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Wave Operators and Analytic Solutions for Systems of Non-Linear Klein–Gordon Equations and of Non-Linear Schrödinger Equations

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Abstract. We consider, in a $1+3$ space time, arbitrary (finite) systems of non-linear Klein–Gordon equations (respectively Schrödinger equations) with an arbitrary local and analytic non-linearity in the unknown and its first and second order space-time (respectively first order space) derivatives, having no constant or linear terms. No restriction is given on the frequency sign of the initial data. In the case of non-linear Klein–Gordon equations all masses are supposed to be different from zero.

We prove, for such systems, that the wave operator (from $t=\infty$ to $t=0$) exists on a domain of small entire test functions of exponential type and that the analytic Cauchy problem, in $\mathbb{R}^+ \times \mathbb{R}^3$, has a unique solution for each initial condition (at $t=0$) being in the image of the wave operator. The decay properties of such solutions are discussed in detail.

1. Introduction

To fix the ideas we first introduce the following systems of non-linear equations:

$$(\square + m_j^2)u_j(t,x) = F_j(u(t,x), Du(t,x), D^2u(t,x)), m_j \neq 0, \quad 1 \leqq j \leqq N \quad \text{(NLKG}')$$

and

$$\left(\frac{\partial}{\partial t} + \varepsilon_j i \Delta\right) u_j(t,x) = F_j(u(t,x), \nabla u(t,x)), \varepsilon_j = \pm 1, \quad 1 \leqq j \leqq N, \quad \text{(NLS)}$$

where $x \in \mathbb{R}^3, t \in \mathbb{R}^+ = \{\eta | \eta \geqq 0\}, \nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3), D = (\partial/\partial t, \nabla)$,

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}, \square = \frac{\partial^2}{\partial t^2} - \Delta, u_j(t,x) \in \mathbb{C}, u = (u_1, \ldots, u_N) \text{ and } N \geqq 1.$$

The non-linearity $F = (F_1, \ldots, F_N)$ is restricted to be an analytic function in a

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neighbourhood $\mathcal{O}$ of zero:

$$F: \mathcal{O} \subset \mathbb{C}^N \times \mathbb{C}^{4N} \times \mathbb{C}^{16N} \to \mathbb{C}^N \quad \text{for} \quad (\text{NLKG}')$$

and

$$F: \mathcal{O} \subset \mathbb{C}^N \times \mathbb{C}^{3N} \to \mathbb{C}^N \quad \text{for} \quad (\text{NLS}),$$

satisfying in both cases $F(0) = F'(0) = 0$ (F' denoting the Fréchet derivative of F). Writing the second time derivatives in the non-linearity of (NLKG′) as $(\partial^2/\partial t^2)u_j = (\Box + m_j^2)u_j + \Delta u_j - m_j^2 u_j$, we can, imposing that $\mathcal{O}$ is sufficiently small and using the implicit function theorem, reduce (NLKG′) to

$$(\Box + m_j^2)u_j(t,x) = F_j(u(t,x), Du(t,x), D\nabla u(t,x)), m_j \neq 0, \quad 1 \leqq j \leqq N, \quad (\text{NLKG})$$

where F is a new analytic function in some neighbourhood $\mathcal{O}$ of zero:

$$F: \mathcal{O} \subset \mathbb{C}^N \times \mathbb{C}^{4N} \times \mathbb{C}^{12N} \to \mathbb{C}^N, F(0) = F'(0) = 0.$$

Remark 1.1.

a) Instead of (NLKG′) one could consider more general systems of non-linear massive local relativistic evolution equations autonomous in t, x. However, under reasonable hypothesis, the Cauchy problem for such evolution equations can be studied through (NLKG′). For example, a non-linear Dirac equation like $(i\gamma^\mu \partial_\mu + m)\varphi = G(u, Du, \varphi, D\varphi)$, with initial condition φ_0 at $t = 0$ is reduced to $(\Box + m^2)\varphi = (i\gamma^\mu \partial_\mu - m)G(u, Du, \varphi, D\varphi)$ with initial condition $(\varphi_0, \dot{\varphi}_0)$, where $(\gamma^0\gamma_j\partial^j + i\gamma^0 m)\varphi_0 - i\gamma^0 G(u_0, Du_0, \varphi_0, (\dot{\varphi}_0, \nabla\varphi_0)) = \dot{\varphi}_0$. The last equation has a solution $\dot{\varphi}_0$ for small initial data.

b) Non-linear Schrödinger equations with real analytic non linearities as $(\partial/\partial t - i\Delta)u(t,x) = G(u(t,x), \overline{u(t,x)}, \nabla u(t,x), \overline{\nabla u(t,x)}), G(0) = G'(0), G$ complex analytic, fall into the class (NLS). (Here $\bar{Z}$ is the complex conjugate of $Z \in \mathbb{C}$). One has only to introduce a new variable $v(t,x) = \overline{u(t,x)}$ and then consider the system

$$\left(\frac{\partial}{\partial t} - i\Delta\right)u = G(u, v, \nabla u, v), \left(\frac{\partial}{\partial t} + i\Delta\right)v = \bar{G}(v, u, \nabla v, \nabla u), \quad \overline{u(0,x)} = v(0,x),$$

where $\bar{G}(\xi) = \sum_{|\alpha| \geqq 2} \bar{a}_\alpha \xi^\alpha$ if $G(\xi) = \sum_{|\alpha| \geqq 2} a_\alpha \xi^\alpha$ is the Taylor development of G around zero. A similar remark is obviously true for NLKG′.

Before outlining the content of the article we introduce certain notations. We denote by $W^{n,p}(\mathbb{R}^3, \mathbb{C}^l), n \geqq 0, 1 \leqq p \leqq \infty, l \geqq 1$, the Sobolev space of functions from $\mathbb{R}^3$ to $\mathbb{C}^l$, being in L^p with their n first derivatives. In the case of (NLKG) (respectively (NLS)) $\mathcal{H} = W^{4,2}(\mathbb{R}^3, \mathbb{C}^N) \oplus W^{3,2}(\mathbb{R}^3, \mathbb{C}^N)$ (respectively $\mathcal{H} = W^{3,2}(\mathbb{R}^3, \mathbb{C}^l)$), the Fourier transform of $f: \mathbb{R}^n \to \mathbb{C}^l, l \geqq 1$, is given by $\hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} dx^{-ik\,x} f(x)$. $D(a)$, $a \geqq 1$ denotes, in the case of (NLKG) (respectively (NLS)), the subset of functions (f, g) (respectively f) in $\mathcal{H}$ such that $\hat{f}, \hat{g} \in C^{14}(\mathbb{R}^3, \mathbb{C}^N)$ and such that the support of $\hat{f}, \hat{g}$ (respectively $\hat{f}$) is contained in $\{k \in \mathbb{R}^3 \mid |k| \leqq a\}$ (respectively $\{k \in \mathbb{R}^3 \mid (a+1)^{-1} \leqq |k| \leqq a\}$). $D(a)$ is given its natural Banach norm. The unitary evolution in $\mathcal{H}$ defined by the linear part of (NLKG) (respectively (NLS)) is denoted V_t, $t\in\mathbb{R}$, i.e.

$$V_t = \bigoplus_{j=1}^{N} \exp\left[t\begin{pmatrix} 0 & I \\ \Delta - m_j^2 & 0 \end{pmatrix}\right]\left(\text{respectively } V_t = \bigoplus_{j=1}^{N} \exp(ti\varepsilon_j\Delta)\right).$$

This is a C_0-group in $D(a)$. The scattering problem, with given scattering data $\varphi_+\in D(a)$ at $t=\infty$, for (NLKG) (respectively (NLS)) is posed by the equation

$$\varphi(t) = V_t\varphi_+ - \int_t^\infty V_{t-s}J(\varphi(s))ds, \quad \varphi(t)\in\mathscr{H}, t\geqq 0, \tag{1.1}$$

where $[\varphi(t)](x) = (u(t,x),(\partial/\partial t)u(t,x))$ (respectively $(\varphi(t))(x) = u(t,x)$) and $(J(\varphi(t)))(x) = (0, F(u(t,x), Du(t,x), D\nabla u(t,x))$, (respectively $(J(\varphi(t)))(x) = F(u(t,x), \nabla u(t,x))$. The corresponding Yang–Feldman equation for the wave operator $\Omega:\varphi_+ \to \varphi(0)$ is

$$\Omega = I - \int_0^\infty V_{-s}J\circ\Omega\circ V_s ds, \quad I = \text{identity}. \tag{1.1'}$$

We prove, in Sects. 2 and 3, that Eq. (1.1′) has a solution Ω being an analytic function from a neighbourhood of zero in $D(a)$, $a\geqq 1$ into $\mathscr{H}$.

Given two Banach spaces X and Y, we denote by $F(X,Y)$, the space of formal power series $A=\sum_{n\geqq 1} A^n$, $A^n\in L_n(X,Y)$, the space of n-linear symmetric continuous maps from X to Y. We do not make the distinction between A^n considered as a monomial from X to Y, or as an element in $L_n(X,Y)$ or as an element in $L(\hat{\otimes}_s^n X, Y)$, where $\hat{\otimes}_s^n$ is the n-fold symmetrized projective tensor product. Equation (1.1′) is solved first by considering Ω as an element of $F(D(a),\mathscr{H})$, which gives the iterative equation

$$\Omega^n = -\sum_{\substack{1\leqq p\leqq n \\ n_1+\dots+n_p=n, n_i\geqq 1}} \int_0^\infty V_{-s}J^p(\Omega^{n_1}\otimes\cdots\otimes\Omega^{n_p})(\otimes^n V_s)ds\sigma, \quad n\geqq 2, \Omega^1 = I, \tag{1.1''}$$

where σ is the normalized symmetrization operator. We show in Sect. 2, by using the method of stationary phase, that the decrease of the $L_n(D(a),\mathscr{H})$ (respectively $L_n(D(a), L^\infty)$) norm of $\Omega^n(\otimes^n V_t)$ is better than $t^{-1/2}$ (respectively t^{-2}) for $n=2$ and t^{-1} (respectively t^{-2}) for $n=3$, as $t\to\infty$. The actual decay of these norms is usually better. For example, the indicated decay for $n=2$ is obtained for (NLKG) when there is a relation of the type $\varepsilon_1 M_1 + \varepsilon_2 M_2 + \varepsilon_3 M_3 = 0$, $\varepsilon_1,\varepsilon_2,\varepsilon_3 = \pm 1$ and $M_1, M_2, M_3\in\{m_1,\dots,m_N\}$, and is obtained for the (NLS) when $\varepsilon_i+\varepsilon_j=0$ for some $1\leqq i,j\leqq N$. Otherwise the decay turns out to be of the order of $t^{-3/2}$ (respectively t^{-3}). The decay of $\Omega^3(\otimes^3 V_t)$ in $L_3(D(a),L^\infty)$ is always better than the one indicated, but a t^{-2} type decay is sufficient for our purpose. Sobolev estimates give now that the time decay for the $L_n(D(a),\mathscr{H})$-norm of $\Omega^n(\otimes^n V_t)$ is at least $t^{-(t+n/8)}$ for $n\geqq 4$ at $t\to\infty$. The linear dependence in n of the exponent is important as it exactly compensates through the integration in (1.1″), the derivative loss due to the fact that J is continuous from $\mathscr{H}$ into $(1-\Delta)^{-1/2}\mathscr{H}$ (but in general not into $\mathscr{H}$). In Sect. 3, the convergence of $\Omega = \sum_{n\geqq 1}\Omega^n$ is proved by using a variation of the iteration scheme proposed in [8]. The solutions of Eq. (1.1) are now given by

$\varphi(t)=\Omega(V_t\varphi_+), t\geqq 0$, for φ_+ in the domain of convergence. $D(a)$ is a space of analytic vectors for the representation $t\to V_t$ (in $D(a)$). From this and the analyticity of Ω we deduce in Sect. 4, that the obtained solutions for (NLKG) and (NLS) are analytic in a neighbourhood of $\mathbb{R}^+\times\mathbb{R}^3$. From the above indicated decay of $\Omega^n(\otimes^n V_t), n\geqq 2$, it is clear that $\|\varphi(t)\|_{L^\infty}$ decays as $t^{-3/2}$, $t\to\infty$.

There is a rich literature on the systems (NLKG) and (NLS), (cf. [5, 9]). Recent contributions on the existence of global solutions and scattering states for (NLKG) are [3, 6] and for (NLS) [1, 3, 6]. These references solve, in the case of $1+3$ dimensions the problem of existence of decaying scattering solutions for the case where F is C^∞ and $F(0)=F'(0)=F''(0)$, i.e. there are no quadratic terms in the non-linearity. The system (NLKG) with $N=1$, and without derivatives in F, was proved [8] to have scattering solutions for a set of scattering states similar to $D(a)$ when certain combinations of masses appear ($N\geqq 1$) this was proved when the scattering states have a definite frequency sign [7].

