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Comments

Absence of Crystalline Ordering in Two Dimensions

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Abstract. We give conditions on the potential of a classical particle system, which imply absence of crystalline ordering in two dimensions. We thereby correct and extent some results in a previous paper.

1. Introduction

In an earlier paper [1] we gave a modified version of Mermin's argument for the absence of crystalline ordering in two-dimensional classical systems of point particles, [2]. In this note we would like to clarify and correct our discussion in [1], following Theorem 1 in that paper, and describe the kind of potentials for which our results apply. Since this note is a complement to [1], we use the same notations as in [1], and we do not repeat the basic definitions.

2. Relative Entropy Argument

We consider a system of point particles in $\mathbb{R}^2$. The configurations of the system are identified with the subsets, ω, of $\mathbb{R}^2$ which are locally finite: $x \in \omega$ means that there is a particle at x, and, for any bounded set Λ, $\omega_\Lambda = \omega \cap \Lambda$ is a finite subset of $\mathbb{R}^2$. The interaction is given by a two-body translation invariant potential $\phi: \mathbb{R}^2 \to \mathbb{R}$, $\phi(x) = \phi(-x)$, and we suppose that

A) ϕ is stable and regular;

B) ϕ is of class C^2, except at the origin.

The energy of a particle at x in the configuration ω is

$$H_\phi(x|\omega) \equiv H(x|\omega) = \sum_{\substack{y: x \neq y \\ y \in \omega}} \phi(x-y)\,, \tag{2.1}$$

J. Fröhlich: Comment on "Absence of Crystalline Ordering in two Dimensions" and the energy of n particles at $x_1, \ldots, x_n$ is denoted by $U(x_1, \ldots, x_n)$. Let P be a Gibbs state (equilibrium state) for an activity z and inverse temperature β. We introduce the n-point correlation function

$$\varrho_P(x_1, \ldots, x_n) = z^n e^{-\beta U(x_1, \ldots, x_n)} \left\langle \prod_{i=1}^{n} e^{-\beta H(x_i|\omega)} \right\rangle_P ; \tag{2.2}$$

(see [3], Chap. 4, and [4]).

We choose a fixed, extremal Gibbs state, P. Let T_a represent the translation, $x \to x + a$, in $\mathbb{R}^2$. We propose to prove that $P = P_a$, with $P_a \equiv T_a^{-1} P$. To show this, we construct a sequence of states, P_n, $n \in \mathbb{N}$, $P_n = T_n^{-1} P$, where T_n is a smooth bijective transformation $\mathbb{R}^2 \to \mathbb{R}^2$, which coincides with T_a on $\Lambda_n = \{x \in \mathbb{R}^2 : |x| \leqq n\}$, and which is the identity transformation outside some bounded region Λ (see [1], p. 284). We choose any fixed number ε in (0, 1) and a non-negative smooth function u on $\mathbb{R}^+$, which is monotone decreasing and has the properties

$$u(x) = 1 \text{ if } x \leqq 1, \left| \frac{du}{dx} \right| \leqq \varepsilon, \; u(x) = 0 \text{ if } x \geqq 2 + 1/\varepsilon .$$

We define

$$T_n : x \to x + a \cdot u\left(\frac{|x|}{n}\right), \; n \geqq 1 .$$

If we can find an upper bound, uniformly in n, for $S(P_n|P)$, the relative entropy of P_n with respect to P, then $P_a = P$. Technically it is easier to estimate $S(P_n|P) + S(P_{-n}|P)$, where $P_{-n} = T_{-n}^{-1} P$, and T_{-n} is given by the same formula as T_n, with a replaced by $-a$. Thus we must find a constant K, independent of n, such that

$$0 \leqq S(P_n|P) + S(P_{-n}|P) \leqq K < \infty . \tag{2.3}$$

The transformation T_n is local, and this implies that P_n is absolutely continuous with respect to P, with density

$$\frac{dP_n}{dP}(\omega) = \left(\prod_{x \in \omega_\Lambda} J_{T_n}(x) \right) \exp \beta(H_\Lambda(\omega_\Lambda|\omega) - H_\Lambda(T_n \omega_\Lambda|\omega)); \tag{2.4}$$

($J_{T_n}(x)$ is the Jacobian ($\geqq 0$) of T_n, and $H_\Lambda(\omega_\Lambda|\omega)$ is the energy of the particles in Λ, taking into account their interactions with the particles outside Λ). This identity permits us to estimate (2.3); (see [1]):

$$\begin{aligned} S(P_n|P) + S(P_{-n}|P) &= -\left\langle \log \frac{dP_n}{dP}(\omega) + \log \frac{dP_{-n}}{dP}(\omega) \right\rangle_P \\ &\leqq 0\left(\frac{\varepsilon^2}{n^2}\right) \left(\langle N_\Lambda(\omega) \rangle_P + \left\langle \sum_{x \in \omega_\Lambda} \sum_{\substack{y \in \omega \\ x \neq y}} \Psi_\varepsilon(x-y) \right\rangle_P \right). \end{aligned} \tag{2.5}$$

The first term comes from the Jacobians, J_{T_n}; $N_\Lambda(\omega)$ counts the number of particles in Λ. In the second term, $\Psi_\varepsilon(x) : \mathbb{R}^2 \to \mathbb{R}$ is defined by (ε is the number used in the definition of T_n)

$$\Psi_\varepsilon(x) = \sup_{\substack{a\in\mathbb{R}^2\\|a|=1}} \sup_{\substack{t\in\mathbb{R}\\|t|\leqq\varepsilon|x|}} \left|\frac{d^2}{dt^2}\Phi(x+ta)\right| |x|^2 . \tag{2.6}$$

