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One Dimensional $1/|j-i|^s$ Percolation Models: The Existence of a Transition for $s \leqq 2$

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Abstract. Consider a one-dimensional independent bond percolation model with p_j denoting the probability of an occupied bond between integer sites i and $i \pm j$, $j \geqq 1$. If p_j is fixed for $j \geqq 2$ and $\lim_{j \to \infty} j^2 p_j > 1$, then (unoriented) percolation occurs for p_1 sufficiently close to 1. This result, analogous to the existence of spontaneous magnetization in long range one-dimensional Ising models, is proved by an inductive series of bounds based on a renormalization group approach using blocks of variable size. Oriented percolation is shown to occur for p_1 close to 1 if $\lim_{j \to \infty} j^s p_j > 0$ for some $s < 2$. Analogous results are valid for one-dimensional site-bond percolation models.

1. Introduction and Main Results

We consider translation-invariant one-dimensional independent site-bond percolation models in which each site $i \in \mathbf{Z}$ is alive (respectively dead) with probability λ (respectively $1-\lambda$) and in which the (non-directed) bond between any distinct $i, j \in \mathbf{Z}$ is occupied (respectively vacant) with probability $p_{|j-i|}$ (respectively $1-p_{|j-i|}$). All the sites and bonds are mutually independent. We will treat both nonoriented and oriented percolation. In ether case the cluster of i, $\mathbf{C}(i)$, consists of those living sites for which there is a path of occupied bonds starting at i, ending at j, and touching only living sites; in particular $i \in \mathbf{C}(i)$ if and only if i is alive. In nonoriented percolation, any such path is allowed; in oriented percolation only paths that move to the right at each step are allowed. Such site-bond models reduce to pure bond models when $\lambda = 1$ and to pure site models when each $p_j = 0$ or 1.

A special case is bond percolation with $\lambda = 1$ and $p_j = 1 - \exp(-\beta |j|^{-s})$ for some s, $\beta \geqq 0$. It is an elementary fact that for $s \leqq 1$, percolation occurs (i.e., $P_\infty \equiv P(\|\mathbf{C}(0)\| = \infty) > 0$, where $\|\mathbf{C}\|$ denotes the number of sites in $\mathbf{C}$) for any $\beta > 0$;

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the fact that moreover (in the nonoriented case) $\mathbf{C}(0)=\mathbf{Z}$ [GKM] will not concern us in this paper. It can also be shown that for $s>2$, percolation does not occur for any β, while for $1<s\leqq 2$, percolation does not occur for small β (i.e., when $2\sum_{j=1}^{\infty} p_j<1$) [Sc]. Our first main result is a proof that for $1<s<2$, oriented (and a fortiori nonoriented) percolation (and hence a phase transition) occurs for sufficiently large β. This is analogous to the occurrence of a phase transition in long-range one-dimensional Ising models having an $|i-j|^{-s}$ interaction with $1<s<2$ [D]. Our second main result (which applies only in the nonoriented case) is that for $s=2$, percolation (and hence a phase transition) occurs for large β. This is analogous to the corresponding Ising model result [FS].

We note that the question of percolation in long-range one-dimensional models was posed by Erdös several years ago [Sh]. In [AN], further results for the $s=2$ case are presented which go beyond what was currently known for Ising models. In [ACCN], the results of [AN] are extended to Ising (and Potts) models; in the process, the relation between the percolation results presented here and the Ising results of [FS] is clarified.

We conclude this section by stating our main results for general one-dimensional site-bond models whose p_j's satisfy appropriate asymptotic hypotheses. Theorem 1.1 corresponds to oriented percolation for $1<s<2$ and Theorem 1.2 to unoriented percolation for $s=2$. Our results on the occurrence of unoriented percolation are, in a certain sense, optimal (see Remarks 1.3 and 2.4 below). Our results on the occurrence of oriented percolation can perhaps be improved. It is not clear, for example, whether oriented percolation can occur for $s=2$. The proofs of Theorems 1.1 and 1.2 are based on a renormalization group analysis. In Sect. 2, this analysis is introduced in terms of natural "continuum-bond" models with simple scaling properties. It is then shown that the proofs of our main results can be reduced to certain asymptotic results concerning iterated mappings on two (for Theorem 1.1) or three (for Theorem 1.2) dimensional parameter spaces. In Sect. 3 (for Theorem 1.1) and Sect. 4 (for Theorem 1.2) these iterated mapping results are shown to be valid. As expected, there are many more technical details involved in the proof of Theorem 1.2 since it treats the critical value of s. It may be worth remarking that although most renormalization group analyses are approximate or heuristic, ours produces rigorous bounds. See [ACCFR] for another (but different) rigorous renormalization group argument in a percolation context. See [AYH] for a pioneering (nonrigorous) renormalization group analysis of $1/|j-i|^2$ Ising models.

In the statements of both theorems, p_1 and λ are regarded as parameters with p_j fixed for $j>1$. λ will be one of the parameters in both parameter spaces introduced in Sect. 2. There will also be a parameter β, essentially $\lim_{j\to\infty} j^s p_j$, which controls the long range behavior of the model. In the two parameter space (for Theorem 1.1) the short range parameter p_1 is dropped while in the three parameter space (for Theorem 1.2) it is replaced by a closely related cutoff parameter ξ.

Theorem 1.1. *If a one-dimensional site-bond model has $\liminf_{j\to\infty} j^s p_j>0$ for some $s<2$, then (with p_j fixed for $j>1$) percolation (both oriented and nonoriented) occurs when p_1 and λ are sufficiently close to* 1.

Theorem 1.2. *If a one-dimensional site-bond model has* $\liminf_{j\to\infty} j^2 p_j > 1$, *then (with* p_j *fixed for* $j > 1$*) nonoriented percolation occurs when* p_1 *and* λ *are sufficiently close to* 1.

Remark 1.3. The hypothesis of Theorem 1.2 is essentially as weak as is possible, since it is proved in [AN] that percolation cannot occur if $\limsup j^2 p_j \leqq 1$. See Remark 2.4 below for further discussion of this point. Note that in the pure bond model with $p_j = p/j^s$ for $j \geqq 1$, where p is necessarily restricted to the interval $[0, 1]$, it follows from Theorem 1.1 that for $s < 2$, the critical value of p for (either oriented or nonoriented) percolation is some $p_c < 1$, while it follows from the result of [AN] mentioned above that for $s = 2$, $p_c = 1$.

Remark 1.4. It can be shown that in the parameter region where the proofs of Theorems 1.1 and 1.2 apply, the conclusion that percolation occurs (i.e., that an infinite cluster exists with probability one) can be strengthened to the conclusion that a positive density infinite cluster occurs with probability one. Note in this regard that by arguments used in [NS], it suffices to show that $\mathbf{C}(0)$ has positive lower density with positive probability, which is indeed shown at the beginning of Sect. 2. Presumably, it is the case in this parameter region that the infinite cluster is unique. This does not exclude the possibility that at the critical point of $s=2$ models, there may occur infinitely many distinct zero-density infinite clusters, a phenomenon first considered (for d-dimensional site percolation models) in [NS].

2. Scaling Arguments and a Renormalization Group Approach

Theorems 1.1 and 1.2 are based on a renormalization group analysis which allows us to estimate the size of certain "block variables" of the original model by comparison with a sequence of independent site-bond models. We remark that the analysis leads to site-bond models even when the original model is pure bond.

In the case of unoriented percolation, we define $\mathbf{B}(K)$ as the largest cluster in the finite model in which the infinite integer lattice is replaced by the block, $\{1, \ldots, K\}$. In the case of oriented percolation $\mathbf{B}(K)$ is the set of sites touched by the longest (oriented) path in $\{1, \ldots, K\}$, i.e., $\mathbf{B}(K)$ is the largest set of living sites of the form $\{i_1, \ldots, i_w\}$ with $1 \leqq i_1 < i_2 < \cdots < i_w \leqq K$ and the bond between i_j and i_{j+1} occupied for each j. In the event of a tie, $\mathbf{B}(K)$ is chosen in either case as the leftmost, but this rule will play no role in our analysis. The number of sites in $\mathbf{B}(K)$ is denoted $\|\mathbf{B}(K)\|$.

Roughly speaking, in order to prove the existence of percolation, we will estimate the (random) density, $\|\mathbf{B}(K)\|/K$, and show that (with positive probability) it does not tend to zero as $K \to \infty$. More precisely, we will show that sequences K_n, ψ_n, and λ_n can be chosen (with $\lim_{n\to\infty} K_n = \infty$ and $\liminf_{n\to\infty} \psi_n \lambda_n > 0$) so that the sequence of random variables, $W_n = \|\mathbf{B}(K_n)\|$, satisfies

$$P(W_n/K_n \geqq \psi_n) \geqq \lambda_n. \tag{2.1}$$

Before beginning our discussion of scaling arguments, we prove the intuitively reasonable fact that such a sequence of inequalities implies percolation. The intuition in, e.g., the unoriented case is that according to (2.1), a fraction λ_n of blocks of size K_n contain clusters of (local) density at least ψ_n so that at least a fraction $\lambda_n \psi_n$

of all sites belong to clusters of (local) density at least ψ_n. Thus the origin should have a probability of at least $\liminf \psi_n\lambda_n$ of belonging to an infinite cluster with (global) density at least $\liminf \psi_n$. The following proof is related to this intuition but has some extra factors of 1/2 needed for technical reasons and for the case of oriented percolation.

For either oriented or nonoriented percolation and for any i in $\{1,\ldots,K\}$, the random variable $\|\mathbf{C}(0)\cap\{0,\ldots,K-1\}\|$ has the same distribution as $\|\mathbf{C}(i)\cap\{i,\ldots,K-1+i\}\|$, which in turn is larger than $\|\mathbf{C}(i)\cap\{i,\ldots,K\}\|$. Hence,

$$\begin{aligned} &P(\|\mathbf{C}(0)\cap\{0,\ldots,K-1\}\|/K \geqq \psi/2) \\ &\geqq P(\|C(i)\cap\{i,\ldots,K\}\|/K \geqq \psi/2 \quad\text{and}\quad i\in\mathbf{B}(K)) \\ &\geqq P(i\in B(K) \quad\text{and}\quad \|\mathbf{B}(K)\cap\{i,\ldots,K\}\| \geqq \psi K/2), \end{aligned} \tag{2.2}$$

where the latter inequality is valid because $\mathbf{C}(i)\cap\{i,\ldots,K\}$ is bigger than $\mathbf{B}(K)\cap\{i,\ldots,K\}$ whenever $i\in\mathbf{B}(K)$. Let us denote by $\tilde{W}$ the number of sites i in $\mathbf{B}(K)$ with $\|\mathbf{B}(K)\cap\{i,\ldots,K)\| \geqq \psi K/2$. Then the sum of the right-hand side of (2.2) over all i in $\{1,\ldots,K\}$ is just $E(\tilde{W})$. If $\|\mathbf{B}(K)\| \geqq \psi K$, then $\tilde{W} \geqq \psi K/2$ and hence $E(\tilde{W}) \geqq (\psi K/2)P(\|\mathbf{B}(K)\| \geqq \psi K)$. Summing (2.2) over i thus yields

$$\text{left-hand side of (2.2)} \geqq K^{-1}E(\tilde{W}) \geqq (\psi/2)P(\|\mathbf{B}(K)\| \geqq \psi K). \tag{2.3}$$

It follows immediately from (2.3) that if (2.1) is valid for all n (with $K_n\to\infty$ and $\liminf_{n\to\infty}\psi_n\lambda_n > 0$), then $\mathbf{C}(0)$ is infinite (in fact has a positive lower density) with positive probability.