2. Decay Properties of Certain Integrals of Products of Free Solutions

We calculate in this paragraph the wave operators for the systems (NLS) and (NLKG) up to order three. This can be done by evaluating, with the help of the method of stationary phase, the decay properties as $t\to\infty$ of integrals of the type

$$\int_t^\infty ds \int dp\, e^{if_k(p)s}u(p,k), \quad u\in C_0^\infty(\mathbb{R}^6), \quad f_k\in C^\infty(\mathbb{R}^3), \quad k\in\mathbb{R}^3.$$

For $m\in\mathbb{R}$ define $\omega_m:\mathbb{R}^3\to\mathbb{R}$ by

$$\omega_m(k)=(m^2+|k|^2)^{1/2}. \tag{2.1}$$

For $m, m_1, m_2\neq 0$, $\varepsilon, \varepsilon_1, \varepsilon_2=\pm 1$, $k\in\mathbb{R}^3$, define $f_k:\mathbb{R}^3\to\mathbb{R}$ by

$$f_k(p)=-\varepsilon\omega_m(k)+\varepsilon_1\omega_{m_1}(p)+\varepsilon_2\omega_{m_2}(p-k). \tag{2.2}$$

As $m_1, m_2\neq 0$, equation $f_k'(p)=0$ (derivative with respect to the variable p) is equivalent to

$$(\varepsilon_1 m_1+\varepsilon_2 m_2)p=\varepsilon_1 m_1 k. \tag{2.3}$$

In the case $\varepsilon_1 m_1+\varepsilon_2 m_2\neq 0$ the values of f_k and $\det f_k''$ are, when $f_k'(p)=0$:

$$f_k(p)=-\varepsilon\omega_m(k)+\frac{\varepsilon_1 m_1+\varepsilon_2 m_2}{|\varepsilon_1 m_1+\varepsilon_2 m_2|}\omega_{(\varepsilon_1 m_1+\varepsilon_2 m_2)^{(k)}}, \tag{2.4}$$

$$\det f_k''(p)=+\left(\frac{\varepsilon_1\varepsilon_2}{m_1 m_2}\right)^3|\varepsilon_1 m_1+\varepsilon_2 m_2|^7(\varepsilon_1 m_1+\varepsilon_2 m_2)(\omega_{(\varepsilon_1 m_1+\varepsilon_2 m_2)^{(k)}}), \tag{2.5}$$

$$\text{if } p=\varepsilon_1 m_1(\varepsilon_1 m_1+\varepsilon_2 m_2)^{-1}k.$$

Lemma 2.1. *Let* $\hat{v}_t(k)=e^{i\varepsilon\omega_m(k)t}$, $f_k(p)=-\varepsilon\omega_m(k)+\varepsilon_1\omega_{m_1}(p)+\varepsilon_2\omega_{m_2}(p-k), m, m_1, m_2>0$, $\varepsilon_1 m_1+\varepsilon_2 m_2\neq 0$, $\varepsilon_1 m_1+\varepsilon_2 m_2\neq\varepsilon m$, $\varepsilon, \varepsilon_1, \varepsilon_2=\pm 1$ *and* $\hat{g}_1, \hat{g}_2\in C_0^\infty(\mathbb{R}^3)$[1]

1 $C_0^k(\mathbb{R}^3)$ is the space of complex valued C^k-functions on $\mathbb{R}^3$ with compact support $\partial_i=\partial/\partial k_i$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, $k^\alpha=k_1^{\alpha_1}k_2^{\alpha_2}k_3^{\alpha_3}$

or let $\hat{v}_t(k) = e^{i\varepsilon|k|^2 t}$, $f_k(p) = -\varepsilon|k|^2 + \varepsilon_1(|p|^2 + |p-k|^2)$, $\varepsilon, \varepsilon_1 = \pm 1$ *and* $\hat{g}_1, \hat{g}_2 \in C_0^\infty(\mathbb{R}^3 - \{0\})$. *If*

$$\hat{h}_{t,T}(k) = \int_t^T ds \int_{\mathbb{R}^3} dp\, e^{if_k(p)s} \hat{g}_1(p)\hat{g}_2(k-p), \quad 0 \leqq t \leqq T, \tag{2.6}$$

then $\hat{h}_{t,T}$ *has a limit* $\hat{h}_t$ *in* $C_0^0(\mathbb{R}^3)$ *as* $T \to \infty$ *and if* $\operatorname{supp} \hat{g}_i = K_i$ *then* $\operatorname{supp} \hat{h}_{t,T} \subset K_1 + K_2$ *and*

$$\|h_t\|_{W^{n,2}} \leqq (1+t)^{-3/2} C(n, f, K_1, K_2) \|\hat{g}_1\|_{W^{6,\infty}} \|\hat{g}_2\|_{W^{6,\infty}},$$

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq (1+t)^{-3} C(n, f, K_1, K_2) \|\hat{g}_1\|_{W^{8,\infty}} \|\hat{g}_2\|_{W^{8,\infty}}.$$

Here C is a constant only dependent on the indicated variables.

Proof. Denote by $p(k)$ the unique solution of $f_k'(p) = 0, k \in \mathbb{R}^3$. The function $k \to p(k)$ is C^∞. Since $\det f_k''(p(k)) \neq 0$ for $k \in \mathbb{R}^3$, Theorems 7.7.5 and 7.7.6 of [2] give

$$\left|\int e^{if_k(p)s} u_k(p)dp - e^{if_k(p(k))s} |\det(f_k''(p(k))/2\pi)|^{-1/2} e^{i\sigma\pi/4} s^{-3/2} \sum_{j=0}^{m-1} (L_{f,j,k} u_k)(p(k)) s^{-j}\right|$$
$$\leqq C(m, f) s^{-m} \|u_k\|_{W^{2m,\infty}} s > 0, \tag{2.7}$$

where $u_k(p) = \hat{g}_1(p)\hat{g}_2(k-p)$, σ the signature of f_k'' and $L_{f,j,k}$ differential operators of order $2j$ in p with C^∞ coefficients in p and k. The supports of $\hat{g}_1$ and $\hat{g}_2$ are compact, so the existence of the limit $\hat{h}_t$ in $C_0^0(\mathbb{R}^3)$ as $T \to \infty$ of $\hat{h}_{t,T}$ follows from (2.7). Introduce $F_\alpha(x) = \int_1^\infty ds e^{ixs} s^{-\alpha}$, $\alpha > 1$. According to (2.7)

$$\left|\hat{h}_t(k) - e^{i\sigma\pi/4} |\det(f_k''(p(k))/2\pi)|^{-1/2} \sum_{j=0}^{m-1} (L_{f,j,k} u_k)(p(k)) F_{3/2+j}(f_k(p(k))t) t^{-(j+1/2)}\right|$$
$$\leqq C(m, f)(m-1)^{-1} t^{-(m-1)} \|u_k\|_{W^{2m,\infty}}, \quad t > 0, m \geqq 2. \tag{2.8}$$

The supports of $\hat{g}_1$ and $\hat{g}_2$ are such that $|f_k(p(k))| \geqq C(K_1, K_2) > 0$ for $(k, k-p(k)) \in \operatorname{supp} \hat{g}_1 \times \operatorname{supp} \hat{g}_2 = K_1 \times K_2$. F_α has the asymptotic expansion $\left|F_\alpha(x) - e^{ix} \sum_{l=0}^{m-1} \rho_{\alpha,l}(ix)^{-(l+1)}\right| \leqq C_m |x|^{-(m+1)}$, where $\rho_{\alpha,0} = -1$ and $\rho_{\alpha,l} = -\alpha(\alpha+1)\dots(\alpha+\alpha-1)$ for $l \geqq 1$. This is obtained from $F_\alpha(x) = (ix)^{-1}(-e^{ix} + \alpha F_{\alpha+1}(x))$. Introducing this into (2.8) and keeping on the left-hand side only terms decreasing slower than $t^{-(m-1)}$ one gets:

$$\left|\hat{h}_t(k) - e^{i\sigma\pi/4} |\det(f_k''(p(k))/2\pi)|^{-1/2} e^{if_k(p(k))t} \sum_{\substack{j,l \geqq 0 \\ 3/2+j+l < m-1}} (L_{f,j,k} u_k)(p(k))\right.$$
$$\left.\cdot \rho_{3/2+j,l}(if_k(p(k)))^{-(l+1)} t^{-(3/2+j+l)}\right| \leqq C(m, f, K_1, K_2) t^{-(m-1)} \|u_k\|_{W^{2m,\infty}}, \quad t > 0. \tag{2.9}$$

One notes that if $K = K_1 + K_2 = \{k = p_1 + p_2 | p_1 \in K_1, p_2 \in K_2\}$, then the support of the function $k \to \|u_k\|_{W^{2m,\infty}}$ is contained in K, so $(\int \|u_k\|^2_{W^{2m,\infty}} dk)^{1/2} \leqq C(K_1, K_2) \|\hat{g}_1\|_{W^{2m,\infty}} \|\hat{g}_2\|_{W^{2m,\infty}}$. Taking $m = 3$, (2.9) gives:

$$\|h_t\|_{W^{n,2}} \leqq C(f, K_1, K_2, n) t^{-3/2} \|\hat{g}_1\|_{W^{6,\infty}} \|\hat{g}_2\|_{W^{6,\infty}}, t \geqq 1. \tag{2.10}$$

It is known that $\|v_t * g\|_{W^{n,\infty}} \leqq C|t|^{-3/2} \|g\|_{W^{n,1}}, t \neq 0$ when $\hat{v}_t(k) = e^{i\varepsilon|k|^2 t}$, and that $\|v_t * g\|_{W^{n,\infty}} \leqq C|t|^{-3/2} \|g\|_{W^{n+3,1}}$, $t \neq 0$ when $\hat{v}_t(k) = e^{i\varepsilon\omega_m(k)t}$ (cf. [4]). If $\hat{g}_{l,j}(k) = e^{i\sigma\pi/4}$ $|\det(f_k''((p(k))/2\pi)|^{-1/2}(L_{f,j,k}u_k)(p(k))\rho_l(if_k(p(k)))^{-(l+1)}$, then $\hat{g}_{l,j} \in C_0^\infty(\mathbb{R}^3)$ and $\|g_{l,j}\|_{W^{n+3,1}} \leqq C_{n,K_1,K_2}|\hat{g}_{l,j}\|_{W^{4,\infty}} \leqq C(n, f, K_1, K_2) \|\hat{g}_1\|_{W^{4+2j,\infty}} \|\hat{g}_2\|_{W^{4+2j,\infty}}$. Consequently the terms in the sum appearing in (2.9) give (with $v_t'(k) = v_t(k)e^{if(p(k))t}$)

$$\|v_t' * g_{l,j} t^{-(3/2+j+l)}\|_{W^{n,\infty}} \leqq C(n, f, K_1, K_2) t^{-(3+l+j)} \cdot \|\hat{g}_1\|_{W^{4+2j,\infty}} \|\hat{g}_2\|_{W^{4+2j,\infty}}, \quad t > 0. \tag{2.11}$$

Using (2.9) with $m = 4$ one gets, by summing over j, $l \geqq 0$, $j + l < 3/2$

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq \sum_{j,l} \|v_t' * g_{l,j} t^{-(3/2+j+l)}\|_{W^{n,\infty}} + \|v_t * (h_t - \sum_{j,l} e^{if(p(\,))t} g_{l,j} t^{-(3/2+j+l)})\|_{W^{n,\infty}} \tag{2.12}$$

As $\|v_t * g\|_{L^\infty} \leqq \|\hat{g}\|_{L^1}$, it follows from (2.9) that the last term in (2.12) is smaller than

$$C(n, f, K_1, K_2) t^{-3} \|\hat{g}_1\|_{W^{8,\infty}} \|\hat{g}_2\|_{W^{8,\infty}}, \quad t > 0.$$

Inserting this and (2.11) in (2.12) one gets:

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq C(n, f, K_1, K_2) t^{-3} \|\hat{g}_1\|_{W^{8,\infty}} \|\hat{g}_2\|_{W^{8,\infty}} \quad t > 1. \tag{2.13}$$

The statement of the lemma follows now from (2.10) and (2.13) by observing that $h_t = h_{t,1} + h_1$,

$$\|v_t * h_{t,1}\|_{W^{n,\infty}} \leqq C(n, K_1, K_2) \|\hat{h}_{t,1}\|_{L^1} \leqq C'(n, K_1, K_2) \|\hat{h}_{t,1}\|_{L^\infty}$$

$$\|h_{t,1}\|_{W^{n,2}} \leqq C(n, K_1, K_2) \|\hat{h}_{t,1}\|_{L^\infty} \text{ and } \|\hat{h}_{t,1}\|_{L^\infty} \leqq C(n, K_1, K_2) \|\hat{g}_1\|_{L^\infty} \|\hat{g}_2\|_{L^\infty}$$

for $0 \leqq t \leqq 1$. (Q.E.D)

If $\varepsilon_1 m_1 + \varepsilon_2 m_2 = 0$, then the limit when $T \to \infty$ of $\hat{h}_{t,T}$ is no longer in $C_0^0(\mathbb{R}^3)$ in general. As in this case, the zeros of $f_k'(p) = 0$ are exactly $k = 0$, $p \in \mathbb{R}^3$ and $f_k(p) \neq 0$ for k, $p \in \mathbb{R}^3$, a bounded singularity appears in general at $k = 0$. However the decay properties of h_t in L^2 and L^∞ are the same as in lemma 2.1.

