\xi}|t,0)\psi$ and used the same symbol for inner products in $L^2(\mathbb{R}^\nu)$ and $L^2(\mathbb{R}^{2\nu})$. Note that the left-hand side of (3.1) is well defined by (2.7). From (3.1) we further get, using Fubini's theorem and formula (2.11), for $\operatorname{im} z > 0$,

$$\begin{aligned}\int\int_0^\infty e^{izt}(S\psi_t(\underset{\sim}{\xi}), CS\psi_t(\underset{\sim}{\xi}))dt dP(\underset{\sim}{\xi}) &= \int_0^\infty e^{izt}(\bar{e}^{iLt}\psi \otimes \bar{\psi} \otimes 1, (S \otimes S \otimes 1)C \otimes 1) \\ &= i((L-z)^{-1}\psi \otimes \bar{\psi} \otimes 1, \quad (S \otimes S \otimes 1)C \otimes 1).\end{aligned} \tag{3.2}$$

Now we define

$$L' = L_0 + (1-\Delta_x)^{n/2} V(\xi, x)(1-\Delta_x)^{-n/2} - (1-\Delta_y)^{n/2} V(\xi, y)(1-\Delta_y)^{-n/2}. \quad (3.3)$$

This is a closed operator on $D(L') = D(L)$, since by our smoothness hypothesis (2.2) $L' - L_0$ is bounded. A standard computation shows

$$(L-z)^{-1}(S^{-1} \otimes S^{-1} \otimes 1) = (S^{-1} \otimes S^{-1} \otimes 1)(L'-z)^{-1},$$

as long as $z \in \rho(L) \cap \rho(L')$, which is surely the case for large $\operatorname{im}(z)$. With the last identity, (3.2) becomes

$$\int \int_0^\infty e^{izt}(S\psi_t, CS\psi_t) dt dP(\underset{\sim}{\xi}) = i((L'-z)^{-1}(S\psi \otimes S\bar{\psi} \otimes 1), C \otimes 1). \quad (3.4)$$

By the estimate (2.7) and a simple approximation argument, formula (3.4) extend to all $C \in L^2(\mathbb{R}^{2\nu})$ and $z \in \rho(L') \cap \{\operatorname{im} z > \beta'\}$. In order to let $z \downarrow 0$ from the upper half plane in (3.4) we need the two following results. The first one is elementary, the second being our main analytical estimate establishing, loosely speaking, (local) boundedness of the kinetic energy (recall that formally $L' = (p^2+1)^{n/2}(k^2+1)^{n/2} \times L(p^2+1)^{-n/2}(k^2+1)^{-n/2}$).

Lemma 3.1. *Let $f(t)$ be a positive measurable function such that the integral*

$$\int_0^\infty e^{-\varepsilon t} f(t) dt$$

exists for big positive ε. Then there is an ε_0 such that this integral converges for $\varepsilon > \varepsilon_0$, diverges for $\varepsilon < \varepsilon_0$, and is an analytic function $g(\varepsilon)$ on the half plane $\{\operatorname{Re} \varepsilon > \varepsilon_0\}$. This function, which may have an analytic continuation across the line $\operatorname{Re} \varepsilon = \varepsilon_0$, *must have a singularity at $\varepsilon = \varepsilon_0$.*

The proof of this lemma being very easy we omit it. Note however the analogy with a well known theorem about analytic functions having positive Taylor coefficients at a point (see [10]).

Theorem 5. $\{\operatorname{im} z > 0\} \subset \rho(L')$ *and* $(L'-z)^{-1}$ *extend to a continuous function from* $\{\operatorname{im} z \geqslant 0\}$ *to*

$$\mathscr{B}(L^2_\delta(\mathbb{R}^{2\nu}) \otimes L^2(E), L^2_{-\delta}(\mathbb{R}^{2\nu}) \otimes L^2(E))$$

for any $\delta > 1/2$.

Assuming C to be positive and such that

$$C_\delta = (1+x^2)^{\delta/2} C (1+x^2)^{\delta/2} \quad (3.5)$$

is also Hilbert Schmidt, we may combine these two results to obtain from (3.4),

$$\int \int_0^\infty (\psi_t, SCS\psi_t) dt dP(\underset{\sim}{\xi}) = i((1+x^2)^{-\delta/2}(1+y^2)^{-\delta/2}(L'-i0)^{-1} S\psi \otimes S\bar{\psi} \otimes 1, C_\delta \otimes 1). \quad (3.6)$$

(Remember $\psi \in \mathscr{S}(\mathbb{R}^\nu)$ so $S\psi \in L^2_\delta(\mathbb{R}^\nu)$).

The last identity further gives for $\phi \in C_0^\infty(\mathbb{R}^\nu)$

$$\int \int_0^\infty (\psi_t, \phi(x)\psi_t)|dtdP(\underset{\sim}{\xi}) \leqq \int \int_0^\infty (\psi_t, |\phi(x)|\psi_t)dtdP(\underset{\sim}{\xi})$$
$$= i((1+x^2)^{-\delta/2}(1+y^2)^{-\delta/2}(L'-i0)^{-1}S\psi \otimes S\bar{\psi} \otimes 1, C_\delta \otimes 1) \qquad (3.7)$$

with

$$C = S^{-1}|\phi(x)|S^{-1}.$$

Since from definition (3.5)

$$C_\delta = [(1+x^2)^{\delta/2}S^{-1}(1+x^2)^{-\delta/2}S]S^{-1}|\phi(x)|(1+x^2)^\delta S^{-1}$$
$$\cdot [S(1+x^2)^{-\delta/2}S^{-1}(1+x^2)^{\delta/2}],$$

with bounded first and last factors, we obtain from standard trace ideals estimates (see [8] or [11])

$$\| C_\delta \|_{\mathrm{H.S.}} \leqq \mathrm{Const} \, \| \phi \|_{L^2_{2\delta}},$$

and thus with Theorem 5, (3.7) becomes

$$\int \int_0^\infty |(\psi_t, \phi(x)\psi_t)|dtdP(\underset{\sim}{\xi}) \leqq (M_0 \| \psi \|^2_{n,\delta}) \| \phi \|_{L^2_{2\delta}}$$

for some constant M_0 and norm $\| \cdot \|_{n,\delta}$. Thus the linear map

$$\mathcal{M}: L^2_{2\delta}(\mathbb{R}^\nu) \to L^1(\mathbb{R}_+ \times \Xi) \qquad (3.8)$$

defined by

$$(\mathcal{M}\phi)(t, \underset{\sim}{\xi}) = (\psi_t(\underset{\sim}{\xi}), \phi(x)\psi_t(\underset{\sim}{\xi})), \qquad (3.9)$$

with fixed $\psi \in \mathcal{S}(\mathbb{R}^\nu)$, is bounded, with norm smaller than $M_0 \| \psi \|^2_{n,\delta}$. On the other hand (3.9) clearly defines a bounded linear map

$$\mathcal{M}: L^\infty(\mathbb{R}^\nu) \to L^\infty(\mathbb{R}_+ \times \Xi) \qquad (3.10)$$

with norm smaller than $\| \psi \|^2$. Interpolating between (3.8) and (3.10) gives

$$\mathcal{M} \in \mathcal{B}(L^{2p}_{2\delta/p}(\mathbb{R}^\nu), L^p(\mathbb{R}_+ \times \Xi)) \qquad (3.11)$$

for $1 \leqslant p \leqslant \infty$ (see [12] for example), the norm in $L^{2p}_{2\delta/p}$ being defined by

$$\| \phi \|_{L^{2p}_{2\delta/p}} = (\int |(1+x^2)^{\delta/p}\phi(x)|^{2p} d^\nu x)^{1/2p}.$$