Although a renormalization group approach can be applied directly to any model satisfying the hypotheses of Theorem 1.1 or 1.2, it is both more convenient and more natural to apply it to a special class of models with

$$p_j = p_j(\beta) \equiv 1 - \exp\left(-\beta\int_0^1\int_j^{j+1} |y-x|^{-s}dydx\right), \quad \text{unless } j=1 \text{ and } s=2, \tag{2.4}$$

$$p_1 = p_1(\beta,\xi) \equiv 1 - \exp\left(-\beta\int_0^1\int_{1,\ |y-x|>\xi}^{2} |y-x|^{-2}dydx\right), \quad \text{for} \quad s=2, \tag{2.5}$$

where $\beta \geqq 0$ and $0<\xi\leqq 1$. The ξ-cutoff is needed for $s=2$ to avoid having $p_1=1$ because of the logarithmic divergence in the integral. Such a model may be obtained by discretizing a continuum bond (or c-bond) model in which the occurrence of a c-bond between the pair of real numbers, $x<y$, corresponds to the occurrence of a particle at (x,y) in a two-dimensional inhomogeneous Poisson point process with

$$\text{density at } (x,y) = \begin{cases} 0 & \text{if } |y-x|\leqq\xi \quad\text{and}\quad s=2 \\ \beta|y-x|^{-s}, & \text{otherwise} \end{cases}$$

We recall that in a Poisson process, the numbers of particles in disjoint spatial regions are independent random variables, each with a Poisson distribution whose mean is the integral of the density over the region. To obtain (2.4)–(2.5), the c-bond model is discretized by declaring that the (discrete) bond between i and j is occupied whenever a c-bond occurs between some x in $[i,i+1)$ and some y in $[j,j+1)$. The key to the success of our approach is the fact that under the scaling transformation, $(x, y) = (Kx', Ky')$, the density $\beta|y-x|^{-s}$ is transformed to $K^{2-s}\beta|y'-x'|^{-s}$, so that for $s<2$, the model is forced toward $\beta=\infty$ as $K\to\infty$. For $s=2$, β is unchanged under this transformation but ξ is transformed to ξ/K so that the model is forced toward $\xi=0$ (or $p_1=1$) as $K\to\infty$.

To prove (2.1), we will not take K_n as C^n for some fixed scale factor C, but rather we will scale at a rate that increases with each step. A block at stage n will consist of C_n "stage-n sites" each of which is a block of C_{n-1} of the next lower level site. Consequently stage-n blocks contain $C_n\, C_{n-1}\cdots C_1 = K_n$ elementary (i.e., stage-1) sites. We will inductively define for stage-n sites the notions of "living" or "occupied bond" by taking each stage-n site to be a block of stage-$(n-1)$ sites and requiring that its largest cluster or oriented path (of stage-$(n-1)$ sites) is sufficiently large or that there be an appropriately located stage-$(n-1)$ occupied bond between the pair of largest clusters or oriented paths. The precise definitions of sufficiently large and appropriately located will be made to insure that a living stage-n site when considered as a block of K_n elementary sites will have $W_n/K_n \geqq \psi_n$. Some light can be shed on these definitions by rewriting (2.1) as

$$P\left(\frac{W_n/[\psi_{n-1}K_{n-1}(\psi_n/\psi_{n-1})^{1/2}]}{C_n} \geqq (\psi_n/\psi_{n-1})^{1/2}\right) \geqq \lambda_n \tag{2.6}$$

for the oriented case and as

$$P\left(\frac{W_n/\psi_{n-1}K_{n-1}}{C_n} \geqq \psi_n/\psi_{n-1}\right) \geqq \lambda_n \tag{2.7}$$

for the nonoriented case. These formulas will soon be seen to have direct meaning in terms of our iterated mappings since it will be the ratios ψ_n/ψ_{n-1} and $C_n = K_n/K_{n-1}$ that will be significant in those mappings. Inequality (2.6) should be regarded as an extension of the simpler (2.7) required by the fact that when a number of oriented paths are connected by occupied bonds, the length of the longest resulting oriented path may be less than the sum of the previous lengths (when the new occupied bond does not connect the end of one oriented path to the beginning of the next).

We now introduce families of mappings $f_{\theta,C}$ (for the oriented case) and $g_{\theta,C}$ (for the unoriented case) indexed by a (positive integer) scale size C and a number $\theta\in(0,1]$ which is essentially the ψ_n/ψ_{n-1} of (2.6)–(2.7). For $s<2$, we will only consider the oriented case (i.e., $f_{\theta,C}$) since nonoriented percolation follows a fortiori. $f_{\theta,C}$ is a mapping on a two-dimensional space of parameters while $g_{\theta,C}$ acts on a three-dimensional space. $f_{\theta,C}$ (respectively $g_{\theta,C}$) should be thought of as mapping from stage-$(n-1)$ to stage-(n) effective values of λ and β (respectively λ, β and ξ).

With fixed $s<2$, θ and C we define $(\lambda', \beta') = f_{\theta,C}(\lambda, \beta)$ by beginning with the independent site-bond percolation model with site parameter λ and bond parameters p_j given by the $p_j(\beta)$ of (2.4) for all $j\geqq 1$, and proceeding as follows. Let $\mathbf{B}_i$ denote the set of living sites touched by the longest oriented path (through living sites) in $H_i \equiv \{(i-1)C+1,\dots,iC\}$ (with the usual convention for ties). Note that $\mathbf{B}_1$ is identical to the previously defined $\mathbf{B}(C)$. We say that the i^{th} block H_i is "alive" if $\|\mathbf{B}_i\|/C \geqq \theta^{1/2}$, and then define

$$\lambda' = P(H_i \text{ is "alive"}) = P(\|\mathbf{B}_i\|/C \geqq \theta^{1/2}). \tag{2.8}$$

For $i < j$, we say that there is an "occupied bond" between H_i and H_j if there is an occupied bond (of the original sort) between some k in $\mathbf{B}_i$ and some l in $\mathbf{B}_j$ with the fraction of sites in $\mathbf{B}_i$ to the right of k and the fraction of sites in $\mathbf{B}_j$ to the left of l each bounded above by $(1-\theta^{1/2})/2$. We next define p'_j for $j \geqq 1$, as the minimum over all site-bond configurations within H_i and H_{i+j} (not including bonds from H_i to H_{i+j}) for which H_i and H_{i+j} are "alive," of the conditional (with respect to the configuration) probability that there is an "occupied bond" between H_i and H_{i+j}. p'_j is a worst case lower bound for the probability that a "living" H_i and a "living" H_{i+j} have an "occupied bond" between them. Finally, we define

$$\beta' = \sup\{b: p'_j \geqq p_j(b) \text{ for all } j \geqq 1\}. \tag{2.9}$$

Theorem 1.1 will be shown later in this section to be a consequence of the following theorem whose proof will be given in Sect. 3.

Theorem 2.1. *Given $s \in (0,2)$, there exist sequences $C_k \in \{2,3,\ldots\}$ and $\theta_k \in (0,1]$ with $\prod_{k=1}^{\infty} \theta_k > 0$ such that*

$$(\lambda_n, \beta_n) \equiv f_{\theta_n, C_n}(\cdots(f_{\theta_2, C_2}(f_{\theta_1, C_1}(\lambda_0, \beta_0)))) \to (1, \infty) \quad \text{as} \quad n \to \infty, \tag{2.10}$$

providing λ_0 is sufficiently close to 1 and β_0 is sufficiently large.

Now, with fixed $s=2$, θ and C, we define $(\lambda', \beta', \xi') = g_{\theta,C}(\lambda, \beta, \xi)$ by beginning with the independent site-bond percolation model with site parameter λ and bond parameters p_j given by the $p_j(\beta)$ of (2.4) for all $j \geqq 2$ with $s=2$ and by the $p_1(\beta, \xi)$ of (2.5) for $j=1$. We now let $\mathbf{B}_i$ denote the largest (unoriented) cluster in the finite model consisting only of sites and bonds within H_i. The block H_i is called "alive" if $\|\mathbf{B}_i\|/C \geqq \theta$, and we define

$$\lambda' = P(H_i \text{ is "alive"}) = P(\|\mathbf{B}_i\|/C \geqq \theta). \tag{2.11}$$

For $i \neq j$, there is said to be an "occupied bond" between H_i and H_j if there is an (original) occupied bond between some k in $\mathbf{B}_i$ and some l in $\mathbf{B}_j$. Next, p'_j is defined as the minimum conditional probability that there is an "occupied bond" between H_i and H_j, conditioned with respect to site-bond configurations of H_i and H_{i+j} for which both blocks are "alive." We then define

$$\beta' = \sup\{b: p'_j \geqq p_j(b) \text{ for all } j \geqq 2\}. \tag{2.12}$$

Finally, we define

$$\xi' = \inf\{x: p'_1 \geqq p_1(\beta', x)\}. \tag{2.13}$$

Note that ξ' may be equivalently defined by requiring $p_1(\beta', \xi') = p'_1$.

Theorem 1.2 will be shown later in this section to be a consequence of the following theorem whose proof will be given in Sect. 4.

Theorem 2.2. *If $\beta_0 > 1$, then there exist sequences $C_k \in \{2,3,\ldots\}$ and $\theta_k \in (0,1]$ with $\prod_{k=1}^{\infty} \theta_k > 0$ such that defining*

$$(\lambda_n, \beta_n, \xi_n) \equiv g_{\theta_n, C_n}(\cdots(g_{\theta_1, C_1}(\lambda_0, \beta_0, \xi_0))), \tag{2.14}$$

we have $(\lambda_n, \xi_n) \to (1, 0)$ *and* $\liminf \beta_n > 1$, *providing* λ_0 *is sufficiently close to* 1 *and* ξ_0 *is sufficiently small.*

Proof that Theorem 2.1 Implies Theorem 1.1. We fix $s \in (0, 2)$ and call the independent site-bond model with $p_j = p_j(\beta)$ for all $j \geqq 1$, the canonical model with parameters λ and β. In the first part of the proof we show via (2.1) or (2.6) that for λ sufficiently close to 1 and β sufficiently large, the canonical model percolates. In the second part of the proof we show that this result for canonical models implies percolation for general models satisfying the hypotheses of Theorem 1.1. A portion of the second part will be stated separately as Proposition 2.3; it is used again in the proof that Theorem 2.2 implies Theorem 1.2.

Using the sequence C_k and θ_k of Theorem 2.1, we choose $K_n = \prod_{k=1}^{n} C_k$, $\psi_n = \prod_{k=1}^{n} \theta_k$, and $(\lambda, \beta) = (\lambda_0, \beta_0)$ such that (2.10) is valid. Since $K_n \to \infty$, and $\liminf \lambda_n \psi_n = \prod_{k=1}^{\infty} \theta_k > 0$, percolation for the canonical model with parameters λ_0, β_0 would be implied by the validity of (2.1) for all n.