Lemma 2.2. *Let $\varepsilon_0, \varepsilon = \pm 1$, $m_0, m > 0, \hat{g}_1, \hat{g}_2 \in C_0^\infty(\mathbb{R}^3)$ and let $f_k(p) = -\varepsilon\omega_m(k) + \varepsilon_0(\omega_{m_0}(p) - \omega_{m_0}(p-k))$. If $\hat{v}_t(k) = e^{i\varepsilon\omega_m(k)t}$ and $\hat{h}_{t,T}(k) = \int_t^T ds \int_{\mathbb{R}^3} dp\, e^{if_k(p)s}$ $\hat{g}_1(p)\hat{g}_2(k-p)$, then for $n = 0, 1\ldots$ $h_{t,T}$ has a limit h_t in $W^{n,2}$ as $T \to \infty$ and if* supp $\hat{g}_i = K_i$ *then* supp $\hat{h}_{t,T} \subset K_1 + K_2$ *and*

$$\|h_t\|_{W^{n,2}} \leqq C(n, K_1, K_2) \|\hat{g}_1\|_{W^{4,\infty}} \|\hat{g}_2\|_{W^{4,\infty}} (1+t)^{-3/2}, \quad t \geqq 0,$$

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq C(n, K_1, K_2) \|\hat{g}_1\|_{W^{4,\infty}} \|\hat{g}_2\|_{W^{4,\infty}} (1+t)^{-3}, \quad t \geqq 0.$$

The proof of this lemma is so similar to that of Lemma 5.1 in [8], that we omit it.

We now consider the cases where the zeros of $f_k(p)$ and $f_k'(p)$ coincide. Decay properties of $\|h_t\|_{L^2}$ and $\|h_t\|_{L^\infty}$ obtained in Lemmas 2.1 and 2.2. are then no more valid.

Lemma 2.3. *Let $\hat{g}_1, \hat{g}_2 \in C_0^\infty(\mathbb{R}^3)$, $\varepsilon, \varepsilon_0 = \pm 1$, $\hat{v}_t(k) = e^{i\varepsilon|k|^2 t}$, and let $f_k(p) = -\varepsilon|k|^2$*

$+\varepsilon_0(|p|^2-|k-p|^2)$. *If*

$$\hat{h}_{t,T}(k)=\int_t^T ds\int dp\, e^{if_k(p)s}\hat{g}_1(p)\hat{g}_2(k-p),\quad 0\leqq t\leqq T$$

and if supp $\hat{g}_i=K_i$, *then* supp $\hat{h}_{t,T}\subset K_1+K_2$, $h_{t,T}$ *has a limit* h_t in $W^{n,2}$, $n=0,1,\ldots$, *as* $T\to\infty$ *and*

$$\|h_t\|_{W^{n,2}}\leqq C(n,K_1,K_2)(1+t)^{-1/2}\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}},\quad t\geqq 0,$$

$$\|v_t*h_t\|_{W^{n,\infty}}\leqq C(n,K_1,K_2)(1+t)^{-2}\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}},\quad t\geqq 0.$$

Proof. $f_k(p)=-(\varepsilon+\varepsilon_0)|k|^2+2\varepsilon_0\, k\cdot p$, so after a change of variables $p\to k-p$ and (or) $p\to -p$ we have only to consider the case $\varepsilon=-\varepsilon_0=-1$. Then $f_k(p)=2k\cdot p$. Let $\varphi\in C_0^\infty(\mathbb{R}^3)$ be equal to one in a neighbourhood of K_1+K_2 and let $\sum_{0\leqq m\leqq 5} v_m(p;k)+r(p,k)$, where $v_m(p;k)$ is a monomial of degree m in k, be the Taylor development of $k\to\hat{g}_1(p)\hat{g}_2(k-p)$ at $k=0$, with a remaining term $k\to r(p,k)$ having a zero of order 6 at $k=0$. Theorem 7.7.1 of [2] gives

$$\left|\int e^{2ik\, ps}r(p,k)\varphi(k)dp\right|\leqq Cs^{-3}\sum_{|\alpha|\leqq 3}\sup_p\left|\frac{\partial^{|\alpha|}}{\partial p^\alpha}r(p,k)\right||k|^{|\alpha|-6}|\varphi(k)|,\quad s>0.$$

So for $0<t\leqq T$

$$\left|\int_t^T ds\int dp\, e^{2isk\, p}r(p,k)\varphi(k)dp\right|\leqq C(K_1,K_2)|\varphi(k)|\,\|\hat{g}_1\|_{W^{9,\infty}}\cdot\|\hat{g}_2\|_{W^{9,\infty}}\left(\frac{1}{t^2}-\frac{1}{T^2}\right).\qquad(2.14)$$

Denoting $F_m(k)=\int e^{2ik\, p}v_m(p;k)$, then $F_m\in S(\mathbb{R}^3)$ and

$$\int e^{2ik\, ps}v_m(p;k)dp=s^{-m}F_m(ks),\quad s>0.$$

The L^q-norm $1\leqq q\leqq\infty$ of the function $k\to s^{-m}F_m(sk)$ is $s^{-(3/q+m)}\|F_m\|_{L^q}$ and $\|F_m\|_{L^q}\leqq C(K_1,K_2)\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}}$, $0\leqq m\leqq 5$, which gives

$$\left\{\int dk\left|\int_t^T ds\int dp\, e^{2isk\, p}v_m(p;k)\varphi(k)\right|^q\right\}^{1/q}\leqq\|\varphi\|_{L^\infty}C(K_1,K_2)\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}}\cdot(t^{-((3-q)/q+m)}-T^{-((3-q)/q+m)}),\ 0\leqq m\leqq 5,\quad 0<t\leqq T, 1\leqq q<\infty.\qquad(2.15)$$

Since $\varphi(k)\left(\sum_{0\leqq m\leqq 5}v_m(p;k)+r(p,k)\right)=\hat{g}_1(p)\hat{g}_2(k-p)$, it follows from (2.14) and (2.15), that

$$\|h_{t,T}\|_{W^{n,2}}\leqq C(n,K_1,K_2)(t^{-1/2}-T^{-1/2})\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}},\quad 1\leqq t\leqq T, n\in\mathbb{N}.\qquad(2.16)$$

$\|h_{t,1}\|_{W^{n,2}}\leqq C(n,K_1,K_2)\|\hat{g}_1\|_{L^\infty}\|\hat{g}_2\|_{L^\infty}$, $0\leqq t\leqq 1$, (2.16) and the definition of $h_{t,T}$ show that the limit h_t exists in $W^{n,2}$ and that

$$\|h_t\|_{W^{n,2}}\leqq C(n,K_1,K_2)\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}}(1+t)^{-1/2},\quad t\geqq 0.$$

Similarly it follows that

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq \|(1+|\,|^2)^{n/2}\hat{h}_t\|_{L^1} \leqq C(n,K_1,K_2)\|\hat{g}_1\|_{W^{14,\infty}}\|\hat{g}_2\|_{W^{14,\infty}}(1+t)^{-2},$$
$$n \geqq 0, \quad \text{as} \quad \|f\|_{L^\infty} \leqq \|\hat{f}\|_{L^1}. \qquad \text{Q.E.D.}$$

Lemma 2.4. *Let* $\varepsilon, \varepsilon_1, \varepsilon_2 = \pm 1$, $m, m_1, m_2 > 0$, $\varepsilon_1 m_1 + \varepsilon_2 m_2 \neq 0$ *and* $\varepsilon m = \varepsilon_1 m_1 + \varepsilon_2 m_2$. *Let* $\hat{v}_t(k) = e^{i\varepsilon\omega_m(k)t}, f_k(p) = -\varepsilon\omega_m(k) + \varepsilon_1\omega_{m_1}(p) + \varepsilon_2\omega_{m_2}(p-k)$, *and let* $\hat{g}_1, \hat{g}_2 \in C_0^\infty(\mathbb{R}^3)$. *If*

$$\hat{h}_{t,T}(k) = \int_t^T ds \int_{\mathbb{R}^3} dp\, e^{if_k(p)s}\hat{g}_1(p)\hat{g}_2(k-p), \quad 0 \leqq t \leqq T,$$

and if $\operatorname{supp}\hat{g}_i = K_i$, *then* $\operatorname{supp}\hat{h}_{t,T} \subset K_1 + K_2$, $\hat{h}_{t,T}$ *has a limit* $\hat{h}_t$ *in* C_0^0 *as* $T \to \infty$ *and*

$$\|h_t\|_{W^{n,2}} \leqq C(n,K_1,K_2)(1+t)^{-1/2}\|\hat{g}_1\|_{W^{4,\infty}}\|\hat{g}_2\|_{W^{4,\infty}}, \tag{2.17}$$

$$\|v_t * h_t\|_{W^{n,\infty}} \leqq C(n,K_1,K_2)(1+t)^{-2}\|\hat{g}_1\|_{W^{8,\infty}}\|\hat{g}_2\|_{W^{8,\infty}}. \tag{2.18}$$

Proof. For $k \in \mathbb{R}^3$, let $p(k)$ be the unique solution of $f_k'(p) = 0$. Then, by (2.5), det $f_k''(p(k)) \neq 0$, therefore, formulas (2.7) and (2.8) are valid. Formula (2.7) gives at once that the limit $\hat{h}_t$ exists in C_0^0. By (2.4), $f_k(p(k)) = 0$, so formula (2.8) reads

$$|\hat{h}_t(k) - e^{i\sigma\pi/4}|\det(f_k''(p(k))/2\pi|^{-1/2}\sum_{j=0}^{m-1}(L_{f,j,k}u_k)(p(k))F_{3/2+j}(0)t^{-(j+1/2)}|$$
$$\leqq C_m(m-1)t^{-(m-1)}\|u_k\|_{W^{2m,\infty}}, t > 0, \quad m \geqq 2. \tag{2.19}$$

If $\hat{l}_j(k)$ is the coefficient in front of $t^{-(j+1/2)}$ in (2.19), then $\hat{l}_j \in C_0^\infty(\mathbb{R}^3)$. Inequality (2.17) follows now from (2.19) with $m = 2$ as the support of $k \to \|u_k\|_{W^{4,\infty}}$ is in $K_1 + K_2$. Taking $m = 3$, one gets:

$$\|v_t * h_t\|_{L^\infty} \leqq \|v_t * (l_0 t^{-1/2} + l_1 t^{-3/2})\|_{L^\infty} + \|v_t * (h_t - l_0 t^{-1/2} - l_1 t^{-3/2})\|_{L^\infty}.$$

As $\|v_t * g\|_{L^\infty} \leqq Ct^{-3/2}\|g\|_{W^{3,1}}$, $t > 0$ and $\|v_t * g\|_{L^\infty} \leqq \|\hat{g}\|_{L^1}$, this gives

$$\|v_t * h_t\|_{L^\infty} \leqq C(K_1,K_2)t^{-3/2}(\|\hat{l}_0\|_{W^{4,\infty}}t^{-1/2} + \|\hat{l}_1\|_{W^{4,\infty}}t^{-3/2})$$
$$+ \|\hat{h}_t - \hat{l}_0 t^{-1/2} - \hat{l}_1 t^{-3/2}\|_{L^1}.$$

Inserting the definition of $\hat{l}_j$ and (2.19) in the last expression, one obtains (2.18) for $n = 0$. For $n \neq 0$, (2.18) follows now from $\operatorname{supp}\hat{h}_t \subset K_1 + K_2$. Q.E.D.

