Commun. Math. Phys. 104, 123–150 (1986)

Communications in
Mathematical
Physics

Integrable Non-Linear σ Models with Fermions

Elcio Abdalla[1,*] and Michael Forger[2,*,**]

1 Instituto de Física, Universidade de São Paulo, Cx. Postal 20516, BR-05508 São Paulo, Brazil
2 CERN, Geneva 23, Switzerland

Abstract. The two-dimensional non-linear σ model on a Riemannian symmetric space $M = G/H$ is coupled to fermions with quartic self-interactions. The resulting hybrid model is presented in a gauge-dependent formulation, with a bosonic field taking values in G and a fermionic field transforming under a given representation of the gauge group H. General criteria for classical integrability are presented: they essentially fix the Lagrangian of the model but leave the fermion representation completely arbitrary. It is shown that by a special choice for the fermion representation (derived from the adjoint representation of G by an appropriate reduction) one arrives naturally at the supersymmetric non-linear σ model on $M = G/H$. The issue of quantum integrability is also discussed, though with less stringent results.

1. Introduction and Summary

Generalized non-linear σ models, also called chiral models, are prime examples of field theories with non-trivial dynamical content which have a geometric origin, and it is well known that they are in many respects closely related to non-abelian gauge theories. (For some of the many aspects of this relation, see for example [1–4].) One of the most attractive features of these models is that classically they provide examples of integrable systems in two-dimensional space-time. Namely, it is known that the non-linear equations of motion are precisely the compatibility conditions for a certain linear system of first-order partial differential equations (Lax pair) containing a spectral parameter, and that this hidden symmetry gives rise to infinite series of local as well as non-local conservation laws [5–7], whenever the field takes values in a Riemannian symmetric space $M = G/H$ [8–10]. (For reviews, see for example [11, 12].) In the quantum theory, these integrability

* Work partially supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil, and KFA Jülich, Federal Republic of Germany
** On leave of absence from Fakultät für Physik der Universität Freiburg, Federal Republic of Germany

properties persist in some models but are spoiled by anomalies in others [13–15, 17–20], the general criterion for quantum integrability being that the stability group H is simple. More precisely, this condition guarantees conservation of the first quantum non-local charge [20], which is sufficient to prove factorization of the S-matrix into two-body amplitudes [14] that can be calculated exactly [21]; see also [22].

The analogies between non-linear σ models and non-abelian gauge theories suggest that they can and should be extended to incorporate fermionic matter fields. One natural way of doing that is to couple the fermions and the bosons supersymmetrically [23–25], and it has been proved that (classical) integrability of the model (more precisely, the existence of a hidden symmetry and of non-local conservation laws [1]) continues to hold in that case [26, 27]. On the other hand, it has been observed in the $\mathbb{C}P^{n-1}$ model [16, 17], and later in the Grassmannian model [18], that this remains true when fermions and bosons are coupled minimally rather than supersymmetrically. General criteria for integrability of non-linear σ models with fermions, however, have so far not been given.

The purpose of the present paper is to fill this gap, i.e., to supply such criteria. It is organized as follows.

In Sect. 2, which can be skipped at a first reading, we recapitulate the general method of coupling matter fields to non-linear σ models. Briefly, σ model fields are maps q from space-time to a given Riemannian manifold M, called the target space, and matter fields are sections Φ of a certain Hermitian complex (or Riemannian real) vector bundle $S \otimes q^* V$ over space-time: this bundle arises by taking the tensor product of an appropriate spinor or tensor bundle S over space-time with the pull-back $q^* V$ to space-time, via the σ model field q, of a given Hermitian complex (or Riemannian real) vector bundle V over M, called the target bundle. (Yes, it is as simple as that!) We shall concentrate here on the special features that appear when the target space is a Riemannian homogeneous ($\equiv$ coset) space $M = G/H$, and the target bundle is an associated vector bundle $V = G x_H V_0$, derived from a given unitary (or orthogonal) representation of the stability group H on a given finite-dimensional complex (or real) vector space V_0. (That seems to complicate things further, but we shall see that, in fact, it does not.) One of these special features is that instead of σ model fields q taking values in M, which are often handled in terms of (arbitrarily chosen, local) co-ordinates for M, we can use σ model fields g taking values in G, defined modulo H. Similarly, instead of matter fields Φ that are sections of $S \otimes q^* V$, which are often handled in terms of (arbitrarily chosen, local) co-ordinates for M and trivializations for V, we can use matter fields ϕ that are sections of $S \otimes V_0$ (i.e., ordinary vector-valued functions if S is trivial, e.g., if space-time is flat), defined modulo H. This procedure is a direct generalization of that in the pure model [8–12, 28], and, of course, it also works only locally (since the principal H-bundle $G \to G/H$ is usually non-trivial). However, the field-theoretical investigations that we have in mind involve only local aspects of geometry, and our gauge dependent formulation of the model, with gauge group H, is perfectly adapted to that kind of problem. (In particular, it gets rid of the aforementioned differential geometric complications.)

1 Higher local conservation laws for non-linear σ models with fermions seem not to have been investigated so far, and we shall disregard them in this paper

In Sect. 3, which forms the core of the paper, we assume the target space to be a Riemannian symmetric space $M = G/H$ (which is the well-known criterion for the bosonic sector to be integrable [8, 12]), and we investigate under what additional conditions the corresponding non-linear σ models with Dirac (or Majorana) fermions are classically integrable[2]. As it turns out, we may allow the fermionic fields to transform under an arbitrary unitary (or orthogonal) representation of the gauge group H, carried by the space V_0, but once this representation has been chosen, the Lagrangian of the model is almost entirely fixed by the requirement of integrability. More specifically, in terms of an invariant scalar product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$ of G, and an orthonormal basis of vectors v_a in V_0, the total Lagrangian L is the sum of three terms: the pure non-linear σ model Lagrangian, $(g^{-1}D_\mu g, g^{-1}D^\mu g)$, the Dirac Lagrangian with minimal coupling, $i/2\bar{\chi}\not{D}\chi$, and a quartic self-interaction term, L_F, which in turn is the sum of a "gauge covariant type" Thirring term, $(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\chi}^b\gamma^\mu\chi^a)$, with a fixed coefficient, and a "gauge invariant type" Thirring term, $(\bar{\chi}^a\gamma_\mu\chi^a)(\bar{\chi}^b\gamma^\mu\chi^b)$, with an arbitrary coefficient. In particular, L is conformally invariant as well as chirally invariant (although these symmetries may, of course, be broken at the quantum level, e.g., by anomalies). The resulting model is therefore a hybrid: its bosonic sector (obtained in the limit where $\chi \equiv 0$) is an integrable non-linear σ model, and its fermionic sector (obtained in the limit where $g \equiv 1$) is an integrable fermionic theory of Thirring and/or chiral Gross-Neveu type (the latter due to Fierz identities)[3]. Moreover, the supersymmetric non-linear σ model on $M = G/H$ is a special case: it arises by discarding the "gauge invariant type" Thirring term in the Lagrangian, and choosing $V_0 = \mathfrak{m}$, which leads to $V = TM$. (Here, $\mathfrak{m} = \mathfrak{g} \ominus \mathfrak{h}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, corresponding to the coset space structure $M = G/H$.) Other choices for the fermion representation, however, will in general lead to integrable non-linear σ models that cannot possibly be supersymmetric, as can be seen by simply counting bosonic versus fermionic degrees of freedom.

In Sect. 4, we analyze the issue of integrability at the quantum level. In contrast with the situation in the pure model [20], the presence of fermion fields leads to a plethora of possible composite operators appearing in the short-distance expansion for the (matrix) commutator of two Noether currents. For the general case, we have not been able to determine these composite operators, let alone the coefficients that multiply them, to a degree sufficiently explicit to allow for defining a quantum version of the first non-local charge and for deciding whether it is conserved or not. It is known, however, that this can be done in certain models, namely in the $\mathbb{C}P^{n-1}$ models and, more generally, in the Grassmannian models. In fact, it has been shown [15, 18] that to all orders in the $1/n$ expansion, the first quantum non-local charge can be defined, that in the pure model it has an explicitly calculable anomaly, and that in the fermionic models this anomaly is cancelled by the Adler-Bardeen anomaly from the fermionic sector. The deeper reasons for this miraculous anomaly cancellation, however, remain obscure, and the question certainly deserves further investigation.

---

2 We do not consider non-linear σ models with chiral fermions, thus avoiding all problems with anomalies that might otherwise render the quantum theory ill-defined [29]. Moreover, our results seem to indicate that integrability forces the model to be chirally invariant; cf. the discussion in the penultimate paragraph of Sect. 3

3 The ordinary (non-chiral) Gross-Neveu model can be made to fit into the picture as well

2. Differential Geometric Setting of Non-Linear σ Models with Matter Fields

To begin with, we recall that a pure non-linear σ model is specified by its target space, which is just a given Riemannian manifold M[4]. A (classical) field configuration is then simply a map q from space-time to M. Similarly, a non-linear σ model with matter fields is specified by its target space and its target bundle: the former is again a given Riemannian manifold M, and the latter is a Hermitian complex (or Riemannian real) vector bundle V over M, together with a fixed compatible linear connection D^V on it[5]. A (classical) field configuration is then a pair (q, Φ), where q is again a map from space-time to M and Φ, or rather each of its spinor or tensor components, is a section of q^*V, the pull-back of V to space-time via q. (We note here that the operation of pulling back may be applied to vector bundles [30, Vol. 1, p. 48f.] as well as to linear connections [30, Vol. 2, p. 324f.], thus producing a Hermitian complex (or Riemannian real) vector bundle q^*V over space-time, together with a compatible linear connection D_q^V on it. The intuitive content of this operation is simple. For bundles, the procedure amounts to a simple relabelling of base points, i.e., the fibre of q^*V at any point in space-time is identical with the fibre of V at the image point in M under q. For connections, the relation is similar: namely, parallel transport in q^*V with respect to D_q^V, along a given curve in space-time, is identical with parallel transport in V with respect to D^V, along the image curve in M under q.)

The most important case, and the only one that we shall be dealing with in this paper, arises when M is homogeneous and V is an associated bundle. More specifically, let M be the quotient space $M=G/H$ of some connected Lie group G, with Lie algebra $\mathfrak{g}$, modulo some compact subgroup $H\subset G$, with Lie algebra $\mathfrak{h}\subset\mathfrak{g}$. Then M is reductive [31, Vol. 2, pp. 190 and 199], which means that there exists an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is the (vector space) direct sum of $\mathfrak{h}$ and $\mathfrak{m}$:

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}. \tag{2.1}$$

Then we have the commutation relations

$$[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}, \qquad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}. \tag{2.2}$$

The decomposition of elements $X\in\mathfrak{g}$ corresponding to (2.1) will be written as follows:

$$X=X_{\mathfrak{h}}+X_{\mathfrak{m}}. \tag{2.3}$$

We also assume that the given left-invariant Riemannian metric $(\cdot,\cdot)$ on M can be obtained, by restriction, from some bi-invariant pseudo-Riemannian metric $(\cdot,\cdot)$ on G for which the direct decomposition (2.1) is orthogonal. (This amounts essentially to requiring that M be naturally reductive; we refer to [2, 12] for a detailed discussion.) Next, we have the natural projection ϱ from G to M

---

4 We exclude indefinite metrics because they would lead to theories that violate the positive energy condition

5 If the theory is to contain matter fields of different spins with different internal symmetry properties, one is forced into using more than one target bundle. Such generalizations will not be considered here

($\varrho(g)=gH$), which defines a principal H-bundle $\varrho: G \to M$ over M, where H acts on the total space G simply by right multiplication [31, Vol. 1, p. 55], [30, Vol. 2, pp. 83 and 194]. This bundle carries additional structures, namely a natural left action of G [given by $g_0 \cdot g = g_0 g$, $g_0 \cdot gH = (g_0 g)H$ for $g_0, g \in G$], as well as a canonical G-invariant connection, given by any of the following:

(a) for any $g \in G$, the horizontal space $\mathrm{Hor}_g G \subset T_g G$ at g is the left translate of $\mathfrak{m} \subset \mathfrak{g}$ by g, just as the vertical space $\mathrm{Ver}_g G \subset T_g G$ at g is the left translate of $\mathfrak{h} \subset \mathfrak{g}$ by g;

(b) the connection form A is the vertical part of the left G-invariant Maurer-Cartan form on G, i.e.

$$A = (g^{-1} dg)_{\mathfrak{h}} . \tag{2.4}$$

Therefore, starting from a unitary (or orthogonal) representation of H on some finite-dimensional complex (or real) vector space V_0, we can construct the associated Hermitian complex (or Riemannian real) vector bundle $\pi: G x_H V_0 \to M$ over M [30, Vol. 2, pp. 198ff.], whose total space $G x_H V_0$ consists of equivalence classes $[g, v]$ of pairs $(g, v) \in G x V_0$ with[6]

$$\begin{aligned}[g_1, v_1] = [g_2, v_2] \ &\Leftrightarrow\ (g_1, v_1) \sim (g_2, v_2) \\ &\Leftrightarrow\ \begin{array}{l}\text{there exists } \ h \in H \ \text{ such that} \\ g_2 = g_1 h \quad \text{and} \quad v_2 = h^{-1} \cdot v_1 .\end{array}\end{aligned} \tag{2.5}$$

This bundle carries additional structures, inherited from the corresponding structures on the principal bundle from which it originates, namely an associated left action of G (given by $g_0 \cdot [g, v] = [g_0 g, v]$, $g_0 \cdot (gH) = (g_0 g)H$ for $g_0, g \in G$, $v \in V_0$), as well as an associated G-invariant linear connection D^{V_0} [30, Vol. 2, p. 406f.]. We may therefore choose $V = G x_H V_0$, $D^V = D^{V_0}$.