We recall the inductive definitions of "living" or "occupied bond" for or between stage-n blocks (which are also stage-$(n+1)$ sites), which were discussed (incompletely) prior to Eq. (2.6). For $n = 1$, the definitions are those used in the definition of $f_{\theta,C}$ with $\theta = \theta_1$ and $C = C_1$. For general $n > 1$, one considers the stage-n block as a union of its stage-n sites for which (as stage-$(n-1)$ blocks) the notions of "living" and "occupied bond" are already defined, and again takes the definitions used for $f_{\theta,C}$ but with $\theta = \theta_n$ and $C = C_n$. To prove (2.1) it suffices to show that for each n the following two statements are true:

$$P(\{1, \ldots, K_n\} \text{ is "alive" as a stage-}n\text{ block}) \geqq \lambda_n. \tag{2.15}$$

$$\text{If } \{1, \ldots, K_n\} \text{ is "alive," then } W_n/K_n \geqq \psi_n. \tag{2.16}$$

If a stage-1 block is "alive," it contains by definition an oriented path touching $(\theta_1)^{1/2} C_1$ living (stage-1) sites. If a stage-n block (with $n > 1$) is "alive," it contains an oriented "stage-n path" touching $(\theta_n)^{1/2} C_n$ living stage-(n) sites by the definition of "occupied bonds" between stage-(n) sites, this means that it contains an oriented "stage-$(n-1)$ path" touching at least $(\theta_n)^{1/2} C_n [1 - 2(1 - (\theta_{n-1})^{1/2})/2] (\theta_{n-1})^{1/2} C_{n-1} = (\theta_n)^{1/2} C_n \theta_{n-1} C_{n-1}$ stage-$(n-1)$ sites. Thus we see inductively that any "living" stage-n block contain an oriented (stage-1) path containing $(\theta_n)^{1/2} C_n \prod_{k=1}^{n-1} \theta_k C_k = (\theta_n)^{-1/2} \psi_n K_n \geqq \psi_n K_n$ (stage-1) sites and thus (2.16) is valid.

The proof of (2.15) is of course based on the definitions (2.8)–(2.10). We denote by H_{ni} the stage-n block, $\{(i-1)K_n + 1, \ldots, iK_n\}$. Inequality (2.15) is equivalent to

$$P(H_{ni} \text{ is "alive"}) \geqq \lambda_n. \tag{2.17}$$

We also define p'_{nj} as the minimum over all configurations of (stage-1) sites and bonds within H_{ni} and $H_{n(i+j)}$ for which H_{ni} and $H_{n(i+j)}$ are "alive" of the conditional probability that there is an "occupied (stage-n) bond" between H_{ni} and $H_{n(i+j)}$. We consider (2.17) together with

$$p'_{nj} \geqq p_j(\beta_n) \text{ for all } j \geqq 1, \tag{2.18}$$

where $p_j(\beta)$ as before denotes the right-hand side of (2.4). We claim that the combined inequalities (2.17)–(2.18) follow from (2.8)–(2.10) by induction on n. The validity of (2.17)–(2.18) for $n = 1$ is clear. Now assume (2.17)–(2.18) for some $(n-1)$. Consider the "stage-n model" consisting of stage-n sites (the $H_{(n-1)i}$'s) and stage-n bonds between them. This is *not* an independent site-bond model; however the sites alone are independent and the bonds are conditionally independent when the configurations (of stage-1 sites and bonds) within each $H_{(n-1)i}$ are specified. It then follows by (2.17)–(2.18) for $(n-1)$ that the stage-n model dominates an independent (canonical) site-bond model with parameters λ_{n-1} and β_{n-1} precisely in the sense that the left-hand sides of (2.17) and (2.18) are no less than the correspondingly defined quantities for the independent model, but by (2.8)–(2.10) the corresponding quantity for (2.17) equals λ_n while the corresponding quantity for (2.18) is no less than the right-hand side of (2.18). The first part of the proof of Theorem 2.1 is now complete.

It remains to show that the occurrence of percolation in the canonical models for arbitrary $s<2$ implies Theorem 1.1. We first note that if $\lim\inf j^{s'} p_j > 0$ for $s' < 2$, then by choosing $s \in (s', 2)$ we have that for any β, $p_j > p_j(\beta)$ for sufficiently large j (depending on β), where again $p_j(\beta)$ denotes the right-hand side of (2.4) with the chosen value of s. Theorem 1.1 now follows immediately from the next proposition.

Proposition 2.3. *Suppose oriented (respectively nonoriented) percolation occurs in a one-dimensional site-bond model with site parameter $\tilde{\lambda}$ in $(0,1]$ and bond parameters $\tilde{p}_j$ in $[0,1)$ (for $j = 1,2,\ldots$). Let $N > 1$ be a fixed integer and consider a second one dimensional site-bond model with parameters λ and p_j such that the p_j's are fixed for $j \geqq 2$ and satisfy $p_j \geqq \tilde{p}_j$ for $j \geqq N$. If $\tilde{\lambda} < 1$, then oriented (respectively nonoriented) percolation occurs in the second model for λ and p_1 in $(0,1)$ sufficiently close to 1. If $\tilde{\lambda} = 1$, then oriented (respectively nonoriented) percolation occurs in the second model for $\lambda = 1$ and p_1 in $(0,1)$ sufficiently close to 1.*

Proof. We suppose $\tilde{\lambda}$ is in $(0,1)$; the proof when $\tilde{\lambda} = 1$ is essentially the same but easier. We define a model which has ordinary bonds between i and j for $|j-i| > 1$ but in which the nearest neighbor bond from l to $l+1$ is replaced for each l by a family of nearest neighbor *links* (indexed by (l,i,j) with $i \leqq l, j \geqq l+1$ and $j-i<N$) and the site at l is replaced for each l by a family of *cubicles* (indexed by (l,i,j) with $i \leqq l, j \geqq l$ and $j-i<N$). All bonds (indexed by (i,j)), links (indexed by (l,i,j)) and cubicles (indexed by (l,i,j)) are independent. The bond occupation probability is $\tilde{p}_{|j-i|}$ for $|j-i| \geqq N$ and zero for $1 < |j-i| < N$; the link occupation probability is $p^{(j-1)}$ (independent of l); and the cubicle occupation probability is $\lambda^{(j-1)}$ (independent of l). For a given l, the probability that at least one (l,i,j)-link is occupied is $1 - \Pi(1 - p^{(j-i)})$, and the probability that at least one (l,i,j)-cubicle is occupied is $1 - \Pi(1 - \lambda^{(j-i)})$, where the product in each case is over the appropriate set of (i,j)'s.

We next construct a related independent site-bond model (of the usual type) by defining site l to be alive if the (l,l,l)-cubicle is occupied, by defining the bond between i and j to be occupied for $j-i \geqq N$ if the corresponding bond is occupied in the original model, and by defining the bond between i and j to be occupied for $1 \leqq$

$j-i<N$ if for each $l=i, i+1, \ldots, j-1$, both the (l,i,j)-link and the (l,i,j)-cubicle are occupied. This constructed model has $\lambda=\lambda^{(0)}$, $p_j=\tilde{p}_j$ for $j \geqq N$, and $p_j=(\lambda^{(j)}p_j^{(j)})^j$ for $1 \leqq j < N$ and thus percolates if $\lambda^{(0)} \geqq \tilde{\lambda}$ and $(\lambda^{(j)}p^{(j)})^j \geqq \tilde{p}_j$ for $1 \leqq j < N$. On the other hand, it is clear that if the constructed model percolates, then so does the model with $\lambda = 1 - \Pi(1-\lambda^{(j-i)})$, $p_j = \tilde{p}_j$ for $j \geqq N$, $p_j = 0$ for $1 < j < N$, and $p_1 = 1 - \Pi(1 - p^{(j-i)})$. Thus, by choosing $\lambda^{(0)} = \tilde{\lambda}$ and $p^{(j)} = \lambda^{(j)} = (\tilde{p}_j)^{(1/2)j}$ for $1 \leqq j < N$, we see that the second model of the theorem percolates providing

$$\lambda \geqq 1-(1-\tilde{\lambda}) \prod_{\substack{i=1 \\ 1 \leqq j-i<N}}^{N} \prod_{j=N}^{2N-1} (1-(\tilde{p}_{j-i})^{1/2(j-i)}),$$

and

$$p_1 \geqq 1 - \prod_{\substack{i=2 \\ 1 \leqq j-i<N}}^{N} \prod_{j=N+1}^{2N-1} (1-(\tilde{p}_{j-i})^{1/2(j-i)}),$$

which completes the proof.

Proof that Theorem 2.2 Implies Theorem 1.2. This proof has essentially the same structure as the proof just given above that Theorem 2.1 implies Theorem 1.1. The details are omitted.

Remark 2.4. In the context of Theorem 1.2, we note that if $\limsup j^2 p_j > 1$, then oriented percolation *may* occur even when $\liminf j^2 p_j < 1$. For a simple example of such a situation, suppose $p_{2k+1} \sim \beta_1(2k+1)^{-2}$, while $p_{2k} \sim \beta_2(2k)^{-2}$; by considering only even integer sites, one sees that percolation will occur (for p_2 and λ close to 1 and hence by Proposition 2.3 for p_1 and λ close to 1) providing $\beta_2 > 4$, even if $\beta_1 < 1$. It is possible to apply the renormalization group arguments of this chapter to obtain a corollary to Theorem 1.2 which yields better results in many such situations. Beginning with a model with parameters λ and p_j, and with some choice of θ, C, we may define λ', p_j' as in the definition of $g_{\theta,C}$. For a given integer C, we choose θ so close to 1 that $(1-\theta)C < 1$; Thus a block will be "alive" only if all its sites are living and belong to a single cluster. Consequently,

$$p_k' = 1 - \prod_{i=1}^{C} \prod_{j=kC+1}^{(k+1)C} (1-p_{j-i}), \tag{2.19}$$

and thus

$$\sum_{i=1}^{C} \sum_{j=kC+1}^{(k+1)C} p_{j-i} \geqq p_k' \geqq 1 - \exp\left(-\sum_{i=1}^{C} \sum_{j=kC+1}^{(k+1)C} p_{j-i}\right). \tag{2.20}$$

It is easy to see that $(\lambda', p_1') \to (1,1)$ if $(\lambda, p) \to (1, 1)$. Thus, if

$$\liminf_{k\to\infty} k^2 \sum_{i=1}^{C} \sum_{j=kC+1}^{(k+1)C} p_{j-i} > 1 \text{ for some } C = 1, 2, \ldots, \tag{2.21}$$

then by Theorem 1.2, percolation occurs in the (λ', p_j')-model and hence in the (λ, p_j)-model for p_1 and λ close to 1. It similarly follows from the results of [AN] (see

Remark 1.3 above) that if

$$\limsup_{k\to\infty} k^2 \sum_{i=1}^{C'} \sum_{j=kC'+1}^{(k+1)C'} p_{j-i} \leqq 1 \text{ for some } C' = 1, 2, \ldots, \tag{2.22}$$

then percolation cannot occur. A sufficient condition for (2.21) to be valid may be obtained by defining a coarse-grained average of p_j,

$$p_k^C = \min_{k\leqq i<k+C} \left(\frac{1}{C} \sum_{j=i}^{i+C} p_j \right), \tag{2.23}$$

and requiring that for some C, $\liminf k^2 p_k^C > 1$; an analogous sufficient condition for (2.22) can be similarly obtained. Applying these arguments (with $C = C' = 2$) to the simple example with $p_{2k+1} \sim \beta_1(2k+1)^{-2}$ and $p_{2k} \sim \beta_2(2k)^{-2}$, one finds that percolation occurs for large p_1 and λ if (and, by [AN] and (2.22), only if) $(\beta_1 + \beta_2)/2 > 1$.