Next, we consider decay properties of third order terms. Sufficiently good decay properties at third order, for the iteration scheme in Sect. 3 to work, are obtained in most cases by combining the decay properties at second order in Lemmas 2.1 and 2.2 with the $t^{-3/2}$ decay of the L^∞ norm of free solutions. Lemmas 2.5 and 2.6 will treat the cases to which this does not apply.

Lemma 2.5. *For* $\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2 = \pm 1, s_1, s_2 \in \mathbb{R}$, $k, k_1, k_2 \in \mathbb{R}^3$ *let* $f_k(k_1,k_2,s_1,s_2) = (-\varepsilon|k|^2 + \varepsilon_1|k_1|^2 + \varepsilon_0|k-k_1|^2)s_1 + (-\varepsilon_0|k-k_1|^2 + \varepsilon_2|k-k_1-k_2|^2)s_2$. *For* $1 \leqq i \leqq 3$, *let* $\hat{g}_i \in C_0^\infty(\mathbb{R}^3 - \{0\})$, *and let* $K_i = \operatorname{supp}\hat{g}_i, K = K_1 + K_2 + K_3$. *If* $\hat{H}_{t,T_1,T_2}(k) = \int_t^{T_1} ds_1 \int_{s_1}^{T_2} ds_2 \int dk_1 dk_2 e^{if_k(k_1,k_2,s_1,s_2)}\hat{g}_1(k_1)\hat{g}_2(k_2)\hat{g}_3(k-k_1-k_2)$, $0 \leqq t \leqq$

$T_1 \leqq T_2$, then $\operatorname{supp} \hat{H}_{t,T_1,T_2} \subset K, \hat{H}_{t,T_1,T_2}$ *has a limit* $\hat{H}_t$ *in* $C_0^0(\mathbb{R}^3)$ *as* $T_1, T_2 \to \infty$ *and*

$$\|H_t\|_{L^2} \leqq C(K)(1+t)^{-1} \|\hat{g}_1\|_{W^{6,\infty}} \|\hat{g}_2\|_{W^{6,\infty}} \|\hat{g}_3\|_{W^{6,\infty}}, \quad t \geqq 0,$$

$$\|v_t * H_t\|_{L^\infty} \leqq C(K_1, K_2, K_3)(1+t)^{-5/2}, \quad t \geqq 0,$$
$$\|\hat{g}_1\|_{W^{10,\infty}} \|\hat{g}_2\|_{W^{10,\infty}} \|\hat{g}_3\|_{W^{10,\infty}},$$

where $\hat{v}_t(k) = e^{i\varepsilon|k|^2 t}$.

Proof. Introduce $x = s_1/s_2$, $y = s_2$ and $D_t = \{(x,y) | 0 \leqq x \leqq 1, \ t \leqq xy\}$. If $h_k(k_1,k_2,x) = (-\varepsilon|k|^2 + \varepsilon_1|k_1|^2 + \varepsilon_0|k-k_1|^2)x + (-\varepsilon_0|k-k_1|^2 + \varepsilon_2|k_2|^2 - \varepsilon_2|k-k_1-k_2|^2)$, then $f_k(k_1,k_2,s_1,s_2) = h_k(k_1,k_2,x)y$ and $h_k'(k_1,k_2,x) = 0$ (h' is the derivative with respect to k_1 and k_2) has only the solution $k_1 = k$, $k_2 = x\varepsilon_1\varepsilon_2 k$. We also have that $\det(h_k''(k, x\varepsilon_1\varepsilon_2 k, x)) = 2^6$ and $h_k(k, x\varepsilon_1\varepsilon_2 k, x) = x(\varepsilon_1 - \varepsilon)|k|^2$. If $u_k(k_1,k_2) = \hat{g}_1(k_1)\hat{g}_2(k_2)\hat{g}_3(k-k_1-k_2)$, then by Theorem 7.7.5 of [2],

$$\Big|\int dk_1 dk_2 e^{ih_k(k_1,k_2,x)y} u_k(k_1,k_2,) - \pi^3 e^{i\pi\sigma/4} e^{ix(\varepsilon_1-\varepsilon)|k|^2 y} y^{-3} \sum_{0\leqq j\leqq m-4} y^{-j}$$
$$\cdot (L_{j,k,x}u_k)(k, x\varepsilon_1\varepsilon_2 k)\Big| \leqq C(K) y^{-m} \|u_k\|_{W^{2m,\infty}}, \quad y > 0, m \geqq 1, \tag{2.20}$$

where $L_{j,k,x}$ is a differential operator of order $2j$ with coefficients C^∞ in all variables.

The existence of the limit $\hat{H}_t$ in $C_0^0(\mathbb{R}^3)$ follows (2.20) with $m = 3$, and we have

$$\hat{H}_t(k) = \int_{D_t} dx dy y \int dk_1 dk_2 e^{ih_k(k_1,k_2,x)y} u_k(k_1,k_2). \tag{2.21}$$

The decay property of $\|H_t\|_{L^2}$, follows now from (2.21) and (2.20) with $m = 3$.

$$\text{Let } v_j(k,z,y) = \int_{y^{-1}}^{1} dx \pi^3 e^{i\pi\sigma/4} e^{ixz} (L_{j,k,x}u_k)(k, x\varepsilon_1\varepsilon_2 k). \tag{2.22}$$

By hypothesis there is $\delta > 0$ such that $K_i \subset \{k \in \mathbb{R}^3 | \delta \leqq |k| \leqq 1/\delta\}, i = 1,2,3$. The support of the function $k \to u_k(k, x\varepsilon_1\varepsilon_2 k)$ is contained in $\{k \in \mathbb{R}^3 | x^{-1}\delta \leqq |k| \leqq \delta^{-1}\}$, which is empty if $x < \delta^2$. Hence $v_j(k,z,y) = v_j(k,z,\delta^{-2})$ if $y \geqq \delta^{-2}$. Put $v_j(k,z,y) = v_j(k,z)$ if $y \geqq \delta^{-2}$. By (2.20), (2.21) and (2.22) we then have

$$\Big|\hat{H}_t(k) - \sum_{0\leqq j\leqq m-4} \int_t^\infty dy v_j(k, (\varepsilon_1 - \varepsilon)|k|^2 y) y^{-(2+j)}\Big|$$
$$\leqq C(K)((m-2)t^{m-2})^{-1} \|u_k\|_{W^{2m,\infty}}, \quad t \geqq \delta^{-2}, m \geqq 3. \tag{2.23}$$

Let $\varepsilon_1 - \varepsilon \neq 0$ and put $\hat{v}_{j,l}(k) = -\pi^3 e^{i\pi\sigma/4}(i(\varepsilon - \varepsilon_1)|k|^2)^{-(l+1)} \partial^l/\partial x^l \times (L_{j,k,x}, u_k(k, \varepsilon_1\varepsilon_2 kx)) | x = 1$, $l, j \geqq 0$. The support of $\hat{v}_{j,l}$ and $k \to v_j(k, (\varepsilon_1 - \varepsilon)|k|^2 y)$ is contained in $K - \{|k| < \delta\}$. Partial integration gives the asymptotic expansion

$$\Big|v_j(k, (\varepsilon_1 - \varepsilon)|k|^2 y) - \sum_{l=0}^{M-1} e^{i(\varepsilon_1-\varepsilon)|k|^2 y} \hat{v}_{j,l}(k) y^{-(l+1)}\Big|$$
$$\leqq C(K,\delta) y^{-M} \|u_k\|_{W^{2j+M,\infty}}, \quad y > 0, M \geqq 1 \tag{2.24}$$

Inequality (2.24) gives after integration in y and introduction of the asymptotic expansion of $F_\alpha(z)=\int\limits_1^\infty e^{izy}y^{-\alpha}dy$, $\alpha>1$(see after (2.8))

$$\left|\int\limits_t^\infty v_j(k,(\varepsilon_1-\varepsilon)|k|^2y)y^{-(2+j)}dy-\sum_{\substack{n,l\geqq 0\\3+j+l+n<M}}\hat v_{j,l}(k)\rho_{3+j+l,n}e^{i(\varepsilon_1-\varepsilon)|k|^2t}\cdot(i(\varepsilon_1-\varepsilon)|k|^2)^{-n}t^{-(2+j+l+n)}\right|\leqq C(\delta,M)t^{-(M+1)}\|u_k\|_{W^{2j+M,\infty}},\quad t>0. \tag{2.25}$$

Inequalities (2.23) and (2.25) show that

$$|\hat H_t(k)-e^{i(\varepsilon_1-\varepsilon)|k|^2t}\sum_{2\leqq j\leqq m-1}\hat h_j(k)t^{-j}|\leqq C(m,\delta)t^{-m}\|u_k\|_{W^{2m+4,\infty}},t\geqq\delta^{-2},\quad \varepsilon_1-\varepsilon\neq 0,$$

where $\hat h_j(k)=\sum\hat v_{r,l}(k)\rho_{3+r+l,n}(i(\varepsilon_1-\varepsilon)|k|^2)^{-n}$ and the last sum is over $j=2+r+l+n$. When $\varepsilon_1=\varepsilon$, it follows directly from (2.23) that

$$|\hat H_t(k)-\sum_{1\leqq j\leqq m-1}\hat h_j(k)t^{-j}|\leqq C(m,K)t^{-m}\|u_k\|_{W^{2m+4,\infty}},\quad t\geqq\delta^{-2},\varepsilon=\varepsilon_1; \tag{2.27}$$

where $\hat h_j(k)=(j+1)^{-1}v_j(k,0)$. Define $\hat h_1=0$, $\hat\mu_t(k)=e^{i\varepsilon_1|k|^2t}$ if $\varepsilon_1-\varepsilon\neq 0$ and $\hat\mu_t(k)=e^{i\varepsilon|k|^2t}$ if $\varepsilon_1-\varepsilon=0$. Equalities (2.26) and (2.27) with $m=3$ give for $\varepsilon_1=\pm\varepsilon$,

$$\begin{aligned}\|v_t*H_t-\sum_{j=1}^2\mu_t*h_jt^{-j}\|_{L^\infty}&\leqq\|\hat v_t\hat H_t-\sum_{j=1}^2\hat\mu_t\hat h_jt^{-j}\|_{L^1}\\&\leqq C(m,\delta)t^{-3}\prod_{i=1}^3\|\hat g_i\|_{W^{10,\infty}},\quad t\geqq\delta^{-2}.\end{aligned} \tag{2.28}$$

Inequalities $\|\mu_t*h_jt^{-j}\|_{L^\infty}\leqq Ct^{-(j+3/2)}\|h_j\|_{L^1}$ and $\|h_j\|_{L^1}\leqq C(K)\|\hat h_j\|_{W^{4,\infty}}\leqq C(\delta)\prod_{i=1}^3\times\|\hat g_i\|_{W^{2j+4,\infty}}$ give together with (2.28) the announced decay of $\|v_t*H_t\|_{L^\infty}$ for $t\geqq\delta^{-2}$. Moreover $\|v_t*H_t\|_{L^\infty}\leqq C\|v_t*H_t\|_{W^{2,2}}=C\|H_t\|_{W^{2,2}}\leqq C(\delta)\prod_{i=1}^3\|\hat g_i\|_{W^{6,\infty}}$ for $0\leqq t\leqq\delta^{-2}$, by the already proved decay property of $\|H_t\|_{L^2}$. The δ-dependence in the constants can finally be replaced by dependence on K_1,K_2,K_3 by taking the smallest δ satisfying its definition. Q.E.D.

Remark 2.1. It follows from the proof that the decay actually is $\|H_t\|_{L^2}\leqq C(1+t)^{-2}$ and $\|v_t*H_t\|_{L^\infty}\leqq C(1+t)^{-7/2}$ if $\varepsilon\neq\varepsilon_1$. The following lemma could have been formulated and proved in a way similar to that of Lemma 2.5.

