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Sufficient Subalgebras and the Relative Entropy of States of a von Neumann Algebra

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Abstract. A subalgebra M_0 of a von Neumann algebra M is called weakly sufficient with respect to a pair (ϕ, ω) of states if the relative entropy of ϕ and ω coincides with the relative entropy of their restrictions to M_0. The main result says that M_0 is weakly sufficient for (ϕ, ω) if and only if M_0 contains the Radon-Nikodym cocycle $[D\phi, D\omega]_t$. Other conditions are formulated in terms of generalized conditional expectations and the relative Hamiltonian.

Introduction

Suppose that an experiment is described by a measurable space $(X, \mathscr{S})$ and the outcome of the experiment is governed by a probability measure μ on $\mathscr{S}$. In mathematical statistics the probability distribution μ belongs to a family Θ of probability distributions. A sub-σ-algebra $\mathscr{R}$ of $\mathscr{S}$ can be considered as application of a statistics or an indirect observation. $\mathscr{R}$ is defined to be sufficient with respect to the family Θ if the conditional distribution of $\mu \in \Theta$ does not depend on μ. More precisely, for every $S \in \mathscr{S}$ there exist an $\mathscr{R}$-measurable function ξ_S such that $\int_R \xi_S \, d\mu = \mu(R \cap S)$ for every $R \in \mathscr{R}$ and $\mu \in \Theta$ (see [12 or 7, 28]). In mathematical statistics the case $|\Theta| = 2$ is called the discrimination between two statistical hypotheses.

Let $\Theta = \{\mu, \nu\}$, and assume that $\mu, \nu \ll \lambda$. Halmos and Savage ([12]) proved that $\mathscr{R}$ is sufficient for $\{\mu, \nu\}$ if and only if the function

$$\frac{d\mu}{d\lambda} \Big/ \frac{d\nu}{d\lambda}$$

is $\mathscr{R}$-measurable. Another equivalent formulation due to Kullback and Leibler [18] is based on the relative entropy of measures. Namely, $\mathscr{R}$ is sufficient if and only if

$$S(\mu, \nu) = S(\mu_0, \nu_0),$$

Permanent address: Mathematical Institute of HAS, Reáltanoda u. 13–15, H-1364 Budapest, P.O. Box. 127, Hungary where $S(\mu, \nu)$ ($S(\mu_0, \nu_0)$) denotes the relative entropy of μ and ν (that of their restrictions to $\mathscr{R}$).

The main goal of this paper is to investigate the equivalencies above in the noncommutative case. In the algebraic approach to quantum mechanics [10] a physical system corresponds to an operator algebra M, and a subalgebra M_0 of M is selected for experimental observation. The determination through the coarse grained algebra M_0 provides only partial information about the state of the system. Mathematically, instead of a state (that is, a positive functional) of M its restriction to M_0 is at our disposal (cf. [31]). Discussions concerning sufficiency of a subalgebra have been made first by Umegaki ([30], see also [11, 15]). The subalgebra M_0 is sufficient with respect to the states ϕ and ω of M, provided there is a conditional expectation E (in the sense of Umegaki, see [30]) onto M_0 preserving ϕ and ω, i.e. $\phi \circ E = \phi$ and $\omega \circ E = \omega$. By Takesaki's theorem ([29], see also [27, 10.1]) projection of norm one onto a subalgebra preserving a given state is an infrequent object in the theory of operator algebras. Hence, the definition above is rather restrictive. On the other hand, in the present shape of quantum probability there exists no satisfactory mathematical formalism for conditioning. Therefore we try to involve the notion of relative entropy of states in the study of sufficiency instead of considering conditional expectations.

The relative entropy in an operator algebra setup was first studied by Umegaki [30] and it has provided a useful tool for mathematical physics in the papers of Lindblad [20, 21]. Araki [4] extended the notion of relative entropy to the case of normal states of arbitrary von Neumann algebra and its relation to the sufficiency problem was discussed also by Hiai et al. [13, 14].