A particular, and important, example arises from letting

$$V_0 = \mathfrak{m} , \tag{2.6}$$

which carries an orthogonal representation of H obtained from the adjoint representation of G on $\mathfrak{g}$ by an appropriate restriction. It can be shown without much effort that (up to natural identifications), V is the tangent bundle of M and D^V is the Levi-Cività connection:

$$V = TM . \tag{2.7}$$

As will become clear at the end of Sect. 3, this choice leads to the supersymmetric non-linear σ model on M; see also [24, 25].

Returning to the general case, we can now extend the gauge dependent formulation of pure non-linear σ models [8–12, 28] to include matter fields. Indeed, this approach is based on (locally) lifting the M-valued field q to a gauge dependent G-valued field g by setting

$$q(x) = g(x)H . \tag{2.8}$$

6 The dot symbolizes the action of elements of H on vectors in V_0 under the given representation of H, as well as the action of elements of $\mathfrak{h}$ on vectors in V_0 under the corresponding (derived) representation of $\mathfrak{h}$

Similarly, we shall (locally) write the matter field Φ, whose spinor or tensor components are sections of q^*V, in terms of a gauge dependent matter field ϕ, whose spinor or tensor components are functions with values in the vector space V_0, by setting

$$\Phi(x)=[g(x),\phi(x)]\,. \tag{2.9}$$

By construction, the fields q and Φ remain invariant when the fields g and ϕ are subjected to gauge transformations

$$g\to gh \quad \text{and} \quad \phi\to h^{-1}\cdot\phi \tag{2.10}$$

with H-valued fields h. On the other hand, the fields q and Φ transform naturally under global symmetry transformations, as expressed through the given left action of G on M and on V, and in terms of the fields g and ϕ, this transformation law takes the form

$$g\to g_0 g \quad \text{and} \quad \phi\to\phi \tag{2.11}$$

with $g_0 \in G$ space-time-independent.

As we are ultimately interested in the question of integrability, we shall assume throughout the rest of this paper that M is not only Riemannian homogeneous but in fact Riemannian symmetric; otherwise, the model would not be integrable even in the pure model limit (where $\Phi\equiv 0$, $\phi\equiv 0$) [8–12]. Essentially, this means that in addition to (2.2), we also have the commutation relation

$$[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}\,. \tag{2.12}$$

Without much loss of generality [9, 12], we may also assume – whenever convenient – that M is either of the compact type or of the non-compact type, and that G is simply connected or that G has finite centre, respectively, so that H will be compact and connected [32, pp. 320f. and 252f.]. For a survey of the properties of these spaces and their classification, see [10, 12], [31, Vol. 2, Chap. 10], and, of course, [32]; in particular, we refer the reader to the tables in [32, pp. 516 and 518].

3. Non-Linear σ Models with Fermions

We now come to the central subject of this paper, namely the construction of non-linear σ models with fermionic matter fields, and the question of their (classical) integrability in two space-time dimensions. The dynamical variables of such a model are a scalar field taking values in a given Riemannian manifold (bosonic sector), plus a Dirac spinor field with spinor components taking values in a given Hermitian complex vector bundle V over M (fermionic sector). As explained in Sect. 2, we shall assume M to be a Riemannian symmetric space $M=G/H$ and V to be the associated bundle $V=Gx_H V_0$ derived from a given unitary representation of the stability group H on a given complex vector space V_0. The dynamical variables of the model will therefore (locally) be represented by a scalar field $g=g(x)$ taking values in G and a Dirac spinor field $\chi=\chi(x)$ with spinor components taking values in V_0; these transform according to

$$g\to gh\,, \qquad \chi\to h^{-1}\cdot\chi \tag{3.1}$$

under gauge transformations (the gauge group being H) and according to

$$g \to g_0 g\,, \quad \chi \to \chi \tag{3.2}$$

under global symmetry transformations (the global symmetry group being G). [For the time being, we shall not need to specify whether our spinors are built on commuting or on anticommuting c-numbers, essentially because the relation between complex conjugation and multiplication, when written in the form

$$(\lambda\mu)^* = \mu^*\lambda^*\,, \tag{3.3}$$

is valid both for commuting and anticommuting c-numbers λ, μ. The distinction will, however, become important when Dirac spinors are replaced by Majorana spinors; we shall have more to say on this later on.]

Continuing our assembly of conventions, we shall suppose all fields to be defined over two-dimensional flat Minkowski space, with metric tensor $g_{\mu\nu}$, determinant tensor $\varepsilon_{\mu\nu}$ and light-cone co-ordinates ξ, η given by $g_{00} = +1$, $g_{11} = -1$, $\varepsilon_{01} = -1$, $\varepsilon_{10} = +1$, $\xi = (x^0 + x^1)/2$, $\eta = (x^0 - x^1)/2$, but the whole analysis that follows can certainly be extended to general two-dimensional space-times (space- and time-oriented Lorentz manifolds). As far as Dirac's γ-matrices are concerned, we shall work in a chiral Majorana representation, i.e., a representation of the anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \tag{3.4}$$

by complex (2×2)-matrices which are unitary,

$$\gamma_\mu^+ = \gamma_\mu^{-1} \quad \text{or} \quad \gamma_0^+ = \gamma_0\,, \quad \gamma_1^+ = -\gamma_1\,, \tag{3.5}$$

and satisfy two additional conditions: first,

$$\gamma_5 = \gamma_0 \gamma_1 \tag{3.6}$$

is diagonal, and second, charge conjugation $.^c$ is simply complex conjugation $.^*$, defined componentwise; this means that the γ-matrices must be purely imaginary. A possible realization is

$$\gamma_0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.7}$$

The invariant scalar product of ordinary spinors ψ, ϕ is as usual:

$$\bar{\psi}\phi = \psi^+ \gamma_0 \phi\,. \tag{3.8}$$

This implies that

$$(\bar{\psi}\gamma_\mu\phi)^* = \bar{\phi}\gamma_\mu\psi\,, \quad (\bar{\psi}\phi)^* = \bar{\phi}\psi\,, \quad (\bar{\psi}\gamma_5\phi)^* = -\bar{\phi}\gamma_5\psi\,, \tag{3.9}$$

where (3.3) has been used, and that

$$\overline{\psi^*}\gamma_\mu\phi^* = \varepsilon\bar{\phi}\gamma_\mu\psi\,, \quad \overline{\psi^*}\phi^* = -\varepsilon\bar{\phi}\psi\,, \quad \overline{\psi^*}\gamma_5\phi^* = \varepsilon\bar{\phi}\gamma_5\psi\,, \tag{3.10}$$

where $\varepsilon = +1$ or $\varepsilon = -1$, according to whether the spinor components are commuting or anticommuting c-numbers. Note also that (3.6) implies the following important identity, which is at the heart of integrability in the fermionic sector:

$$-\gamma_5\gamma_\mu=\varepsilon_{\mu\nu}g^{\nu\kappa}\gamma_\kappa=\gamma_\mu\gamma_5\,,\qquad \gamma_\mu\gamma_\nu=g_{\mu\nu}-\varepsilon_{\mu\nu}\gamma_5\,. \tag{3.11}$$

Next, we proceed to define various composite fields made up from the basic fields g and χ. For example, the bosonic sector provides vector fields A_μ and k_μ, taking values in $\mathfrak{h}$ and in $\mathfrak{m}$, respectively, defined as in the pure model [8–12],

$$A_\mu=(g^{-1}\partial_\mu g)_{\mathfrak{h}}\,,\qquad k_\mu=(g^{-1}\partial_\mu g)_{\mathfrak{m}} \tag{3.12}$$

[cf. (2.1) and (2.3)]. On the other hand, the fermionic sector gives rise to various fields which are bilinear in the spinors and are built by inserting either generators of the representation or operators which commute with all such generators. More specifically, we define a vector field B_μ, a scalar field B and a pseudoscalar field B_5, all of which take values in $\mathfrak{h}$, by requiring that for all $T\in\mathfrak{h}$,

$$(B_\mu,T)=-\frac{i}{2}\bar{\chi}\gamma_\mu T\cdot\chi\,,\quad (B,T)=-\frac{i}{2}\bar{\chi}T\cdot\chi\,,\quad (B_5,T)=\frac{1}{2}\bar{\chi}\gamma_5 T\cdot\chi\,, \tag{3.13}$$

where $(\cdot,\cdot)$ denotes the non-degenerate $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{h}$ obtained from the given non-degenerate $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ by restriction. More explicitly, in terms of an arbitrary basis of generators $T_j\in\mathfrak{h}$, with $g_{jk}=(T_j,T_k)$ and $(g^{jk})=(g_{jk})^{-1}$, this means that

$$\begin{aligned} B_\mu&=B_\mu^jT_j\,, & B_\mu^j&=-\frac{i}{2}g^{jk}\bar{\chi}\gamma_\mu T_k\cdot\chi\,,\\ B&=B^jT_j\,, & B^j&=-\frac{i}{2}g^{jk}\bar{\chi}T_k\cdot\chi\,,\\ B_5&=B_5^jT_j\,, & B_5^j&=\frac{1}{2}g^{jk}\bar{\chi}\gamma_5T_k\cdot\chi\,. \end{aligned} \tag{3.14}$$

Similarly, we define a vector field C_μ, a scalar field C and a pseudoscalar field C_5, all of which are isoscalars, as follows:

$$C_\mu=\frac{1}{2}\bar{\chi}\gamma_\mu\chi\,,\qquad C=\frac{1}{2}\bar{\chi}\chi\,,\qquad C_5=\frac{i}{2}\bar{\chi}\gamma_5\chi\,. \tag{3.15}$$

More generally, if the given unitary representation of H on V_0 is reducible and

$$V_0=\bigoplus_r V_0^{(r)} \tag{3.16}$$

is the orthogonal direct decomposition of V_0 into irreducible subspaces $V_0^{(r)}$ under H, then writing $\pi_0^{(r)}$ for the orthogonal projection of V_0 onto $V_0^{(r)}$, we define vector fields $C_\mu^{(r)}$, scalar fields $C^{(r)}$, and pseudoscalar fields $C_5^{(r)}$, all of which are isoscalars, as follows:

$$C_\mu^{(r)}=\frac{1}{2}\bar{\chi}\gamma_\mu\pi_0^{(r)}\chi\,,\qquad C^{(r)}=\frac{1}{2}\bar{\chi}\pi_0^{(r)}\chi\,,\qquad C_5^{(r)}=\frac{i}{2}\bar{\chi}\gamma_5\pi_0^{(r)}\chi\,. \tag{3.17}$$

[Note that the generators $T\in\mathfrak{h}$ being represented by anti-Hermitian linear transformations on V_0 and the projection operators $\pi_0^{(r)}$ being Hermitian linear transformations on V_0, (3.9) shows that the expressions in (3.13), (3.15), and (3.17) are real, so that B_μ, B, and B_5 all take values in the stability algebra $\mathfrak{h}$, rather than just its complexification, while C_μ, C, and C_5 and the $C_\mu^{(r)}$, $C^{(r)}$, $C_5^{(r)}$ all are real isoscalars, rather than just complex ones.] Now as in the pure model, A_μ is a gauge potential [$A_\mu \to h^{-1}A_\mu h + h^{-1}\partial_\mu h$ under gauge transformations (3.1)], while the other $\mathfrak{m}$-valued and $\mathfrak{h}$-valued fields are gauge covariant [$k_\mu \to h^{-1}k_\mu h$, $B_\mu \to h^{-1}B_\mu h$, $B \to h^{-1}Bh$, $B_5 \to h^{-1}B_5 h$ under gauge transformations (3.1)], and the isoscalar fields are gauge invariant; moreover, all these composite fields remain invariant under global symmetry transformations (3.2). We therefore introduce the gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3.18}$$

and the covariant derivatives

$$D_\mu g = \partial_\mu g - gA_\mu, \qquad D_\mu D_\nu g = \partial_\mu D_\nu g - D_\nu g A_\mu, \tag{3.19}$$

$$D_\mu \chi = \partial_\mu \chi + A_\mu \cdot \chi, \tag{3.20}$$

$$D_\mu k_\nu = \partial_\mu k_\nu + [A_\mu, k_\nu], \tag{3.21}$$

$$D_\mu B_\nu = \partial_\mu B_\nu + [A_\mu, B_\nu], \tag{3.22}$$

$$D_\mu B = \partial_\mu B + [A_\mu, B], \qquad D_\mu B_5 = \partial_\mu B_5 + [A_\mu, B_5]. \tag{3.23}$$

Conjugating the gauge covariant and globally invariant fields k_μ, B_μ, B, $B_{5\mu}$, and $F_{\mu\nu}$ by means of the bosonic field g, we obtain the following gauge invariant and globally covariant composite fields:

$$j_\mu = -gk_\mu g^{-1}, \tag{3.24}$$

$$j_\mu^M = gB_\mu g^{-1}, \tag{3.25}$$

$$j^M = gBg^{-1}, \qquad j_5^M = gB_5 g^{-1}, \tag{3.26}$$

and

$$G_{\mu\nu} = gF_{\mu\nu}g^{-1}. \tag{3.27}$$

Note that

$$k_\mu = g^{-1}D_\mu g, \qquad j_\mu = -D_\mu g g^{-1}. \tag{3.28}$$

Moreover, as a consequence of the symmetric space structure of M, we have the identities [8–12]

$$[k_\mu, k_\nu] = -F_{\mu\nu}, \tag{3.29}$$

and

$$D_\mu k_\nu - D_\nu k_\mu = 0, \tag{3.30}$$

which, after conjugation by g, take the form

$$[j_\mu, j_\nu] = -G_{\mu\nu}, \tag{3.31}$$

and

$$\partial_\mu j_\nu - \partial_\nu j_\mu + 2[j_\mu, j_\nu] = 0, \tag{3.32}$$

respectively.

