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Cauchy Problems for the Conformal Vacuum Field Equations in General Relativity

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Abstract. Cauchy problems for Einstein's conformal vacuum field equations are reduced to Cauchy problems for first order quasilinear symmetric hyperbolic systems. The "hyperboloidal initial value" problem, where Cauchy data are given on a spacelike hypersurface which intersects past null infinity at a spacelike two-surface, is discussed and translated into the conformally related picture. It is shown that for conformal hyperboloidal initial data of class H^s, $s \geqq 4$, there is a unique (up to questions of extensibility) development which is a solution of the conformal vacuum field equations of class H^s. It provides a solution of Einstein's vacuum field equations which has a smooth structure at past null infinity.

1. Introduction

In contrast with the field equations of other gauge theories Einstein's field equations are not conformally invariant: rescalings of the metric create Ricci curvature. However, important substructures of the field and the field equations, the conformal Weyl tensor and the vacuum Bianchi identity, written as an equation for the rescaled Weyl tensor, are conformally invariant. It is this very particular behaviour of the field equations under conformal rescalings which allows one to impose conditions on the global conformal structure of the field without restricting the freedom to prescribe asymptotic initial data for the field. Global conditions of this type are inherent in Penrose's concept of an "asymptotically empty and simple space-time" [1]. The essential idea is to stipulate for a given space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ the existence of a surface $\mathscr{I}$ such that $M = \tilde{M} \cup \mathscr{I}$ forms a manifold with three-dimensional boundary $\mathscr{I}$ and on M the existence of a function Ω such that $\Omega > 0$ on $\tilde{M}$; $\Omega \equiv 0, d\Omega \neq 0$ on $\mathscr{I}$ and such that $g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}$ extends to a smooth ("non-physical") Lorentz metric on M. Further global requirements imply that $\mathscr{I}$ consists of two components $\mathscr{I}^-$ respectively $\mathscr{I}^+$ ("past respectively future null infinity") each being diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$. The appropriateness of the fall-off conditions implicit in these assumptions was suggested in particular by preceding investigations of the field equations and the radiative behaviour of gravitational fields by Bondi et al. [2], Sachs [3], Newman and Penrose [4] and others. These authors studied formal ("Bondi-type") expansions of the field to analyse what in retrospect may be called the "asymptotic characteristic initial value problem" [5]. Their results implied in particular that the conditions of "asymptotic flatness at null infinity" as introduced by Penrose fix a metrical structure at null infinity similar to that of Minkowski space while retaining essentially the same degree of freedom to prescribe data on past null infinity and on an incoming null hypersurface as one may dispose of in the case of the regular characteristic initial value problem, where data are given on two intersecting null hypersurfaces (which are thought of being embedded in space-time). As a further outcome of their investigations the authors above were able to formulate various notions of physical significance like "radiation field," "Bondi-mass," etc. which found their natural place in the concept of asymptotically empty and flat space-times. For these reasons and because of its technical simplicity and elegance, Penrose's idea was taken as a starting point for further investigations of the asymptotic behaviour of various fields. It entered the discussion of diverse global issues, and provided a basis for a variety of considerations concerned with the relation between quantum field theory and general relativity.

While the differential geometric properties of asymptotically flat spaces are well understood, basic questions concerned with the interplay of the global conditions imposed by asymptotic emptiness and flatness and the requirements of the field equations are still open:

—What is the structure of the sources which are compatible with the existence of a smooth structure at null infinity?

—Given a field with a smooth past null infinity, under which conditions will it evolve such as to develop a smooth future null infinity?

—What is the relation between past and future null infinity?

Problems of this type cannot be investigated by analysing Bondi-type expansions, and considerations concerning these and related problems which involve approximation methods are plagued by conceptual difficulties and lead to conflicting results [6–8]. On the other hand the rigorous methods developed in the study of the standard Cauchy problem for Einstein's equations seem at present not capable of dealing with any questions concerned with the existence or the detailed structure of null infinity.