In [23] we used the following definition. The subalgebra M_0 of M is weakly sufficient for the pair (ϕ, ω) if the relative entropy of ϕ and ω coincides with the relative entropy of their restrictions to M_0. Applying a method of Holevo [15] we characterized weakly sufficient commutative subalgebras of finite dimensional operator algebras. In the present paper we drop the assumption on commutativity and use a different method based on quasi-entropies (see [23, 24]. We prove that M_0 is weakly sufficient for (ϕ, ω) if and only if the Radon-Nikodym cocycle $[D\phi, D\omega]_t$ is in M_0. This is equivalent to the condition $E_\phi = E_\omega$, where E_ψ denotes the ψ-conditional expectation of Accardi and Cecchini [1] with respect to a state Ψ.

Preliminaries

Let M be a von Neumann algebra with a faithful normal state ω. We can assume that M acts on a Hilbert space H and ω is determined by a cyclic separating vector Ω. If ϕ is another normal state, then the densely defined quadratic form $a\Omega \to \phi(aa^*)$ ($a \in M$) is closable and there exists an associated positive selfadjoint operator Δ. It is characterized by the following properties. $M\Omega$ is a core for $\Delta^{1/2}$ and $\|\Delta^{1/2} a\Omega\|^2 = \phi(aa^*)$ [17, VI, Sect. 2]. Δ was called by Araki the relative modular operator of ϕ and ω and it is usually denoted by $\Delta(\phi, \omega)$ (see [4, 5]).

The modular group of ω is a one-parameter group of automorphisms of M and it looks like

$$\sigma_t^\omega(a) = \Delta(\omega, \omega)^{it} a \Delta(\omega, \omega)^{-it}.$$

Another Radon-Nikodym derivative-like object for comparison of two states is the Radon-Nikodym cocycle discovered by Connes ([6], see also [5]). If ϕ is a faithful normal semifinite weight, then $[D\phi, D\omega]_t = u_t$ is a σ^ω-cocycle and $\sigma_t^\phi = u_t \sigma_t^\omega u_t^*$. The formula $[D\phi, D\omega]_t = \Delta(\phi, \omega)^{it} \Delta(\omega, \omega)^{-it}$ forms a bridge between the two objects (see [5]).

The inner perturbation of a state was studied by Araki [2]. If $h = h^* \in M$, then there is a state ω^h determined by

$$[D\omega^h, D\omega]_t = e^{it(H+h)} e^{-itH},$$

where H stands for $\log \Delta(\omega, \omega)$. The main properties of the perturbed state are summarized in [3].

Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function. The quasi-entropy $S_f(\phi, \omega)$ is defined as $\langle f(\Delta(\phi, \omega))\Omega, \Omega \rangle$. It possesses nice convexity properties when f is operator convex (see [23, 24]). The choice $f(t) = -\log t$ gives the relative entropy:

$$S(\phi, \omega) = -\langle \log \Delta(\phi, \omega)\Omega, \Omega \rangle .$$

Monotonicity of quasi-entropies means the following assertion. Assume that M_0 and M are von Neumann algebras and $\alpha: M_0 \to M$ is a 2-positive unit preserving mapping. If ϕ and ω are faithful normal states on M, then $S_f(\phi \circ \alpha, \omega \circ \alpha) \leqq S_f(\phi, \omega)$, provided that $-f$ is operator monotone ([24]).

If M_0 is a subalgebra of M, then ω-preserving conditional expectation of M onto M_0 does not exists always. However, there exists a very natural completely positive mapping $E_\omega: M \to M_0$, which is defined by Accardi and Cecchini ([1]) and may be called generalized conditional expectation for several reasons. The ω-conditional expectation E_ω has a convenient description using standard forms [22]. Assume that $M(M_0)$ has the standard form $(M, H, J, \mathscr{P})$ $((M_0, H_0, J_0, \mathscr{P}_0))$, and let $\iota: M_0 \to M$ be the embedding. If $\Omega \in \mathscr{P}$ $(\Omega_0 \in \mathscr{P}_0)$ is the vector representative for the state ω (ω restricted to M_0) then

$$V_\omega a \Omega_0 = \iota(a)\Omega \qquad (a \in M_0)$$

defines a contraction of H_0 into H. The ω-conditional expectation is defined as

$$E_\omega(b) = J_0 V_\omega^* J b J V_\omega J_0 \qquad (b \in M).$$

Concerning the ω-conditional expectation we refer to the papers [1] and [20].