With all these preliminaries out of the way, we can write down the general conformally invariant Lagrangian for this type of model: it reads

$$L=\frac{1}{2}g^{\mu\nu}(D_\mu g, D_\nu g)+\frac{i}{4}\bar{\chi}\overleftrightarrow{\not{D}}\chi+\frac{1}{2}L_F, \tag{3.33}$$

where the fermionic self-interaction term takes the form

$$L_F=ag^{\mu\nu}(B_\mu, B_\nu)+b(B,B)+b_5(B_5,B_5)+cg^{\mu\nu}C_\mu C_\nu+dC^2+d_5C_5^2 \tag{3.34}$$

with coupling constants a, b, b_5, c, d, d_5 [cf. (3.15)], or more generally,

$$\begin{aligned}L_F=&ag^{\mu\nu}(B_\mu, B_\nu)+b(B,B)+b_5(B_5,B_5)\\&+\sum_{r,s}\{c_{rs}g^{\mu\nu}C_\mu^{(r)}C_\nu^{(s)}+d_{rs}C^{(r)}C^{(s)}+(d_5)_{rs}C_5^{(r)}C_5^{(s)}\}\end{aligned} \tag{3.35}$$

with coupling constants a, b, b_5, c_{rs}, d_{rs}, $(d_5)_{rs}$, symmetric in r and s [cf. (3.17)]. [It should be noted that representing L_F in this form contains a certain amount of redundancy since one may use Fierz identities to relate, e.g., $C^2+C_5^2$ to $g^{\mu\nu}(B_\mu, B_\nu)$ and $(B,B)+(B_5,B_5)$ to $g^{\mu\nu}C_\mu C_\nu$. The specific form of any such relation, however, depends on the group H and on the representation of H on V_0.] The resulting equations of motion split into a bosonic field equation,

$$g^{\mu\nu}(D_\mu k_\nu-D_\mu B_\nu+[B_\mu, k_\nu])=0, \tag{3.36}$$

and a fermionic field equation,

$$\not{D}\chi=ag^{\mu\nu}\gamma_\mu B_\nu\cdot\chi+bB\cdot\chi+ib_5\gamma_5B_5\cdot\chi+icg^{\mu\nu}\gamma_\mu\chi C_\nu+id\chi C-d_5\gamma_5\chi C_5 \tag{3.37}$$

[from (3.34)], or more generally,

$$\begin{aligned}\not{D}\chi=&ag^{\mu\nu}\gamma_\mu B_\nu\cdot\chi+bB\cdot\chi+ib_5\gamma_5B_5\cdot\chi\\&+\sum_{r,s}\{ic_{rs}g^{\mu\nu}\gamma_\mu\pi_0^{(r)}\chi C_\nu^{(s)}+id_{rs}\pi_0^{(r)}\chi C^{(s)}-(d_5)_{rs}\gamma_5\pi_0^{(r)}\chi C_5^{(s)}\}\end{aligned} \tag{3.38}$$

[from (3.35)]. Moreover, we note that as a consequence of the fermionic field equation, the composite field B_μ has vanishing covariant divergence,

$$g^{\mu\nu}D_\mu B_\nu=0, \tag{3.39}$$

while its covariant curl is, in general, considerably more complicated: namely, in terms of an arbitrary basis of generators $T_j\in\mathfrak{h}$, with $g_{jk}=(T_j, T_k)$ and $(g^{jk})=(g_{jk})^{-1}$,

$$\begin{aligned}4\varepsilon^{\mu\nu}(D_\mu B_\nu-a[B_\mu, B_\nu], T_j)=&-bg^{kl}(\bar{\chi}\gamma_5[T_j, T_k]_+\cdot\chi)(\bar{\chi}T_l\cdot\chi)\\&+b_5g^{kl}(\bar{\chi}[T_j, T_k]_+\cdot\chi)(\bar{\chi}\gamma_5T_l\cdot\chi)\\&-2d(\bar{\chi}\gamma_5T_j\cdot\chi)(\bar{\chi}\chi)\\&+2d_5(\bar{\chi}T_j\cdot\chi)(\bar{\chi}\gamma_5\chi)\end{aligned} \tag{3.40}$$

[from (3.37)], or more generally,

$$\begin{aligned}4\varepsilon^{\mu\nu}(D_\mu B_\nu-a[B_\mu, B_\nu], T_j)=&-bg^{kl}(\bar{\chi}\gamma_5[T_j, T_k]_+\cdot\chi)(\bar{\chi}T_l\cdot\chi)\\&+b_5g^{kl}(\bar{\chi}[T_j, T_k]_+\cdot\chi)(\bar{\chi}\gamma_5T_l\cdot\chi)\\&-\sum_{r,s}2d_{rs}(\bar{\chi}\gamma_5\pi_0^{(r)}T_j\cdot\chi)(\bar{\chi}\pi_0^{(s)}\chi)\\&+\sum_{r,s}2(d_5)_{rs}(\bar{\chi}\pi_0^{(r)}T_j\cdot\chi)(\bar{\chi}\gamma_5\pi_0^{(s)}\chi)\end{aligned} \tag{3.41}$$

[from (3.38)], where $[\cdot, \cdot]_+$ denotes the anticommutator with respect to the given representation of H on V_0. (The proof of (3.39)–(3.41) proceeds by straightforward calculation, making use of the total antisymmetry of the structure constants c_{jkl}, defined by $c_{jkl} = g_{jm} c^m_{kl}$ and $[T_k, T_l] = c^m_{kl} T_m$, and of (3.11); we leave it to the reader to fill in the details.) In particular, it follows that a sufficient condition for the equation

$$D_\mu B_\nu - D_\nu B_\mu - 2a[B_\mu, B_\nu] = 0 \tag{3.42}$$

to hold is that

$$b = 0 = b_5, \quad d = 0 = d_5 \tag{3.43}$$

in (3.34), or more generally,

$$b = 0 = b_5, \quad d_{rs} = 0 = (d_5)_{rs} \tag{3.44}$$

in (3.35). (More precisely, it suffices that L_F may be rewritten in this form by making use of Fierz identities.) Next, the Noether current of the theory corresponding to the global symmetry under G [cf. (3.2)] is precisely the gauge invariant and globally covariant vector field

$$J_\mu = j_\mu + j^M_\mu \tag{3.45}$$

(which explains the notation j^M_μ, standing for "matter field contribution to the Noether current"). Its conservation,

$$g^{\mu\nu} \partial_\mu J_\nu = 0, \tag{3.46}$$

is easily checked to be a consequence of (and in fact equivalent to) the bosonic field Eq. (3.36). More specifically, combining (3.36) and (3.39) gives two equations, of which (3.46) is the sum, namely

$$g^{\mu\nu}(\partial_\mu j_\nu - [j_\mu, j^M_\nu]) = 0, \tag{3.47}$$

$$g^{\mu\nu}(\partial_\mu j^M_\nu + [j_\mu, j^M_\nu]) = 0. \tag{3.48}$$

If in addition, (3.42) also holds, then

$$\partial_\mu j^M_\nu - \partial_\nu j^M_\mu + [j_\mu, j^M_\nu] - [j_\nu, j^M_\mu] - 2a[j^M_\mu, j^M_\nu] = 0. \tag{3.49}$$

Our goal is now to find conditions on the fermionic self-interaction term L_F that will guarantee the model to be classically integrable. By classical integrability we shall mean the existence of a non-trivial one-parameter family of $\mathfrak{g}$-valued "gauge potentials" $A_\mu(\lambda)$ ($\lambda \in \mathbb{R}$) which a) are linear combinations

$$A_\mu(\lambda) = R(\lambda)^\kappa_\mu j_\kappa + S(\lambda)^\kappa_\mu j^M_\kappa \tag{3.50}$$

of the two contributions j_μ and j^M_μ to the Noether current J_μ, with field-independent coefficients $R(\lambda)$ and $S(\lambda)$, and b) satisfy the requirement that by virtue of the equations of motion, the $\mathfrak{g}$-valued "gauge fields" $F_{\mu\nu}(\lambda)$ ($\lambda \in \mathbb{R}$) vanish identically, where, of course,

$$F_{\mu\nu}(\lambda) = \partial_\mu A_\nu(\lambda) - \partial_\nu A_\mu(\lambda) + [A_\mu(\lambda), A_\nu(\lambda)]. \tag{3.51}$$

As usual, this condition is equivalent to the integrability, for any value of the parameter λ, of the following linear system of first-order differential equations,

$$\partial_\mu U^{(\lambda)} = U^{(\lambda)} A_\mu(\lambda)\,, \tag{3.52}$$

where $U^{(\lambda)}$ is a G-valued field which serves as a generating functional for an infinite sequence of non-local charges.

To analyze this situation, we return to (3.50) and note first that Lorentz covariance forces the (2×2)-matrices $R(\lambda)$ and $S(\lambda)$ to take the form

$$R(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \quad \text{and} \quad S(\lambda) = \begin{pmatrix} \gamma(\lambda) & \delta(\lambda) \\ \delta(\lambda) & \gamma(\lambda) \end{pmatrix}. \tag{3.53}$$

Then, after some calculation, we obtain from (3.32), (3.47), and (3.48).

$$\begin{aligned} \varepsilon^{\mu\nu} F_{\mu\nu}(\lambda) = & (\beta - \delta + \alpha\delta - \beta\gamma)(\lambda)\, g^{\mu\nu} [j_\mu, j^M_\nu] \\ & + \tfrac{1}{2}(\alpha^2 - \beta^2 - 2\alpha)(\lambda)\, \varepsilon^{\mu\nu} [{}^j{}_\mu A\, \lambda_\nu] \\ & + \gamma(\lambda)\, \varepsilon^{\mu\nu} \partial_\mu j^M_\nu \\ & + (\alpha\gamma - \beta\delta)(\lambda)\, \varepsilon^{\mu\nu} [j_\mu, j^M_\nu] + \tfrac{1}{2}(\gamma^2 - \delta^2)(\lambda)\, \varepsilon^{\mu\nu} [j^M_\nu, j^M_\nu]\,. \end{aligned}$$

Thus we should have

$$\beta - \delta + \alpha\delta - \beta\gamma = 0\,, \qquad \alpha^2 - \beta^2 - 2\alpha = 0\,.$$

If in addition, (3.42) also holds, then (3.49) shows that the last two equations plus the two equations

$$\gamma = \alpha\gamma - \beta\delta = -\frac{1}{2a}(\gamma^2 - \delta^2)$$

will make $\varepsilon^{\mu\nu} F_{\mu\nu}(\lambda)$ vanish. We are therefore left with four equations for four unknowns which turn out to have a solution if and only if

$$-2a = 1\,. \tag{3.54}$$

In that case, the ansatz $\beta = -\sinh\lambda$ gives

$$\alpha = 1 \mp \cosh\lambda\,, \quad \beta = -\sinh\lambda\,, \quad \gamma = \tfrac{1}{2}(1 - \cosh 2\lambda)\,, \quad \delta = \mp\tfrac{1}{2}\sinh 2\lambda \tag{3.55}$$

(either sign is possible).