Lemma 2.6. *Let* E, $E_0=\pm 1$, $M,M_0>0$, $\hat g_0\in C_0^\infty(\mathbb{R}^3)$, supp $\hat g_0=K_0$ *and let* $\hat\mu_t(k)=e^{iE\omega_M(k)t}$. *In the situation of Lemma* 2.4, *let*

$$\hat H_{t,T}(k)=\int\limits_t^T dse^{-iE\omega_M(k)s}\int dpe^{iE_0\omega_{M_0}(p)s}\hat g_0(p)(\hat v_s\hat h_s)(k-p).$$

Then supp $\hat H_{t,T}\subset K=K_0+K_1+K_2$, $\hat H_{t,T}$ *has a limit* $\hat H_t$ in L^2 as $T\to\infty$ *and*

$$\|H_t\|_{L^2}\leqq C(K)(1+t)^{-1}\prod_{i=1}^3\|\hat g_i\|_{W^{4,\infty}},\quad t\geqq 0$$

$$\|\mu_t*H_t\|_{L^\infty}\leqq C(K)(1+t)^{-5/2}\prod_{i=1}^3\|\hat g_i\|_{W^{8,\infty}},\quad t\geqq 0.$$

Proof. By Lemma 2.4, $\hat{h}_s \in C_0^0$, and supp $\hat{h}_s \in K_1 + K_2$. This shows that $\hat{H}_{t,T} \in C_0^\infty$, that supp $\hat{H}_{t,T} \subset K_0 + K_1 + K_2$ and that $\|\mu_t * H_{t,1}\|_{L^\infty}$ and $\|H_{t,1}\|_{L^2}$ are smaller than $C(K)\|\hat{g}_0\|_{L^\infty} \sup_{0 \leqq s \leqq 1} \|\hat{h}_s\|_{L^2}$ for $0 \leqq t \leqq 1$. By (2.17), one than get that $\|H_{t,1}\|_{L^2}$ and $\|\mu_t * H_{t,1}\|_{L^\infty}$ are bounded by

$$C(K)\|\hat{g}_0\|_{L^\infty}\|\hat{g}_1\|_{W^{4,\infty}}\|\hat{g}_2\|_{W^{4,\infty}}, \quad 0 \leqq t \leqq 1. \tag{2.29}$$

Put $v_t(p) = -E\omega_M(k) + E_0\omega_{M_0}(p) + \varepsilon\omega_m(p-k)$. It follows from (2.19) that

$$\left| \int_t^\infty ds \int dp\, e^{iv_k(p)s} \hat{g}_0(p) \left(\hat{h}_s(k-p) - \sum_{j=0}^{m-1} \hat{l}_j(k-p) s^{-(j+1/2)} \right) \right| \tag{2.30}$$

$$\leqq C(m,K) t^{-(m-2)} \sup_{p \in \mathbb{R}^3} (|\hat{g}_0(p)| \, \|u_{k-p}\|_{W^{2m,\infty}}), \quad m \geqq 3, \quad t > 0,$$

where $u_k(p) = \hat{g}_1(p)\hat{g}_2(k-p)$.

First let $E_0M_0 + \varepsilon m = 0$. Then there is $\lambda > 0$, such that $|v_k(p)| \geqq \lambda$, for $k, p \in \mathbb{R}^3$. It follows from (2.30) with $m = 5$, the definition of $\hat{l}_j$ and the asymptotic expansion of F_α after (2.8) that

$$\left| \int_t^\infty ds \int dp\, e^{iv_k(p)s} \hat{g}_0(p) \hat{h}_s(k-p) - \int dp\, e^{iv_k(p)t} \sum_{j,n} \rho_{(1/2+j),n} \hat{g}_0(p) \hat{l}_j(k-p) \right.$$

$$\left. \cdot (iv_k(p))^{-(l+1)} t^{-(1/2+j+l)} \right| \leqq C(K) t^{-3} \|v_k\|_{W^{8,\infty}}, \quad t \geqq 1, \tag{2.31}$$

where the sum is taken over $1/2 + j + l < 3$ and $v_k(k_1, k_2) = \hat{g}_0(k_1)\hat{g}_1(k_2) \times \hat{g}_3(k - k_1 - k_2)$. It follows now as in the proof of Lemma 5.1 of [8] that if $\hat{q}_{l,j,t}(k) = \int dp\, \hat{g}_0(p) \hat{l}_j(k-p) e^{iv_k(p)t} (iv_k(p))^{-(l+1)}$, then

$$\|q_{l,j,t}\|_{L^2} \leqq C(K,l)(1+t)^{-3/2} \|\hat{g}_0\|_{W^{4,\infty}} \|\hat{l}_j\|_{W^{4,\infty}},$$

and

$$\|\mu_t * q_{l,j,t}\|_{L^\infty} \leqq C(K,l)(1+t)^{-3} \|\hat{g}_0\|_{W^{4,\infty}} \|\hat{l}_j\|_{W^{4,\infty}}.$$

As $\|\hat{l}_j\|_{W^{4,\infty}} \leqq C(K)\|\hat{g}_1\|_{W^{2j+4,\infty}}\|\hat{g}_2\|_{W^{2j+4,\infty}}$, the two last inequalities prove, together with (2.31), the existence of the limit H_t in L^2 and the decay properties in the case $E_0M_0 + \varepsilon m = 0$.

Second, if $E_0M_0 + \varepsilon m \neq 0$, then $\int dp\, e^{iv_k(p)s} \hat{g}_0(p) \hat{l}_j(k-p) = R_{j,s}(k)$ can be estimated by formula (2.7), which leads to an asymptotic expansion of the type (2.8) of $\int_t^\infty R_{j,s}(k) s^{-(j+1/2)} ds$. If $EM \neq E_0M_0 + \varepsilon m$, one then proceeds as in the proof of Lemma 2.1, and if $EM = E_0M_0 + \varepsilon m$, one proceeds as in the proof of Lemma 2.4, to determine the decay of the terms $\|r_{j,t}\|_{L^2}$ and $\|\mu_t * r_{j,t}\|_{L^\infty}$, where $\hat{r}_{j,t}(k) = \int_t^\infty R_{j,s} s^{-(j+1/2)} ds$. Substitution of these results into (2.30) gives then the announced results. Q.E.D.

The following lemma is well-known and is a trivial consequence of decay properties of free solutions.

Lemma 2.7. *Let* $\varepsilon_0, \varepsilon_i = \pm 1$, $m_0, m_i > 0$, $\hat{g}_i \in C_0^\infty$, supp $\hat{g}_i = K_i$, $1 \leqq i \leqq 3$. *Moreover let* $f_k(k_1, k_2) = \varepsilon_0 \omega_{m_0}(k) - \varepsilon_1 \omega_{m_1}(k_1) - \varepsilon_2 \omega_{m_2}(k_2) - \varepsilon_3 \omega_{m_3}(k_1 + k_2 - k)$, *or let* $f_k(k_1, k_2) = \varepsilon_0 |k|^2 - \varepsilon_1 |k_1|^2 - \varepsilon_2 |k_2|^2 - \varepsilon_3 |k_1 + k_2 - k|^2$. *If*

$$\hat{H}_{t,T}(k) = \int_t^T ds \int dk_1 dk_2 e^{i f_k(k_1,k_2)s} \hat{g}_1(k_1) \hat{g}_2(k_2) \hat{g}_3(k - k_1 - k_2),$$

then $H_{t,T}$ *has a limit* H_t in L^2 *when* $T \to \infty$, supp $\hat{H}_{t,T} \subset K = K_1 + K_2 + K_3$ *and* $\|H_t\|_{L^2} \leqq C(K)(1+t)^{-2} \prod_{i=1}^{3} \|\hat{g}_i\|_{W^{4,\infty}}$, *for* $t \geqq 0$.

Proof. If $\hat{v}_{j,t}(k) = e^{i\varepsilon_j \omega_{m_j}(k)t}$ (respectively $\hat{v}_{j,t}(k) = e^{i\varepsilon_j |k|^2 t}$), then

$$H_{t,T} = \int_t^T ds v_{0,-s} * \left(\prod_{i=1}^{3} v_{i,s} * g_i \right) \cdot \text{As } \| v_{i,t} g_i \|_{W^{n,\infty}} \leqq C(n, K_i) \| g_i \|_{L^1} |t|^{-3/2}, \text{ one gets}$$

$$\| v_{i,t} * g_i \|_{W^{n,\infty}} \leqq C(n, K_i) \| \hat{g}_i \|_{W^{4,\infty}} (1 + |t|)^{-3/2}.$$

The lemma follows now from

$$\| H_{t,T} \|_{L^2} \leqq \int_t^T ds \| v_{1,s} * g_1 \|_{L^\infty} \| v_{2,s} * g_2 \|_{L^\infty} \| g_3 \|_{L^2} (1+s)^{-1}. \qquad \text{Q.E.D.}$$

Lemmas 2.1 to 2.7 lead immediately to the existence and decay properties of the wave operators, at second and third order. We note that by Eq. (1″)

$$\Omega^2 \circ V_t = - \int_t^\infty ds V_{t-s} J^2 \circ V_s,$$

$$\Omega^3 \circ V_t = - \int_t^\infty ds_1 V_{t-s_1} \left(J^3 \circ V_{s_1} - \int_{s_1}^\infty ds_2 V_{s_1 - s_2} J^2 (I \otimes J^2 + J^2 \otimes I) \circ V_{s_2} \right). \tag{2.32}$$

Proposition 2.8. *Let* V_t, $t \geqq 0$, *be defined as before and* $v_1, v_2, v_3 \in D(a)$, $a \geqq 1$. *The strong Riemann integrals* $\Omega^2(v_1 \otimes v_2)$, $\Omega^3(v_1 \otimes v_2 \otimes v_3)$ *exist then in* $\mathscr{H}$, $\Omega^i \in L_i(D(a), \mathscr{H})$, $i = 2, 3$, supp $\Omega^2(v_1 \otimes v_2)^\wedge \subset K$, supp $\Omega^3(v_1 \otimes v_2 \otimes v_3)^\wedge \subset 3K$, *where* $K = \{x \in \mathbb{R}^3 | \, |x| \leqq a\}$, *and for* $t \geqq 0$,

$$\| \Omega^2 (V_t v_1) \otimes (V_t v_2)) \|_{L^2} \leqq C(a)(1+t)^{-1/2} \| v_1 \|_{D(a)} \| v_2 \|_{D(a)},$$

$$\| \Omega^3 ((V_t v_1) \otimes (V_t v_2) \otimes (V_t v_3)) \|_{L^2} \leqq C(a)(1+t)^{-1} \| v_1 \|_{D(a)} \| v_2 \|_{D(a)} \| v_3 \|_{D(a)},$$

$$\| \Omega^2 ((V_t v_1) \otimes (V_t v_2)) \|_{L^\infty} \leqq C(a)(1+t)^{-2} \| v_1 \|_{D(a)} \| v_2 \|_{D(a)},$$

$$\| \Omega^3 ((V_t v_1) \otimes (V_t v_2) \otimes (V_t v_3)) \|_{L^\infty} \leqq C(a)(1+t)^{-2} \| v_1 \|_{D(a)} \| v_2 \|_{D(a)} \| v_3 \|_{D(a)}.$$

Proof. The Fourier transform $(V_{-t} \Omega_2((V_t v_1) \otimes (V_t v_2)))^\wedge$ is just a sum of terms $\hat{h}$ (with C^∞ coefficients in the case of NLKG) defined in Lemmas 2.1–2.4. In the case of Ω_3 the analog is true, by Lemmas 2.5–2.7. The lemma follows now, if one use, in the case of the terms figuring in Lemma 2.7, that $\| V_t f \|_{L^\infty} \leqq C(a) \; \| f \|_{L^2}$, $f \in D(a)$.

Q.E.D.

3. Construction of Wave Operators

Let $(\tau_b f)(x) = f(x+b)$, $x, b \in \mathbb{R}^3$. Introduce the space $\mathscr{B}(a)$, $a \geqq 1$ of formal power series $A = \sum_{n \geqq 2} A^n$, $A^n \in L_n(D(a), \mathscr{H})$, $\tau_b A^n = A^n(\otimes^n \tau_b)$. Let $\mathscr{B}_\lambda(a)$, $\lambda > 0$ be the Banach space of elements $A \in \mathscr{B}(a)$ for which the norm $p_\lambda(A) = \sum_{n \geqq 2} \lambda^n \, |A^n|_n < \infty$, where

$$|A^2|_2 = \sup_{t \geqq 0} \left((1+t)^{1/2} \left(\|A^2 \circ V_t\|_{L_2(D(a),\mathscr{H})} + (1+t)^{3/2} \|A^2 \circ V_t\|_{L_2(D(a),L^\infty)}\right)\right), \tag{3.1}$$

$$|A^3|_3 = \sup_{t \geqq 0} \left((1+t) \left(\|A^3 \circ V_t\|_{L_3(D(a),\mathscr{H})} + (1+t) \|A^3 \circ V_t\|_{L_3(D(a),L^\infty)}\right)\right), \tag{3.2}$$

and

$$|A^n|_n = \sup_{t \geqq 0} \left((1+t)^{1+n/8} \|A^n \circ V_t\|_{L_n(D(a),\mathscr{H})}\right), \quad n \geqq 4. \tag{3.3}$$

In this paragraph, we shall show that the equation

$$B = -\int_0^\infty V_{-s} J \circ (I + \Omega^2 + \Omega^3 + B) \circ V_s ds - \Omega^2 - \Omega^3 \tag{3.4}$$

has a unique solution $B \in \mathscr{B}_\lambda(a)$, for some $\lambda > 0$; i.e. there is a unique wave operator $\Omega = I + \Omega^2 + \Omega^3 + B$ satisfying Eq. (1′). Let $\bar{\mathscr{B}}(a)$ be the subspace of elements $A \in \mathscr{B}(a)$, with $A^2 = A^3 = 0$, and let $\bar{\mathscr{B}}_\lambda(a) = \mathscr{B}_\lambda(a) \cap \bar{\mathscr{B}}(a)$. Introduce

$$N_T(B) = -\int_0^T V_{-s}(J \circ (I + \Omega^2 + \Omega^3 + B) - J \circ (I + \Omega^2 + \Omega^3)) \circ V_s ds, \tag{3.6}$$

for $T \geqq 0$ and $B \in \bar{\mathscr{B}}_\lambda(a)$ and denote by $N(B) = N_\infty(B)$ and $N_T(B)^n$ (respectively $N(B)^n$) the n-homogeneous part of $N_T(B)$ (respectively $N(B)$). We note that $N_T(B)^n = 0$ for $n \leqq 4$.