3. Proof of Theorem 2.1

We choose $q > 1$, $\Theta \in (0, 1)$, and take

$$\theta_k = 1 - \Theta k^{-q}, \quad C_k = 2k^r. \tag{3.1}$$

It will be seen (from (3.3) below) that for the given $s \in (0, 2)$, r (which for simplicity is taken to be an integer) must be chosen to satisfy

$$r > 2q/(2 - s). \tag{3.2}$$

We will prove (2.10) by showing that for λ_0, β_0 sufficiently large, the following are valid for all n:

$$\beta_n \geqq (H')^n K_n^{2-s}(n!)^{-2q} = (2H')^n (n!)^{r(2-s)-2q} \tag{3.3}$$

$$\lambda_n \geqq 1 - (\Theta/4)(n+1)^{-q}, \tag{3.4}$$

where $H' > 0$ depends (only) on s, q, and r.

Since $f_{\theta,C}(\lambda, \beta) \to (1, \infty)$ as $(\lambda, \beta) \to (1, \infty)$, it follows that for fixed s, q, r, H' and Θ, and any $N < \infty$, (3.3)–(3.4) can be made valid for all $n \leqq N$ by choosing λ_0, β_0 sufficiently large. It consequently suffices to prove that for some $H' > 0$ and all N *sufficiently large*, the validity of (3.3)–(3.4) for $n = N$ implies its validity for $n = N + 1$. The proof of this large N induction step is an elementary consequence of the following proposition, as can be seen by setting $\lambda = \lambda_N$, $\beta = \beta_N$, $\theta = \theta_{N+1}$, $C = C_{N+1}$, $\lambda' = \lambda_{N+1}$, and $\beta' = \beta_{N+1}$. In fact a much improved version of (3.4) follows from (3.6) below.

Proposition 3.1. *Let $(\lambda', \beta') = f_{\theta,C}(\lambda, \beta)$ with $1 - \theta^{1/2} \geqq (3/2)(1 - \lambda)$ and $\theta \geqq 1/2$. Then*

$$\beta' \geqq \beta H C^{2-s}(\theta^{1/2}(1-\theta^{1/2})/2)^2 \geqq \beta H C^{2-s}((1-\theta^{1/2})/2\sqrt{2})^2 \geqq \beta (H/32) C^{2-s}(1-\theta)^2 \tag{3.5}$$

for some $H > 0$ depending only on s and

$$1 - \lambda' \leqq \exp(-C(1-\lambda)/16) + \binom{C}{2} \exp(-\beta(C+1)^{-s}). \tag{3.6}$$

Proof. Inequality (3.5) is a consequence of the fact that in the canonical model, the probability that there is no occupied bond between two disjoint sets of sites S and S' is

$$\prod_{k\in S}\prod_{l\in S'}(1-p_{|k-l|}(\beta)). \tag{3.7}$$

If we take $i<j$, assume that H_i and H_j are "alive," and let S_i (respectively S_j) denote the rightmost (respectively leftmost) subset of $\mathbf{B}_i$ (respectively $\mathbf{B}_j$) containing (approximately) a fraction $(1-\theta^{1/2})/2$ of the sites of $\mathbf{B}_i$ (respectively $\mathbf{B}_j$), then since $\|\mathbf{B}_i\|$, $\|\mathbf{B}_j\| \geqq \theta^{1/2}C$, we see that (3.7) with $S=S_i$ and $S'=S_j$ is bounded above by

$$\exp[-\beta I(\theta^{1/2}C,(j-i)C+(1-\theta^{1/2})C,\theta^{1/2}(1-\theta^{1/2})C/2)], \tag{3.8}$$

where, for $x_1 \leqq y_1$ and $\Delta \geqq 0$,

$$I(x_1,y_1,\Delta)=\int\limits_{x_1-\Delta}^{x_1}\int\limits_{y_1}^{y_1+\Delta}(y-x)^{-s}dydx. \tag{3.9}$$

Thus (3.8) is an upper bound for $1-p'_{j-i}$. Lemma 3.2 below implies that (for $\theta \geqq 1/2$)

$$p'_j \geqq p_j(HC^{2-s}(\theta^{1/2}(1-\theta^{1/2})/2)^2\beta), \tag{3.10}$$

and thus, by the definition (2.9), yields (3.5).

To obtain (3.6), we denote by V the number of dead sites in H_1 and use the fact that if $V<(1-\theta^{1/2})C$ and if there is an occupied bond between *every* pair of (living) sites in H_1, then H_1 is "alive." This leads to the simple estimate (but sufficient for our purposes) that

$$1-\lambda' \leqq P(V \geqq (1-\theta^{1/2})C)+\sum_{1\leqq l<k\leqq C}(1-p_{k-l}(\beta)). \tag{3.11}$$

We obtain (3.6) by noting that for each l, k appearing in (3.11), $(1-p_{k-l}(\beta)) \leqq 1-p_{C-1}(\beta) \leqq \exp(-\beta(C+1)^{-s})$, and by using Lemma 3.3 below to control the first term on the right-hand side of (3.11) involving the binomial random variable V with parameters C and $1-\lambda$.

Lemma 3.2. Let $s\in(0,2)$ be fixed. There exists $H>0$ (depending only on s) so that for any $\theta \geqq 1/2$, $C>0$, and positive integer j,

$$I(\theta^{1/2}C, jC+(1-\theta^{1/2})C, \theta^{1/2}(1-\theta^{1/2})C/2) \geqq HC^{2-s}(\theta^{1/2}(1-\theta^{1/2})/2)^2 I(1,j,1). \tag{3.12}$$

Proof. By the change of variables $x=\theta^{1/2}C+\theta^{1/2}(1-\theta^{1/2})C(x'-1)/2$ (and similarly for y) we may rewrite the desired inequality as

$$I\left(1,\frac{j+1-2\theta^{1/2}+\theta^{1/2}(1-\theta^{1/2})/2}{\theta^{1/2}(1-\theta^{1/2})/2},1\right) \geqq H(\theta^{1/2}(1-\theta^{1/2})/2)^s I(1,j,1). \tag{3.13}$$

The left-hand side of this inequality is bounded below by

$$I(1,j(\theta^{1/2}(1-\theta^{1/2})/2)^{-1},1),$$

since I is decreasing in y_1 for fixed x_1 and Δ, and since $1-2\theta^{1/2}+\theta^{1/2}(1-\theta^{1/2})/2 \leqq 0$ for $\theta^{1/2} \geqq (-3+\sqrt{17})/2$ and hence for $\theta \geqq 1/2$. Finally, since $I(1,y,1)$ is asymptotic to y^{-s} as $y\to\infty$, we have $I(1,\kappa j,1)\geqq H\kappa^{-s}I(1,j,1)$ for all κ, $j\geqq 1$, where

$$H=[\inf_{y\geqq 1} y^s I(1,y,1)]/[\sup_{y\geqq 1} y^s I(1,y,1)],$$

which completes the proof of Lemma 3.2.

Lemma 3.3. *Let V have a binomial distribution with parameters n and ρ; then for $h\geqq 1/2$,*

$$P(V/n\geqq(1+h)\rho)\leqq\exp(-n\rho/16). \tag{3.14}$$

Proof. It is a standard large deviation estimate, that

$$P(V\geqq n(1+h)\rho)\leqq\inf_{t\geqq 0}E(e^{t(V-n(1+h)\rho)})=\exp[n\inf_{t\geqq 0}\{\ln(1-\rho+\rho e^t)-(1+h)\rho t\}]. \tag{3.15}$$

The desired result is now obtained by using (with $h=1/2$)

$$\begin{aligned}\inf_{t\geqq 0}\{\ln(1-\rho+\rho e^t)-(1+h)\rho t\}&\leqq\inf_{t\geqq 0}\{\rho(e^t-1)-(1+h)\rho t\}\\ &=\rho(h-(1+h)\ln(1+h))\leqq\rho(h-(1+h)(h-h^2/2))=-h^2(1-h)/2.\end{aligned}$$

4. Proof of Theorem 2.2

We choose $q>1$, $\Theta\in(0,1)$, and take

$$\theta_k=1-\theta k^{-q},\quad C_k=[2k^r], \tag{4.1}$$

where $[y]$ denotes the greatest integer less than or equal to y. We will find it useful to choose r (and q) so that

$$1<q<r\leqq 4/3, \tag{4.2}$$

e.g., $r=4/3$ and $q\in(1,4/3)$; these two parameters are henceforth fixed. We define

$$\psi_n=\prod_{k=1}^{n}\theta_k,\quad \psi=\prod_{k=1}^{\infty}\theta_k. \tag{4.3}$$

Θ will be chosen sufficiently small (depending on β_0) so that the θ_k's, ψ_n's and ψ are all uniformly close to 1. We will prove Theorem 2.2 by showing that if $\beta_0>1$, then for Θ sufficiently small (depending on β_0), λ_0 sufficiently close to 1 (depending on β_0 and Θ) and ξ_0 sufficiently small (depending on β_0 and Θ), the following are valid for all n:

$$\beta_n\geqq\theta_n^6\beta_{n-1}\geqq\psi_n^6\beta_0, \tag{4.4}$$

$$\xi_n\leqq 2e(1-\theta_n)=2e\Theta n^{-q}, \tag{4.5}$$

$$\lambda_n\geqq 1-(\Theta/4)(n+1)^{-q}. \tag{4.6}$$

Since $g_{\theta,C}(\lambda,\beta,\xi)\to(1,\beta',0)$ for some $\beta'>0$ as $(\lambda,\xi)\to(1,0)$ with fixed β, θ and C, it follows that for fixed q,r,β_0, and Θ,and any $N<\infty$,(4.5)–(4.6) can be made valid for all $n\leqq N$ by choosing λ_0 sufficiently close to 1 and ξ_0 sufficiently small. It consequently suffices to prove first that (4.4) is true for all n and second that if $\beta_0>1$, then for some Θ sufficiently small (depending on β_0), and all N *sufficiently large*, the validity of (4.4)–(4.6) for $n=N$ implies the validity of (4.5)–(4.6) for $n=N+1$. The first proof and half of the second proof (i.e., the validity of (4.5) for $n=N+1$) are immediate consequences of the following Proposition 4.1. By (4.4), given $\beta_0>1$, one may choose Θ so small that $\psi^6\beta_0$ (and hence each β_n) is bounded below by some $\beta>1$. The rest of the second proof is thus an immediate consequence of Proposition 4.2 below.