To summarize, we have shown that the model under consideration, with total Lagrangian L given by

$$L = \frac{1}{2} g^{\mu\nu}(D_\mu g, D_\nu g) + \frac{i}{4}\bar{\chi}\overleftrightarrow{\not{D}}\chi + \frac{1}{2} L_F \tag{3.33}$$

will be integrable if (by making use of Fierz identities if necessary) the fermionic self-interaction term L_F can be brought into the form

$$L_F = -\tfrac{1}{2} g^{\mu\nu}(B_\mu, B_\nu) + c\, g^{\mu\nu} C_\mu C_\nu \tag{3.56}$$

with an arbitrary coupling constant c [cf. (3.15), (3.34)], or more generally,

$$L_F = -\tfrac{1}{2} g^{\mu\nu}(B_\mu, B_\nu) + \sum_{r,s} c_{rs}\, g^{\mu\nu} C^{(r)}_\mu C^{(s)}_\nu \tag{3.57}$$

with arbitrary coupling constants c_{rs} [cf. (3.17), (3.35)]. In fact, we have for this case established the integrability, for any value of the parameter λ, of the following linear system of first-order differential equations [26, 27]

$$\begin{aligned}\partial_\mu U^{(\lambda)} = U^{(\lambda)}\{&(1 \mp \cosh(\lambda))j_\mu - \sinh(\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa \\ &+\tfrac{1}{2}(1-\cosh(2\lambda))j_\mu^M \mp \tfrac{1}{2}\sinh(2\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa^M\}\end{aligned} \tag{3.58}$$

(either sign is possible). As a consequence, there exists a whole one-parameter family of $\mathfrak{g}$-valued "Noether currents" $J_\mu^{(\lambda)}$ ($\lambda \in \mathbb{R}$), given by

$$\begin{aligned}J_\mu^{(\lambda)} = U^{(\lambda)}\{&\pm\cosh(\lambda)j_\mu + \sinh(\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa \\ &\pm\cosh(2\lambda)j_\mu^M + \sinh(2\lambda)\varepsilon_{\mu\nu}g^{\nu\kappa}j_\kappa^M\}\,U^{(\lambda)^{-1}}\end{aligned} \tag{3.59}$$

(either sign is possible). Their conservation,

$$g^{\mu\nu}\partial_\mu J_\nu^{(\lambda)} = 0\,, \tag{3.60}$$

follows from the identity (3.32), from (3.47)–(3.49), and from (3.58). In light-cone coordinates ξ, η, and with the convention $\gamma = \pm e^{\mp\lambda}$, the linear system (3.58) becomes

$$\begin{aligned}\partial_\xi U^{(\gamma)} &= U^{(\gamma)}\{(1-\gamma^{-1})j_\xi + \tfrac{1}{2}(1-\gamma^{-2})j_\xi^M\}\,, \\ \partial_\eta U^{(\gamma)} &= U^{(\gamma)}\{(1-\gamma)j_\eta + \tfrac{1}{2}(1-\gamma^2)j_\eta^M\}\,,\end{aligned} \tag{3.61}$$

while the definition (3.59) becomes

$$\begin{aligned}J_\xi^{(\gamma)} &= U^{(\gamma)}\{\gamma^{-1}j_\xi + \gamma^{-2}j_\xi^M\}U^{(\gamma)^{-1}}\,, \\ J_\eta^{(\gamma)} &= U^{(\gamma)}\{\gamma j_\eta + \gamma^2 j_\eta^M\}U^{(\gamma)^{-1}}\,.\end{aligned} \tag{3.62}$$

Now just as in the pure model [6, 8–12], expanding (3.60) around $\lambda = 0$ gives an infinite series of $\mathfrak{g}$-valued conservation laws which (except for the very first) are non-local. In particular, the first non-local charge is given by

$$\begin{aligned}Q^{(1)}(t) = &\int dy_1 dy_2 \theta(y_1-y_2)[J_0(t,y_1), J_0(t,y_2)] \\ &-\int dy(J_1 + j_1^M)(t,y)\,.\end{aligned} \tag{3.63}$$

Its conservation (i.e., time-dependence) can also be checked directly from (3.46) and the equation

$$\partial_\mu(J_\nu + j_\nu^M) - \partial_\nu(J_\mu + j_\mu^M) + 2[J_\mu, J_\nu] = 0\,, \tag{3.64}$$

which follows from combining (3.32) with (3.49) and (3.54).

Throughout the preceding discussion, we have been working with Dirac spinors which transform according to a given unitary representation of H on a complex vector space V_0, but there are, of course, situations where one wants to replace these by Majorana spinors which transform according to a given orthogonal representation of H on a real vector space W_0. This can be achieved, e.g., by viewing Majorana spinors as Dirac spinors satisfying an additional reality constraint,

$$\chi^* = \chi\,; \tag{3.65}$$

then V_0 is the complexification of W_0, and the unitary representation of H on V_0 is the complex extension of the orthogonal representation of H on W_0. Therefore, the generators $T \in \mathfrak{h}$ are represented by real antisymmetric linear transformations on V_0, and the projection operators $\pi_0^{(r)}$ are real symmetric linear transformations on V_0, so that combining (3.10) with (3.65) and (3.14), (3.15), and (3.17), we get

$$2\bar{\chi}\overleftrightarrow{\not{D}}\chi = g^{\mu\nu}\partial_\mu(\bar{\chi}\gamma_\nu\chi), \quad B_\mu = 0, \quad B_5 = 0, \quad C = 0 \quad \text{or} \quad C^{(r)} = 0$$
$$\text{for commuting Majorana spinors.} \tag{3.66}$$

$$B = 0, \quad C_\mu = 0 \quad \text{or} \quad C_\mu^{(r)} = 0, \quad C_5 = 0 \quad \text{or} \quad C_5^{(r)} = 0$$
$$\text{for anticommuting Majorana spinors.} \tag{3.67}$$

But this shows that the case of commuting Majorana spinors is dynamically trivial (the fermions have trivial kinetic Lagrangian and moreover they decouple from the bosons), so only the case of anticommuting Majorana spinors is physically acceptable. This situation is, of course, reversed if the H-invariant symmetric scalar product on W_0 used to contract the internal symmetry indices of the Majorana spinors is replaced by an H-invariant antisymmetric scalar product, i.e., an H-invariant symplectic form, on W_0 – which can be done if, and only if, the real dimension of W_0 is even, say $\dim W_0 = 2N$. In other words, δ_{ab} is replaced by ε_{ab}, and the $SO(2N)$-symmetry of the fermionic sector is replaced by an $Sp(2N, \mathbb{R})$-symmetry; this is a well-known procedure in handling the ordinary Gross-Neveu model [33, 34]. However, H is compact, so the image of H under the given representation on W_0 must then be contained in a maximal compact subgroup of $Sp(2N, \mathbb{R})$, which is a $U(N)$. But this means that we are effectively dealing with Dirac spinors which transform according to a unitary representation of H on W_0, which is really a complex vector space of complex dimension N. [Of course, the fermionic model by itself could be required to have a full $Sp(2N, \mathbb{R})$-symmetry, but without reducing this symmetry to a $U(N)$-symmetry, it could not be coupled to a non-linear σ model, or to a gauge theory, without using a non-compact stability group H, i.e., without violating the positive energy condition.]

In view of these arguments, we shall from now on disregard commuting Majorana spinors and deal exclusively with anticommuting Majorana spinors. (There are, of course, several other good reasons to consider commuting spinors, even in the Dirac case, as being physically irrelevant, but we shall not dwell on them here.) Then it is clear that the fermionic self-interaction term L_F in the Lagrangian L [cf. (3.33)–(3.35)] can be rewritten in the form

$$L_F = a g^{\mu\nu}(B_\mu, B_\nu) + b_5(B_5, B_5) + dC^2 \tag{3.68}$$

[cf. (3.34)], or more generally,

$$L_F = a g^{\mu\nu}(B_\mu, B_\nu) + b_5(B_5, B_5) + \sum_{r,s} d_{rs} C^{(r)} C^{(s)} \tag{3.69}$$

[cf. (3.35)], and according to (3.67), this is manifestly chirally invariant.

An interesting aspect of the results obtained in this section is that integrability of the model under consideration apparently forces its Lagrangian to be chirally invariant even in the Dirac case. One may object that this conclusion seems to be contradict the well-established integrability of the ordinary Gross-Neveu model [33, 34]. A closer look, however, reveals that integrability of the ordinary Gross-Neveu model requires an extension from N-component Dirac spinors, with $U(N)$-

symmetry, to $2N$-component Majorana spinors, with $\mathrm{Sp}(2N,\mathbb{R})$-symmetry (for commuting spinors) respectively $\mathrm{SO}(2N)$-symmetry (for anticommuting spinors): this happens because the fermionic current j_μ^M takes values in the Lie algebra $\mathrm{sp}(2N,\mathbb{R})$ respectively $\mathrm{so}(2N)$, and not in the Lie subalgebra $\mathrm{u}(N)$. But on this level, chiral invariance is obvious. In fact, the ordinary Gross-Neveu model has a hidden chiral invariance, hidden by the phenomenon that the corresponding γ_5 does not commute with the complex structure needed to reduce the $2N$ real components to N complex components, but apparent when one doubles the number of complex components.

To conclude this section, let us have a look at the integrable non-linear σ model on an irreducible Riemannian symmetric space $M=G/H$ with (anticommuting Majorana) fermions, which according to (3.33), (3.56), and (3.67) has Lagrangian

$$L=\frac{1}{2}g^{\mu\nu}(D_\mu g, D_\nu g)+\frac{i}{4}\bar{\chi}\overleftrightarrow{\not{D}}\chi-\frac{1}{4}g^{\mu\nu}(B_\mu, B_\nu), \tag{3.70}$$

in the orthogonal representation of H on the real vector space

$$W_0=\mathfrak{m}, \tag{3.71}$$

derived from the adjoint representation of G on $\mathfrak{g}$ by restriction [cf. (2.1)]. We claim that this is precisely the supersymmetric non-linear σ model on $M=G/H$. Namely, the variation of the fields g and χ under an infinitesimal supersymmetry transformation, parametrized by an anticommuting Majorana spinor ε, is

$$\delta_\varepsilon g = g\bar{\varepsilon}\chi\,. \tag{3.72}$$

$$\delta_\varepsilon \chi = -i\not{k}\varepsilon\,. \tag{3.73}$$

Moreover, if $M=G/H$ is a Hermitian symmetric space, rather than just a Riemannian one, then there exists a generator I in the centre of the stability algebra $\mathfrak{h}$ which, via ad, induces the invariant complex structure on $M=G/H$ [31, p. 261 f.]:

$$[I,X]=0 \quad \text{for} \quad X\in\mathfrak{h}, \qquad [I,[I,X]]=-X \quad \text{for} \quad X\in\mathfrak{m}\,. \tag{3.74}$$

This generator can be used to show that the model actually admits an $N=2$ extended supersymmetry (see also [25, 27]): namely, the variation of the fields g and χ under an infinitesimal supersymmetry of the second type, again parametrized by an anticommuting Majorana spinor ε, is

$$\delta'_\varepsilon g = g[I,\bar{\varepsilon}\chi]\,. \tag{3.75}$$

$$\delta'_\varepsilon \chi = +i[I,\not{k}]\varepsilon\,. \tag{3.76}$$

[The proof that the transformations (3.72) and (3.73) leave the Lagrangian (3.70) invariant, and that they satisfy the correct commutation relations, is delegated to an appendix; the corresponding proof for the transformations (3.75) and (3.76) is entirely analogous.] An alternative approach, systematically developed in [27], is to start with a superfield

$$g(x,\theta)=g(x)+\bar{\theta}\psi(x)+\tfrac{1}{2}\bar{\theta}\theta G(x) \tag{3.77}$$

and to eliminate the auxiliary field $G(x)$ by using its equation of motion, which follows from variation of the superspace Lagrangian density. Then setting

$$\psi = g\chi , \tag{3.78}$$

this superspace Lagrangian density, after integration over the Grassmann variables, reduces to (3.70).

4. On Quantization of the Non-Local Charge

In order to discuss the fate of the first non-local charge (3.63) in the quantum theory, we shall proceed along the same lines as in the pure model, which has been treated in [20].

We begin by collecting a couple of group-theoretical conventions. First of all, we assume that $M = G/H$ is an irreducible Riemannian symmetric space of the compact type, and we perform an orthogonal, Ad(H)-invariant direct decomposition

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r \oplus \mathfrak{m} \tag{4.1}$$

of $\mathfrak{g}$ into Ad(H)-invariant irreducible subspaces (some of which may be $\{0\}$), where $\mathfrak{h}_0$ is the (at most one-dimensional) centre of $\mathfrak{h}$, and $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ are the simple ideals in $\mathfrak{h}$ [20]; the corresponding representations of H or $\mathfrak{h}$ on $\mathfrak{h}_i$ respectively $\mathfrak{m}$ will be denoted by D_i respectively D. Moreover, all $\mathfrak{h}$-valued fields are further decomposed according to (4.1), i.e.,

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)} + \ldots + A_\mu^{(r)} , \tag{4.2}$$

$$F_{\mu\nu} = F_{\mu\nu}^{(0)} + F_{\mu\nu}^{(1)} + \ldots + F_{\mu\nu}^{(r)} , \tag{4.3}$$

$$B_\mu = B_\mu^{(0)} + B_\mu^{(1)} + \ldots + B_\mu^{(r)} , \tag{4.4}$$

etc., and we set

$$G_{\mu\nu}^{(i)} = g F_{\mu\nu}^{(i)} g^{-1} , \tag{4.5}$$

$$j_\mu^{(M)(i)} = g B_\mu^{(i)} g^{-1} , \tag{4.6}$$

etc. Next, M being of the compact type, we may assume that G is compact semi-simple[7], and we shall fix a representation of G by unitary $(N \times N)$-matrices, which induces a representation of generators in $\mathfrak{g}$ by anti-Hermitian $(N \times N)$-matrices: this representation is required to be faithful, and for simplicity, the Ad(G)-invariant scalar product $(\cdot , \cdot)$ on $\mathfrak{g}$ is supposed to be the corresponding trace form (up to normalization). We shall also find it convenient to fix an orthonormal basis of vectors v_a in V_0, so that the fermionic sector carries a representation of H by unitary $(N_0 \times N_0)$-matrices which induces a representation of generators T in $\mathfrak{h}$ by anti-Hermitian $(N_0 \times N_0)$-matrices T_{ab} $(N_0 = \dim V_0)$; this representation of H or $\mathfrak{h}$ will be denoted by D_0. In addition, we require that G and $\mathfrak{g}$ respectively H and $\mathfrak{h}$ (or rather their images under the respective representations) are stable under the operations $.^*$ of complex conjugation and $.^T$ of transposition of matrices: this will

7 Certain non-compact cases can be handled in a similar way; cf. the footnote on p. 185 of [20]

enable us to accommodate, in a natural way, the operation of charge conjugation. [Note that G respectively H is stable if and only if $\mathfrak{g}$ respectively $\mathfrak{h}$ is, and that stabilities under $.^*$ and under $.^T$ are mutually equivalent. In particular, on $\mathfrak{g}$, $.^*$ and $.^T$ preserve the $\mathrm{Ad}(G)$-invariant scalar product $(\cdot,\cdot)$.] Finally, we demand that on $\mathfrak{h}$, the two operations of complex conjugation and the two operations of transposition thus defined in terms of the two given matrix representations are in fact identical, which means that for generators T in $\mathfrak{h}$,

$$(T^*)_{ab}=(T_{ab})^*\,, \qquad (T^T)_{ab}=T_{ba}\,, \tag{4.7}$$

so we may simply write T^*_{ab}.