Main Result

Let M be a von Neumann algebra with faithful normal states ϕ and ω. When M_0 is a von Neumann subalgebra of M, then ϕ_0 and ω_0 stand for the the restriction of ϕ and ω, respectively. By the monotonicity property of the relative entropy, we have $S(\phi_0, \omega_0) \leqq S(\phi, \omega)$. We say that M_0 is weakly sufficient with respect to the pair (ϕ, ω) if $S(\phi_0, \omega_0) = S(\phi, \omega)$ (see [23]). Recall that M_0 is sufficient if there exists a conditional expectation $E: M \to M_0$ such that $\phi \circ E = \phi$ and $\omega \circ E = \omega$. In particular, sufficiency implies weak sufficiency.

In the rest of this section we assume that M acts on a Hilbert space H and ω is given by the cyclic and separating vector $\Omega \in H$. We consider the natural representation of M_0 on $H_0 = [M_0\Omega]$, and P denotes the orthogonal projection of H onto H_0. We shall shorten our notation to Δ (Δ_0) for the relative modular operator $\Delta(\phi, \omega)$ ($\Delta(\phi_0, \omega_0)$).

Lemma 1. *Assume that M_0 is weakly sufficient for (ϕ, ω) and $S(\phi, \omega)$ is finite. Then*

$$S_{f_t}(\phi, \omega) = S_{f_t}(\phi_0, \omega_0),$$

where $f_t(\lambda) = 1/\lambda + t$ $(0 < t < \infty)$.

Proof. Using the formula

$$\log x = \int_0^\infty (1+t)^{-1} - (x+t)^{-1}\, dt,$$

we write

$$\begin{aligned} S(\phi, \omega) &= -\langle \log \Delta \Omega, \Omega \rangle = \int_0^\infty \langle (\Delta + t)^{-1}\Omega, \Omega\rangle - (1+t)^{-1}\, dt \\ &= \int_0^\infty S_{f_t}(\phi, \omega) - (1+t)^{-1}\, dt. \end{aligned}$$

Since $-f_t$ is operator monotone, by the monotonicity of the quasi-entropies, we have $S_{f_t}(\phi_0, \omega_0) \leqq S_{f_t}(\phi, \omega)$ and $S(\phi, \omega) = S(\phi_0, \omega_0)$ implies that $S_{f_t}(\phi, \omega) = S_{f_t}(\phi_0, \omega_0)$ for almost all $t \in \mathbb{R}^+$. By continuity we obtain the equality for all $t \in \mathbb{R}^+$.

Lemma 2. *For $\lambda > 0$, we have*

$$P(\lambda + P\Delta_0 P)^{-1} P \leqq P(\lambda + \Delta)^{-1} P.$$

Proof. Δ is the associated positive selfadjoint operator to the densely defined closable quadratic form $q: a\Omega \to \phi(aa^*)$, $(a \in M)$ and Δ_0 is associated to the restriction $q|M_0\Omega$. Let $\int_0^\infty \lambda\, dE(\lambda)$ be the spectral decomposition of Δ, and define $H_n = \int_0^n \lambda\, dE(\lambda)$. Then $(\lambda + H_n)^{-1} \to (\lambda + \Delta)^{-1}$ strongly for all $\lambda > 0$.

The set $\{a\Omega + \eta : \eta \in H,\ P\eta = 0,\ a \in M_0\}$ is a core for $(P\Delta_0 P)^{1/2} = P\Delta_0^{1/2} P$. Evidently,

$$\begin{aligned} \|(P\Delta_0 P)^{1/2}(a\Omega + \eta)\|^2 &= \|\Delta_0^{1/2} a\Omega\|^2 = \phi(aa^*) = \|\Delta^{1/2} a\Omega\|^2 \\ &\geqq \|(PH_n P)^{1/2}(a\Omega + \eta)\|^2, \end{aligned}$$

and we establish $PH_nP \leqq P\Delta_0 P$. Then

$$(\lambda + P\Delta_0 P)^{-1} \leqq (\lambda + PH_n P)^{-1}.$$

Since

$$P(\lambda + PH_n P)^{-1} P \leqq P(\lambda + H_n)^{-1} P$$

(see [8]) we arrive at

$$P(\lambda + P\Delta_0 P)^{-1} P \leqq P(\lambda + H_n)^{-1} P,$$

and letting $n \to \infty$, we complete the proof.