Applying this result to the special case $\phi(x) = (1+x^2)^{-\alpha}$ we obtain finally, optimizing over δ:

Theorem 3. *Let* $p \geqslant 2$ *and* $\alpha p > \nu/2 + 1$, *then for any* $\psi \in \mathcal{S}(\mathbb{R}^\nu)$

$$\int_0^\infty \| (1+x^2)^{-\alpha/2} U(\underset{\sim}{\xi}|t,0)\psi \|^p \, dt < \infty \quad P-\text{a.s.}.$$

Remark. We proved Theorem 3 for $t > 0$, but clearly an analogous result holds for $t < 0$, which we will use without further comments.

4. Asymptotic Completeness

Let $\mathscr{S}_0 \subset \mathscr{S}(\mathbb{R}^\nu)$ be a denumerable dense set in $L^2(\mathbb{R}^\nu)$, and choose γ and p such that

$$\frac{2}{s} < \gamma < 1 \quad p \geqq 2 \quad p > \frac{\frac{\nu}{2}+1}{(1-\gamma)s}. \tag{4.1}$$

Then from Theorem 3 we can find a set Ξ_0 of full P measure such that

$$\int\limits_{-\infty}^{\infty} \|(1+x^2)^{-(1-\gamma)s/2} U(\underset{\sim}{\xi}|t,0) f\|^p dt < \infty \ \forall \underset{\sim}{\xi} \in \Xi_0 \text{ and } \forall f \in \mathscr{S}_0. \tag{4.2}$$

To get sufficient control on the free evolution we will also need the

Lemma 4.1. *There are two bounded self adjoint operators P_+, P_- on $L^2(\mathbb{R}^\nu)$ such that*

$$P_+ + P_- = I, \|P_\pm\| = 1,$$

$$s\text{-}\lim_{t\to\pm\infty} P_\mp e^{-iH_0 t} = 0 \ (\textit{we set } H_0 = -\Delta),$$

$\phi(x) e^{\mp iH_0 t} P_\pm$ are compact if $t>0$ and ϕ bounded and vanishing at infinity. Furthermore if $\nu \geqq 3$ and $\sigma > 2$ there is a constant C such that

$$\|(1+x^2)^{-\sigma/2} e^{\mp iH_0 t} P_\pm\| \leqq C(1+t)^{-1} \quad \text{for} \quad t>0.$$

P_- (respectively P_+) have to be interpreted as projections on the incoming (respectively outgoing) scattering states (see [2]).

A similar result has already been used by Jafaev in [5], however since our norm estimate is not contained in [5] the lemma is proved in the appendix.

From now on we consider a fixed path $\underset{\sim}{\xi} \in \Xi_0$, and drop any reference to it. Assume

$$\phi \perp \operatorname{Ran} \Omega^-(0). \tag{4.3}$$

There is some $f \in \mathscr{S}_0$ with $\|\phi - f\| < \varepsilon$, and by (4.2) a sequence $\{t_n\}$ such that

$$t_n \to +\infty \text{ and } \|F(|x|<n) U(t_n, 0) f\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.4}$$

We set $f_n = U(t_n, 0) f$, then

$$\begin{aligned} \|f\|^2 = \|f_n\|^2 &= (f_n, (P_+ + P_-) f_n) \\ &= (f_n, (1-\Omega^+(t_n)) P_- f_n) + (f_n, \Omega^+(t_n) P_- f_n) \\ &\quad + (f_n, (1-\Omega^-(t_n)) P_+ f_n) + (f_n, \Omega^-(t_n) P_+ f_n). \end{aligned}$$

We now estimate the four terms in the right-hand side of the last identity as $n \to \infty$.

(i) $$\begin{aligned} (f_n, \Omega^-(t_n) P_+ f_n) &= (f_n, U(t_n, 0) \Omega^-(0) e^{iH_0 t_n} P_+ f_n) \text{ by (2.8)} \\ &= (f, \Omega^-(0) e^{iH_0 t_n} P_+ f_n) \\ &= (\phi, \Omega^-(0) e^{iH_0 t_n} P_+ f_n) + (f - \phi, \Omega^-(0) e^{iH_0 t_n} P_+ f_n). \end{aligned}$$

The first term vanishes by assumption (4.3), and the second is bounded by $\varepsilon \|f\|$, thus

$$|(f_n, \Omega^-(t_n)P_+ f_n)| \leqq \varepsilon \| f \|$$

(ii) $$\begin{aligned}(f_n, \Omega^+(t_n)P_- f_n) &= (f_n, U(t_n,0)\Omega^+(0)e^{iH_0 t_n}P_- f_n) \\ &= (f, \Omega^+(0)\, e^{iH_0 t_n}P_- f_n) \\ &= (P_- e^{-iH_0 t_n}(\Omega^+(0))^* f, f_n) = o(1),\end{aligned}$$

since by Lemma 4.1 $\underset{n\to\infty}{s\text{-lim}}\, P_- e^{-iH_0 t_n} = 0.$

(iii) $$\begin{aligned}&(f_n, (1-\Omega^-(t_n))P_+ f_n) \\ &= \lim_{t\to\infty} (f_n, (1 - U(t,t_n)^* e^{-iH_0(t-t_n)})P_+ f_n) \\ &= \lim_{t\to\infty} \int_{t_n}^{t} \left(f_n, \frac{d}{d\tau} U(\tau, t_n)^* e^{-iH_0(\tau - t_n)} P_+ f_n \right) d\tau \\ &= -i \lim_{t\to\infty} \int_{t_n}^{t} (f_n, U(\tau,t_n)^*(1+x^2)^{-s/2} W(\tau) e^{-iH_0(\tau-t_n)} P_+ f_n) d\tau \\ &= -i \lim_{t\to\infty} \int_{t_n}^{t} ((1+x^2)^{-(1-\gamma)s/2} U(\tau,0) f, W(\tau)(1+x^2)^{-\gamma s/2} e^{-iH_0(\tau-t_n)} P_+ f_n) d\tau,\end{aligned}$$