With these conventions, the basic fields of the quantized model are the $(N\times N)$-matrix fields g, g^+ (which have dimension 0)[8] and the N_0-vector fields $\chi, \bar{\chi}$ (which have dimension $\frac{1}{2}$), together with their covariant derivatives. In order to give a definite meaning to products of such operators at the same point (including the products involved in the definition of the covariant derivatives), we shall assume, as in the pure model, the existence of a normal product prescription $\mathcal{N}[\ldots]$, which is compatible with the constraints and also with the internal symmetries of the model [cf. (3.1) and (3.2)]; see [20] for more details. Then the existence of a quantum version of the first non-local charge (3.63), and the question of whether it is actually time-dependent or not, is governed by a Wilson expansion for the (matrix) commutator of two currents at nearby (spacelike separated) points

$$[J_\mu(x+\varepsilon), J_\nu(x-\varepsilon)] \sim \sum_k C^{(k)}_{\mu\nu}(\varepsilon)\mathcal{N}[\mathcal{O}_k(x)] \qquad (\varepsilon^2<0)\,, \tag{4.8}$$

where $\sim$ means equality up to terms that go to zero as $\varepsilon\to 0$, and k labels a complete set of independent composite local operators of (canonical) dimension $\leqq 2$. More specifically, these operators $\mathcal{N}[\mathcal{O}_k(x)]$, as well as their conjugates $\mathcal{N}[(g\mathcal{O}_k g^+)(x)]$ under the bosonic field g, must be products, of (canonical) dimension $\leqq 2$, of the fields g, g^+, χ, $\bar{\chi}$ and their covariant derivatives, and contraction of internal symmetry indices among the factors must be performed in such a way that the resulting product takes values in $\mathfrak{g}$ and is globally G-covariant and locally H-invariant (for $\mathcal{N}[\mathcal{O}_k(x)]$) respectively globally G-invariant and locally H-covariant (for $\mathcal{N}[(g\mathcal{O}_k g^+)(x)]$), as well as chirally invariant. Therefore, we arrive at the following list of candidates for admissible gauge covariant operators, whose gauge invariant counterparts are obtained by conjugating back with the bosonic field g^+ (for simplicity, we have omitted the normal product symbol):

dimension 0: –;

dimension 1:

no fermion fields $\to$ one derivative $\to$ k_μ [20],

two fermion fields $\to$ no derivative $\to$ $R_{ab}\bar{\chi}^a\gamma_\mu\chi^b$, $R^{(i)}_{ab}\bar{\chi}^a\gamma_\mu\chi^b$;

dimension 2:

no fermion fields $\to$ two derivatives $\to$ $D_\mu k_\nu$, $F^{(i)}_{\mu\nu}$ [20],

8 The symbol $.^+$ denotes Hermitian adjunction of matrices; in particular, $g^{-1}=g^+$

two fermion fields → one derivative →

a) derivative acts on fermion fields → $R_{ab}D_\mu(\bar{\chi}^a\gamma_\nu\chi^b)$, $R^{(i)}_{ab}D_\mu(\bar{\chi}^a\gamma_\nu\chi^b)$,
$iS_{ab}\bar{\chi}^a\gamma_\nu\overleftrightarrow{D}_\mu\chi^b$, $iS^{(i)}_{ab}\bar{\chi}^a\gamma_\nu\overleftrightarrow{D}_\mu\chi^b$,

b) derivative acts on boson fields → $[k_\mu, R_{ab}\bar{\chi}^a\gamma_\nu\chi^b]$, $[k_\mu, R^{(i)}_{ab}\bar{\chi}^a\gamma_\nu\chi^b]$,

four fermion fields → no derivative → $U_{abcd}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\chi}^c\gamma_\nu\chi^d)$,
$U^{(i)}_{abcd}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\chi}^c\gamma_\nu\chi^d)$.

[In compiling this list, we have used two facts. First, projecting the matrix product AB or BA of two anti-Hermitian matrices A and B to its anti-Hermitian part will automatically produce their commutator,

$$A^+=-A\,,\quad B^+=-B \;\Rightarrow\; +AB-(AB)^+=[A,B]=-BA+(BA)^+\,;$$

this has been applied to $A=k_\mu$ and $B=R_{ab}\bar{\chi}^a\gamma_\nu\chi^b$ or $B=R^{(i)}_{ab}\bar{\chi}^a\gamma_\nu\chi^b$. Second, due to a Fierz identity, the chirally invariant product

$$V_{abcd}((\bar{\chi}^a\chi^b)(\bar{\chi}^c\chi^d)-(\bar{\chi}^a\gamma_5\chi^b)(\bar{\chi}^c\gamma_5\chi^d))$$

can be rewritten as

$$V_{adcb}g^{\mu\nu}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\chi}^c\gamma_\nu\chi^d)\,.]$$

The tensors R and S respectively $R^{(i)}$ and $S^{(i)}$ appearing in the list above represent linear maps from $V_0^*\otimes V_0$ to $\mathfrak{m}^c$ respectively $\mathfrak{h}_i^c$ which, in order to achieve H-covariance, must intertwine the representations $D_0^*\otimes D_0$ and D^c respectively D_i^c, and take the Hermitian part of $V_0^*\otimes V_0$ to the real part $\mathfrak{m}$ respectively $\mathfrak{h}_i$[9]. Similarly, the tensors U respectively $U^{(i)}$ appearing in the list above represent linear maps from $V_0^*\otimes V_0\otimes V_0^*\otimes V_0$ to $\mathfrak{m}^c$ respectively $\mathfrak{h}_i^c$ which, in order to achieve H-covariance, must intertwine the representations $D_0^*\otimes D_0\otimes D_0^*\otimes D_0$ and D^c respectively D_i^c, and take the Hermitian part of $V_0^*\otimes V_0\otimes V_0^*\otimes V_0$ to the real part $\mathfrak{m}$ respectively $\mathfrak{h}_i$[9]. Explicitly, the reality condition means that

$$\begin{aligned} A^{ba*}=A^{ab} \quad&\Rightarrow\quad R_{ab}A^{ab}\in\mathfrak{m}\,, \qquad R^{(i)}_{ab}A^{ab}\in\mathfrak{h}_i\,,\\ A^{ba*}=A^{ab} \quad&\Rightarrow\quad S_{ab}A^{ab}\in\mathfrak{m}\,, \qquad S^{(i)}_{ab}A^{ab}\in\mathfrak{h}_i\,,\\ A^{badc*}=A^{abcd} \quad&\Rightarrow\quad U_{abcd}A^{abcd}\in\mathfrak{m}\,, \qquad U^{(i)}_{abcd}A^{abcd}\in\mathfrak{h}_i\,. \end{aligned} \tag{4.9}$$

A further constraint on these intertwining operators, derived from the condition that both sides of (4.8) must have the same transformation law under charge conjugation [which takes, e.g. J_μ to $J_\mu^*=-J_\mu^T$, $\bar{\chi}^a\gamma_\mu\chi^b$ to $-(\bar{\chi}^a\gamma_\mu\chi^b)^*=-\bar{\chi}^b\gamma_\mu\chi^a$ and $\bar{\chi}^a\gamma_\nu\overleftrightarrow{D}_\mu\chi^b$ to $-(\bar{\chi}^a\gamma_\nu\overleftrightarrow{D}_\mu\chi^b)^*=\bar{\chi}^b\gamma_\nu\overleftrightarrow{D}_\mu\chi^a$], is that they intertwine not only the representations of H or $\mathfrak{h}$ but also, up to certain well-defined signs, the operations $.^*$ of complex conjugation and (equivalently $.^T$ of transposition on the various space involved. More explicitly, this condition reads

$$\begin{aligned} R^*_{ab}&=-R_{ab}\,, \qquad & R^{(i)*}_{ab}&=-R^{(i)}_{ab}\,,\\ S^*_{ab}&=+S_{ab}\,, \qquad & S^{(i)*}_{ab}&=+S^{(i)}_{ab}\,,\\ U^*_{abcd}&=+U_{abcd}\,, \qquad & U^{(i)*}_{abcd}&=+U^{(i)}_{abcd}\,, \end{aligned} \tag{4.10}$$

9 The superscript $.^c$ denotes complexification

and (equivalently)

$$
\begin{aligned}
R^T_{ab} &= +R_{ba}\,, & R^{(i)T}_{ab} &= +R^{(i)}_{ba}\,, \\
S^T_{ab} &= -S_{ba}\,, & S^{(i)T}_{ab} &= -S^{(i)}_{ba}\,, \\
U^T_{abcd} &= -U_{badc}\,, & U^{(i)T}_{abcd} &= -U^{(i)}_{badc}\,.
\end{aligned} \tag{4.11}
$$

From these arguments, it is clear that the task of finding a complete set of independent composite local operators for the right-hand side of the Wilson expansion (4.8) is reduced to a purely group-theoretical problem: namely that of determining the complete set of intertwining operators R, S, U, $R^{(i)}$, $S^{(i)}$, $U^{(i)}$ which satisfy the additional conditions (4.9)–(4.11). In particular, one has to know the multiplicity with which the irreducible representations D and D_i occur in the tensor product representations $D_0^*\otimes D_0$ and $D_0^*\otimes D_0\otimes D_0^*\otimes D_0$. When considered in full generality, this group-theoretical problem is known to be a notoriously complicated one, and we shall not pursue this issue here. Note, however, that we can always set

$$
R^{(i)}_{ab} = -\frac{i}{2} g^{jk} (T^{(i)}_k)_{ab} T^{(i)}_j\,. \tag{4.12}
$$

$$
U^{(i)}_{abcd} = -\frac{1}{4} c^{jkl} (T^{(i)}_k)_{ab} (T^{(i)}_l)_{cd} T^{(i)}_j\,. \tag{4.13}
$$

[Here, we have used a basis of generators T_j in $\mathfrak{h}$, each of which belongs either to the centre $\mathfrak{h}_0$ of $\mathfrak{h}$ or to precisely one of the simple ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ of $\mathfrak{h}$, so that $T^{(i)}_j$ is either T_j or 0, and $g_{jk} = (T_j, T_k)$, $(g^{jk}) = (g_{jk})^{-1}$, $c^{jkl} = g^{km} g^{ln} c^j_{mn}$, $[T_m, T_n] = c^j_{mn} T_j$; of course, the expressions in (4.12) and (4.13) do not depend on the specific choice of basis made.] In fact, it is easy to check that this gives intertwining operators $R^{(i)}$ and $U^{(i)}$ which satisfy the additional conditions (4.9)–(4.11); they lead to the following admissible gauge covariant operators:

dimension 1: $\quad B^{(i)}_\mu = R^{(i)}_{ab} \bar{\chi}^a \gamma_\mu \chi^b\,;$

dimension 2:

$$
\begin{aligned}
D_\mu B^{(i)}_\nu &= R^{(i)}_{ab} D_\mu (\bar{\chi}^a \gamma_\nu \chi^b)\,, \\
[k_\mu, B^{(i)}_\nu] &= [k_\mu, R^{(i)}_{ab} \bar{\chi}^a \gamma_\nu \chi^b]\,, \\
[B^{(i)}_\mu, B^{(i)}_\nu] &= U^{(i)}_{abcd} (\bar{\chi}^a \gamma_\mu \chi^b)(\bar{\chi}^c \gamma_\nu \chi^d)\,.
\end{aligned}
$$

Inserting their gauge invariant counterparts into the Wilson expansion (4.8) and slightly rearranging terms, we get

$$
\begin{aligned}
[J_\mu(x+\varepsilon), J_\nu(x-\varepsilon)] \sim{}& C^\varrho_{\mu\nu}(\varepsilon) J_\varrho(x) + \sum_{i=0}^r \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon) j^{M(i)}_\varrho(x) \\
&+ D^{\sigma\varrho}_{\mu\nu}(\varepsilon)(\partial_\sigma J_\varrho)(x) + \sum_{i=0}^r \hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon)(\partial_\sigma j^{M(i)}_\varrho)(x) \\
&+ \sum_{i=0}^r E^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) G^{(i)}_{\sigma\varrho}(x) \\
&+ \sum_{i=0}^r F^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) \mathcal{N}[[J_\sigma, j^{M(i)}_\varrho](x)] \\
&+ \sum_{i=0}^r G^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) \mathcal{N}[[j^{M(i)}_\sigma, j^{M(i)}_\varrho](x)] \\
&+ \ldots\,,
\end{aligned} \tag{4.14}
$$

where the dots indicate additional terms that will be present if there are other intertwining operators which satisfy the additional conditions (4.9)–(4.11), besides the ones given by (4.12) and (4.13). (The specific form of such additional terms will depend on the choice of G, H and the fermion representation D_0.)