Proposition 4.1. *Let* $(\lambda', \beta', \xi') = g_{\theta,C}(\lambda, \beta, \xi)$ *with* $\theta\in(0,1)$ *and* $\xi\in(0,1)$. *Then*

$$\beta' \geqq \theta^6\beta. \tag{4.7}$$

If in addition, $(1-\theta)C\geqq 1$ *and* $(1-\theta)2e\leqq 1$, *then*

$$\xi' \leqq 2e(1-\theta). \tag{4.8}$$

Proposition 4.2. *With fixed* β, Θ, q *and* r *satisfying* $\beta>1$, $\Theta>0$ *and* $1<q<r\leqq 4/3$, *let* $(\lambda'_N, \beta'_N, \xi'_N) = g_{\theta,C}(\lambda, \beta, \xi)$, *where* $\theta = 1-\Theta N^{-q}$, $C=[2N^r]$, $\lambda = 1-(\Theta/4)(N+1)^{-q}$ *and* $\xi = 2e\Theta N^{-q}$. *Then*

$$\lim_{n\to\infty} N^q(1-\lambda'_N) = 0, \tag{4.9}$$

and hence for all sufficiently large N,

$$\lambda'_N \geqq 1-(\Theta/4)(N+2)^{-q}.$$

Proof of Proposition 4.1. This proof is similar to that of (3.5) above. Using the notation of (3.9) with $s=2$, we have

$$1-p'_{j-i} \leqq \exp(-\beta I(\theta C, (j-i+1-\theta)C, \theta C)). \tag{4.10}$$

This inequality is valid for $j-i\geqq 2$ and if $(1-\theta)C\geqq 1$, then it is also valid for $j-i=1$. Now

$$I(\theta C, (k+1-\theta)C, \theta C) = I\left(1, \frac{k+1-\theta}{\theta}, 1\right) = I\left(1, k+\frac{(k+1)(1-\theta)}{\theta}, 1\right)$$

$$= \int_0^1\int_k^{k+1}\left(y+\frac{(k+1)(1-\theta)}{\theta}-x\right)^{-2} dydx \geqq HI(1,k,1) \text{ for } k\geqq 2, \tag{4.11}$$

where

$$H = \inf_{k\geqq 2}\ \inf_{k-1\leqq y-x\leqq k+1} \frac{(y-x)^2}{\left(y-x+\frac{(k+1)(1-\theta)}{\theta}\right)^2} = \inf_{k\geqq 2}\left(1+\frac{(k+1)(1-\theta)}{\theta(k-1)}\right)^{-2}$$
$$= (1+3(1-\theta)/\theta)^{-2}$$
$$= \left[\frac{\theta}{1+2(1-\theta)}\right]^2 \geqq \frac{\theta^2}{[1+(1-\theta)]^4}$$
$$\geqq \theta^2[1-(1-\theta)]^4 = \theta^6. \tag{4.12}$$

Combining the $s=2$ case of (2.4), (4.11) and (4.12) we have

$$p'_k \geqq p_k(\theta^6\beta) \quad \text{for all } k\geqq 2, \tag{4.13}$$

which, by the definition (2.12), yields (4.7). Next, if $(1-\theta)C\geqq 1$, we may use (4.10) with $j-i=1$ and the identity (for $s=2$, $\Delta>0$ and $x_1<y_1$),

$$I(x_1,y_1,\Delta)=\int\limits_{x_1-\Delta}^{x_1}[(y_1-x)^{-1}-(y_1+\Delta-x)^{-1}]dx=\ln\left[\frac{(y_1-x_1+\Delta)^2}{(y_1-x_1)(y_1-x_1+2\Delta)}\right], \tag{4.14}$$

to obtain

$$1-p_1'\leqq\exp\left\{-\beta\ln\left[\frac{(2-\theta)^2}{2(2-2\theta)}\right]\right\}\leqq\left\{\frac{4(1-\theta)}{[1+(1-\theta)]^2}\right\}^{\beta'}\leqq[4(1-\theta)]^{\beta'}, \tag{4.15}$$

where the second inequality is due to the fact that (when $s=2$) $\beta'\leqq\beta$ since $\beta'=\beta$ for $\theta=1$ by the scaling invariance of I. On the other hand, combining (2.5) with the identity (for $0<\xi<1$)

$$\int\limits_0^1\int\limits_{|y-x|>\xi}{}_1^2(y-x)^{-2}dydx=\int\limits_0^{1-\xi}[(1-x)^{-1}-(2-x)^{-1}]dx$$
$$+\int\limits_{1-\xi}^1[\xi^{-1}-(2-x)^{-1}]dx=-\ln(2\xi)+1=-\ln(2\xi/e), \tag{4.16}$$

we have

$$p_1(\beta',x)=1-(2x/e)^{\beta'}. \tag{4.17}$$

Comparing (4.15) and (4.17), we see that if $2e(1-\theta)\leqq 1$, then (4.8) follows from the definition (2.13). This completes the proof of Proposition 4.1.

The remainder of the paper is devoted to a (lengthy) proof of Proposition 4.2. The proof is analogous to that of (3.4) and (3.6) above but it is considerably more difficult because, whereas $\beta_n\to\infty$ for $s<2$ (see (3.3) and (3.5)), $\beta_n\not\to\infty$ for $s=2$ (see (4.4) and (4.7)). With $\beta_n\not\to\infty$, we must rely on the fact that $\xi_n\to 0$ (see (4.5) and (4.8)), i.e., in the context of Proposition 4.2, β is fixed while $\xi\to 0$ as $N\to\infty$, so that $p_k(\beta,\xi)$ is fixed for $k\geqq 2$ and only $p_1(\beta,\xi)\to 1$ as $N\to\infty$. The lower bound needed (in order to obtain (4.9)) for λ_N', the probability of the block $H_1\equiv\{1,2,\ldots,C\}$ being "alive," must be based largely on the high probability of *nearest neighbor* bonds being occupied. This contrasts with the previous proof of (3.4) and (3.6) which utilized the high probability (due to $\beta_n\to\infty$) of *all* bonds in H_1 being occupied.

Proof of Proposition 4.2. Throughout this proof we suppress the subscript N. Recalling the definition of λ' given by (2.11), we must obtain an appropriate upper bound for the probability $1-\lambda'$ that H_1 is not "alive." This will require a number of new definitions.

We denote by $\bar{V}$ the *number* of dead sites in H_1, by $\bar{R}$ the *set* of living sites in H_1. A nonempty interval, $\bar{\Lambda}=\{k,k+1,\ldots,k+l\}$, in H_1 will be called a *run* (of length $\|\Lambda\|=l+1$) if the site $k+i$ is alive for each $i=0,\ldots,l$ and the (nearest neighbor) bond from $k+i$ to $k+i+1$ is occupied for each $i=0,\ldots,l-1$; it will be called a *maximal run* if in addition neither $\{k-1,\ldots,k+l\}$ nor $\{k,\ldots,k+l+1\}$ is a run (in H_1). We parition $\bar{R}$ into maximal runs, $\bar{R}=\bar{\Lambda}_1\cup\cdots\cup\bar{\Lambda}_M$ with $\bar{\Lambda}_i$ to the left of $\bar{\Lambda}_{i+1}$ for each $i=1,\ldots,M-1$. An *obstruction* will be said to occur between $\bar{\Lambda}_i$ and $\bar{\Lambda}_{i+1}$ if there is no occupied bond between any site in $\bar{\Lambda}_i$ and any site in $\bar{\Lambda}_{i+1}$; $\bar{\Lambda}_i$ and $\bar{\Lambda}_{i+1}$ will be said to be *involved* in the obstruction. Such an obstruction will be said to be *bridged* if there is an occupied bond between some site in $\bar{\Lambda}_1 \cup \cdots \cup \bar{\Lambda}_i$ and some site in $\bar{\Lambda}_{i+1} \cup \cdots \cup \bar{\Lambda}_M$.

A $\bar{\Lambda}_i$ will be called *short* (respectively *long*) if $\|\bar{\Lambda}_i\| \leqq N^{q_1}$ (repectively $\|\bar{\Lambda}_i\| > N^{q_1}$), where q_1 will be a fixed constant satisfying

$$0 < q_1 < r - q, \tag{4.18}$$

so that $N^{q_1} \leqq ((1-\theta)C/2)$ for large N. In addition to (4.18) there will be another restriction on q_1 depending on β (see (4.87) and (4.89) below). An obstruction between $\bar{\Lambda}_i$ and $\bar{\Lambda}_{i+1}$ will be called an *interior obstruction* unless both $\max(\bar{\Lambda}_i)$ and $\min(\bar{\Lambda}_{i+1})$ are in $\{1, 2,\ldots,[N^{q_1}]\} \cup \{C-[N^{q_1}]+1, C-[N^{q_1}]+2,\ldots,C\}$.

Our bound on $1-\lambda'$ will be based on estimating the probability that none of the following occur, and hence that H_1 is "alive": $\bar{V} > (1-\theta)C/2$, more than one obstruction occurs between a long $\bar{\Lambda}_i$ and a long $\bar{\Lambda}_{i+1}$, more than one of the short $\bar{\Lambda}_i$'s is involved in an obstruction (but a single short $\bar{\Lambda}_i$ may be involved in two obstructions, one to its left and one to its right), there is simultaneously an obstruction involving a long $\bar{\Lambda}_i$ and a long $\bar{\Lambda}_{i+1}$ and an obstruction involving a short $\bar{\Lambda}_{i'}$, and finally some interior obstruction is not bridged. We claim that if none of these events occur, then H_1 will contain a cluster at least as big as $C-(1-\theta)C/2-N^{q_1}$, hence at least as big as θC (for large N) and thus H_1 will be "alive."

To see why this is so, suppose that none of these events occur. If no obstructions occur, then all $C-\bar{V}$ living sites belong to a single cluster and $\bar{V} \leqq (1-\theta)C/2$. If an obstruction involving two long maximal runs occurs then it is necessarily an interior obstruction, hence bridged, and no other obstruction occurs; thus again all $C-\bar{V}$ living sites belong to a single cluster. If an obstruction involving a short $\bar{\Lambda}_i$ occurs, then there can be either one or two obstructions involving that $\bar{\Lambda}_i$ (each of which can be interior or not) but no other obstructions. There are now several possibilities. If there is one obstruction and it is bridged then again all $C-\bar{V}$ living sites belong to a single cluster. If there is one obstruction and it is not bridged, then it cannot be an interior obstruction and hence there is a cluster containing at least $C-\bar{V}-N^{q_1}$ sites. If there are two obstructions and both are bridged, then all maximal runs except (possibly) one short $\bar{\Lambda}_i$ belong to a single cluster which must therefore have at least $C-\bar{V}-\|\bar{\Lambda}_i\| \geqq C-\bar{V}-N^{q_1}$ sites. If there are two obstructions and only one is bridged, then the unbridged one is not interior and hence either there is a cluster containing all maximal runs except a single short $\bar{\Lambda}_i$ or else there is a cluster containing all (including those in $\bar{\Lambda}_i$) except at most N^{q_1} living sites (either in $\{1,\ldots,[N^{q_1}]\}$ or in $\{C-[N^{q_1}]+1,\ldots,C\}$). Finally if there are two unbridged obstructions, then neither is interior and (for large N) either both are in $\{1,\ldots,[N^{q_1}]\}$ or both are in $\{C-[N^{q_1}]+1,\ldots,C\}$ so that there is a cluster containing all except at most N^{q_1} living sites.