which can be estimated by

$$\begin{aligned}&\int_{t_n}^{\infty} \|(1+x^2)^{-(1-\gamma)s/2} U(\tau,0) f\| \, \|W\|_\infty \|(1+x^2)^{-\gamma s/2} e^{-iH_0(\tau - t_n)} P_+\| \quad \|f\| d\tau \\ &\quad \leqq \|W\|_\infty \|f\| \left(\int_{t_n}^{\infty} \|(1+x^2)^{-(1-\gamma)s/2} U(\tau,0) f\|^p d\tau \right)^{1/p} \\ &\quad \cdot \left(\int_0^{\infty} \|(1+x^2)^{-\gamma s/2} e^{-iH_0\tau} P_+\|^q d\tau \right)^{1/q},\end{aligned}$$

with $q^{-1} = 1 - p^{-1}$ by Hölders inequality. The second integral is finite by Lemma 4.1 since $q > 1$ and $\gamma s > 2$ by (4.1). The first one vanishes as $n \to \infty$ by (4.2). Thus

$$(f_n, (1-\Omega^-(t_n))P_+ f_n) = o(1).$$

(iv) $$\begin{aligned}(f_n, (1-\Omega^+(t_n))P_- f_n) &= (f_n, (1-\Omega^+(t_n))P_- F(|x|<n) f_n) \\ &\quad + (f_n (1-\Omega^+(t_n))P_- F(|x|>n) f_n).\end{aligned}$$

The first term being bounded by $2\|f\| \; \|F(|x|<n)\; U(t_n,0)f\| = o(1)$ by (4.4), the second term can be handled in the same way as in (iii) to yield the bound

$$\begin{aligned}&\|W\|_\infty \|f\| \left(\int_{-\infty}^{0} \|(1+x^2)^{-(1-\gamma)s/2} U(\tau,0) f\|^p d\tau \right)^{1/p} \\ &\cdot \left(\int_0^{\infty} \|(1+x^2)^{-\gamma s/2} e^{iH_0\tau} P_- F(|x|>n)\|^q d\tau \right)^{1/q}.\end{aligned}$$

The first integral is finite by (4.2). The integrand of the second is dominated by

$$\|(1+x^2)^{-\gamma s/2} e^{iH_0\tau} P_-\|^q,$$

which is integrable by Lemma 4.1. Now by the same $(1+x^2)^{-\gamma s/2}e^{iH_0\tau}P_-$ is compact, and since $F(|x|>n)$ strongly vanishes as $n\to\infty$, the integrand vanishes pointwise as $n\to\infty$. Application of the dominated convergence theorem gives

$$(f_n,(1-\Omega^+(t_n))P_-f_n)=o(1).$$

Collecting our four estimates (i)–(iv),

$$\|f\|^2=\|f_n\|^2\leqq\varepsilon\|f\|+o(1)\quad \text{as } n\to\infty,$$

and thus

$$\|\phi\|\leqq\|f\|+\|\phi-f\|\leqq 2\varepsilon.$$

Since ε was arbitrary we conclude $\phi=0$ and $\operatorname{Ran}\Omega^-(0)=L^2(\mathbb{R}^\nu)$. This holds for any $\underset{\sim}{\xi}\in\Xi_0$, therefore

$$P[\operatorname{Ran}\Omega^-(\underset{\sim}{\xi}|0)=L^2(\mathbb{R}^\nu)]=1.$$

The same analysis clearly applies to $\Omega^+(\underset{\sim}{\xi}|0)$. We have proven

Theorem 4. *The wave operators $\Omega^\pm(\underset{\sim}{\xi}|0)$ are unitary with probability one.*

5. Resolvent Estimate

5.1. Resolvent Formulae. We set

$$E_1=(1-\Delta_x)^{n/2}V(\xi,x)(1-\Delta_x)^{-n/2},$$
$$E_2=(1-\Delta_y)^{n/2}V(\xi,y)(1-\Delta_y)^{-n/2}.$$

They are bounded operators on $L^2(\mathbb{R}^{2\nu}\times E)$. Also

$$F_1=(1-\Delta_x)^{n/2}(1+x^2)^{-s/4}(1-\Delta_x)^{-n/2},$$
$$F_2=(1-\Delta_y)^{n/2}(1+y^2)^{-s/4}(1-\Delta_y)^{-n/2},$$
$$G_1=(1-\Delta_x)^{n/2}(1+x^2)^{-s/4}W(\xi,x)(1-\Delta_x)^{-n/2},$$
$$G_2=-(1-\Delta_y)^{n/2}(1+y^2)^{-s/4}W(\xi,y)(1-\Delta_y)^{-n/2},$$

all are bounded, and

$$E_j=F_jG_j=G_jF_j\quad (j=1,2). \tag{5.1}$$

Finally let

$$R_0(z)=(L_0-z)^{-1},$$
$$R_j(z)=(L_0+E_j-z)^{-1}=(L_j-z)^{-1}\quad (j=1,2),$$
$$R(z)=(L_0+E_1+E_2-z)^{-1}=(L'-z)^{-1}.$$

Since by (5.1) $\|E_j\|\leqq\|F_j\|\;\|G_j\|$, we have

$$\{z|\operatorname{im}z>\|F_1\|\;\|G_1\|+\|F_2\|\;\|G_2\|\}\subset\rho(L_0)\cap\rho(L_1)\cap\rho(L_2)\cap\rho(L'),$$

and on this set a simple computation shows

$$R_j(z)=R_0(z)-R_0(z)G_j(1+Q_j(z))^{-1}F_jR_0(z)\quad (j=1,2), \tag{5.2}$$

$$R(z)=R_1(z)-R_1(z)G_2(1-Q(z))^{-1}(1+Q_2(z))^{-1}F_2R_1(z), \tag{5.3}$$

where we have defined the Birman–Schwinger kernels as:

$$Q_j(z) = F_j R_0(z) G_j \quad (j = 1, 2), \tag{5.4}$$

$$Q(z) = (1 + Q_2(z))^{-1} F_2 R_0(z) G_1 (1 + Q_1(z))^{-1} F_1 R_0 G_2 . \tag{5.5}$$

Formulae (5.2) and (5.3) will allow us to control the resolvent $R(z)$ as z becomes real from the upper half plane.

5.2. The Birman–Schwinger Kernels $Q_j(z)$. Clearly we only need to consider $Q_1(z)$, the case of $Q_2(z)$ being completely analogous. We use the fibration

$$L^2(\mathbb{R}^\nu \times \mathbb{R}^\nu \times E) = L^2(\mathbb{R}^\nu, d^\nu k; L^2(\mathbb{R}^\nu, d^\nu x) \otimes L^2(E, d\mu)),$$

which reduces $Q_1(z)$ according to

$$\left.\begin{aligned} (Q_1(z)f)(k) &= q(k^2 + z) f(k) \\ q(z) &= F_1(-\Delta - iA - z)^{-1} G_1 \end{aligned}\right\} . \tag{5.6}$$

Lemma 5.1. *$q(z)$ is a compact valued analytic function on the open upper half plane, continuous on the closed upper half plane and*

$$\lim_{\substack{z \to \infty \\ \operatorname{im} z \geqslant 0}} \| q(z) \| = 0 .$$