The expansion (4.14) is still somewhat redundant because the constraints and the field equations produce relations between the normal products appearing on the right-hand side of (4.14). For example, the analogues of the classical identities (3.31) and (3.32) in the quantum theory read

$$\begin{aligned} \mathcal{N}[[j_\sigma, j_\varrho](x)] &= -G_{\sigma\varrho}(x)\,, \\ \varepsilon^{\sigma\varrho}\mathcal{N}[[j_\sigma, j_\varrho](x)] &= c_0\varepsilon^{\sigma\varrho}(\partial_\sigma j_\varrho)(x)\,, \end{aligned} \tag{4.15}$$

where c_0 is a renormalization-scheme-dependent constant. Similarly, the analogue of the classical curl equation (3.49) [with (3.54)], split up into components along the $\mathfrak{h}_i$ [cf. (4.1)–(4.6)], in the quantum theory reads

$$\begin{aligned} &2\varepsilon^{\sigma\varrho}\mathcal{N}[[j_\sigma, j_\varrho^{M(i)}](x)] + c_1^{(i)}\varepsilon^{\sigma\varrho}\mathcal{N}[[j_\sigma^{M(i)}, j_\varrho^{M(i)}](x)] \\ &\quad = -c_2^{(i)}\varepsilon^{\sigma\varrho}(\partial_\sigma j_\varrho^{M(i)})(x) + c_3^{(i)}\varepsilon^{\sigma\varrho}G_{\sigma\varrho}^{(i)}(x)\,, \end{aligned} \tag{4.16}$$

where the $c_1^{(i)}$, $c_2^{(i)}$, $c_3^{(i)}$ are renormalization-scheme-dependent constants, and the last term in (4.16) is nothing but the Adler-Bardeen anomaly.

We have now reached the point where we can state a criterion for quantum integrability of our model. Namely, the coefficients in the Wilson expansion (4.14) and in the identities (4.15) and (4.16) must conspire in such a way that after insertion of the latter into the former, all unwanted, anomalous terms cancel, i.e., we are left with a simplified Wilson expansion

$$\begin{aligned} [J_\mu(x+\varepsilon), J_\nu(x-\varepsilon)] \sim{} & C_{\mu\nu}^{\varrho}(\varepsilon)J_\varrho(x) + \sum_{i=0}^{r} \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon)j_\varrho^{M(i)}(x) \\ & + D_{\mu\nu}^{\sigma\varrho}(\varepsilon)(\partial_\sigma J_\varrho)(x) + \sum_{i=0}^{r} \hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon)(\partial_\sigma j_\varrho^{M(i)})(x) \end{aligned} \tag{4.17}$$

[with possibly modified coefficients $C_{\mu\nu}^{\varrho}$, $\hat{C}^{(i)\varrho}_{\mu\nu}$, $D_{\mu\nu}^{\sigma\varrho}$, $\hat{D}^{(i)\sigma\varrho}_{\mu\nu}$, as compared to (4.14)]. Indeed, if this is the case, we can define the first quantum non-local charge Q as the limit

$$Q(t) = \lim_{\delta\to 0} Q_\delta(t) \tag{4.18}$$

of a cut-off charge Q_δ, which reads

$$\begin{aligned} Q_\delta(t) = & \int_{|y_1-y_2|\geqq\delta} dy_1 dy_2 \theta(y_1-y_2)[J_0(t,y_1), J_0(t,y_2)] \\ & - Z(\delta)\int dy J_1(t,y) - \sum_{i=0}^{r} \hat{Z}^{(i)}(\delta)\int dy\, j_1^{M(i)}(t,y) \end{aligned} \tag{4.19}$$

[cf. (3.63)], with coefficients $Z(\delta)$, $\hat{Z}^{(i)}(\delta)$ to be determined in such a way that the limit in (4.18) exists (at least in a weak sense); it can then be proved by combining (4.17) with general principles that Q is conserved.

We shall briefly review the basic arguments of how this is done, mainly because the original reference [14] contains an error[10] and also because it applies to the pure model only.

We begin by exploiting locality, which tells us that ε being spacelike, the matrix commutators $[J_\mu(x+\varepsilon), J_\nu(x-\varepsilon)]$ and $[J_\nu(x-\varepsilon), J_\mu(x+\varepsilon)]$ add up to zero (just as they would for commuting c-number fields). Therefore,

$$\begin{aligned} C^{\varrho}_{\mu\nu}(\varepsilon) &= -C^{\varrho}_{\nu\mu}(-\varepsilon), \qquad \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon) = -\hat{C}^{(i)\varrho}_{\nu\mu}(-\varepsilon), \\ D^{\sigma\varrho}_{\mu\nu}(\varepsilon) &= -D^{\sigma\varrho}_{\nu\mu}(-\varepsilon), \qquad \hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) = -\hat{D}^{(i)\sigma\varrho}_{\nu\mu}(-\varepsilon). \end{aligned} \tag{4.20}$$

On the other hand, covariance under PT [which is implemented by an antiunitary operator on Hilbert space and defined on the basic field operators by $g(x)\to g(-x)$, $\chi(x)\to\gamma_5\chi(-x)$] requires

$$\begin{aligned} C^{\varrho}_{\mu\nu}(\varepsilon) &= -C^{\varrho}_{\mu\nu}(-\varepsilon), \qquad \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon) = -\hat{C}^{(i)\varrho}_{\mu\nu}(-\varepsilon), \\ D^{\sigma\varrho}_{\mu\nu}(\varepsilon) &= +D^{\sigma\varrho}_{\mu\nu}(-\varepsilon), \qquad \hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) = +\hat{D}^{(i)\sigma\varrho}_{\mu\nu}(-\varepsilon). \end{aligned} \tag{4.21}$$

In particular, we infer that the C-coefficients are symmetric and the D-coefficients are antisymmetric in μ and ν.

The next step uses covariance under Lorentz transformations and under parity, which strongly restricts the tensorial nature of the coefficients, leading to

$$\begin{aligned} C^{\varrho}_{\mu\nu}(\varepsilon) &= C_1(-\varepsilon^2) g_{\mu\nu}\varepsilon^{\varrho} + C_2(-\varepsilon^2)(\delta^{\varrho}_{\mu}\varepsilon_\nu + \delta^{\varrho}_{\nu}\varepsilon_\mu) + C_3(-\varepsilon^2)\varepsilon_\mu\varepsilon_\nu\varepsilon^{\varrho}, \\ D^{\sigma\varrho}_{\mu\nu}(\varepsilon) &= D_1(-\varepsilon^2)\varepsilon_{\mu\nu}\varepsilon^{\sigma\varrho} + D_2(-\varepsilon^2)(\delta^{\sigma}_{\mu}\varepsilon_\nu\varepsilon^{\varrho} - \delta^{\sigma}_{\nu}\varepsilon_\mu\varepsilon^{\varrho}), \end{aligned} \tag{4.22}$$

$$\begin{aligned} \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon) &= \hat{C}^{(i)}_1(-\varepsilon^2) g_{\mu\nu}\varepsilon^{\varrho} + \hat{C}^{(i)}_2(-\varepsilon^2)(\delta^{\varrho}_{\mu}\varepsilon_\nu + \delta^{\varrho}_{\nu}\varepsilon_\mu) + \hat{C}^{(i)}_3(-\varepsilon^2)\varepsilon_\mu\varepsilon_\nu\varepsilon^{\varrho}, \\ \hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) &= \hat{D}^{(i)}_1(-\varepsilon^2)\varepsilon_{\mu\nu}\varepsilon^{\sigma\varrho} + \hat{D}^{(i)}_2(-\varepsilon^2)(\delta^{\sigma}_{\mu}\varepsilon_\nu\varepsilon^{\varrho} - \delta^{\sigma}_{\nu}\varepsilon_\mu\varepsilon^{\varrho}), \end{aligned} \tag{4.23}$$

with invariant functions C_1, C_2, C_3, D_1, D_2 and $\hat{C}^{(i)}_1$, $\hat{C}^{(i)}_2$, $\hat{C}^{(i)}_3$, $\hat{D}^{(i)}_1$, $\hat{D}^{(i)}_2$ that remain to be determined. [The spectrum condition has been used to eliminate terms containing the invariant functions $\theta(\pm\varepsilon^1)$, such as $\hat{D}_1(-\varepsilon^2)\,\mathrm{sign}(\varepsilon^1)\varepsilon_{\mu\nu}g^{\sigma\varrho}$, for example.] Conditions on these functions follow from current conservation, i.e., from rewriting (4.17) in the form

$$\begin{aligned} [J_\mu(x+2\varepsilon), J_\nu(x)] \sim\; & C^{\varrho}_{\mu\nu}(\varepsilon)J_\varrho(x) + \sum_{i=0}^{r} \hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon) j^{M(i)}_{\varrho}(x) \\ & + (D^{\sigma\varrho}_{\mu\nu}(\varepsilon) + \varepsilon^{\sigma} C^{\varrho}_{\mu\nu}(\varepsilon))(\partial_\sigma J_\varrho)(x) \\ & + \sum_{i=0}^{r} (\hat{D}^{(i)\sigma\varrho}_{\mu\nu}(\varepsilon) + \varepsilon^{\sigma}\hat{C}^{(i)\varrho}_{\mu\nu}(\varepsilon))\,(\partial_\sigma j^{M(i)}_{\varrho})\,(x), \end{aligned} \tag{4.24}$$

and applying $\partial/\partial\varepsilon_\mu$, which must give zero. This forces

$$D_2 = 2D_1' + \frac{c}{\varepsilon^2}, \qquad C_1 = 6D_1' - 4\varepsilon^2 D_1'', \qquad C_2 = -2D_1', \qquad C_3 = 4D_1'' + \frac{2c}{(\varepsilon^2)^2}, \tag{4.25}$$

$$\hat{D}^{(i)}_2 = 2\hat{D}^{(i)\prime}_1, \qquad \hat{C}^{(i)}_1 = 6\hat{D}^{(i)\prime}_1 - 4\varepsilon^2\hat{D}^{(i)\prime\prime}_1, \qquad \hat{C}^{(i)}_2 = -2\hat{D}^{(i)\prime}_1, \qquad \hat{C}^{(i)}_3 = 4\hat{D}^{(i)\prime\prime}_1, \tag{4.26}$$

---

10 We are indebted to M. Lüscher for communicating to us an (unpublished) revision [35] of [14] on which this presentation is largely based

where c is an arbitrary constant and the prime denotes differentiation with respect to the positive real variable $-\varepsilon^2$. Note that the invariant functions D_1 and $\hat{D}_1^{(i)}$ are still undetermined. Their explicit knowledge, however, is not required, because one can directly show that the limit in (4.18) exists, and that it is time-independent, if one puts

$$Z(\delta)=2D_1(\delta^2/4)+\delta^2 D_1'(\delta^2/4)+c\,, \tag{4.27}$$

$$\hat{Z}^{(i)}(\delta)=2\hat{D}_1^{(i)}(\delta^2/4)+\delta^2\hat{D}_1^{(i)\prime}(\delta^2/4)\,. \tag{4.28}$$

In the literature, it is taken for granted that, modulo terms that go to zero as $\delta\to 0$,

$$D_1(-\varepsilon^2)=a\ln(-4\mu^2\varepsilon^2)+b\,, \tag{4.29}$$

and hence

$$\begin{gathered} D_2(-\varepsilon^2)=\frac{c-2a}{\varepsilon^2}\,, \\ C_1(-\varepsilon^2)=-\frac{2a}{\varepsilon^2}\,,\qquad C_2(-\varepsilon^2)=+\frac{2a}{\varepsilon^2}\,,\qquad C_3(-\varepsilon^2)=\frac{2c-4a}{(\varepsilon^2)^2}\,. \end{gathered} \tag{4.30}$$

(In [14], for example, $a=-(n-2)/8\pi$, $b=2a$, $c=4a$.) Moreover, equations such as (4.29) and (4.30) have been checked, e.g., in the $\mathbb{C}P^{n-1}$ models [15–17] or Grassmannian models [18], within the $1/n$ expansion. A renormalization group analysis shows, however, that these equations cannot be exact [35], and probably there are log log corrections. Fortunately, the definition and the conservation of the first quantum non-local charge are insensitive to such complications.

Appendix: Proof of Supersymmetry for Integrable Models with a Special Choice of Fermion Representation

In the following, we shall, for the sake of completeness, give an explicit proof of the claims made in the last paragraph of Sect. 3. Wherever convenient, we use an (arbitrarily chosen) orthonormal basis of generators T_a in $\mathfrak{m}$ and also an (arbitrarily chosen) orthonormal basis of generators T_j in $\mathfrak{h}$.