Lemma 3. *Assume that*

$$\langle(\lambda+P\Delta_0P)^{-1}\Omega,\Omega\rangle=\langle(\lambda+\Delta)^{-1}\Omega,\Omega\rangle$$

for all $\lambda>0$. *Then* $f(\Delta)\Omega=f(\Delta_0)\Omega$ *for every bounded continuous function* $f:\mathbb{R}^+\to\mathbb{C}$.

Proof. Using Lemma 2 we can estimate

$$\|P(\lambda+\Delta)^{-1}\Omega-(\lambda+\Delta_0)^{-1}\Omega\|^2=\|(P(\lambda+\Delta)^{-1}P-P(\lambda+P\Delta_0P)^{-1}P)\Omega\|^2$$
$$\leqq\|(P(\lambda+\Delta)^{-1}P-P(\lambda+P\Delta_0P)^{-1}P)^{1/2}\|^2\ \langle P(\lambda+\Delta)^{-1}\Omega-(\lambda+\Delta_0)^{-1}\Omega,\Omega\rangle,$$

whenever $\lambda>0$. By the hypothesis, we have $P(\lambda+\Delta)^{-1}\Omega=(\lambda+\Delta_0)^{-1}\Omega$ for all $\lambda>0$, and through analytic continuation for all $\lambda\in\mathbb{C}-\mathbb{R}^-$. Derivating by λ we obtain $P(\lambda+\Delta)^{-2}\Omega=(\lambda+\Delta_0)^{-2}\Omega$. Hence

$$\|(\lambda+\Delta)^{-1}\Omega\|^2=\langle(\lambda+\Delta)^{-2}\Omega,\Omega\rangle=\langle(\lambda+\Delta_0)^{-2}\Omega,\Omega\rangle$$
$$=\langle P(\lambda+\Delta)^{-1}\Omega,(\lambda+\Delta)^{-1}\Omega\rangle,$$

and $(\lambda+\Delta)^{-1}\Omega$ must be in H_0. So $(\lambda+\Delta)^{-1}\Omega=(\lambda+\Delta_0)^{-1}\Omega$, $(\lambda\in\mathbb{C}-\mathbb{R}^-)$. Looking at the formula for spectral resolution [9, p. 1202], we can establish $E((s,t),\Delta)\Omega=E((s,t),\Delta_0)\Omega$ for every open interval $(s,t)\subset\mathbb{R}$. Reference to the spectral theorem makes the proof complete.

Theorem 4. *Assume that* $S(\phi,\omega)$ *is finite. For a subalgebra* M_0 *of* M *the following conditions are equivalent.*

(i) $[D\phi,D\omega]_t\in M_0$ *for every* $t\in\mathbb{R}$.
(ii) *There exists a subalgebra of* M_0 *which is sufficient for* (ϕ,ω).
(iii) M_0 *is weakly sufficient for* (ϕ,ω).
(iv) $[D\phi,D\omega]_t=[D\phi_0,D\omega_0]_t$ *for every* $t\in\mathbb{R}$.
(v) $E_\phi=E_\omega$.
(vi) $\phi\circ E_\omega=\phi$.
(vii) $\omega\circ E_\phi=\omega$.

Proof. Suppose (i) and let N be the subalgebra generated by $\{[D\phi,D\omega]_t:t\in\mathbb{R}\}$. Then N is stable under the modular group of ω and there is a conditional expectation $E:M\to N$ preserving ω. The converse of Connes' theorem [6, 27, 5.1] guarantees a weight ϕ' on N such that $[D\phi,D\omega]_t=[D\phi',D(\omega|N)]_t$. On the other hand $[D(\phi'\circ E),D\omega]_t=[D\phi',D(\omega|N)]_t$ (see [27, 10.5]) and $\phi'\circ E$ must be ϕ. Consequently, E preserves also ϕ.

(ii) implies (iii) according to the monotonicity theorem.

To prove (iii)→(iv), we first note that

$$[D\phi,D\omega]_t\Omega=\Delta(\phi,\omega)^{it}\,\Delta(\omega,\omega)^{-it}\Omega=\Delta^{it}\Omega,$$

and similarly $[D\phi_0,D\omega_0]_t\Omega=\Delta_0^{it}\Omega$. Due to Lemma 3 $[D\phi,D\omega]_t\Omega=[D\phi_0,D\omega_0]_t\Omega$, and we conclude (iv).