We define three random variables related to obstructions. L^1 is the number of obstructions involving a long $\bar{\Lambda}_i$ and a long $\bar{\Lambda}_{i+1}$. L^2 is the number of short $\bar{\Lambda}_i$'s involved in obstructions. L^3 is the number of interior obstructions which are *not* bridged. Our basic bound which will eventually lead to (4.9) is

$$1-\lambda' \leqq P(\bar{V} > (1-\theta)C/2) + P(L^1+L^2 \geqq 2) + P(L^3 \geqq 1). \tag{4.19}$$

In order to prove (4.9), we will show that for $\beta > 1$, q_1 can be chosen so that each term on the right-hand side of (4.19) is $o(N^{-q})$ as $N \to \infty$. As to the first term, $\bar{V}$ is a binomial random variable with parameters $C = [2N^r]$ and $(1-\lambda) = (\Theta/4)(N+1)^{-q}$, so by Lemma 3.3

$$\begin{aligned} P(\bar{V} > (1-\theta)C/2) &= P(\bar{V}/C > \Theta N^{-q}/2) \leqq P(\bar{V}/C \geqq 2(1-\lambda)) \\ &\leqq \exp(-C(1-\lambda)/16) = o(N^{-q}) \quad \text{as} \quad N \to \infty, \end{aligned} \tag{4.20}$$

since $C(1-\lambda) \sim \text{const}\, N^{r-q}$ and $q < r$. It remains to prove

$$N^q P(L^1 + L^2 \geqq 2) \to 0, \tag{4.21}$$

$$N^q P(L^3 \geqq 1) \to 0. \tag{4.22}$$

We will first prove (4.22) and then return to (4.21). We denote by U_k^3 ($k = 1, 2, \ldots, C$) the event that for some $i < M$, there is an unbridged obstruction between $\bar{\Lambda}_i$ and $\bar{\Lambda}_{i+1}$ and either $k = \max(\bar{\Lambda}_i)$ or $k = \min(\bar{\Lambda}_{i+1})$. The indicator random variable of an event U is denoed by 1_U. Then

$$L^3 \leqq \sum_{k=[N^{q_1}]+1}^{C-[N^{q_1}]} 1_{U_k^3}, \tag{4.23}$$

and so

$$P(L^3 \geqq 1) \leqq E(L^3) \leqq \sum_{k=[N^{q_1}]+1}^{C-[N^{q_1}]} P(U_k^3). \tag{4.24}$$

For $k \neq 1$ or C, the event that $k = \max(\bar{\Lambda}_i)$ or $\min(\bar{\Lambda}_i)$ for some i is identical to the event U_k, that k is a living site and either $k-1$ is dead or $k+1$ is dead or the bond between $k-1$ and k is vacant or the bond between k and $k+1$ is vacant. Thus by (4.17),

$$P(U_k) = \lambda[1 - \lambda^2(p_1(\beta,\xi))^2] < \lambda[1 - \lambda^2(1 - 4\Theta N^{-q})^2] \sim \text{const}\, N^{-q}. \tag{4.25}$$

In order that U_k^3 occur it is necessary that k be the endpoint of some $\bar{\Lambda}_i$ and that every bond between a living site (in H_1) to the left of k and a living site (in H_1) to the right of k be vacant. Let us denote by X_k the indicator random variable of the event that k is a living site. Then for $k \neq 1, 2, C-1$ or C,

$$\begin{aligned} P(U_k^3) &\leqq P(U_k) \cdot E\left(\prod_{i=1}^{k-2} \prod_{j=k+2}^{C} (1 - p_{j-i}(\beta))^{X_i X_j}\right) \\ &\leqq \text{const}\, N^{-q} E\left(\exp\left[-\beta \sum_{i=1}^{k-2} \sum_{j=k+2}^{C} J_{j-i} X_i X_j\right]\right), \end{aligned} \tag{4.26}$$

where

$$J_j = \int_0^1 \int_j^{j+1} |y-x|^2 \, dy\, dx \sim j^{-2} \quad \text{as} \quad j \to \infty. \tag{4.27}$$

Let us define for $n, n' \geqq 1$,

$$Y_{n,n'} = \sum_{i=-n}^{-2} \sum_{j=2}^{n'} J_{j-i} X_i X_j; \tag{4.28}$$

then by translation invariance,

$$P(U_k^3) \leqq \text{const } N^{-q} E(\exp[-\beta Y_{k-1,C-k}]). \tag{4.29}$$

We further note that $Y_{n,n'}$ is increasing in both n and n'. Hence if we define

$$Y_n = Y_{n,n} = \sum_{i=-n}^{-2} \sum_{j=2}^{n} J_{j-i} X_i X_j, \tag{4.30}$$

then $Y_{n,n'} \geqq Y_{\min(n,n')}$ which, together with (4.24) and (4.29) implies that

$$P(L^3 \geqq 1) \leqq \text{const } N^{-q} \sum_{i=[N^{q_1}]}^{[C/2]} E(\exp[-\beta Y_i]). \tag{4.31}$$

It immediately follows from (4.31) that in order to prove (4.22), it suffices (by dominated convergence) to show that for some N_0 sufficiently large,

$$\sum_{n=2}^{\infty} \sup_{N \geqq N_0} \{E(\exp[-\beta Y_n])\} < \infty. \tag{4.32}$$

This follows easily from the next lemma.

Lemma 4.3. *Let* $\ldots X_{-1}, X_0, X_1, \ldots$ *be independent random variables with* $P(X_i = 1) = \lambda$, $P(X_i = 0) = 1 - \lambda$ *for each i. Let* $J_j \geqq 0$ *be a numerical sequence such that* $\lim_{j\to\infty} j^2 J_j = 1$ *and define* Y_n *by* (4.30). *Then for fixed* n, $\{J_j\}$ *and* β, $E(\exp[-\beta Y_n])$ *is decreasing in* λ. *Moreover if* $\lambda^2\beta > 1$, *then*

$$\sum_{n=2}^{\infty} E(\exp[-\beta Y_n]) < \infty. \tag{4.33}$$

Proof of Lemma 4.3. To see the monotonicity in λ, Let Z_i be independent and uniformly distributed on $(0, 1)$ and write $X_i = 1_{Z_i \leqq \lambda}$. To obtain (4.33), it will be convenient to define for $l \geqq 1$,

$$Y_n^l = \sum_{i'=l}^{n} \sum_{j=l}^{n} J_{j+i'} X_{-i'} X_j,$$

so that $Y_n = Y_n^2$. Since Y_n^l is decreasing in l and is independent of J_j for $j < 2l$, it is easy to see that to obtain (4.33) (for general J_j) whenever $\lambda^2\beta > 1$, it suffices to take $J_j = j^{-2}$ and show that for every fixed $l \geqq 1$,

$$\sum_{n=l}^{\infty} E(\exp(-\beta Y_n^l)) < \infty \tag{4.34}$$

whenever $\lambda^2\beta > 1$.

To do this, we use the identity (where $\min(j, n)$ is denoted $j \wedge n$)

$$\begin{aligned}\sum_{i'=l}^{n} \sum_{j=l}^{n} J_{j+i'} x_{-i'} x_j &= \sum_{i'=l}^{n} x_{-i'} \sum_{k=l}^{\infty} (J_{k+i'} - J_{k+i'+1}) \left(\sum_{j=l}^{k\wedge n} x_j \right) \\ &= \sum_{k=l}^{\infty} \sum_{k'=l}^{\infty} \tilde{J}_{k'+k} \left(\sum_{i'=l}^{k'\wedge n} x_{-i'} \right) \left(\sum_{j=l}^{k\wedge n} x_j \right),\end{aligned} \tag{4.35}$$

where

$$\tilde{J}_k = (J_k - J_{k+1}) - (J_{k+1} - J_{k+2}) = \frac{6k^2 + 12k + 4}{k^2(k+1)^2(k+2)^2} > 0.$$

For l_n in $\{l,\dots,n\}$ and $\varepsilon < 1$, we define the event $A_n(\varepsilon, l_n)$ as

$$\sum_{j=l}^{k} X_j < (1-\varepsilon)\lambda(k-l+1) \text{ or } \sum_{i'=l}^{k} X_{-i'} < (1-\varepsilon)\lambda(k-l+1)$$
$$\text{for some } k \geqq l_n.$$

In the complement of $A_n(\varepsilon, l_n)$, one has (by using (4.35) twice)

$$\begin{aligned} Y_n^l &\geqq \sum_{k=l_n}^{\infty} \sum_{k'=l_n}^{\infty} \tilde{J}_{k'+k}(1-\varepsilon)^2\lambda^2((k' \wedge n) - l + 1)((k \wedge n) - l + 1) \\ &\geqq (1-\varepsilon)^2\lambda^2 \sum_{k=l_n}^{\infty} \sum_{k'=l_n}^{\infty} \tilde{J}_{k'+k}\left(\sum_{i'=l_n}^{k'\wedge n} 1\right)\left(\sum_{j=l_n}^{k\wedge n} 1\right) \\ &= (1-\varepsilon)^2\lambda^2 \sum_{i'=l_n}^{n} \sum_{j=l_n}^{n} J_{j+i'}. \end{aligned}$$

Thus,

$$E(\exp(-\beta Y_n^l)) \leqq P(A_n(\varepsilon, l_n)) + \exp\left(-\beta(1-\varepsilon)^2\lambda^2 \sum_{i'=l_n}^{n} \sum_{j=l_n}^{n} (j+i')^{-2}\right).$$

The second term on the right-hand side of the last inequality can be made summable over n by choosing ε so small that $\beta(1-\varepsilon)^2\lambda^2 > 1$ and choosing l_n so that

$$\lim_{n\to\infty} l_n/n^\delta = 0 \text{ for every } \delta > 0.$$

This is a consequence of the elementary fact (based on (4.14)) that $l_n = O(n^\delta)$ for every $\delta > 0$ implies

$$\lim_{n\to\infty} (\ln(n))^{-1} \sum_{i'=l_n}^{n} \sum_{j=l_n}^{n} (j+i')^{-2} = 1.$$

To prove (4.34) and complete the proof of Lemma 4.3, it only remains to show that there is a sequence l_n (which is $O(n^\delta)$ for every $\delta > 0$) such that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(A_n(\varepsilon, l_n)) < \infty. \tag{4.36}$$

From the definition of $A_n(\varepsilon, l_n)$ and (3.15), we have

$$\begin{aligned} P(A_n(\varepsilon, l_n)) &\leqq 2 \sum_{k=l_n}^{\infty} P\left(\sum_{j=1}^{k} X_j < (1-\varepsilon)\lambda(k-l+1)\right) \\ &= 2 \sum_{k=l_n}^{\infty} P\left(\sum_{j=l}^{k} (1-X_j) > (k-l+1)\left(1 + \varepsilon\frac{\lambda}{1-\lambda}\right)(1-\lambda)\right) \\ &\leqq 2 \sum_{k=l_n}^{\infty} \exp(-(k-l+1)W) = 2\exp(-(l_n-l+1)W)\cdot(1-e^{-W})^{-1}, \end{aligned}$$

where

$$W = -\inf_{t \geqq 0}\{\ln(\lambda + (1-\lambda)e^t) - (1-\lambda+\varepsilon\lambda)t\} > 0.$$

Since $P(A_n(\varepsilon, l_n))$ is $O(\exp(-l_n W))$ and $W > 0$, (4.36) will be valid providing

$$\lim_{n\to\infty} l_n/\ln(n) = \infty.$$

A choice of $l_n = [\ln(n)]^2$ will satisfy this and be $O(n^\delta)$ for every $\delta > 0$ which completes the proof of (4.36) and thus of Lemma 4.3.