To begin with, we note that the fermionic self-interaction term L_F in the Lagrangian (3.70) can be written in the form

$$L_F=-\tfrac{1}{4}g^{\mu\nu}(B_\mu,B_\nu)=\tfrac{1}{16}g^{\mu\nu}R_{ab,cd}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\chi}^c\gamma_\nu\chi^d) \tag{A.1}$$

with

$$\begin{aligned} R_{ab,cd}&=g^{jk}(T_a,[T_j,T_b])(T_c,[T_k,T_d]) \\ &=g^{jk}([T_a,T_b],T_j)(T_k,[T_c,T_d]) \\ &=([T_a,T_b],[T_c,T_d]) \end{aligned} \tag{A.2}$$

(where we have used, in the last step, a completeness relation, together with the commutation relation $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$). In other words, the coefficients $R_{ab,cd}$ are precisely the components of the curvature tensor of M [32, p. 215] which, in addition to having the usual antisymmetry properties $R_{ba,cd}=-R_{ab,cd}=R_{ab,dc}$, is cyclic:

$$R_{ab,cd}+R_{bc,ad}+R_{ca,bd}=0\,. \tag{A.3}$$

This property leads to the following special identities:

$$R_{ab,cd}\{2(\bar{\chi}^a\gamma_5\chi^b)\gamma_5\chi^c-(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\chi^c\}=0\,, \tag{A.4}$$

$$R_{ab,cd}\{2(\bar{\chi}^a\gamma_5\chi^b)\chi^c-(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\gamma_5\chi^c\}=0\,, \tag{A.5}$$

$$R_{ab,cd}(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\gamma^\mu\chi^c=0\,. \tag{A.6}$$

In fact, let $\Gamma=1$ or $\Gamma=\gamma_5$ or $\Gamma=\gamma^\mu$. Then by applying a Fierz identity, we get

$$\begin{aligned}
&R_{ab,cd}(\bar{\chi}^a\gamma_\varrho\chi^c)\gamma^\varrho\Gamma\chi^b\\
&\quad=R_{ab,cd}\{-\tfrac{1}{2}(\bar{\chi}^a\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\chi^c-\tfrac{1}{2}(\bar{\chi}^a\gamma_5\chi^b)\gamma^\varrho\Gamma\gamma_5\gamma_\varrho\chi^c\\
&\qquad-\tfrac{1}{2}(\bar{\chi}^a\gamma_\sigma\chi^b)\gamma^\varrho\Gamma\gamma^\sigma\gamma_\varrho\chi^c\}\\
&\quad=R_{ab,cd}\{-\tfrac{1}{2}(\bar{\chi}^a\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\chi^c+\tfrac{1}{2}(\bar{\chi}^a\gamma_5\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma_5\chi^c\\
&\qquad-(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^c+\tfrac{1}{2}(\bar{\chi}^a\gamma_\sigma\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma^\sigma\chi^c\}\,.
\end{aligned}$$

On the other hand, using (A.3) and the Majorana condition on χ [cf. (3.10) and (3.65)], we can apply a Fierz identity in a different manner:

$$\begin{aligned}
&R_{ab,cd}(\bar{\chi}^a\gamma_\varrho\chi^c)\gamma^\varrho\Gamma\chi^b\\
&\quad=-(R_{bc,ad}+R_{ca,bd})(\bar{\chi}^a\gamma_\varrho\chi^c)\gamma^\varrho\Gamma\chi^b\\
&\quad=R_{ab,cd}\{-(\bar{\chi}^c\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^a-(\bar{\chi}^b\gamma_\varrho\chi^a)\gamma^\varrho\Gamma\chi^c\}\\
&\quad=R_{ab,cd}\{(\bar{\chi}^b\gamma_\varrho\chi^c)\gamma^\varrho\Gamma\chi^a+(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^c\}\\
&\quad=R_{ab,cd}\{-\tfrac{1}{2}(\bar{\chi}^b\chi^a)\gamma^\varrho\Gamma\gamma_\varrho\chi^c-\tfrac{1}{2}(\bar{\chi}^b\gamma_5\chi^a)\gamma^\varrho\Gamma\gamma_5\gamma_\varrho\chi^c\\
&\qquad-\tfrac{1}{2}(\bar{\chi}^b\gamma_\sigma\chi^a)\gamma^\varrho\Gamma\gamma^\sigma\gamma_\varrho\chi^c+(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^c\}\\
&\quad=R_{ab,cd}\{-\tfrac{1}{2}(\bar{\chi}^a\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\chi^c-\tfrac{1}{2}(\bar{\chi}^a\gamma_5\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma_5\chi^c\\
&\qquad-\tfrac{1}{2}(\bar{\chi}^a\gamma_\sigma\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma^\sigma\chi^c+2(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^c\}\,.
\end{aligned}$$

Comparing the two expressions, we deduce that

$$R_{ab,cd}\{(\bar{\chi}^a\gamma_5\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma_5\chi^c+(\bar{\chi}^a\gamma_\sigma\chi^b)\gamma^\varrho\Gamma\gamma_\varrho\gamma^\sigma\chi^c-3(\bar{\chi}^a\gamma_\varrho\chi^b)\gamma^\varrho\Gamma\chi^c\}=0\,,$$

which, upon inserting $\Gamma=1$, $\Gamma=\gamma_5$, and $\Gamma=\gamma^\mu$ (and using $\gamma^\varrho\gamma^\mu\gamma_\varrho=0$) gives (A.4), (A.5), and (A.6), respectively.

With these technicalities out of the way, we can proceed to compute the variation of the Lagrangian (3.70) under the infinitesimal supersymmetry transformations (3.72) and (3.73) and to show that the algebra closes on solutions of the field equations.

First of all, (3.72) implies

$$\begin{aligned}
\delta_\varepsilon(g^{-1}\partial_\mu g)&=-g^{-1}\delta_\varepsilon g\,g^{-1}\partial_\mu g+g^{-1}\partial_\mu\delta_\varepsilon g\\
&=-(\bar{\varepsilon}\chi)(g^{-1}\partial_\mu g)+(g^{-1}\partial_\mu g)(\bar{\varepsilon}\chi)+\bar{\varepsilon}\partial_\mu\chi\\
&=[k_\mu,\bar{\varepsilon}\chi]+\bar{\varepsilon}D_\mu\chi\,.
\end{aligned}$$

Projecting out the components along $\mathfrak{h}$ and along $\mathfrak{m}$, we get

$$\delta_\varepsilon A_\mu=[k_\mu,\bar{\varepsilon}\chi]\,,\qquad \delta_\varepsilon k_\mu=\bar{\varepsilon}D_\mu\chi\,. \tag{A.7}$$

Next, we have from (3.73)

$$\begin{aligned}
&\frac{i}{4}(\delta_\varepsilon\bar{\chi}\overleftrightarrow{\not{D}}\chi+\bar{\chi}\overleftrightarrow{\not{D}}\delta_\varepsilon\chi)\\
&\quad=-\tfrac{1}{4}k_\nu^a\,\bar{\varepsilon}\gamma^\nu\gamma^\mu D_\mu\chi^a+\tfrac{1}{4}D_\mu k_\nu^a\,\bar{\varepsilon}\gamma^\nu\gamma^\mu\chi^a\\
&\qquad-\tfrac{1}{4}D_\mu\bar{\chi}^a\gamma^\mu\gamma^\nu\varepsilon\,k_\nu^a+\tfrac{1}{4}\bar{\chi}^a\gamma^\mu\gamma^\nu\varepsilon\,D_\mu k_\nu^a\\
&\quad=-\tfrac{1}{4}\partial_\mu(k_\nu^a\{\bar{\varepsilon}\gamma^\nu\gamma^\mu\chi^a+\bar{\chi}^a\gamma^\mu\gamma^\nu\varepsilon\})+\tfrac{1}{2}D_\mu k_\nu^a\{\bar{\varepsilon}\gamma^\nu\gamma^\mu\chi^a+\bar{\chi}^a\gamma^\mu\gamma^\nu\varepsilon\}\\
&\quad=-\tfrac{1}{4}\partial_\mu(k_\nu^a\{g^{\mu\nu}(\bar{\varepsilon}\chi^a+\bar{\chi}^a\varepsilon)+\varepsilon^{\mu\nu}(\bar{\varepsilon}\gamma_5\chi^a-\bar{\chi}^a\gamma_5\varepsilon)\})\\
&\qquad+\tfrac{1}{2}D_\mu k_\nu^a\{g^{\mu\nu}(\bar{\varepsilon}\chi^a+\bar{\chi}^a\varepsilon)+\varepsilon^{\mu\nu}(\bar{\varepsilon}\gamma_5\chi^a-\bar{\chi}^a\gamma_5\varepsilon)\}\\
&\quad=g^{\mu\nu}(D_\mu k_\nu,\bar{\varepsilon}\chi)-\tfrac{1}{2}\partial_\mu\{g^{\mu\nu}(k_\nu,\bar{\varepsilon}\chi)+\varepsilon^{\mu\nu}(k_\nu,\bar{\varepsilon}\gamma_5\chi)\}\quad\text{by (3.30)}\\
&\quad=-g^{\mu\nu}(k_\nu,\bar{\varepsilon}D_\mu\chi)+\tfrac{1}{2}\partial_\mu\{g^{\mu\nu}(k_\nu,\bar{\varepsilon}\chi)-\varepsilon^{\mu\nu}(k_\nu,\bar{\varepsilon}\gamma_5\chi)\},
\end{aligned}$$

and therefore, due to

$$\delta_\varepsilon(\tfrac{1}{2}g^{\mu\nu}(D_\mu g,D_\nu g))=g^{\mu\nu}(k_\nu,\delta_\varepsilon k_\mu),$$

$$\delta_\varepsilon\left(\frac{i}{4}\bar{\chi}\overleftrightarrow{\not{D}}\chi\right)=\frac{i}{4}(\delta_\varepsilon\bar{\chi}\overleftrightarrow{\not{D}}\chi+\bar{\chi}\overleftrightarrow{\not{D}}\delta_\varepsilon\chi)+\frac{i}{2}g^{\mu\nu}\bar{\chi}\gamma_\mu[\delta_\varepsilon A_\nu,\chi],$$

we obtain

$$\begin{aligned}
&\delta_\varepsilon\left(\frac{1}{2}g^{\mu\nu}(D_\mu g,D_\nu g)+\frac{i}{4}\bar{\chi}\overleftrightarrow{\not{D}}\chi\right)\\
&\quad=\frac{i}{2}g^{\mu\nu}\bar{\chi}\gamma_\mu[[k_\nu,\bar{\varepsilon}\chi],\chi]+\frac{1}{2}\partial_\mu\{g^{\mu\nu}(k_\nu,\bar{\varepsilon}\chi)-\varepsilon^{\mu\nu}(k_\nu,\bar{\varepsilon}\gamma_5\chi)\}.
\end{aligned}\tag{A.8}$$

On the other hand, for any generator T in $\mathfrak{h}$ (with ad(T) on $\mathfrak{m}$ represented by $T_{ab}=(T_a,[T,T_b])=-(T_b,[T,T_a])=-T_{ba}$), we have

$$\begin{aligned}
(\delta_\varepsilon B_\mu,T)=\delta_\varepsilon(B_\mu,T)&=-\frac{i}{2}(\delta_\varepsilon\bar{\chi}\gamma_\mu[T,\chi]+\bar{\chi}\gamma_\mu[T,\delta_\varepsilon\chi])\\
&=\tfrac{1}{2}T_{ba}k_\nu^b(\bar{\varepsilon}\gamma^\nu\gamma_\mu\chi^a+\bar{\chi}^a\gamma_\mu\gamma^\nu\varepsilon)\\
&=\tfrac{1}{2}T_{ba}g^{\nu\lambda}k_\lambda^b\{g_{\mu\nu}(\bar{\varepsilon}\chi^a+\bar{\chi}^a\varepsilon)+\varepsilon_{\mu\nu}(\bar{\varepsilon}\gamma_5\chi^a-\bar{\chi}^a\gamma_5\varepsilon)\}\\
&=T_{ba}k_\mu^b\bar{\varepsilon}\chi^a+T_{ba}\varepsilon_{\mu\nu}g^{\nu\lambda}k_\lambda^b\bar{\varepsilon}\gamma_5\chi^a\\
&=(k_\mu,[T,\bar{\varepsilon}\chi])+(\varepsilon_{\mu\nu}g^{\nu\lambda}k_\lambda,[T,\bar{\varepsilon}\gamma_5\chi]),
\end{aligned}$$

i.e.,

$$\delta_\varepsilon B_\mu=-[k_\mu,\bar{\varepsilon}\chi]-[\varepsilon_{\mu\nu}g^{\nu\lambda}k_\lambda,\bar{\varepsilon}\gamma_5\chi],\tag{A.9}$$

and therefore we obtain

$$\begin{aligned}
&\delta_\varepsilon(-\tfrac{1}{4}g^{\mu\nu}(B_\mu,B_\nu))\\
&\quad=-\frac{i}{4}g^{\mu\nu}\bar{\chi}\gamma_\mu[[k_\nu,\bar{\varepsilon}\chi],\chi]-\frac{i}{4}\varepsilon^{\mu\nu}\bar{\chi}\gamma_\mu[[k_\nu,\bar{\varepsilon}\gamma_5\chi],\chi].
\end{aligned}\tag{A.10}$$

Adding up (A.8) and (A.10), we are left with a total divergence, since the other terms add up to

$$\begin{aligned}&\frac{i}{4}g^{\mu\nu}\bar{\chi}\gamma_\mu[[k_\nu,\bar{\varepsilon}\chi],\chi]-\frac{i}{4}\varepsilon^{\mu\nu}\bar{\chi}\gamma_\mu[[k_\nu,\bar{\varepsilon}\gamma_5\chi],\chi]\\&=\frac{i}{4}(T_a,[[T_d,T_c],T_b])\,k_\nu^d\{g^{\mu\nu}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\varepsilon}\chi^c)-\varepsilon^{\mu\nu}(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\varepsilon}\gamma_5\chi^c)\}\\&=\frac{i}{4}R_{ab,cd}\,k_\nu^d\,(\bar{\chi}^a\gamma_\mu\chi^b)(\bar{\varepsilon}\gamma^\mu\gamma^\nu\chi^c),\end{aligned}$$

which vanishes, according to (A.6).