Proposition 3.1. *There is $r_0 > 0$ (depending on a) such that, if $0 < \lambda < r_0$, $0 < r < r_0$, B, B_1, $B_2 \in \bar{\mathscr{B}}_\lambda(a)$ and $p_\lambda(B)$, $p_\lambda(B_1)$, $p_\lambda(B_2) \leqq r$, then*

i) $\sum_{n \geqq 5} \lambda^n \sup_{t \geqq 0} (1+t)^{\chi + n/8} \|N_{T_1}(B \circ V_t)^n - N_{T_2}(B \circ V_t)^n\|_{L_n(D(a),\mathscr{H})} \leqq C T^{-(1-\chi)}$, for $0 \leqq \chi < 1$, T_1, $T_2 \leqq T$ and for some $C > 0$,
ii) $N(B) \in \bar{\mathscr{B}}_\lambda(a)$
iii) $p_\lambda(B_1 - B_2) \leqq C(\lambda + r)\, p_\lambda(B_1 - B_2)$, for some $C > 0$.

Proof. By hypothesis, the function F in (NLKG) (respectively (NLS)) is analytic in a neighbourhood of zero in $E = \mathbb{C}^N \times \mathbb{C}^{4N} \times \mathbb{C}^{12N}$ (respectively $E = \mathbb{C}^N \times \mathbb{C}^{3N}$). Let $F = \sum_{n \geqq 2} F^n$ be the Taylor expansion of F around zero, where $F^n \in L_n(E, \mathbb{C}^N)$. There is $\rho > 0$ such that the series $G(z) = \sum_{n \geqq 2} z^n \|F^n\|_{L_n(E,\mathbb{C}^N)}$ converges for $|z| \leqq \rho$, $z \in \mathbb{C}$.

Denote by $A^1 = I$, $A^2 = \Omega^2$, $A^3 = \Omega^3$, $A = \sum_{n=1}^{3} A^n$ and $N_1(B_1, B_2) = J \circ (A + B_1) -$

$J\circ(A+B_2)$, for B_1, $B_2\in\bar{\mathscr{B}}(a)$. $N_1(B_1,B_2)$ is an element of $\bar{\mathscr{B}}(a)$ as $\bar{\mathscr{B}}(a)$ is stable under composition with ∇ and under pointwise multiplication. In order to estimate the $L_n(D(a),\mathscr{H})$-norm of $N_1(B_1,B_2)^n$, the n-homogeneous part of $N_1(B_1,B_2)$, we note first that

$$N_1(B_1,B_2)=\int_0^1 J'\circ(A+xB_1+(1-x)B_2).(B_1-B_2)dx. \tag{3.7}$$

Introduce (as a formal power series in v)

$$q_v(\cdot)=\sum_{n=1}^{3} v^n\|\cdot\|_{L_n(D(a),W^{4,\infty})}+\sum_{n=4}^{\infty} v^n\|\cdot\|_{L_n(D(a),\mathscr{H})'}, \quad v\geqq 0,$$

and introduce the functions $f_n:\mathbb{R}^+\to\mathbb{R}^+$, $n\geqq 5$ by the following equality of formal power series in v:

$$\sum_{n\geqq 5} v^n f_n(s,x)=C_1G'(C_1q_v(A\circ V_s)+C_1q_v(B_1\circ V_s)+C_1q_v(B_2\circ V_s))q_v((B_1-B_2)\circ V_s). \tag{3.8}$$

In the case of NLS (respectively NLKG) the n-homogeneous part $J_{1,x}^n$ of $J'\circ(A+xB_1+(1-x)B_2).(B_1-B_2)$ is a sum of products of factors, each factor being a term in A, B_1, B_2 or B_1-B_2 composed with not more than one (respectively two for the $W^{4,2}$-valued part and one for the $W^{3,2}$-valued part) derivative. Taking $W^{2,\infty}$-norm of factors involving a term from A and $W^{2,2}$-norm of the other factors, one gets after composition with V_s that

$$\|J_{1,x}^n\circ V_s\|_{L_n(D(a),(1-\Delta)^{-1/2}\mathscr{H})}\leqq f_n(s,x), \quad s\geqq 0, n\geqq 5$$

if C_1 is sufficiently big. It follows then from the definition of $D(a)$ and the translation invariance of $J_{1,x}^n\circ V_s$ that

$$\|J_{1,x}^n\circ V_s\|_{L_n(D(a),\mathscr{H})}\leqq C_2 an f_n(s,x), \quad s\geqq 0, n\geqq 5 \tag{3.9}$$

for some $C_2>0$. By the definition of $D(a)$ and by Proposition 2.8, there is $C_a>0$ such that $C_1q_v(A\circ V_s)\leqq C_a(v+v^2+v^3)$ and $q_v(B_i\circ V_s)\leqq(1+s)^{-1}\,p_{v(1+s)^{-1/8}}(B_i\circ V_s)$, $i=1,2$, by the definition of q_v and p_v. It follows now from the positivity of the coefficients in the expansion of G and (3.8) that

$$f_n(s,x)\leqq g_n(s)(1+s)^{-(2+n/8)}\leqq g_n(0)(1+s)^{-(2+n/8)},$$

where

$$\begin{aligned}\sum_{n\geqq 5} v^n g_n(s)(1+s)^{-(2+n/8)}&=C_aG'(C_a((1+s)^{-3/2}v+(1+s)^{-2}(v^2+v^3)\\&\quad+(1+s)^{-1}(p_{v(1+s)^{-1/8}}(B_1)+p_{v(1+s)^{-1/8}}(B_2))))(1+s)^{-1}\\&\quad\cdot p_{v(1+s)^{-1/8}}(B_1-B_2), \quad s\geqq 0.\end{aligned} \tag{3.10}$$

and C_a is sufficiently big. Let the argument in G' be smaller than ρ for $s=0$, so that (3.10) is an equality of convergent series for small positive v. Inequality (3.9) give then (with new C_a)

$$\|J_{1,x}^n\circ V_s\|_{L_n(D(a),\mathscr{H})}\leqq C_a n g_n(0)(1+s)^{-(2+n/8)}, \quad n\geqq 5, s\geqq 0. \tag{3.11}$$

Equations (3.7) and (3.11) give for $B_1 = B$ and $B_2 = 0$,

$$\begin{aligned}
&\sup_{t\geqq 0}(1+t)^{\chi+n/8}\|N_{T_1}(B\circ V_t)^n - N_{T_2}(B\circ V_t)^n\|_{L_n(D(a),\mathscr{H})} \\
&\leqq \sup_{t\geqq 0}(1+t)^{\chi+n/8}\left|\int_{T_1}^{T_2} C_a n g_n(0)(1+s+t)^{-(2+n/8)}ds\right| \\
&\leqq g_n(0)16C_a \sup_{t\geqq 0}(1+t)^{\chi+n/8}(1+t+T)^{-(1+n/8)} \\
&\leqq 32C_a g_n(0)\left(\frac{\chi+n/8}{1+n/8}\right)^{\chi+n/8}(1+n/8)^{1-\chi}\left(\frac{T}{1-\chi}\right)^{-(1-\chi)} \\
&\leqq g_n(0)16C_a(1-\chi)^{1-\chi}T^{-(1-\chi)} \quad \text{for} \quad 0\leqq\chi<1, \quad T_1, T_2 \leqq T. \qquad (3.12)
\end{aligned}$$

Summation of inequality (3.12) for $n \geqq 5$ and the definition (3.10) of g_n give (with new C_a):

$$\begin{aligned}
&\sum_{n\geqq 5}\sup_{t\geqq 0} v^n\|N_{T_1}(B\circ V_t)^n - N_{T_2}(B\circ V_t)^n\|_{L_n(D(a),\mathscr{H})} \\
&\leqq C_a(1-\chi)^{1-\chi}T^{-(1-\chi)}G'(C_a((v+v^2+v^3)+p_v(B)))p_v(B), \qquad (3.13)
\end{aligned}$$

for $0 \leqq \chi < 1$, T_1, $T_2 \leqq T$. One can choose r_0 small enough so that $C_a(r_0 + r_0^2 + r_0^3) + r_0 < \rho$. Together with (3.13) this proves part i) of the proposition.

In analogy with the derivation of estimate (3.12) one finds, using (3.7), (3.11) and definition (3.3), that (with new C_a)

$$p_\lambda(N(B_1) - N(B_2)) \leqq C_a G'(C_a((\lambda+\lambda^2+\lambda^3) + p_\lambda(B_1) + p_\lambda(B_2)))p_\lambda(B_1 - B_2). \qquad (3.14)$$

Take r_0 small enough so that $C_a(\lambda+\lambda^2+\lambda^3)+2r<\rho$. The right-hand side of inequality (3.14) is then finite for $p_\lambda(B_i) \leqq r$, and $B_i \in \bar{\mathscr{B}}_\lambda(a)$, $i = 1, 2$. This proves point ii) of the proposition by taking $B_2 = 0$. If r_0 is sufficiently small then $C_a G'((\lambda+\lambda^2+\lambda^3)+2r)) \leqq C'(\lambda + r)$, which proves iii). Q.E.D.

Remark 3.2.

i) The fact that J is a continuous map from neighbourhood of zero in $\mathscr{H}$ into $(1-\Delta)^{-1/2}\mathscr{H}$ (but not generally into $\mathscr{H}$) gives rise to the factor n on the right-hand side of (3.9). This loss of radious of analyticity is compensated by the integration in $N(B)$, which is seen in the second inequality in (3.12). If the nonlinearities in NLKG or NLS involved more derivatives (or if $a = \infty$), then the situation would be different and the above proof would not permit us to conclude that $N(B) \in \bar{\mathscr{B}}_\lambda(a)$.

ii) The first part of Proposition 3.1 assures the convergence of $N_T(B)$ to $N_\infty(B)$ in the space of functions analytic on the ball of radius λ in $D(a)$ into $\mathscr{H}$.

Proposition 3.3. *There is $\lambda > 0$ such that Eq. (3.4) has a unique solution B in $\bar{\mathscr{B}}_\lambda(a)$.*

Proof. Equation (3.4) reads $B = N(B) + X$, where

$$X = -\int_0^\infty V_{-s}J\circ(I+\Omega^2+\Omega^3)\circ V_s ds - \Omega^2 - \Omega^3. \qquad (3.15)$$

Let $r_0 > 0$ be such that the conclusions of Proposition 3.1. hold. Choose λ and r in the interval $]0, r_0[$ such that $C(\lambda + r) \leqq \frac{1}{2}$, where C is given by Proposition 3.1. iii). Then $p_\lambda(N(B)) \leqq r/2$ and $p_\lambda(N(B_1) - N(B_2)) \leqq \frac{1}{2} p_\lambda(B_1 - B_2)$. It will be proved in Lemma 3.4 that $X \in \bar{\mathscr{B}}_\nu(a)$, for some $\nu > 0$, so $p_\nu(X) \leqq r/2$ if ν is sufficiently small. Now, let λ also satisfy $\lambda \leqq \nu$. The map $B \to N(B) + X$ is then a contraction of the ball of radius r around zero in $\bar{\mathscr{B}}_\lambda(a)$ into itself. This proves that the equation $B = N(B) + X$ has a unique solution in that ball. As the solution of this equation is unique in $\bar{\mathscr{B}}(a)$, it is also unique in $\bar{\mathscr{B}}_\lambda(a)$. Q.E.D.