We have now proved (4.22). In order to complete the proof of Proposition 4.2, it remains to prove (4.21). Toward this end, we will find it convenient to compare the $\bar{\Lambda}_i$'s ($i = 1, \ldots, M$), which are the successive maximal runs in the block H_1, to the successive maximal runs in the half space, $\{1, 2, 3, \ldots\}$, which we denote by Λ_i ($i = 1, 2, \ldots$). Both M and the $\bar{\Lambda}_i$'s may be constructed from the Λ_i's:

$$M = \min\{i: \Lambda_{i+1} \cap H_1 = \varnothing\}, \tag{4.37}$$

$$\bar{\Lambda}_i = \begin{cases} \Lambda_i & \text{if } i < M \\ \Lambda_i \cap H_1 & \text{if } i = M \end{cases}. \tag{4.38}$$

The lengths $\|\Lambda_i\|$ of the Λ_i's and the spacings between them, $G_i \equiv \min(\Lambda_{i+1}) - \max(\Lambda_i)$ (with $G_0 \equiv \min(\Lambda_1)$), are independent random variables but this is not the case for the $\bar{\Lambda}_i$'s. Note that $G_i - 1$ is the number of dead sites between Λ_i and Λ_{i+1}. The distributions of these random variables are given by:

$$P(\|\Lambda_i\| = k) = (1 - p_1\lambda)(p_1\lambda)^{k-1}, \quad k = 1, 2, \ldots \tag{4.39}$$

for all i, where $p_1 = p_1(\beta, \xi)$ (see (2.5)),

$$P(G_0 = k) = \lambda(1-\lambda)^{k-1}, \quad k = 1, 2, \ldots, \tag{4.40}$$

$$P(G_i = k) = \begin{cases} \lambda(1-p_1)/(1-p_1\lambda), & k = 1 \\ [(1-\lambda)/(1-p_1\lambda)]\lambda(1-\lambda)^{k-2}, & k = 2, 3, \ldots \end{cases} \tag{4.41}$$

for all $i \geqq 1$. These formulas are all easily derived; e.g., the $k = 1$ case of (4.41) is simply the (conditional) probability that site $j+1$ is alive conditioned on the complement of the following event: site $j+1$ is alive and the nearest neighbor bond from j to $j+1$ is occupied. We note that by (4.17),

$$p_1 = p_1(\beta, \xi) = 1 - (4\Theta N^{-q})^\beta,$$

so that

$$1 - p_1\lambda \sim \text{const } N^{-q}. \tag{4.42}$$

We define two sequences of events involving the Λ_i's which are closely related to the L^1 and L^2 defined above in terms of the $\bar{\Lambda}_i$'s. W_i^1 is the event that $\|\Lambda_i\| > N^{q_1}$, $\|\Lambda_{i+1}\| > N^{q_1}$ and an obstruction occurs between Λ_i and Λ_{i+1}. W_i^2 is the event that $\|\Lambda_i\| \leqq N^{q_1}$ and an obstruction occurs involving Λ_i (for $i = 1$ this must involve Λ_1 and Λ_2). We also define the corresponding events involving the $\bar{\Lambda}_i$'s. $\bar{W}_i^1$ is the event that $M > i$, $\|\bar{\Lambda}_i\| > N^{q_1}$, $\|\bar{\Lambda}_{i+1}\| > N^{q_1}$ and an obstruction occurs between $\bar{\Lambda}_i$ and

$\bar{\Lambda}_{i+1}$. $\bar{W}_i^2$ is the event that $M \geqq i$, $\|\bar{\Lambda}_i\| \leqq N^{q_1}$ and an obstruction occurs involving $\bar{\Lambda}_i$ (for $i=1$ this must involve $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ while for $M=i$ thus must involve $\bar{\Lambda}_{M-1}$ and $\bar{\Lambda}_M$). Thus

$$L^j = \sum_{i=1}^{\infty} 1_{\bar{W}_i^j} = \sum_{i=1}^{M} 1_{\bar{W}_i^j} \quad \text{for} \quad j=1, 2.$$

Returning to (4.21), we define the (nonrandom) integer $\bar{m}$ by

$$\bar{m} = \min\{m : m \geqq (3/2)C(1-p_1\lambda)\},$$

and note that by (4.42)

$$\bar{m} \sim \text{const}\, N^{r-q} \to \infty. \tag{4.43}$$

We use the estimate

$$\begin{aligned} P(L^1+L^2 \geqq 2) &\leqq P(M>\bar{m}) + P(L^1+L^2 \geqq 2, M \leqq \bar{m}) \\ &\leqq P(M>\bar{m}) + P(L^1 \geqq 2,\, M<\bar{m}) + P(L^2 \geqq 2, M \leqq \bar{m}) + P(L^1 \geqq 1, L^2 \geqq 1, M \leqq \bar{m}) \\ &\leqq P(M>\bar{m}) + \sum_{j=1}^{2}\sum_{j'=1}^{2} P(\bar{W}_i^j \text{ and } \bar{W}_k^{j'} \text{ occur for some } i<k \leqq M,\ M \leqq \bar{m}). \end{aligned} \tag{4.44}$$

Thus, to prove (4.21), it suffices to prove the following limits:

$$N^q P(M>\bar{m}) \to 0, \tag{4.45}$$

$$N^q P(\bar{W}_i^j \text{ and } \bar{W}_k^{j'} \text{ occur for some } i<k \leqq M, M \leqq \bar{m}) \to 0 \quad \text{for } j, j' = 1, 2. \tag{4.46}$$

We use the fact that $\bar{\Lambda}_i = \Lambda_i$ for $i<M$ and the (left-right) symmetry between $\{\bar{\Lambda}_i\}$ and $\{\bar{\Lambda}_{M-i+1}\}$ to bound the probability in (4.46) by

$$\begin{aligned} &\sum_{i=2}^{\bar{m}-3}\sum_{k=i+1}^{\bar{m}-2} P(W_i^j, W_k^{j'}) + P\left(\bigcup_{i=3}^{M-2}\{\bar{W}_i^j, \bar{W}_{M-1}^{j'}, M \leqq \bar{m}\}\right) \\ &+ P\left(\bigcup_{i=3}^{M-1}\{\bar{W}_i^j, \bar{W}_M^{j'}, M \leqq \bar{m}\}\right) \\ &+ P\left(\bigcup_{k=2}^{M-2}\{\bar{W}_1^j, \bar{W}_k^{j'}, M \leqq \bar{m}\}\right) + P(\bar{W}_2^j, \bar{W}_{M-1}^{j'}) + P(\bar{W}_2^j, \bar{W}_M^{j'}) \\ &+ P(\bar{W}_1^j, \bar{W}_{M-1}^{j'}) + P(\bar{W}_1^j, \bar{W}_M^{j'}) \leqq \sum_{i=2}^{\bar{m}-3}\sum_{k=i+1}^{\bar{m}-2} P(W_i^j, W_k^{j'}) \\ &+ \sum_{k=3}^{\bar{m}-2} P(W_2^{j'}, W_k^j) + \sum_{k=2}^{\bar{m}-2}[P(W_1^{j'}, W_k^j) + P(W_1^j, W_k^{j'})] + \sum_{i,i'=1}^{2} P(\bar{W}_i^j, \bar{W}_{M-i'+1}^{j'}). \end{aligned} \tag{4.47}$$

We then use the following estimates

$$P(W_i^1, W_k^1) = P(W_i^1)P(W_k^1) = [P(W_1^1)]^2 \quad \text{if} \quad k-i>1, \tag{4.48}$$

$$P(W_i^1, W_{i+1}^1) = P(W_1^1, W_2^1), \tag{4.49}$$

$$P(W_i^j, W_k^{j'}) = P(W_i^j)P(W_k^{j'})$$

$$\leq \begin{cases} P(W_1^1)P(W_2^2) & \text{if } k-i>2 \text{ and exactly one of } (j,j')=2 \\ [P(W_2^2)]^2 & \text{if } k-i>2 \text{ and } j=j'=2 \end{cases} \tag{4.50}$$

$$P(W_i^2, W_k^2) \leqq P(\|\Lambda_i\| \leqq N^{q_1}, \|\Lambda_k\| \leqq N^{q_1}) = [P(\|\Lambda_1\| \leqq N^{q_1})]^2 \quad \text{if} \quad k-i=1 \text{ or } 2, \tag{4.51}$$

$$P(W_i^j, W_k^{j'}) \leqq P(\|\Lambda_1\| \leqq N^{q_1})P(W_1^1) \quad \text{if} \quad k-i=1 \text{ or } 2 \text{ and exactly one of } (j,j')=2. \tag{4.52}$$

$$\begin{aligned}
P(\bar{W}_i^j, \bar{W}_{M-i'+1}^{j'}) &\leqq P\left(\sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)>C/2\right)+P\left(\sum_{k'=M-2}^{M}(G_{k'}+\|\Lambda_{k'}\|)>C/2\right)\\
&\quad +P\left(\bar{W}_i^j, \bar{W}_{M-i'+1}^{j'}, \sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)\leqq C/2, \sum_{k'=M-2}^{M}(G_{k'}+\|\Lambda_{k'}\|)\leqq C/2\right)\\
&=2P\left(\sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)>C/2\right)\\
&\quad +P\left(W_i^j, \sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)\leqq C/2\right)\cdot P\left(W_{i'}^{j'}, \sum_{k'=1}^{3}(G_{k'-1}+\|\Lambda_{k'}\|)\leqq C/2\right)\\
&\leqq 2P\left(\sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)>C/2\right)+P(W_i^j)P(W_{i'}^{j'})\\
&\leqq 2P\left(\sum_{k=1}^{3}(G_{k-1}+\|\Lambda_k\|)>C/2\right)\\
&\quad +\begin{cases}[P(W_1^1)]^2 & \text{if } j=j'=1\\ P(W_1^1)\cdot P(W_2^2) & \text{if exactly one of } (j,j')=2.\\ [P(W_2^2)]^2 & \text{if } j=j'=2\end{cases}
\end{aligned} \tag{4.53}$$

Combining (4.45)–(4.53) and the fact that $\bar{m}\to\infty$ (see (4.43)), we see that (4.21) would be a consequence of (4.45) and the following limits:

$$N^q\bar{m}^2[P(W_1^1)]^2\to 0, \tag{4.54}$$

$$N^q\bar{m}^2[P(W_2^2)]^2\to 0, \tag{4.55}$$

$$N^q\bar{m}P(W_1^1, W_2^1)\to 0, \tag{4.56}$$

$$N^q\bar{m}[P(\|\Lambda_1\|\leqq N^{q_1})]^2\to 0, \tag{4.57}$$

$$N^q\bar{m}P(\|\Lambda_1\|\leqq N^{q_1})P(W_1^1)\to 0, \tag{4.58}$$

$$N^qP\left(\sum_{i=1}^{3}(G_{i-1}+\|\Lambda_i\|)>C/2\right)\to 0. \tag{4.59}$$

Note that all these limits involve the half-space maximal runs, Λ_i (rather than the statistically more complicated H_1 maximal runs, $\bar{\Lambda}_i$), and their related random variables M, $\|\Lambda_i\|$, G_i and W_i^j. We first prove (4.45), (4.59) and (4.57).

To prove (4.45), we note that by (4.37),

$$P(M>\bar{m})\leqq P(\|\Lambda_1\|+\cdots+\|\Lambda_{\bar{m}}\|<C). \tag{4.60}$$

The $\|\Lambda_i\|$'s are i.i.d. random variables with, according to (4.39), a geometric distribution of parameter $1-p_1\lambda$; thus the left-hand side of (4.60) equals $P(V \geqq \bar{m})$, where V is a binomial random variable with parameters C and $1-p_1\lambda$. Consequently,

$$P(M > \bar{m}) \leqq P(V/C \geqq \tfrac{3}{2}(1-p_1\lambda) \leqq \exp(-C(1-p_1\lambda)/16) \tag{4.61}$$

by the definition of $\bar{m}$ and by Lemma 3.3. By (4.42), this yields (4.45) since $r > q$.

To prove (4.59), we note that by (4.39)–(4.42), the probability in (4.59) is bounded by

$$\sum_{i=1}^{3} [P(G_{i-1} \geqq C/12) + P(\|\Lambda_i\| \geqq C/12)] = O((1-\lambda)^{C/12-2} + (p_1\lambda)^{C/12-1})$$
$$= O(\exp[-\text{const}\, N^{r-q}]),$$

which yields (4.59).