To check that the algebra closes, we compute

$$\begin{aligned}\delta_{\varepsilon_2}\delta_{\varepsilon_1}g=\delta_{\varepsilon_2}(g\bar{\varepsilon}_1\chi)&=g(\bar{\varepsilon}_2\chi)(\bar{\varepsilon}_1\chi)+g(-i\bar{\varepsilon}_1 \not{k}\varepsilon_2)\\&=g(\bar{\varepsilon}_2\chi)(\bar{\varepsilon}_1\chi)-i\bar{\varepsilon}_1\gamma^\mu\varepsilon_2 D_\mu g\,,\end{aligned}$$

$$\begin{aligned}\delta_{\varepsilon_2}\delta_{\varepsilon_1}\chi&=\delta_{\varepsilon_2}(-ik_\mu\gamma^\mu\varepsilon_1)=-i(\bar{\varepsilon}_2 D_\mu\chi)\gamma^\mu\varepsilon_1\\&=\frac{i}{2}\{(\bar{\varepsilon}_2\varepsilon_1)\gamma^\mu D_\mu\chi+(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\gamma^\mu\gamma_5 D_\mu\chi+(\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)\gamma^\mu\gamma^\varrho D_\mu\chi\}\\&=i(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)D_\mu\chi+\frac{i}{2}(\bar{\varepsilon}_2\varepsilon_1)\not{D}\chi\\&\quad-\frac{i}{2}\{(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\gamma_5\not{D}\chi+(\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)\gamma^\varrho\not{D}\chi\}\,,\end{aligned}$$

where we have applied a Fierz identity. Using the fermionic field equation (3.37),

$$\not{D}\chi=-\tfrac{1}{2}[\not{B},\chi] \tag{A.11}$$

[cf. (3.54)], together with

$$\begin{aligned}([B_\mu,\chi],T_d)=(B_\mu,[T_c,T_d])\chi^c&=-\frac{i}{2}\bar{\chi}\gamma_\mu[[T_c,T_d],\chi]\chi^c\\&=-\frac{i}{2}(T_a,[[T_c,T_d],T_b])(\bar{\chi}^a\gamma_\mu\chi^b)\chi^c\\&=\frac{i}{2}R_{ab,cd}(\bar{\chi}^a\gamma_\mu\chi^b)\chi^c\,,\end{aligned}$$

$$\begin{aligned}([B_5,\chi],T_d)=(B_5,[T_c,T_d])\chi^c&=\tfrac{1}{2}\bar{\chi}\gamma_5[[T_c,T_d],\chi]\chi^c\\&=\tfrac{1}{2}(T_a,[[T_c,T_d],T_b])(\bar{\chi}^a\gamma_5\chi^b)\chi^c\\&=-\tfrac{1}{2}R_{ab,cd}(\bar{\chi}^a\gamma_5\chi^b)\chi^c\,,\end{aligned}$$

and the relations (A.5) and (A.6), we can transform the last term in curly brackets (or rather its scalar product with any generator T_d in $\mathfrak{m}$) as follows:

$$-\frac{i}{2}(\{(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\gamma_5 \not{D}\chi + (\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)\gamma^\varrho \not{D}\chi\}, T_d)$$

$$= \frac{i}{4}\{(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\gamma_5\gamma^\mu([B_\mu, \chi], T_d) + (\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)\gamma^\varrho\gamma^\mu([B_\mu, \chi], T_d)\}$$

$$= -\frac{1}{8}R_{ab,cd}(\bar{\chi}^a\gamma_\mu\chi^b)\{(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\gamma_5\gamma^\mu\chi^c + (\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)\gamma^\varrho\gamma^\mu\chi^c\}$$

$$= \frac{1}{4}R_{ab,cd}(\bar{\varepsilon}_2\gamma_5\varepsilon_1)(\bar{\chi}^a\gamma_5\chi^b)\chi^c - \frac{1}{4}R_{ab,cd}(\bar{\varepsilon}_2\gamma_\varrho\varepsilon_1)(\bar{\chi}^a\gamma^\varrho\chi^b)\chi^c$$

$$= -\frac{1}{2}(\bar{\varepsilon}_2\gamma_5\varepsilon_1)([B_5, \chi], T_d) + \frac{i}{2}(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)([B_\mu, \chi], T_d).$$

Putting everything together, we arrive at

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]g = 2i(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)D_\mu g + g[\bar{\varepsilon}_2\chi, \bar{\varepsilon}_1\chi], \tag{A.12}$$

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\chi = 2i(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)D_\mu\chi + i(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)[B_\mu, \chi] - (\bar{\varepsilon}_2\gamma_5\varepsilon_1)[B_5, \chi], \tag{A.13}$$

[cf. (3.10)], which is the desired result: the first terms in (A.12) and (A.13) represent the action of translation generators (with translation vector $\bar{\varepsilon}_2\gamma_\mu\varepsilon_1$), while the other terms represent the action of gauge transformation generators.

References

1. Forger, M.: Instantons in non-linear σ models, gauge theories, and general relativity. In: Differential geometrical methods in mathematical physics. Proceedings, Clausthal, Germany 1978. Doebner, H.D. (ed.). Lecture Notes in Physics, Vol. 139. Berlin, Heidelberg, New York: Springer 1981
2. Forger, M.: Differential geometric methods in non-linear σ models and gauge theories. Ph.D. Thesis, Freie Universität Berlin (1980) (unpublished)
3. Dubois-Violette, M., Georgelin, Y.: Gauge theory in terms of projector valued fields. Phys. Lett. **82** B, 251–254 (1979)
4. Polyakov, A.M.: String representations and hidden symmetries for gauge fields. Phys. Lett. **82** B, 247–250 (1979)
5. Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through quadratic constraints. Commun. Math. Phys. **46**, 207–221 (1976)
6. Lüscher, M., Pohlmeyer, K.: Scattering of massless lumps and non-local charges in the two-dimensional non-linear σ model. Nucl. Phys. B **137**, 46–54 (1978)
7. Zakharov, V.E., Mikhailov, A.V.: Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Sov. Phys. JETP **47**, 1017–1027 (1978)
8. Eichenherr, H., Forger, M.: On the dual symmetry of the non-linear σ models. Nucl. Phys. B **155**, 381–393 (1979)
9. Eichenherr, H., Forger, M.: More about non-linear σ models on symmetric spaces. Nucl. Phys. B **164**, 528–535 (1980)

10. Eichenherr, H., Forger, M.: Higher local conservation laws for non-linear σ models on symmetric spaces. Commun. Math. Phys. **82**, 227–255 (1981)
11. Eichenherr, H.: Geometric analysis of integrable σ models. In: Integrable quantum field theories. Proceedings, Tvärminne, Finland 1981. Hietarinta, J., Montonen, C. (eds.). Lecture Notes in Physics, Vol. 151. Berlin, Heidelberg, New York: Springer 1982
12. Forger, M.: Non-linear σ models on symmetric spaces. In: Non-linear partial differential operators and quantization procedures. Proceedings, Clausthal, Germany 1981. Andersson, S.I., Doebner, H.D. (eds.). Lecture Notes in Mathematics, Vol. 1037. Berlin, Heidelberg, New York: Springer 1983
13. Polyakov, A.M.: Hidden symmetry of the two-dimensional chiral fields. Phys. Lett. **72** B, 224–226 (1977);
Lowenstein, J.H., Speer, E.R.: Conservation of local currents in the quantum chiral model. Nucl. Phys. B **158**, 397–409 (1979);
Goldschmidt, Y.Y., Witten, E.: Conservation laws in some two-dimensional models. Phys. Lett. **91** B, 392–396 (1980)
14. Lüscher, M.: Quantum non-local charges and absence of particle production in the two-dimensional non-linear σ model. Nucl. Phys. B **135**, 1–19 (1978)
15. Abdalla, E., Abdalla, M.C.B., Gomes, M.: Anomaly in the non-local quantum charge of the $\mathbb{C}P^{n-1}$ model. Phys. Rev. D **23**, 1800–1805 (1981)
16. Abdalla, E., Abdalla, M.C.B., Gomes, M.: Anomaly cancellations in the supersymmetric $\mathbb{C}P^{n-1}$ model. Phys. Rev. D **25**, 452–460 (1982)
17. Abdalla, E., Abdalla, M.C.B., Gomes, M.: Non-local charge of the $\mathbb{C}P^{n-1}$ model and its supersymmetric extension to all orders. Phys. Rev. D **27**, 825–836 (1983)
18. Abdalla, E., Forger, M., Lima Santos, A.: Non-local charges for non-linear σ models on Grassmann manifolds. Nucl. Phys. B **256**, 145–180 (1985)
19. de Vega, H.J., Eichenherr, H., Maillet, J.M.: Classical and quantum algebras of non-local charges in σ models. Commun. Math. Phys. **92**, 507–524 (1984);
Maillet, J.M.: Kac-Moody algebra and extended Yang-Baxter relations in the $O(N)$ non-linear σ model, preprint PAR-LPTHE-85/22 (June 1985)
20. Abdalla, E., Forger, M., Gomes, M.: On the origin of anomalies in the quantum non-local charge for the generalized non-linear σ models. Nucl. Phys. B **210** [FS6], 181–192 (1982)
21. Zamolodchikov, A.B., Zamolodchikov, A.B.: Relativistic factorized S-matrix in two dimensions having $O(N)$ isotopic symmetry. Nucl. Phys. B **133**, 525–535 (1978)
22. Abdalla, E., Abdalla, M.C.B., Lima Santos, A.: On the exact S-matrix of the principal chiral model. Phys. Lett. **140** B, 71–75 (1984); **146** B, 457 (1984);
Abdalla, E., Abdalla, M.C.B., Forger, M.: Exact S-matrices for anomaly-free non-linear σ models on symmetric spaces (in preparation)
Abdalla, E.: Non-linear sigma models: a geometrical approach in quantum field theory. In: Non-linear equations in classical and quantum field theory. Proceedings, Meudon and Paris VI, France 1983/1984.
Sanchez, N. (ed.): Lecture Notes in Physics, Vol. 226. Berlin, Heidelberg, New York: Springer 1985
23. Witten, E.: Supersymmetric form of the non-linear σ model in two dimensions. Phys. Rev. D **16**, 2991–2994 (1977);
D'Adda, A., Di Vecchia, P., Lüscher, M.: Confinement and chiral symmetry breaking in $\mathbb{C}P^{n-1}$ models with quarks. Nucl. Phys. B **152**, 125–144 (1979);
Aoyama, S.: The supersymmetric $U_{N,r}\sigma$ model and its O_2 extended supersymmetry. Nuovo Cimento **57** A, 176–184 (1980)
24. Freedman, D.Z., Townsend, P.K.: Antisymmetric tensor gauge theories and non-linear σ models. Nucl. Phys. B **177**, 282–296 (1981)
25. Alvarez-Gaumé, L., Freedman, D.Z.: Kähler geometry and the renormalization of supersymmetric σ models. Phys. Rev. D **22**, 846–853 (1980); Geometrical structure and ultra-violet finiteness in the supersymmetric σ model. Commun. Math. Phys. **80**, 443–451 (1981)
26. Curtright, T.L., Zachos, C.K.: Non-local currents for supersymmetric non-linear models. Phys. Rev. D **21**, 411–417 (1980)

27. Schonfeld, J.F.: Dynamical conservation laws in supersymmetric theories of symmetric-space-valued fields. Nucl. Phys. B **169**, 49–67 (1980)
28. Faddeev, L.D., Semenov-Tjan-Shansky, M.A.: On the theory of non-linear quantum fields. Vestnik LGU **13**, 81–88 (1977) (in Russian)
29. Moore, G., Nelson, P.: Anomalies in non-linear σ models. Phys. Rev. Lett. **53**, 1519–1522 (1984); The aetiology of σ model anomalies. Commun. Math. Phys. **100**, 83–132 (1985)
30. Greub, W., Halperin, S., Vanstone, R.: Connections, curvature, and cohomology, Vols. 1–3. New York: Academic Press 1972, 1973, 1976
31. Kobayashi, S., Nomizu, K.: Foundations of differential geometry, Vols. 1, 2. New York: Interscience 1963, 1969
32. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. New York: Academic Press 1978
33. Neveu, A., Papanicolaou, N.: Integrability of the classical $[\bar{\psi}_i\psi_i]_2^2$ and $[\bar{\psi}_i\psi_i]_2^2-[\bar{\psi}_i\gamma_5\psi_i]_2^2$ interactions. Commun. Math. Phys. **58**, 31–64 (1978)
34. de Vega, H.J., Eichenherr, H., Maillet, J.M.: Canonical charge algebras for integrable fermionic theories. Phys. Lett. **132** B, 337–341 (1983)
35. Lüscher, M.: Private communication

Communicated by G. Mack

Received August 26, 1985