The expression on the right side of (2.5) is given by

$$0\left(\frac{\varepsilon^2}{n^2}\right)\left(\int_\Lambda dx\varrho_P(x) + \int_\Lambda dx \int_{\mathbb{R}^2} dy\,\varrho_P(x,y)\Psi_\varepsilon(x-y)\right). \tag{2.7}$$

Theorem. *Let Φ be a translation invariant potential satisfying conditions A and B. Let P an extremal Gibbs state such that $\varrho_P(x)$ and $\varrho_P(x,y)$ are well-defined, and let $\Psi_\varepsilon(x)$ be given by* (2.6). *If there exist two finite constants C_1 and C_2 such that, for all bounded $\Lambda \subseteqq \mathbb{R}^2$,*

$$\int_\Lambda dx\,\varrho_P(x) = \langle N_\Lambda(\omega)\rangle_P \leqq C_1|\Lambda|,$$

and

$$\int_\Lambda dx \int_{\mathbb{R}^2} dy\,\varrho_P(x,y)\;\Psi_\varepsilon(x-y) \leqq C_2|\Lambda|,$$

($|\Lambda|$ = area of Λ), then P is translation invariant.

Remarks.

1) The theorem is an immediate consequence of (2.7), since, in this expression, $|\Lambda| = 0(n^2)$, and therefore (2.3) follows.

2) The theorem is equivalent to Theorem 1 in [1]. Indeed, for any bounded Λ',

$$\int_{\Lambda'} dy\,\varrho_P(x,y)\Psi_\varepsilon(x-y) = z\langle H_{\Psi_\varepsilon}(x|\omega_{\Lambda'})\,e^{-\beta H(x|\omega)}\rangle_P . \tag{2.8}$$

The proof of (2.8) is quite similar to the proof of Lemma 2.3 in [1]. The equivalence follows then by the monotone convergence theorem.

We now suppose that P is a Gibbs state for which all correlation functions are well-defined, and that there exists a ξ such that, for all n,

$$\varrho_P(x_1,\dots,x_n) \leqq \xi^n . \tag{2.9}$$

Let a:$\mathbb{R}^2 \to \mathbb{R}^+$. For any bounded subset Λ,

$$\begin{aligned}\left\langle \exp\left(\sum_{x\in\omega_\Lambda}\alpha(x)\right)\right\rangle_P &= 1 + \sum_{n\geqq 1}\frac{1}{n!}\int_\Lambda dx_1 \dots \int_\Lambda dx_n \varrho_P(x_1,\dots,x_n)\prod_{i=1}^n (e^{\alpha(x_i)}-1)\\ &\leqq 1 + \sum_{n\geqq 1}\frac{\xi^n}{n!}\left(\int_\Lambda dx(e^{\alpha(x)}-1)\right)^n = \exp\left(\xi\int_\Lambda dx(e^{\alpha(x)}-1)\right).\end{aligned} \tag{2.10}$$

The proof of (2.10) is accomplished by writing

$$\prod_{i=1}^n e^{\alpha(x_i)} = \prod_{i=1}^n ((e^{\alpha(x_i)}-1)+1) = \prod_{i=1}^n (f(x_i)+1) = \sum_{Y\subset\{x_1,\dots,x_n\}} f_Y , \tag{2.11}$$

where $f(x) \equiv e^{\alpha(x)} - 1$, $f_\phi \equiv 1$, $f_Y \equiv \prod_{x \in Y} f(x)$. Using (2.10) and the regularity of Φ, we get

$$\left\langle \prod_{i=1}^{2} e^{-\beta H(x_i|\omega_\Lambda)} \right\rangle_P \leqq \prod_{i=1}^{2} \langle e^{-2\beta H(x_i|\omega_\Lambda)} \rangle_P^{1/2} \leqq \exp\left(\xi \int_{\mathbb{R}^2} dx (e^{2\beta|\Phi_-(x)|} - 1) \right) < \infty , \tag{2.12}$$

where $\Phi_-(x)$ is the negative part of the potential Φ. Thus, the hypotheses of the theorem are satisfied if

$$\int_{\mathbb{R}^2} dx \, \Psi_\varepsilon(x) \, e^{-\beta\Phi(x)} \leqq C < \infty ;$$

(see (2.2), (2.7), (2.12)).

Corollary. *Let Φ be a potential satisfying the hypotheses of the theorem, and suppose that P is an extremal Gibbs state with correlation functions satisfying*

$$\varrho_P(x_1, \ldots, x_n) \leqq \xi^n, \ \forall n .$$

If

$$\int_{\mathbb{R}^2} dx \, \Psi_\varepsilon(x) \, e^{-\beta\Phi(x)} < \infty , \tag{2.13}$$

then P is translation invariant (in two dimensions).

Remarks.

1) The condition (2.13) is essentially a condition of integrability of Ψ_ε at infinity. Indeed, for large $|x|$, $\exp(-\beta\Phi(x))$ is almost one and, for small $|x|$, the divergence which may appear in $\Psi_\varepsilon(x)$ is in general compensated by $\exp(-\beta\Phi(x))$. Hence, for a potential $\Phi(x) = \Phi(|x|)$, the main condition on Φ in the corollary is roughly speaking the integrability of $\Phi''(|x|)|x|^2$, for large $|x|$.

2) The conditions on the correlation functions are satisfied by the equilibrium states, whose existence has been proven by Ruelle, [4]. Therefore if Φ satisfies the hypotheses of the corollary, is superstable, and if P is a tempered Gibbs state, then P is translation invariant.

3) The above corollary replaces the corollary on p. 282 in [1].

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