To prove (4.57) we use (4.39) and (4.42) to obtain

$$P(\|\Lambda_1\| \leqq N^{q_1}) = 1 - P(\|\Lambda_1\| > N^{q_1}) \leqq 1 - (p_1\lambda)^{N^{q_1}}$$
$$\leqq 1 - [1 - N^{q_1}(1-p_1\lambda)] = O(N^{q_1-q}). \tag{4.62}$$

By (4.43) and (4.62), the left-hand side of (4.57) is $O(N^{q+r-q+2(q_1-q)}) = O(N^{-2(q-r/2-q_1)})$. Thus (4.57) will be valid providing

$$q_1 < q - r/2. \tag{4.63}$$

But it follows from (4.2) that $r - q < q - r/2$, so that (4.63) is a consequence of (4.18). We have now proved (4.45), (4.59) and (4.57). It remains to prove (4.54), (4.55), (4.56) and (4.58).

We begin by writing

$$P(W_1^1) \leqq P(G_1 > 3) + P(W_1^1, G_1 \leqq 3). \tag{4.64}$$

$$P(W_2^2) \leqq P(G_1 > 3 \text{ or } G_2 > 3) + P(W_2^2, G_1 \leqq 3, G_2 \leqq 3)$$
$$\leqq 2P(G_1 > 3) + P(W_2^2, G_1 \leqq 3, G_2 \leqq 3), \tag{4.65}$$

$$P(W_1^1, W_2^2) \leqq 2P(G_1 > 3) + P(W_1^1, W_2^2, G_1 \leqq 3, G_2 \leqq 3, \tag{4.66}$$

By (4.41),

$$P(G_1 > 3) = O((1-\lambda)^{4-2}) = O(N^{-2q}), \tag{4.67}$$

so that by (4.43), $N^q\bar{m}\ P(G_1 > 3) = O(N^{q+r-q-2q}) = O(N^{r-2q})$ while $N^q\bar{m}^2[P(G_1 > 3)]^2 = O(N^{q+2(r-q)-4q}) = O(N^{2r-5q})$. Since (4.2) implies that $r - 2q < 0$ and $2r - 5q < 0$, it follows from (4.43), (4.62) and (4.64)–(4.67) that in order to obtain (4.54), (4.55), (4.56) and (4.58) (and hence obtain (4.21) and complete the proof of Proposition 4.2), it suffices to prove the following four limits:

$$N^{r-q/2}P(W_1^1, G_1 \leqq 3) \to 0, \tag{4.68}$$

$$N^{r-q/2}P(W_2^2, G_1 \leqq 3, G_2 \leqq 3) \to 0, \tag{4.69}$$

$$N^r P(W_1^1, W_2^1, G_1 \leqq 3, G_2 \leqq 3) \to 0, \tag{4.70}$$

$$N^{r+q_1-q}P(W_1^1, G_1 \leqq 3) \to 0. \tag{4.71}$$

We have, as in the proof of Lemma 4.2 (see particularly (4.14)), that

$$P(\text{every bond between } \Lambda_i \text{ and } \Lambda_{i+1} \text{ is vacant} \mid \|\Lambda_i\| = l, G_i = d, \|\Lambda_{i+1}\| = l')$$

$$= \begin{cases} \exp\left(-\beta \int\limits_0^l \int\limits_{l+d-1}^{l+d+l'-1} (y-x)^{-2}(1 - 1_{x>l-1, y<l+1}) dy dx \right) & \text{if } d = 1 \\ \exp\left(-\beta \int\limits_0^l \int\limits_{l+d-1}^{l+d+l'-1} (y-x)^2 dy dx \right) & \text{if } d > 1 \end{cases}$$

$$\leqq \exp\left(-\beta \int\limits_0^{l-1} \int\limits_{l+d}^{l+d+l'-1} (y-x)^{-2} dy dx \right)$$

$$= \exp(-\beta[\ln(l+d) + \ln(d+l') - \ln(l+d+l'-1) - \ln(d+1)])$$

$$= \left[\frac{(d+1)(l+l'+d-1)}{(l+d)(l'+d)} \right]^\beta \leqq \begin{cases} \left[\dfrac{2(d+1)}{\min(l,l')} \right]^\beta, & \text{if } 2(d+1) < \min(l,l') \\ 1 \quad , & \text{otherwise} \end{cases} \tag{4.72}$$

Similarly,
P(every bond between Λ_1 and Λ_2 and between Λ_2 and Λ_3 is vacant|

$$\|\Lambda_1\| = l_1,\ G_1 = d_1,\ \|\Lambda_2\| = l_2,\ G_2 = d_2,\ \|\Lambda_3\| = l_3)$$
$$\leqq \left[\frac{2(d_1+1)}{\min(l_1, l_2)} \right]^\beta \cdot \left[\frac{2(d_2+1)}{\min(l_2, l_3)} \right]^\beta . \tag{4.73}$$

It follows that if we define $Q_i = \min(\|\Lambda_i\|, \|\Lambda_{i+1}\|)$, and $\tilde{Q}_i = \min(\|\Lambda_i\|, \|\Lambda_{i+1}\|, \|\Lambda_{i+2}\|)$, then

$$P(W_1^1, G_1 \leqq 3) \leqq E\left\{ \left[\frac{8}{Q_1} \right]^\beta \cdot 1_{Q_1 > N^{q_1}} \right\}, \tag{4.74}$$

$$P(W_2^2, G_1 \leqq 3,\ G_2 \leqq 3) \leqq E\left\{ \left(\left[\frac{8}{Q_1} \right]^\beta + \left[\frac{8}{Q_2} \right]^\beta \right) \cdot 1_{\|\Lambda_2\| \leqq N^{q_1}} \right\} \leqq 2E\left\{ \left[\frac{8}{Q_1} \right]^\beta \right\}, \tag{4.75}$$

$$P(W_1^1, W_2^1,\ G_1 \leqq 3,\ G_2 \leqq 3) \leqq E\left\{ \left[\frac{64}{Q_1 Q_2} \right]^\beta \cdot 1_{Q_1 > N^{q_1}} \cdot 1_{Q_2 > N^{q_1}} \right\}$$
$$\leqq E\left\{ \left[\frac{8}{\tilde{Q}_1} \right]^{2\beta} \cdot 1_{\tilde{Q}_1 > N^{q_1}} \right\} \tag{4.76}$$

The $\|\Lambda_i\|$'s are independent random variables with a common geometric distribution given by (4.39). It is a standard fact that consequently $Q = Q_1$ or $\tilde{Q}_1$ has a geometric distribution,

$$P(Q = k) = \gamma(1-\gamma)^{k-1}, \quad k = 1, 2, \ldots, \tag{4.77}$$

where (see (4.42))

$$\gamma = \begin{cases} 1 - (p_1 \lambda)^2 = O(N^{-q}), & \text{for} \quad Q = Q_1 \\ 1 - (p_1 \lambda)^3 = O(N^{-q}), & \text{for} \quad Q = \tilde{Q}_1 \end{cases} . \tag{4.78}$$

Estimates for the expectations appearing in (4.74)–(4.76) follow from Lemma 4.4

below which immediately implies that

$$E([Q_1]^{-\beta}\cdot 1_{Q_1>N^{q_1}}) = O(N^{-q-(\beta-1)q_1}), \tag{4.79}$$

$$E([Q_1]^{-\beta}) = O(N^{-q}), \tag{4.80}$$

$$E([\tilde{Q}_1]^{-2\beta}\cdot 1_{\tilde{Q}_1>N^{q_1}}) = O(N^{-q-(2\beta-1)q_1}). \tag{4.81}$$

Combining (4.74)–(4.76) with (4.79)–(4.81) yields

$$P(W_1^1, G_1 \leqq 3) = O(N^{-q-(\beta-1)q_1}), \tag{4.82}$$

$$P(W_2^2, G_1 \leqq 3,\ G_2 \leqq 3) = O(N^{-q}), \tag{4.83}$$

$$P(W_1^1, W_2^1, G_1 \leqq 3,\ G_2 \leqq 3) = O(N^{-q-(2\beta-1)q_1}). \tag{4.84}$$

Thus we see that

$$r-(3q/2)-(\beta-1)q_1 < 0 \text{ implies (4.68) which implies (4.54)}, \tag{4.85}$$

$$r-(3q/2) < 0 \text{ implies (4.69) which implies (4.55)}, \tag{4.86}$$

$$r-q-(2\beta-1)q_1 < 0 \text{ implies (4.70) which implies (4.56)}, \tag{4.87}$$

$$r-2q-(\beta-2)q_1 < 0 \text{ implies (4.71) which implies (4.58)}. \tag{4.88}$$

The inequality in (4.85) is a consequence of the inequality in (4.86), since $\beta > 1$ and $q_1 > 0$, which in turn follows from (4.2). The inequality in (4.87) is equivalent (for $\beta > 1/2$) to

$$q_1 > (r-q)/(2\beta-1). \tag{4.89}$$

Since $\beta > 1$ implies $2\beta - 1 > 1$, q_1 can be chosen so that (4.89) and (4.18) are satisfied simultaneously. Since $2q - r > 0$ (by (4.2)), the inequality in (4.88) is automatically valid for $\beta \geqq 2$; for $1 < \beta < 2$, it is equivalent to $q_1 < 2(q - r/2)/(2-\beta)$ which follows from (4.63) (which itself is a consequence of (4.2) and (4.18)) and the fact that $2/(2-\beta) \geqq 1$.

With q_1 chosen to satisfy (4.18) and (4.89), all the inequalities in (4.85)–(4.88) are valid. We thus have (modulo (4.79)–(4.81) which follow from Lemma 4.4 below) proved (4.54), (4.55), (4.56) and (4.58) which in turn (together with the already proved (4.45), (4.57) and (4.59)) imply (4.21). The desired (4.9) is finally a consequence of (4.21) and the already proved (4.19), (4.20) and (4.22). The proof of Proposition 4.2 will be complete after the following lemma is stated and proved.

Lemma 4.4. *Suppose Q is a random variable with the geometric distribution (4.77) and $\bar{\beta} > 1$. Then*

$$E(Q^{-\bar{\beta}}) \leqq \gamma\bar{\beta}/(\bar{\beta}-1), \tag{4.90}$$

$$E(Q^{-\bar{\beta}}1_{Q>z}) \leqq \gamma/[(\bar{\beta}-1)(z-1)^{\bar{\beta}-1}] \quad \text{for} \quad z > 1. \tag{4.91}$$

Proof of Lemma 4.4. Inequality (4.91) is a consequence of

$$E(Q^{-\bar{\beta}}1_{Q>z}) = \sum_{k=[z+1]}^{\infty} \gamma(1-\gamma)^{k-1}\, k^{-\bar{\beta}} \leqq \gamma \sum_{k=[z+1]}^{\infty} k^{-\bar{\beta}} \leqq \gamma \int_{z-1}^{\infty} u^{-\bar{\beta}} du.$$

To obtain (4.90), we use (4.91) to write

$$E(Z^{-\bar{\beta}}) = P(Z=1) + E(Z^{-\bar{\beta}} 1_{Z>z}) \leqq \gamma + \gamma/[(\bar{\beta}-1)(z-1)^{\bar{\beta}-1}] \quad \text{for} \quad 1 < z < 2,$$

and then let z tend to 2 from below. This completes the proof of Lemma 4.4 and of Proposition 4.2.

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