Proof. Recall $S = (1 - \Delta)^{n/2}$ and let $T = (1 + x^2)^{s/4}$, then

$$q(z) = ST^{-1}S^{-1}T\{T^{-1}(-\Delta - iA - z)^{-1}T^{-1}\}TST^{-1}S^{-1}(SW(\xi, x)S^{-1}).$$

The last factor is bounded by (2.2). The two operators $ST^{-1}S^{-1}T$ and $TST^{-1}S^{-1}$ are also easily shown to be bounded, thus it suffices to consider

$$T^{-1}(-\Delta - iA - z)^{-1}T^{-1} = \bigoplus_n \{T^{-1}(-\Delta - i\lambda_n - z)^{-1}T^{-1}\}. \tag{5.7}$$

Analyticity in the open upper half plane is clear; to go further we need the

Lemma 5.2. *Let f, $g \in L^p(\mathbb{R}^\nu)$ $(2 \leqq p \leqq \infty)$ and $\phi \in L^2(\mathbb{R}^\nu)$, then*

$$\| f(x) e^{i\Delta t} g(x) \phi \| \leqq (2\pi |t|)^{-\nu/p} \| f \|_p \| g \|_p \| \phi \|$$

for $\operatorname{im} t \leqq 0$.

This result is well known, at least for real t, but its proof by interpolation easily extends to $\operatorname{im} t \leqq 0$, see for example [9]. Now from the representation

$$T^{-1}(-\Delta - z)^{-1}T^{-1} = i \int_0^\infty e^{izt} T^{-1} e^{i\Delta t} T^{-1} dt$$

Lemma 5.2 implies via dominated convergence continuity of each summand in (5.7) in the closed upper half plane. Continuity of the sum follows from the estimate

$$\| \bigoplus_{n \geqq N} \{T^{-1}(-\Delta - i\lambda_n - z)^{-1}T^{-1}\} \| = \sup_{n \geqq N} \| T^{-1}(-\Delta - i\lambda_n - z)^{-1}T^{-1} \| \leqq \lambda_N^{-1} \to 0 \tag{5.8}$$

as $N \to \infty$. Compactness follows from the same argument, since each summand in

(5.7) is compact for $\operatorname{im} z > 0$. From (5.8) and

$$\lim_{R\to\infty} \| F(|x|>R)(1+x^2)^{-\varepsilon} \| = 0 \quad \forall \varepsilon > 0,$$

we easily see that

$$\lim_{\substack{z\to\infty \\ \operatorname{im} z \geqq 0}} \| F(|x|<R)(-\Delta - z)^{-1} F(|x|<R) \| = 0 \quad \forall R > 0$$

suffices to prove the last statement of the lemma. But

$$\begin{aligned} F(|x|<R)(-\Delta-z)^{-1}F(|x|<R) &= i\int_0^\infty e^{izt}F(|x|<R)e^{i\Delta t}F(|x|<R)dt \\ &= -i\int_0^\infty e^{iz(t-\pi/z)}F(|x|<R)e^{i\Delta t}F(|x|<R)dt. \end{aligned}$$

Using analyticity of the free evolution in $\{\operatorname{im} t < 0\}$ we may deform the integration contour in the last integral, obtaining

$$\begin{aligned} F(|x|<R)(-\Delta-z)^{-1}F(|x|<R) &= \frac{i}{2}\int_0^{\pi/2} e^{izt}F(|x|<R)e^{i\Delta t}F(|x|<R)dt \\ &+\frac{i}{2}\int_0^\infty e^{izt}F(|x|<R)\{e^{i\Delta t}-e^{i\Delta(t+\pi/z)}\}F(|x|<R)dt. \end{aligned}$$

The first integral is bounded in norm by $\pi/2|z|$. By the dominated convergence theorem the second will also vanish as $z\to\infty$ if $F(|x|<R)e^{i\Delta t}F(|x|<R)$ is norm continuous in $\mathbb{C}\backslash\{0\}$. But

$$\begin{aligned} &\| F(|x|<R)(e^{i\Delta t}-e^{i\Delta s})F(|x|<R)f \|^2 \\ &= \int_{|x|<R} \left| \int_{|y|<R} \left\{ \frac{e^{i((x-y)^2/4t)}}{(4\pi i t)^{\nu/2}} - \frac{e^{i((x-y)^2/4s)}}{(4\pi i s)^{\nu/2}} \right\} f(y) d^\nu y \right|^2 d^\nu x \\ &\leqq \int_{|x|<R} \left\{ \int_{|y|<R} \left| \frac{e^{i((x-y)^2/4t)}}{(4\pi i t)^{\nu/2}} - \frac{e^{i((x-y)^2/4s)}}{(4\pi i s)^{\nu/2}} \right|^2 d^\nu y \| f \|^2 \right\} d^\nu x \\ &\leqq \text{Const} \sup_{0\leqq u\leqq R^2} \left\{ \left| \frac{e^{i(u/t)}}{t^{\nu/2}} - \frac{e^{i(u/s)}}{s^{\nu/2}} \right|^2 \right\} \| f \|^2, \end{aligned}$$

and thus

$$\lim_{s\to t\neq 0} \| F(|x|<R)(e^{i\Delta t}-e^{i\Delta s})F(|x|<R) \| = 0,$$

which achieves the proof. □

Next we claim that $(1+q(z))^{-1}$ exists for all z in the closed upper half plane. Assume for z_0 this fails to be true; then there is a $\psi \in L^2(\mathbb{R}^\nu \times E)$ such that

$$q(z_0)\psi + \psi = \lim_{\varepsilon\downarrow 0} q(z_0 + i\varepsilon)\psi + \psi = 0.$$

Using $[S^{-1}, (-\Delta - iA - z_0 - i\varepsilon)^{-1}] = 0$ in formula (5.6) and multiplying the last identify with S^{-1} it becomes

$$\lim_{\varepsilon \downarrow 0} T^{-1}(-\Delta - iA - z_0 - i\varepsilon)^{-1} T^{-1} W S^{-1}\psi + S^{-1}\psi = 0. \tag{5.9}$$

We now take the imaginary part of the inner product of (5.9) with $WS^{-1}\psi$ to obtain

$$\lim_{\varepsilon \downarrow 0} \sum_n (\lambda_n + \operatorname{im} z_0 + \varepsilon) \| (-\Delta - i\lambda_n - z_0 - i\varepsilon)^{-1} (T^{-1} W S^{-1}\psi)_n \|^2 = 0.$$