A recent analysis of the formal structure of Einstein's vacuum field equations showed, however, that these equations retain just sufficient conformal invariance as to imply well posed initial value problems on initial surfaces which comprise part of null infinity. This result is due to the concurrence of two distinct properties of the field equations [5, 9, 10]:

—There exists a technique to reduce initial value problems for the field equations of gauge theories on a given spacetime to initial value problems for "symmetric hyperbolic systems" [11]. By writing Einstein's vacuum field equations as a first order system for a frame field, the connection coefficients, and the Weyl tensor such that they look similar to other gauge field equations, this technique applies in particular to Einstein's equations. Here geometrical gauge conditions for the coordinates and the frame are used instead of the "harmonic gauge" which so far was basic for all rigorous existence proofs for initial value problems in relativity [12].

—Instead of studying pairs $(\tilde{M}, \tilde{g}_{\mu\nu})$ with $\tilde{g}_{\mu\nu}$ satisfying Einstein's equations, one may work completely in terms of the nonphysical quantities $(M, g_{\mu\nu}, \Omega)$ such that Ω, $g_{\mu\nu}$ satisfy the "conformal vacuum field equations"

$$R_{\mu\nu}[\Omega^{-2} g_{\lambda\sigma}] = 0,$$
$$R[g_{\lambda\sigma}] = 0,$$

considered as differential equations for Ω and $g_{\mu\nu}$. The equation for Ω thus obtained as well as the equation for $g_{\mu\nu}$, however, degenerate where Ω vanishes. But if this system is written in a fashion similar to that indicated for the vacuum field equations before, it turns out that by some identity it can be represented by a new system which is formally regular everywhere, even at points where Ω vanishes.

These two facts work together such that initial value problems for the new system can be reduced to initial value problems for symmetric hyperbolic systems. The freedom to prescribe Cauchy data for the conformal factor on the initial surface is sufficient to take care of Penrose's conditions at past or future null infinity.

This result shows that Penrose's requirements on the asymptotic behaviour of the field near past or future null infinity are in perfect agreement with the properties of Einstein's vacuum field equations. Furthermore it follows that not only differential geometric investigations of the asymptotic behaviour of the field may be carried out in terms of local differential geometry if asymptotic conditions are formulated as requirements on the conformal structure, but also some global questions concerning the propagation of the field may be studied in terms of local initial value problems.

The regular representation of the conformal vacuum field equations has been used to reduce the asymptotic characteristic initial value problem, where data are given on part of past null infinity and on an incoming null hypersurface which intersects null infinity at a spacelike two-surface, to a characteristic initial value problem for a symmetric hyperbolic system [9]. Thus the properties of past null infinity are built into the problem from the outset. Applying a modified Cauchy–Kovalevskaja technique to the system obtained, one finds that all Bondi-type expansions for arbitrary analytic data are in fact convergent and define analytic solutions of Einstein's vacuum field equations which are asymptotically flat at null infinity [13]. The formulation of the problem, however, is designed to serve also as a basis for an existence proof for initial data of low differentiability.

One may try to find some answer to the first of the problems mentioned before by extending the data surface which propagates from past null infinity into space time as far as possible to connect it to an inner solution. However, now a difficult problem arises. The null hypersurface will inevitably start to develop caustics and selfintersections. The location of these depends on the data, which, however, must not be given in the future of caustics or selfintersections. The analysis of this problem leads to a complicated system of singular partial differential equations [14] which makes it difficult to decide whether the solution constructed near null infinity connects at all to some inner solution.

To circumvent the difficulties posed by the occurrence of caustics, one may consider Cauchy problems for the conformal vacuum field equations. There are essentially two distinct types of Cauchy problems which can be posed in an asymptotically empty and flat spacetime.

One of these is the "hyperboloidal initial value problem" which will be investigated in this paper. To control what comes in from the infinite timelike past one may prescribe Cauchy data on a spacelike hypersurface S which intersects past null infinity at a spacelike two-dimensional surface Z. Since in Minkowski-space hypersurfaces of this type are provided by the spacelike hyperboloids, these surfaces and the Cauchy data implied on them will be called "hyperboloidal." The corresponding Cauchy problem for Einstein's field equation will be called the hyperboloidal initial value problem. As a solution of this problem one will obtain part of the domain of dependence of the initial surface S. The past Cauchy horizon will coincide with a part of past null infinity near Z and the future Cauchy horizon N will be given near Z by the null hypersurface extending from Z orthogonally to Z into space-time. A more detailed description of this situation will be given in Chap. 2 and its complete translation into "non-physical" terms will be obtained in Chap. 4. In Chap. 6 it will be shown that for sufficiently smooth hyperboloidal initial data there exists a unique (up to questions of extensibility) solution of the hyperboloidal initial value problem (Theorem (6.5)). The solution is obtained by working entirely in terms of the non-physical quantities. The corresponding initial value problem for the conformal vacuum field equations is formulated in Chap. 3, 4, 5 as a Cauchy problem for a symmetric hyperbolic system. The solution of the hyperboloidal initial value problem provides the "correct" data on N for the asymptotic characteristic initial value problem.