Lemma 3.4. *X defined by (3.15) is an element of $\bar{\mathscr{B}}_\lambda(a)$ if $\lambda > 0$ is sufficiently small.*

Proof. Let the function G be as in the proof of Proposition 3.1 and define H by $H(z) = G(z) - z^2 \|F^2\|_{L_2(E,\mathbb{C}^N)} - z^3 \|F^3\|_{L_3(E,\mathbb{C}^N)}$, where $z \in \mathbb{C}$ and $|z| \leqq \rho$. Introduce the norm $p_\nu = \sum_{n \geqq 1} \nu^n \| \ \|_{L_n(D(a),\mathscr{H})}$, $\nu \geqq 0$, and let $A = I + \Omega^2 + \Omega^3$, $R = J - J^2 - J^3$. If λ is sufficiently small then $R \circ A \in \bar{\mathscr{B}}_\lambda(a)$, and we have

$$R \circ A = \int_0^1 R' \circ (xA).A dx. \tag{3.16}$$

Denote by $(R \circ A)^n$ (respectively $(R' \circ (xA) \cdot A)^n$) the n-homogeneous part of $R \circ A$ (respectively $R' \circ (xA) \cdot A$), and introduce the functions $f_n : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^+$ by the equality of formal power series (in ν)

$$\sum_{n \geqq 4} \nu^n f_n(s,x) = C_1 H'(C_1 x q_\nu(A \circ V_s)) p_\nu(A \circ V_s), \tag{3.17}$$

$$x \in [0,1], \quad C_1 > 0,$$

where q_ν is as in the proof of Proposition 3.1. One proves, in the same way as in equality (3.9), that

$$\|(R' \circ (xA).A)^n \circ V_s\|_{L_n(D(a),\mathscr{H})} \leqq C_2 a n f_n(s,x), \quad n \geqq 4, s \geqq 0, \tag{3.18}$$

$x \in [0,1]$, for some $C_2 > 0$. It follows from Proposition 2.8 and decay properties of V_s that

$$q_\nu(A \circ V_s) \leqq C_a(\nu(1+s)^{-3/2} + (\nu^2 + \nu^3)(1+s)^{-2}), \quad s \geqq 0, \tag{3.19}$$

and that

$$p_\nu(A \circ V_s) \leqq C_a(\nu + \nu^2(1+s)^{-1/2} + \nu^3(1+s)^{-1}), \quad s \geqq 0, \tag{3.20}$$

for some $C_a > 0$. Let the functions $g_n : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by the equality of formal powerseries:

$$\sum_{n \geqq 4} \nu^n (1+s)^{-(2+n/8)} g_n(s) = C_a H'(C_a(\nu(1+s)^{-3/2} + (\nu^2 + \nu^3)(1+s)^{-2}) \cdot (\nu + \nu^2(1+s)^{-1/2} + \nu^3(1+s)^{-1}), \quad s \geqq 0. \tag{3.21}$$

The functions g_n satisfy $g_n(s) \leqq g_n(0)$, for $s \geqq 0$. It follows then from the definition of H, and formulas (3.17) and (3.19)–(3.21) that if C_a in (3.21) is sufficiently big then $f_n(s,x) \leqq (1+s)^{-(2+n/8)} g_n(s) \leqq g_n(0)(1+s)^{-(2+n/8)}$ This together with (3.18) and

(3.16), leads to

$$\|(R\circ A)^n\circ V_s\|_{L_n(D(a),\mathscr{H})}\leqq C'_a n g_n(0)(1+s)^{-(2+n/8)},\quad s\geqq 0, n\geqq 4. \tag{3.22}$$

The definition of p_λ and (3.22) show that

$$p_\lambda\left(\int_0^\infty V_{-s}R\circ A\circ V_s ds\right)\leqq \sum_{n\geqq 4}\sup_{t\geqq 0}(1+t)^{(1+n/8)}\int_0^\infty \|(R\circ A)^n\circ V_{s+t}\|_{L_n(D(a),\mathscr{H})}ds$$

$$\leqq \sum_{n\geqq 4} C'_a g_n(0)8n(8+n)^{-1}\leqq \sum_{n\geqq 4} 8C'_a g_n(0).$$

By the definition (3.21) of g_n it follows from the last inequality that

$$p_\lambda\left(\int_0^\infty V_{-s}(R\circ A\circ V_s)ds\right)\leqq 8C'_a C_a H'(C_a(\lambda+\lambda^2+\lambda^3))(\lambda+\lambda^2+\lambda^3),$$

where $\lambda>0$ is so small that the right-hand side is finite. Hence

$$\int_0^\infty V_{-s}(R\circ A\circ V_s)ds\in\bar{\mathscr{B}}_\lambda(a),\quad \text{for some } \lambda>0. \tag{3.23}$$

Let $Y=-\int_0^\infty ds V_{-s}(J\circ(I+A)-J^2-(J^2(I\otimes A^2+A^2\otimes I)+J^3))\circ V_s$ and let $\Omega^1=I$. The integrand can be written as

$$V_{-s}R\circ(\Omega^1+\Omega^2+\Omega^3)\circ V_s+\sum_{\substack{p_1+p_2\geqq 4\\ 1\leqq p_i\leqq 3}} V_{-s}J^2(\Omega^{p_1}\otimes\Omega^{p_2})\circ V_s$$

$$+\sum_{\substack{p_1+p_2+p_3\geqq 4\\ 1\leqq p_i\leqq 3}} V_{-s}J^3(\Omega^{p_1}\otimes\Omega^{p_2}\otimes\Omega^{p_3})\circ V_s.$$

From (3.23) and Proposition 2.8 it follows that $Y\in\bar{\mathscr{B}}_\lambda(a)$, for $\lambda>0$ sufficiently small. From the definition of Ω^2,Ω^3 it then follows that $X=Y\in\bar{\mathscr{B}}_\lambda(a)$, for such λ.

Q.E.D.

Corollary 3.4. *The equation*

$$\Omega=I-\int_0^\infty ds V_{-s}J\circ\Omega\circ V_s$$

has, for some $\lambda>0$, a unique solution $\Omega-I\in\bar{\mathscr{B}}_\lambda(a)$. If $u\in D(a)$, $\|u\|_{D(a)}<\lambda$, then $\Omega(u)$ is an analytic function on $\mathbb{R}^3$.

Proof. The uniqueness and existence follow from Proposition 3.3 by the substitution $B=\Omega-I-\Omega^2-\Omega^3$ and the fact that $\Omega^2+\Omega^3\in\bar{\mathscr{B}}_\lambda(a)$ by Proposition 2.8. For $x\in\mathbb{R}^3$ the function $x\to\tau_x u\in D_a$ is analytic and so is the function $x\to\Omega(\tau_x u)\in\mathscr{H}$, as $\Omega-I\in\bar{\mathscr{B}}_\lambda(a)$. $(\Omega(u))(x)$ is well defined for every $x\in\mathbb{R}^3$, as $\Omega(u)\in\mathscr{H}$) and $\mathscr{H}$ consists of continuous functions. $x\to(\Omega(u))(x)$ is then analytic as the function $\delta\to(\Omega(u))(x+\delta)=(\tau_\delta\Omega(u))(x)=(\Omega(\tau_\delta u))(x)$ is analytic in a neighbourhood of zero.

Q.E.D.

4. Properties of Solutions of the Nonlinear Equations

Through this paragraph Ω will denote the solution of the equation in Corollary 3.4.

The following simple uniqueness result, for the analytic Cauchy problems (NLKG) and (NLS), is certainly well-known.

Proposition 4.1. *Let $u^{(1)}$ and $u^{(2)}$ be two solutions of* (NLKG) (*respectively* (NLS)) *analytic in a connected open neighbourhood $\mathcal{O}$ of $\{0\} \times \mathbb{R}^3$. If $u^{(1)}(0,x) = u^{(2)}(0,x)$, $(\partial/\partial t)u^{(1)}(0,x) = (\partial/\partial t)u^{(2)}(0,x)$ (respectively $u^{(1)}(0,x) = u^{(2)}(0,x)$) for every $x \in \mathbb{R}^3$, then $u^{(1)} = u^{(2)}$ in $\mathcal{O}$.*

Proof. The proofs for the two cases (NLKG) and (NLS) are analogous, so we only consider the first one. Let $u = u^{(1)} - u^{(2)}$. By hypothesis $(\nabla^n u)(0,x) = 0$ and $(\nabla^n Du)(0,x) = 0$ for $x \in \mathbb{R}^3$, $n \geqq 0$. By equation (NLKG) it follows that

$$\left(\frac{\partial^2}{\partial t^2} u_j\right)(0,x) = m_j^2(\Delta u)(0,x) + F_j(u^{(1)}(0,x), Du^{(1)}(0,x), \nabla Du^{(1)}(0,x))$$
$$- F_j(u^{(2)}(0,x), Du^{(2)}(0,x), \nabla Du^{(2)}(0,x)), \quad 1 \leqq j \leqq N, x \in \mathbb{R}^3.$$

Thus $((\partial^2/\partial t^2)u)(0,x) = 0$ for $x \in \mathbb{R}^3$. Repeating this argument for the time derivatives of (NLKG) gives that $((\partial^n/\partial t^n)u)(0,x) = 0$ for $n \geqq 0$, $x \in \mathbb{R}^3$. In particular it follows that u is zero in a neighbourhood of zero in $\mathbb{R}^4$, which by analytic continuation proves that $u = 0$ in $\mathcal{O}$. Q.E.D.

We introduce a set E of scattering states φ_+, invariant under the linear evolution V_t, $t \geqq 0$, on which the series $\Omega(\varphi_+)$ converges and $\Omega^n(V_t\varphi_+)$, $n \geqq 2$, has the decay properties described by the norms $|\ \ |_n$ in (3.1)–(3.3) for $t \to \infty$. For this purpose let

$$E' = \{\varphi_+ \in D(a) \,|\, \|\varphi_+\|_{D(a)} < \lambda, a \geqq 1, \lambda > 0 \quad \text{and} \quad \Omega - I \in \mathscr{B}_\lambda(a)\}.$$

We define $E = \bigcup_{t \geqq 0} V_t E'$. Obviously $V_t E \subset E$ for $t \geqq 0$, and it follows from Corollary 3.4 that E is non-empty. We notice that $E \cap D(a)$ is open in $D(a)$.

For convenience we give the decay properties of $\Omega^n(V_t\varphi_+)$, $\varphi_+ \in E$, which are rather obvious from the definition of E. For $f: \mathbb{R}^+ \to \mathscr{H}$ let

$$\rho_1(f) = \sup_{t \geqq 0}(\|f(t)\|_{L^2} + (1+t)^{3/2}\|f(t)\|_{L^\infty}), \tag{4.1}$$

$$\rho_2(f) = \sup_{t \geqq 0}((1+t)^{1/2}\|f(t)\|_{L^2} + (1+t)^2\|f(t)\|_{L^\infty}), \tag{4.2}$$

$$\rho_3(f) = \sup_{t \geqq 0}((1+t)\|f(t)\|_{L^2} + (1+t)^2\|f(t)\|_{L^\infty}), \tag{4.3}$$

and

$$\rho_n(f) = \sup_{t \geqq 0}((1+t)^{1+n/8}\|f(t)\|_{L^2}), \quad n \geqq 4. \tag{4.4}$$

Proposition 4.2. *Let $\varphi_+ \in E$ and let $\varphi_{m,n}(t) = (1-\Delta)^{m/2}\Omega^n(V_t\varphi_+)$. Then*

i) $\sum_{n \geqq 1} \rho_n(\varphi_{m,n}) < \infty$ *for $m \geqq 0$,*

ii) *there is $\varepsilon > 0$ such that $V_{-\varepsilon}\varphi_+ \in E$ and*

iii) *the function* $\psi \to \Omega(V_t\varphi_+ + \psi) \in \mathscr{H}$, $t \geqq 0$ *is analytic in a neighbourhood of zero in* $D(a)$, *for every* $a \geqq 1$.