If $\operatorname{im} z_0 > 0$ this clearly implies $T^{-1}WS^{-1}\psi = 0$ and by (5.9) $\psi = 0$. If $\operatorname{im} z_0 = 0$ we only have $(T^{-1}WS^{-1}\psi)_n = 0$ for $n > 0$, but then

$$\phi = (-\Delta - iA - z_0 - i0)^{-1} T^{-1} W S^{-1}\psi \tag{5.10}$$

is independent of ξ (i.e. $\phi_n = 0$ for $n > 0$), and $\phi \in L^2_{-s/2}(\mathbb{R}^\nu)$, since by (5.9)

$$T^{-1}\phi = -S^{-1}\psi \in L^2(\mathbb{R}^\nu). \tag{5.11}$$

Using the distributional inverse to (5.10),

$$(-\Delta - z_0)\phi = T^{-1} W S^{-1}\psi,$$

we obtain, multiplying (5.11) by $WT^{-1} = VT$

$$(-\Delta + V(\xi, x))\phi(x) = z_0 \phi(x). \tag{5.12}$$

Since this distributional Schrödinger equation has to hold for μ—a.a. $\xi \in E$, (2.3)′ clearly implies

$$\phi_{\restriction \mathcal{O}} = 0. \tag{5.13}$$

Further $\phi \in L^2_{-s/2} \subset L^2_{\mathrm{loc}}$ and by (5.11),

$$\phi \in H^n_{\mathrm{loc}}(\mathbb{R}^\nu). \tag{5.14}$$

We are now in position to apply the

Lemma 5.3. *Assume $V \in L^\infty(\mathbb{R}^\nu)$ and let $\phi \in H^2_{\mathrm{loc}}(\mathbb{R}^\nu)$ satisfy*

$$(-\Delta + V)\phi = 0,$$

then if ϕ vanishes on some non-empty open set, it vanishes everywhere.

This is a special case of Theorem XIII.63 in [13]. Thus (5.12)–(5.14) imply $\phi = 0$ and (5.11) $\psi = 0$; the claim is proved. Now, since $\sup_{\operatorname{im} z \geq 0} \| (1 + q(z))^{-1} \| < \infty$ by the last statement of Lemma 5.1, $(1 + Q_1(z))^{-1}$ also exists for all z in the closed upper half plane and

$$\sup_{\operatorname{im} z \geq 0} \| (1 + Q_1(z))^{-1} \| = \sup_{\operatorname{im} z \geq 0} \sup_{k \in \mathbb{R}^\nu} \| (1 + q(z + k^2))^{-1} \| < \infty.$$

Further

$$\begin{aligned} \| F(k^2 > E)(1 - (1 + Q_1(z))^{-1}) f \|^2 &= \int_{k^2 > E} \| (1 - (1 + q(z + k^2))^{-1}) f(k) \|^2 d^\nu k \\ &\leq \sup_{k^2 > E} \| 1 - (1 + q(z + k^2))^{-1} \|^2 \| f \|^2, \end{aligned}$$

and thus applying once again the last statement of Lemma 5.1,

$$\lim_{E\to\infty} \| F(k^2 > E)(1-(1+Q_1(z))^{-1}) \| = 0,$$

uniformly on compact subsets of the closed upper half plane. Since clearly

$$F(k^2 < E)(1+Q_1(z))^{-1}$$

is norm continuous on the same set, we obtain norm continuity of $(1+Q_1(z))^{-1}$. On the open upper half plane we have

$$\frac{(1+Q_1(z))^{-1}-(1+Q_1(z'))^{-1}}{z-z'} = (1+Q_1(z))^{-1}\frac{Q_1(z')-Q_1(z)}{z-z'}(1+Q_1(z'))^{-1}$$
$$= (1+Q_1(z))^{-1}F_1\frac{R_0(z')-R_0(z)}{z-z'}G_1(1+Q_1(z'))^{-1}$$

from which

$$\frac{d}{dz}(1+Q_1(z))^{-1} = -(1+Q_1(z))^{-1}F_1R_0(z)^2G_1(1+Q_1(z))^{-1}$$

follows in the uniform topology. We just proved the

Proposition 5.4. *$(1+Q_j(z))^{-1}$ are bounded continuous functions from the closed upper half plane to the bounded operators, analytic in the open upper half plane.*

5.3. The Birman–Schwinger Kernel $Q(z)$. Recall

$$Q(z) = (1+Q_2(z))^{-1}(F_2R_0(z)G_1)(1+Q_1(z))^{-1}(F_1R_0(z)G_2). \tag{5.15}$$

The first and third factors are controled by Proposition 5.4, for the second and fourth we need the

Lemma 5.5. *Let $\phi\in C_0^\infty(\mathbb{R}^\nu)$, then the operator*

$$\phi(x)R_0(z)\phi(y)$$

is compact for $\operatorname{im} z > 0$.

Proof. As in the proof of Lemma 5.1 we have

$$\phi(x)R_0(z)\phi(y) = \bigoplus_n \phi(x)(p^2-k^2-i\lambda_n-z)^{-1}\phi(y),$$

and it will suffice to prove compactness of each summand. This is done in two steps:

(i) $\phi(x)(p^2-k^2-z)^{-1}F(p^2<E)\,\phi(y)$ is compact for im $z>0$.

We may write

$$\phi(x)(p^2-k^2-z)^{-1}F(p^2<E)\phi(y)$$
$$= (\phi(x)F(p^2<E))\left(\frac{k^2+1}{p^2-k^2-z}F(p^2<E)\right)((k^2+1)^{-1}\phi(y))$$
$$= (C_1\otimes I)B(p,k)(I\otimes C_2), \tag{5.16}$$

where C_1 and C_2 are compact while $B(p,k)\in L^\infty(\mathbb{R}^{2\nu})$. Assume for a while C_1, C_2 to be rank one

$$C_j=(f_j,\cdot)g_j. \quad (j=1,2).$$

Then (5.16) will be Hilbert–Schmidt since its kernel

$$\bar{f}_1(p')g_1(p)B(p,k')\bar{f}_2(k')g_2(k)$$

clearly is square integrable. The result now follows by approximating C_1 and C_2 by finite rank operators.

(ii) $\lim_{E\to\infty} \|\phi(x)(p^2-k^2-z)^{-1}F(p^2>E)\phi(y)\|=0.$

The proof of this statement is a simple modification of the proof of Lemma 3.8 in [14], we omit it. □

Now, consider for example

$$\begin{aligned} F_1R_0(z)G_2 &= -(1-\Delta_x)^{n/2}(1+x^2)^{-s/4}(1-\Delta_x)^{-n/2}\\ &\quad\cdot R_0(z)(1-\Delta_y)^{n/2}(1+y^2)^{-s/4}W(\xi,y)(1-\Delta_y)^{-n/2}\\ &= -(S_xT_x^{-1}S_x^{-1}T_x)(T_x^{-1}R_0(z)T_y^{-1})(T_yS_yT_y^{-1}S_y^{-1})(S_yW(\xi,y)S_y^{-1}) \end{aligned}$$

with obvious notation. The first, third and forth factors are bounded, and choosing $\phi\in C_0^\infty(\mathbb{R}^\nu)$ to be one near $x=0$ we clearly have

$$T_x^{-1}R_0(z)T_y^{-1}=\lim_{R\to\infty} T_x^{-1}\phi\left(\frac{x}{R}\right)R_0(z)\phi\left(\frac{y}{R}\right)T_y^{-1}$$

in the norm. Thus application of Lemma 5.5 gives compactness of $F_1R_0(z)G_2$, the same of course being true for $F_2R_0(z)G_1$. From (5.15) we obtain compactness and analyticity of $Q(z)$ in the open upper half plane. To see what happens as z becomes real, we note that combining (5.2) with (5.15),

$$Q(z)=(1+Q_2(z))^{-1}F_2(R_0(z)-R_1(z))G_2. \tag{5.17}$$

The critical term on the right-hand side of this identity can be controlled in exactly the same way as we do in Lemma 5.1,

$$\begin{aligned} F_2(R_1(z)-R_0(z))G_2 &= (S_yT_y^{-1}S_y^{-1}T_y)T_y^{-1}(R_0(z)-R_1(z))T_y^{-1}\\ &\quad\cdot(T_yS_yT_y^{-1}S_y^{-1})(S_yW(\xi,y)S_y^{-1}), \end{aligned}$$

and we need only to consider

$$T_y^{-1}(R_1(z)-R_0(z))T_y^{-1}=-i\int_0^\infty e^{izt}(e^{-i(-\Delta+V-iA)t}-e^{-i(-\Delta-A)t})\otimes T^{-1}e^{i\Delta t}T^{-1}dt. \tag{5.18}$$