In order to prove (iv)→(v), let $(M,H,J,\mathscr{P})$ ($(M_0,H_0,J_0,\mathscr{P}_0)$) be the standard form of M and M_0, respectively. Define the contractions $V_\omega,V_\phi:H_0\to H$ by the formulas

$$V_\omega a\Omega=\iota(a)\Omega\quad\text{and}\quad V_\phi a\Phi_0=\iota(a)\Phi\qquad(a\in M_0),$$

where $\Omega \in \mathscr{P}$ and $\Phi \in \mathscr{P}$ ($\Omega \in \mathscr{P}_0$ and $\Phi_0 \in \mathscr{P}_0$) are the vector representatives for the states ω and ϕ (ω_0 and ϕ_0), moreover $\imath : M_0 \to M$ is the inclusion. By assumption

$$V_\omega(a[D\phi_0, D\omega_0]_t \Omega) = \imath(a)\,[D\phi, D\omega]_t \Omega$$

for all $a \in M_0$ and $t \in \mathbb{R}$. Since $[D\phi_0, D\omega_0]_t \Omega = \Delta(\phi_0, \omega_0)^{it} \Omega$, and $\Omega \in D(\Delta(\phi_0, \omega_0)^{1/2})$ [in fact, $\Delta(\phi_0, \omega_0)^{1/2} \Omega = \Phi_0$] the function

$$F_a(t) : t \to V_\omega(a[D\phi_0, D\omega_0]_t \Omega)$$

has continuous extension to the strip $S = \{z \in \mathbb{C} : -1/2 \leqq \operatorname{Im} z \leqq 0\}$, which is analytic on $S^0 = \{z \in \mathbb{C} : -1/2 < \operatorname{Im} z < 0\}$ [26, 9.15]. Similarly, $G_a(z) = a\Delta(\phi_0, \omega_0)^{it} \Omega$ is continuous on S and analytic on S^0. Since $F_a(t) = G_a(t)$ for every $t \in \mathbb{R}$, we can conclude

$$V_\omega(a\Phi_0) = F_a(-i/2) = G_a(-i/2) = a\Phi \qquad (a \in M_0),$$

and this means that $V_\phi = V_\omega$. Therefore, $E_\phi = E_\omega$.

The implications (v)→(vi) and (v)→(vii) are obvious and both (vi) and (vii) imply (iii) by the monotonicity theorem. Finally, (iv)→(i) is trivial.

It seems worth noting that weak sufficiency for a pair (ϕ, ω) is not symmetrical by definition, but it turns out to be such according to Theorem 4. The subalgebra generated by $\{[D\phi, D\omega]_t : t \in \mathbb{R}\}$ is the smallest weakly sufficient subalgebra and it is sufficient.

Corollary 5. *If $S(\phi, \omega)$ is finite, M_0 is weakly sufficient for (ϕ, ω) and there is a conditional expectation of M onto M_0 which preserves ϕ or ω, then M_0 is sufficient.*

Proof. If, for example, $\phi \circ E = \phi$, then $E = E_\phi$ must hold, and according to condition (viii) in Theorem 4 $\omega \circ E = \omega$.

Theorem 6. *Let M_0 be a von Neumann subalgebra of the von Neumann algebra M. Assume that ω is a faithful normal state on M and denote by ω^h the perturbation of ω by $h = h^* \in M$. Then the following conditions are equivalent.*

(i) $\sigma_s^\omega([D\omega^h, D\omega]_t) \in M_0$ $(s, t \in \mathbb{R})$.
(ii) $\sigma_s^\omega(h) \in M_0$ $(s \in \mathbb{R})$.
(iii) $E_\omega(h) = h$, *where E_ω denotes the ω-conditional expectation of M into M_0.*