Proof.
i) By the definition of E there is $\lambda > 0$, $a \geqq 1$ and $T \geqq 0$ such that $\varphi_+ \in D(a)$, $\Omega - I \in \mathscr{B}_\lambda(a)$ and $\|V_{-T}\varphi_+\|_{D(a)} < \lambda$. Let $\psi_{m,n}(t) = (1-\Delta)^{m/2}\Omega^n(V_t(V_{-T}\varphi_+))$. By the definition of ρ_n it follows now that $\rho_n(\varphi_{m,n}) \leqq \rho_n(\psi_{m,n})$, and then by the definition of the norm $|\ |_n$ (see (3.1)–(3.3)) that $\rho_n(\psi_{m,n}) \leqq (1+n^2a^2)^{m/2}|\Omega^n|_n \|V_{-T}\varphi_+\|^n_{D(a)}$, for $n \geqq 2$. The series $\sum_{n\geqq 2} \lambda^n|\Omega^n|_n$ is convergent as $\Omega - I \in \mathscr{B}_\lambda(a)$. The series $\sum_{n\geqq 2} (1+n^2a^2)^{m/2}|\Omega^n|_n\|V_{-T}\varphi_+\|^n_{D(a)}$ is then also convergent as $\|V_{-T}\varphi_+\|_{D(a)} < \lambda$. It is well known that $\rho_1(\psi_{m,1}) < \infty$. This proves that $\sum_{n\geqq 1} \rho_n(\varphi_{m,n}) < \infty$.
ii) The second statement of the proposition follows from the fact that there is $\varepsilon > 0$, such that $\|V_{-T}(V_{-\varepsilon}\varphi_+)\|_{D(a)} < \lambda$ as V is strongly continuous in $D(a)$.
iii) As $\Omega - I \in \mathscr{B}_\lambda(a)$, it follows from formulas (3.1)–(3.3) that $(\Omega - I)\circ V_t \in \mathscr{B}_\lambda(a)$ for every $t \geqq 0$. The map $\psi \to V_t\psi$ map open neighbourhoods of zero in $D(a')$, $a' \geqq 1$ onto open neighbourhoods of zero in $D(a')$, for $t \in \mathbb{R}$. By the definition of T it follows then that the map $\psi \to \Omega(V_t\varphi_+ + \psi) = (\Omega \circ V_{t+T})(V_{-T}\varphi_+ + V_{-t-T}\psi) \in \mathscr{H}$ is analytic in an open neighbourhood of zero in $D(a')$. Q.E.D.

The next proposition leads to analyticity properties of the solutions we are going to construct for (NLKG) and (NLS). We recall that $(\Omega(\varphi_+))(x)$ is in $\mathbb{C}^{2N}$ (respectively $\mathbb{C}^N$) for the case of (NLKG) (respectively (NLS)).

Proposition 4.3. *Let* $\varphi_+ \in E$. *There is then* $\varepsilon > 0$ *such that the function* $(t,x) \to (\Omega(V_t\varphi_+))(x)$ *is analytic in* $\mathcal{O}_\varepsilon =]-\varepsilon, \infty[\times \mathbb{R}^3$.

Proof. By Proposition 4.2 there is $\varepsilon > 0$ such that $V_{-\varepsilon}\varphi_+ \in E$. Let $b = (b_1, b_2, b_3)$, $(\tau_b f)(x) = f(x+b)$ and $b_i \in \mathbb{R}$ for $0 \leqq i \leqq 3$. Then $(b_0, b) \in \mathbb{R}^4 \to V_{b_0}\tau_b\varphi_+ \in D(a)$, is an analytic function, where a has been chosen such that $\varphi_+ \in D(a)$. It follows that the function $(b_0, b) \to \Omega(V_{t+b_0}\tau_b\varphi_+) \in \mathscr{H}$, $t > -\varepsilon$, is analytic in a neighbourhood of zero in $\mathbb{R}^4$ as $\psi \to \Omega(V_t\varphi_+ + \psi) \in \mathscr{H}$ is analytic in a neighbourhood of zero in $D(a)$, by Proposition 4.2. The function $(b_0, b) \to (\Omega(V_{t+b_0}\tau_b\varphi_+))(x) = (\Omega(V_{t+b_0}\varphi_+))(x+b) \in \mathbb{C}^{2N}$ (in the case of (NLKG) and $\mathbb{C}^N$ in the case of (NLS)) is then analytic in a neighbourhood of zero in $\mathbb{R}^4$, by Sobolev embedding, for $t > -\varepsilon$, $x \in \mathbb{R}^3$.

We next prove the fact that the wave operator Ω composed by the linear evolution V_t is the non-linear evolution.

Proposition 4.4. *Let* $\varphi_+ \in E$. *Then the function* $t \in [-\varepsilon, \infty[\to \varphi(t) = \Omega(V_t\varphi_+) \in \mathscr{H}$ *is a solution of the equation*

$$\varphi(t) = V_t\varphi_+ - \int_t^\infty V_{t-s}J(\varphi(s))ds, \tag{4.5}$$

and $\varphi:]-\varepsilon, \infty[\to \mathscr{H}$ *is analytic, for some* $\varepsilon > 0$.

Proof. By Proposition 4.2 there is $\varepsilon > 0$ such $V_{-\varepsilon}\varphi_+ \in E$; so $V_t\varphi_+ \in E$ for $t > -\varepsilon$, by the definition of E. As $\Omega - I \in \mathscr{B}_\lambda(a)$ implies that $(\Omega - I)\circ V_s \in \mathscr{B}_\lambda(a)$ for $s > 0$, it follows from Corollary 3.4 that φ satisfies (in $\mathscr{H}$) the equation

$$\varphi(t)=V_t\varphi_+-\int_0^\infty V_{-s}J(\varphi(s+t))ds,\quad t\geqq -\varepsilon.$$

Equation (4.5) follows after the substitution $s+t\to s$. Then as in the proof of Proposition 4.3, $\varphi\colon]-\varepsilon,\infty[\to\mathscr{H}$ is analytic if $\varepsilon>0$ is sufficiently small. Q.E.D.

The set $F=\Omega E$ will play the role of initial data at $t=0$ for the Cauchy problems (NLKG) and (NLS).

When we say that $\varphi(t)=\Omega\circ V_t\varphi_+$ is a solution of (NLKG) we mean that the first component of the function $(t,x)\to(\varphi(t))(x)=(u(t,x),(\partial/\partial t)u(t,x))$ is a solution of (NLKG).

Corollary 4.5. *Let $\varphi_0\in F$. There is then $\varepsilon>0$, such that the equation* (NLKG) *(respectively* (NLS)*) has a unique analytic solution $(t,x)\colon]-\varepsilon,\infty]\times\mathbb{R}^3\to(\varphi(t))(x)\in\mathbb{C}^{2N}$ (respectively $\mathbb{C}^N$), with $\varphi_0=\varphi(0)$. Further there is $\varphi_+\in E$ such that $\varphi(t)=\Omega(V_t\varphi_+)$.*

Proof. By Corollary 3.4, $x\in\mathbb{R}^3\to\varphi_0(x)\in\mathbb{C}^{2N}$ (respectively $\mathbb{C}^N$) is an analytic function, and by Proposition 4.1 the solution φ of (NLKG) (respectively (NLS)) is unique as an analytic function, when it exists. The definition of F means that there exists $\varphi_+\in E$ such that $\varphi_0=\Omega(\varphi_+)$. Let $\varphi(t)=\Omega(V_t\varphi_+)$. By Proposition 4.3 there is $\varepsilon>0$ such that $(t,x)\to(\varphi(t))(x)\in\mathbb{C}^{2N}$ (respectively $\mathbb{C}^N$) is an analytic function in $\mathcal{O}_\varepsilon=]-\varepsilon,\infty[\times\mathbb{R}^3$, and in view of Proposition 4.4 the analytic function $t\in]-\varepsilon,\infty[\to\varphi(t)\in\mathscr{H}$ satisfies Eq. (4.5) if $\varepsilon>0$ is sufficiently small. But an analytic solution of (4.5) is also a solution of (NLKG) (respectively (NLS)).

Q.E.D.

The following proposition shows that we have as many initial data in F at $t=0$ as in E at $t=\infty$.

Proposition 4.6. *The map $\Omega\colon E\to\mathscr{H}$ is injective.*

Proof. Let $\varphi_0^{(1)},\ \varphi_0^{(2)}\in F$. There is then $\varphi_+^{(1)},\ \varphi_+^{(2)}\in E$ such that $\varphi_0^{(i)}=\Omega(\varphi_+^{(i)})$, $i=1,2$. If $\varphi^{(i)}(t)=\Omega(V_t\varphi_+^{(i)})$, then $\lim_{t\to\infty}V_{-t}\varphi^{(i)}(t)=\varphi_+^{(i)}$ in $\mathscr{H}$, which follows from Proposition 4.2.i. Thus if $\varphi_+^{(1)}\neq\varphi_+^{(2)}$, then there is $T\in\mathbb{R}^+$ such that $\varphi^{(1)}(T)\neq\varphi^{(2)}(T)$. The function $(t,x)\in]-\varepsilon,\infty[\times\mathbb{R}^3\to(\varphi^{(i)}(t))(x)\in\mathbb{C}^{2N}$ (respectively $\mathbb{C}^N$), $i=1,2$, is, by Corollary (4.5) an analytic solution of (NLKG) (respectively (NLS)). Proposition 4.1 applied to the data $\varphi^{(1)}(T)$ and $\varphi^{(2)}(T)$ gives that $\varphi^{(1)}\neq\varphi^{(2)}$. This shows, once more by Proposition 4.1 that $\varphi_0^{(1)}\neq\varphi_0^{(2)}$ if $\varphi_+^{(1)}\neq\varphi_+^{(2)}$. Q.E.D.

One can reformulate certain properties of the solutions of the analytic Cauchy problems (NLKG) and (NLS) in F in terms of an abstract evolution operator.

Theorem 4.7. *There is a unique evolution operator $U\colon\mathbb{R}^+\times F\to\mathscr{H}$ such that the function $t\to\varphi(t)=U_t(\varphi_0)\in\mathscr{H}$, $\varphi_0\in\mathscr{H}$ is analytic in a neighbourhood of $\mathbb{R}^+$ and such that*

$$\varphi(t)=V_t\varphi_0+\int_0^t V_{t-s}J(\varphi(s))\,ds. \tag{4.6}$$

Further U has the following properties:

i) $U:\mathbb{R}^+ \times F \to F, U_{t+t'} = U_t \circ U_{t'}, U_t = \Omega \circ V_t \circ \Omega^{-1}$, *for* $t, t' \geqq 0$.
ii) *The function* $(t,x) \to (U_t(\varphi_0))(x)$ *is analytic in* $\mathcal{O}_\varepsilon =]-\varepsilon, \infty[\times \mathbb{R}^3$ *for every* $\varphi_0 \in F$ *and some* $\varepsilon > 0$ *(dependent of* φ_0*).*
iii) *For* $\varphi_0 \in F$ *let* φ_+ *be the unique element in* E *such that* $U_t(\varphi_0) = \sum_{n\geqq 1} \Omega^n(V_t\varphi_+)$, $t \geqq 0$, *and let* $\varphi_{m,n}(t) = (1-\Delta)^{m/2}\Omega^n(V_t\varphi_+)$. *Then* $\sum_{n\geqq 1} \rho_n(\varphi_{m,n}) < \infty$, *for every* $m \geqq 0$.

Proof. Let $\varphi_0 \in F$. By Proposition 4.6 there is then a unique $\varphi_+ \in E$ such that $\varphi_0 = \Omega(\varphi_+)$. In light of Proposition 4.2 there is $\varepsilon > 0$ such that the function $t \to \varphi(t) = \Omega(V_t\varphi_+) \in \mathscr{H}$ is analytic in $]-\varepsilon, \infty[$. By Proposition 4.4, φ is a solution of Eq. (4.5). From $V_t\varphi_0 = V_t\left(\varphi_+ - \int_0^\infty V_{-s}J(\varphi(s))ds\right)$ it follows:

$$\varphi(t) = V_t\varphi_+ - \int_t^\infty V_{t-s}J(\varphi(s))\,ds = V_t\varphi_0 + \int_0^t V_{t-s}J(\varphi(s))\,ds.$$

This gives Eq. (4.6). The uniqueness of the solution φ of Eq. (4.6) as an analytic function in t, follows now by successive derivations in t as in the proof of Proposition 4.1. We define $U_t(\varphi_0) = \Omega(V_t\varphi_+)$. Then $U:\mathbb{R}^+ \times F \to F$ and $U_t = \Omega \circ V_t \circ \Omega^{-1}$ (Proposition 4.6), which proves the existence of U as well as i). Point ii) (respectively iii)) are just a reformulation of Proposition 4.3 (respectively 4.2. i)). Q.E.D.

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Note added in proof After submission of the present article, the authors have taken knowledge of the following results:

[1′] Kleinerman, S.: Global existence of small amplitude solutions to nonlinear Klein–Gordon equations in four space-time dimensions Preprint

[2′] Ginibre, J , Velo, G.: The global Cauchy problem for the nonlinear Schrödinger equation revisited, and: The global Cauchy problem for the nonlinear Klein–Gordon equation Preprints

The case of quadratic nonlinearities is covered by these references under various hypothesis, supplementary (or different) to ours. The spaces of initial conditions considered for these cases are larger than ours