Lemma 5.2 and dominated convergence theorem together imply continuity of (5.18) in the closed upper half plane.

Summarizing. $Q(z)$ is an analytic function on the open upper half plane, continuous on its closure with values in the compact operators.

As in subsection 5.2 we now claim $(1-Q(z))^{-1}$ to exist for all z in the closed upper half plane. Thus let us assume $\psi \in L^2(\mathbb{R}^{2\nu} \times E)$ such that

$$Q(z_0)\psi = \psi.$$

Multiplying with $(1 + Q_2(z_0))$ we obtain by (5.17),

$$F_2 R_1(z_0) G_2 \psi + \psi = \lim_{\varepsilon \downarrow 0} F_2 R_1(z_0 + i\varepsilon) G_2 \psi + \psi = 0,$$

or more explicitly and after multiplication by S_y^{-1},

$$\lim_{\varepsilon \downarrow 0} T_y^{-1} R_1(z_0 + i\varepsilon) T_y^{-1} W(\xi, y) S_y^{-1} \psi = S_y^{-1} \psi. \tag{5.19}$$

Taking the imaginary part of the inner product of (5.19) with $W(\xi, y)S_y^{-1}\psi$, we arrive at

$$\lim_{\varepsilon \downarrow 0} \sum_n (\lambda_n + \operatorname{im} z_0 + \varepsilon) \| (R_1(z_0 + i\varepsilon) T_y^{-1} W(\xi, y) S_y^{-1} \psi)_n \|^2 = 0.$$

If $\operatorname{im} z_0 > 0$ this implies $R_1(z_0) T_y^{-1} W(\xi, y) S_y^{-1} \psi = 0$ and thus $\psi = 0$. If $\operatorname{im} z_0 = 0$ we have only that

$$\phi = R_1(z_0 + i0) T_y^{-1} W(\xi, y) S_y^{-1} \psi \tag{5.20}$$

is independent of ξ. We now may write (5.19) as

$$T_y^{-1} \phi = S_y^{-1} \psi, \tag{5.21}$$

from which we conclude

$$\phi \in L^2(\mathbb{R}^\nu) \otimes H^n_{\mathrm{loc}}(\mathbb{R}^\nu) \tag{5.22}$$

(which is a shorthand for $f(y)\phi \in L^2(\mathbb{R}^\nu) \otimes H^n(\mathbb{R}^\nu)$ $\forall f \in C_0^\infty(\mathbb{R}^\nu)$.) Multiplying (5.21) by $T_y^{-1} W(\xi, y)$ and using

$$(p^2 - k^2 - iA + V(\xi, x) - z_0)\phi = T_y^{-1} W(\xi, y) S_y^{-1} \psi,$$

which is the distributional inverse of (5.20), we easily obtain

$$(-\Delta_x + V(\xi, x) + \Delta_y - V(\xi, y))\phi(x, y) = z_0 \phi(x, y). \tag{5.23}$$

From this and (5.22) we further obtain $\phi \in H^2(\mathbb{R}^\nu) \otimes H^{n-2}_{\mathrm{loc}}(\mathbb{R}^\nu)$. As in subsection 5.2 we conclude from (5.23) and (2.3)

$$\phi \restriction_{\mathbb{R}^\nu \times \mathcal{O}} = 0. \tag{5.24}$$

We are ready to apply

Lemma 5.6. *Assume $V \in L^\infty(\mathbb{R}^\nu)$ and let $\phi \in H^2(\mathbb{R}^\nu) \otimes H^2_{\mathrm{loc}}(\mathbb{R}^\nu)$ satisfy $(-\Delta_x + V(x))\phi(x, y) - (-\Delta_y + V(y))\phi(x, y) = z\phi(x, y)$, then if for some non-empty open set $\mathcal{O} \subset \mathbb{R}^\nu$*

$$\phi_{\restriction \mathbb{R}^\nu \times \mathcal{O}} = 0$$

ϕ vanishes everywhere.

Proof. $H = -\Delta_x + V(x)$ is self adjoint on $H^2(\mathbb{R}^\nu)$, and there is a measure space (M, ν) such that for some function f on M

$$L^2(\mathbb{R}^\nu) \simeq L^2(M, d\nu),$$
$$H^2(\mathbb{R}^\nu) \simeq \{u(m) \mid f(m)u(m) \in L^2(M, d\nu)\},$$

H acting on $L^2(M)$ by multiplication with f. (This is the spectral theorem, see [15].) Thus

$$L^2(\mathbb{R}^\nu) \otimes H^2_{\mathrm{loc}}(\mathbb{R}^\nu) \simeq L^2(M) \otimes H^2_{\mathrm{loc}}(\mathbb{R}^\nu)$$
$$\phi(x, y) \leftrightarrow \tilde{\phi}\,(m, y)$$

and

$$(-\Delta_y + V(y))\tilde{\phi}(m, y) = (f(m) + z)\tilde{\phi}(m, y)$$
$$\tilde{\phi}_{\restriction M \times \mathcal{O}} = 0.$$

Application of Lemma 5.3 gives the result. □

From (5.22)–(5.24) we get $\phi = 0$, and from (5.21) $\psi = 0$ proving the claim.