Proof. We abbreviate $\log \Delta(\omega, \omega)$ as H. By straightforward computation, we have for $\xi \in D(H)$,

$$\lim (e^{it(H+h)} e^{-itH} \xi - \xi)/it = h\xi$$

as $t \to 0$. We show that

$$(e^{it(H+h)} e^{-itH} - I)/t = (e^{it(H+h)} - e^{itH}) e^{-itH}/t$$

is bounded in a neighbourhood of 0. We need the norm convergent expansion

$$e^{it(H+h)} = \sum_{n=0}^{\infty} U_n(t),$$

where $U_0(t) = e^{itH}$, $U_{n+1}(t) = -i \int_0^t e^{i(t-s)H} h U_n(s)\, ds$ and the estimate

$$\|U_n(t)\| \leqq t^n \|h\|^n / n!$$

(see [17, IX, Sect. 2]). So

$$\|(e^{it(H+h)}-e^{itH})/t\| = \left\|\sum_{n=1}^{\infty} U_n(t)/t\right\| \leqq \sum_{n=1}^{\infty} t^{n-1}\|h\|^n/n! = (e^{t\|h\|}-1)/t$$

is a bounded function of t in a neighbourhood of 0. Consequently, the limit above exists for every $\xi \in H$.

Now we prove (i)→(ii). Since

$$\sigma_s^\omega([D\omega^h, D\omega]_t) - I = e^{isH}(e^{it(H+h)}e^{-ith} - I)e^{-isH}$$

by the previous considerations, it follows that the strong derivative of $\sigma_s^\omega([D\omega^h, D\omega]_t)$ at $t=0$ is $-i\sigma_s^\omega(h)$.

Conversely, we refer to the expansion

$$[D\omega^h, D\omega]_t = \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \sigma_{t_n}^\omega(h) \dots \sigma_{t_1}^\omega(h)$$

(see [2]) which shows immediately the implication (ii)→(i).

The equivalence (ii)↔(iii) is in Theorem 5.1 of [1].

Corollary. *The subalgebra M_0 is is weakly sufficient for the pair (ω^h, ω) (where ω^h is an inner perturbation of ω) if and only if h is a fixed point of the ω-conditional expectation of M into M_0.*

Discussions

Let M be a von Neumann algebra with faithful normal states ϕ and ω. In [25] Raggio introduced a transition probability $P_A(\phi, \omega)$ which turned out to be a quasi-entropy by the function $f(t)=t^{1/2}$ (see [23]). Suppose that M_0 is a subalgebra of M. The equality $P_A(\phi, \omega) = P_A(\phi|M_0, \omega|M_0)$ can be studied by our methods, and it is equivalent to the conditions (i)–(vii) in Theorem 4.

The outstanding problem of joint convexity of the relative entropy functional was proved by Lieb [19]. We recall the argument in [24] deducing convexity from monotonicity. If ϕ and ω normal states on M $(i=1,2)$, then consider the following states on $N = M \oplus M$:

$$\phi_{12} = \lambda\phi_1 + (1-\lambda)\phi_2 \quad \text{and} \quad \omega_{12} = \lambda\omega_1 + (1-\lambda)\omega_2 .$$

So

$$S(\phi_{12}, \omega_{12}) = \lambda S(\phi_1, \omega_1) + (1-\lambda)\, S(\phi_2, \omega_2) .$$

On the other hand,

$$S(\phi_{12}|N_0, \omega_{12}|N_0) = S(\lambda\phi_1 + (1-\lambda)\phi_2, \lambda\omega_1 + (1-\lambda)\omega_2),$$

where $N_0 = \{a \oplus a : a \in M\}$ is a subalgebra of N. The monotonicity yields

$$S(\lambda\phi_1 + (1-\lambda)\phi_2, \lambda\omega_1 + (1-\lambda)\omega_2) \leqq \lambda S(\phi_1, \omega_1) + (1-\lambda)\, S(\phi_2, \omega_2) .$$

We are going to find a condition for the equality. By Theorem 4 the equality holds if $[D\phi_{12}, D\omega_{12}]_t \in N_0$ for $t \in \mathbb{R}$. Since

$$[D\phi_{12}, D\omega_{12}]_t = [D\phi_1, D\omega_1]_t \oplus [D\phi_2, D\omega_2]_t ,$$

the equality $[D\phi_1, D\omega_1]_t = [D\phi_2, D\omega_2]_t$ is a necessary and sufficient condition. In terms of densities this reads

$$\varrho_1^{-it}\varrho_2^{it} = \nu_1^{-it}\nu_2^{it},$$

where ν_i and ϱ_i are the densities of ϕ_i and ω_i, respectively $(i=1,2)$.

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