Proposition 5.7. $(1 + Q(z))^{-1}$ *is an analytic function from the open upper half plane to the bounded operators, continuous on the closed upper half plane.*

5.4 The Resolvent $R(z)$. Proposition 5.4 and 5.7 together with formulae (5.2) (5.3) clearly imply analyticity of $R(z)$ in the open upper half plane, i.e.

$$\{\operatorname{im} z > 0\} \subset \rho(L').$$

For $\delta > 1$

$$(1 + y^2)^{-\delta/2} R_1(z)(1 + y^2)^{-\delta/2} = i \int_0^\infty e^{izt} e^{-i(-\Delta + V - iA)t} \otimes (1 + y^2)^{-\delta/2} e^{-i\Delta t}(1 + y^2)^{-\delta/2} dt$$

has, by Lemma 5.2, a continuous extension to $\{\operatorname{im} z \geqq 0\}$. In the same way, the same property holds for

$$(1 + y^2)^{-\delta/2} R_1(z) G_2 \quad \text{and} \quad F_2 R_1(z)(1 + y^2)^{-\delta/2}.$$

Thus by formula (5.3)

$$(1 + y^2)^{-\delta/2} R(z)(1 + y^2)^{-\delta/2} \tag{5.25}$$

is continuous on $\{\operatorname{im} z \geqq 0\}$ for $\delta > 1$. Clearly the same is true if we replace y by x in (5.25). Interpolating between the two cases we obtain our

Theorem 5. *The resolvent* $R(z) = (L' - z)^{-1}$ *is analytic in the open upper half plane, and extend to a continuous function from its closure to*

$$\mathscr{B}(L^2_\delta(\mathbb{R}^{2\nu}) \otimes L^2(E), L^2_{-\delta}(\mathbb{R}^{2\nu}) \otimes L^2(E))$$

for any $\delta > \frac{1}{2}$.

6. Confined Systems

In this section we consider the random time dependence of the potential as a perturbation, the unperturbed potential satisfying:

$$U(x) \geqq 0, \tag{6.1}$$

$$U \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{\nu}), \tag{6.2}$$

$$\lim_{|x| \to \infty} U(x) = +\infty. \tag{6.3}$$

Since the argument of this section is very close to [1], we will be very sketchy. The unperturbed Hamiltonian

$$H_0 = -\Delta + U(x), \tag{6.4}$$

well defined as form sum, is a selfadjoint positive operator with compact resolvent (see for example [9] and [13]). We perturb it in the usual way with a potential $V(\xi, x)$ which we assume continuous in the first, and two times continuously differentiable in the second argument, with bounded derivatives. We also assume the non-triviality condition (2.3), and the usual hypotheses on the process $\xi(\cdot)$ (see [1]). Then the general results of [1] may be applied to the quantum evolution generated by

$$H(t) = H_0 + V(\xi(t), x). \tag{6.5}$$

The only problem is to verify the spectral condition ($\sum$). This may be done as in the appendix of [1], provided we can extend the unique continuation theorem used there to the operators

$$H(\xi) = H_0 + V(\xi, x) \quad (\xi \in E).$$

The proof of such a result is an easy modification of that given in [13] once we note, as a simple consequence of (6.1)–(6.3):

$$D(H_0) = D(H(\xi)) \subset H^2_{\mathrm{loc}}(\mathbb{R}^{\nu}).$$

Thus under our assumptions

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \| C U(t,0) f \|^2 dt = 0$$

with probability one for any compact C and any state f. Applying this to the spectral projection

$$C = F(H_0 < E),$$

which is compact for any finite energy E, since H_0 has compact resolvent, we obtain the desired result:

> Any state f has, under the time evolution $U(t,s)$ generated by (6.4)–(6.5), an unbounded H_0-energy (with probability one).

For similar but stronger results on the perturbed harmonic oscillator see [17].

Appendix. The Asymptotic Projectors $P_{\pm}$; Proof of Lemma 4.1.

Here we start with the momentum representation

$$\mathscr{H} = L^2(\mathbb{R}^{\nu}, d^{\nu}p),$$

and thus look at $\vec{x}$ as a differential operator

$$\vec{x} = i\frac{\partial}{\partial \vec{p}}.$$

All Sobolev spaces over $\mathbb{R}^\nu$ in this appendix have to be understood in this setting, and we will drop any mention of the independent variable $\vec{p}$. The formula

$$(Jf)(\lambda,\omega) = \begin{cases} 0 & \text{if } \lambda < 0 \\ \lambda^{(\nu/2-1)/2} f(\lambda^{1/2}\omega) & \text{else} \end{cases} \tag{A.1}$$

clearly defines a partial isometry

$$J\colon \mathscr{H} \to \mathscr{K} = L^2(\mathbb{R}, d\lambda) \otimes L^2(S^{\nu-1}, d\omega) \tag{A.2}$$

whose range $\mathscr{K}_+ = L^2(\mathbb{R}_+, d\lambda) \otimes L^2(S^{\nu-1}, d\omega)$ is nothing but the space of spectral representation for the free Hamiltonian p^2, thus

$$\begin{aligned} J^+J &= I, \\ JJ^+ &= F(\lambda > 0). \end{aligned} \tag{A.3}$$

Let us also define

$$S = i\frac{\partial}{\partial\lambda} \tag{A.4}$$

on $\mathscr{K}$. Then λ and S are canonically conjugated operators and in particular the free evolution e^{-ip^2t} acts as a shift on S:

$$SJe^{-ip^2t} = Se^{-i\lambda t}J = e^{-i\lambda t}(S+t)J. \tag{A.5}$$

This strongly suggests to set

$$P_\pm = J^+F(S\in\mathbb{R}_\pm)J.$$

The first immediate consequences are:

$$\begin{aligned} P_+ + P_- &= J^+\{F(S>0) + F(S<0)\}J = J^+J = I \\ \|P_\pm\| &\leqq 1, \end{aligned}$$

and a short computation using (A.3) gives

$$P_\pm^* = P_\pm.$$

Also very easy is:

$$\begin{aligned} \|P_\pm e^{-ip^2t}f\| &= \|J^+F(S \lessgtr 0)Je^{-ip^2t}f\| \\ &\leqq \|F(S \lessgtr 0)e^{-i\lambda t}Jf\| \\ &\leqq \|e^{-i\lambda t}F(S+t \lessgtr 0)Jf\| \quad \text{by (A.5)} \\ &\leqq \|F(|S| > |t|)Jf\| \to 0 \quad \text{as} \quad t \to \pm\infty, \end{aligned}$$

i.e.

$$\operatorname*{s-lim}_{t\to\pm\infty} P_\pm e^{-ip^2t} = 0.$$

The norm estimate of Lemma 4.1 requires some more work, but the idea is very simple.

We claim for $v \geqq 3$ and $\sigma > 2$:

$$J \in \mathscr{B}(H^\sigma(\mathbb{R}^v), H^1(\mathbb{R}) \otimes L^2(S^{v-1})). \tag{A.6}$$

From this let us prove our norm estimate; for simplicity we only consider the case $t \to +\infty$:

$$\begin{aligned}
\| P_- e^{-ip^2t}(1+x^2)^{-\sigma/2} f \| &= \| J^+ F(S<0) J e^{-ip^2t}(1+x^2)^{-\sigma/2} f \| \\
&\leqq \| F(S<0) e^{-i\lambda t} J(1+x^2)^{-\sigma/2} f \| \\
&\leqq \| F(S<-t) J(1+x^2)^{-\sigma/2} f \| \\
&\leqq \| F(S<-t) S^{-1} S J(1+x^2)^{-\sigma/2} f \| \\
&\leqq \| F(S<-t) S^{-1} \| \, \| S J(1+x^2)^{-\sigma/2} f \| \\
&\leqq t^{-1} |||J||| \, \| f \|,
\end{aligned}$$

where $|||J|||$ is the norm corresponding to (A.6), and we are done. Let us now prove the claim in three simple steps:

Step 1.

$$J \in \mathscr{B}(H^\sigma(\mathbb{R}^v), H^1(\mathbb{R}_+) \otimes L^2(S^{v-1})).$$

Let $f \in C_0^\infty(\mathbb{R}^v)$, then a simple computation shows

$$(i\partial_\lambda Jf)(\lambda, \omega) = (JTf)(\lambda, \omega) \quad \text{for} \quad \lambda > 0, \tag{A.7}$$

where T is the formal time operator

$$T = \tfrac{1}{4}\{p^{-2}\vec{p}\cdot\vec{x} + \vec{x}\cdot\vec{p}p^{-2}\} = \frac{i}{4}(v-2)p^{-2} + \frac{i}{2}p^{-1}\partial_p.$$

But $\partial_p = \hat{p}\cdot\vec{x} \in \mathscr{B}(H^\sigma(\mathbb{R}^v), H^{\sigma-1}(\mathbb{R}^v))$ and a simple estimate gives $p^{-r} \in \mathscr{B}(H^s(\mathbb{R}^v), L^2(\mathbb{R}^v))$ if $r < \operatorname{Min}(v, s)$. Thus with our assumptions $v \geqq 3$ and $\sigma > 2$ we obtain

$$T \in \mathscr{B}(H^\sigma(\mathbb{R}^v), L^2(\mathbb{R}^v)),$$

and the first step is achieved by (A.7) and a density argument.

Step 2.

$$J \in \mathscr{B}(H^\sigma(\mathbb{R}^v), H_0^1(\mathbb{R}_+) \otimes L^2(S^{v-1})).$$

By a well known characterisation of $H_0^1(\mathbb{R}_+)$ (see for example [16]) it suffices to show

$$\lambda^{-1}(Jf)(\lambda, \omega) \in L^2(\mathbb{R}_+ \times S^{v-1}) \quad \text{for} \quad f \in H^\sigma(\mathbb{R}^v).$$

But

$$\lambda^{-1} Jf = Jp^{-2} f,$$

and by the same estimate as in step 1 $p^{-2} \in \mathscr{B}(H^\sigma, L^2)$, which prove step 2.

Step 3. Use the following standard fact in Sobolev technology (see also [16]), i being the imbedding of $L^2(\mathbb{R}_+)$ in $L^2(\mathbb{R})$:

$$i \in \mathscr{B}\ (H_0^1(\mathbb{R}_+), H^1(\mathbb{R})),$$

from which the claim clearly follows.

Finally let us prove the compactness statement of Lemma 4.1. We consider only the case $t > 0$, i.e.

$$P_- e^{-ip^2 t}\phi(x) = P_- e^{-ip^2 t}F(|x| < R)\phi(x) + P_- e^{-ip^2 t}F(|x| > R)\phi(x).$$

Since the second term in the right-hand side vanishes in norm as $R \to \infty$ by our assumptions on ϕ, we need only to show the first term to be compact for all $R > 0$, this can be further written as

$$P_- e^{-ip^2 t}(1 + x^2)^{-1/2}\{(1 + x^2)^{1/2}F(|x| < R)\phi(x)\},$$

and we need only to consider the case $\phi(x) = (1 + x^2)^{-1/2}$. To get a further decomposition assume a function $\chi \in C^\infty(\mathbb{R})$ to be given with the properties:

(i) $0 \leqq \chi \leqq 1$

(ii) $\chi(\lambda) = \begin{cases} 0 \text{ if } \lambda \leqq 1 \\ 1 \text{ if } \lambda \geqq 2 \end{cases}$

(iii) $\sup\limits_{\lambda} |\chi'(\lambda)| \leqq 2.$

We then set $\chi_E(\lambda) = \chi(\lambda/E)$ for any $E > 0$ and note by (iii)

$$\|\chi_E'\|_\infty \leqq 2E^{-1}. \tag{A.8}$$

Then

$$\begin{aligned} P_- e^{-ip^2 t}(1 + x^2)^{-1/2} = {} & P_- e^{-ip^2 t}\chi_E(p^2)(1 + x^2)^{-1/2} \\ & + P_- e^{-ip^2 t}(1 - \chi_E(p^2))(1 + x^2)^{-1/2}, \end{aligned}$$

and the second term in the right-hand side being clearly compact, we only need to prove the vanishing in norm of the first term as $E \to \infty$. To do that we first look at

$$\begin{aligned} \| |S| J\chi_E(p^2)(1 + x^2)^{-1/2} f \| &= \| S\chi_E(\lambda)J(1 + x^2)^{-1/2} f \| \\ &= \| \{\chi_E(\lambda)S + [S, \chi_E(\lambda)]\} J(1 + x^2)^{-1/2} f \| \\ &\leqq \| \chi_E(\lambda)SJ(1 + x^2)^{-1/2} f \| + \| \chi_E'(\lambda)J(1 + x^2)^{-1/2} f \|, \end{aligned}$$

and use (A.7):

$$F(\lambda > 0)\, SJ = JT$$

i.e.

$$\chi_E(\lambda)F(\lambda > 0)SJ = \chi_E(\lambda)SJ = \chi_E(\lambda)JT = J\chi_E(p^2)T$$

from which

$$\begin{aligned} \| |S| J\chi_E(p^2)(1 + x^2)^{-1/2} f \| &\leqq \| \chi_E(p^2)T(1 + x^2)^{-1/2} f \| + \| \chi_E'(p^2)(1 + x^2)^{-1/2} f \| \\ &\leqq \left\| \chi_E(p^2)\left\{\frac{i}{4}(\nu - 2)p^{-2} + \frac{i}{2}p^{-1}\hat{p}\cdot\vec{x}\right\}(1 + x^2)^{-1/2} f \| + \| \chi_E'(p^2)(1 + x^2)^{-1/2} f \right\| \\ &\leqq \left(\frac{\nu - 2}{4}E^{-1} + \tfrac{1}{2}E^{-1/2}\right)\| f \| + 2E^{-1}\| f \| < \text{Const}\, E^{-1/2}\| f \|, \end{aligned}$$

where we have used the properties (i)–(iii) and in particular (A.8). Then we obtain:

$$\begin{aligned}\| P_- e^{-ip^2t}\chi_E(p^2)(1+x^2)^{-1/2} \| &\leqq \| F(S<0)e^{-i\lambda t}J\chi_E(p^2)(1+x^2)^{-1/2} \| \\ &\leqq \| F(S<-t)J\chi_E(p^2)(1+x^2)^{-1/2} \| \\ &\leqq \left\| \left| \frac{S}{t} \right| J\chi_E(p^2)(1+x^2)^{-1/2} \right\| \leqq \frac{\text{Const}}{tE^{1/2}} \to 0. \qquad \square\end{aligned}$$

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Note added in proof. It is possible to replace condition 2.3 by the more natural one $\mathrm{Var}(V(x,\cdot)) \not\equiv \text{Const}$.