In the second type of Cauchy problem, which will not be considered in detail in this paper, data are prescribed on a space-time Cauchy surface. The interest in this problem here arises not so much from the possibility to derive, with the methods introduced in this paper, existence theorems similar to those already available [12, 15–17], but from the prospect that an analysis of this problem in terms of the conformal field equations may help to eludidate the relationship between past and future null and spatial infinity. The structure of spatial infinity consistent with the requirements of the field equations has been investigated in the context of the standard Cauchy problem, in particular in the study of the constraint equations [18, 19], see also the literature given in [15]. Christodoulou and O'Murchandha [17] extended the domain of validity of preceding considerations on spatial infinity by proving the existence of developments of asymptotically flat (at spatial infinity) initial data which include complete spacelike surfaces boosted relative to the initial surface. In spite of this result it still seems difficult to obtain any statements on the existence of (a smooth structure at) "null infinity" in this treatment of the Cauchy problem. Beig and Schmidt therefore studied Einstein's equations near spatial infinity by techniques reminiscent of Bondi's approach to null infinity [20]. Ashtekar and Hansen [21], see also [22], arguing mainly from a geometric point of view, suggested a unifying picture by introducing the notion of an "AEFANSI" spacetime. This allows discussion of the relationship of the different asymptotic regimes. It imposes, in terms of the conformal structure, conditions on the asymptotic behaviour of the field on spacelike surfaces as well as global conditions on the propagation of the field. It appears natural to study the propagation near spatial infinity in this context by analysing further the conformal field equations as formulated in Chaps. 3, 4, 5.

2. The Hyperboloidal Initial Value Problem

If a solution $(\tilde{M}, \tilde{g}_{\mu\nu})$ of Einstein's field equations possesses a "smooth structure at null infinity." this is essentially uniquely determined by the conformal geometry of $(\tilde{M}, \tilde{g}_{\mu\nu})$ [23–25]. Nevertheless it may be quite difficult to decide for a given spacetime whether it has this property and to describe it explicitly in terms of $(\tilde{M}, \tilde{g}_{\mu\nu})$. On the other hand, in the formulation given by Penrose [1] theoretical questions concerning properties of null infinity are easily analysed in terms of the "non-physical quantities" (M, g, Ω), which are available by definition. In the initial value problem which will be discussed here, part of the desired structure of null infinity will be built in from the outset. This can be done most conveniently by formulating the problem immediately in non-physical terms.

The type of spaces which are to be constructed are characterized by

Definition (2.1). A triple $(M, g_{\mu\nu}, \Omega)$ is called an "asymptote of a solution of Einstein's vacuum field equation which is asymptotically flat at past null infiniy" if

i) M is a four-dimensional manifold with boundary $\mathscr{I}^-$, $\mathscr{I}^-$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.

ii) $g_{\mu\nu}$ is a Lorentz metric on M, $(M, g_{\mu\nu})$ is time- and space-oriented and strongly causal.

iii) Ω is a function ("the conformal factor") on M with:

$$\Omega > 0 \text{ on } \tilde{M} = M \backslash \mathscr{I}^-; \quad \Omega \equiv 0, \quad d\Omega \neq 0 \text{ on } \mathscr{I}^-.$$

iv) $\mathscr{I}^-$ is a null hypersurface with respect to $g_{\mu\nu}$ in the past of $\tilde{M}$.

v) The metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$, defined on $\tilde{M}$, satisfies Einstein's vacuum field equations

$$R_{\mu\nu}[\tilde{g}_{\lambda\sigma}] = 0. \tag{2.1}$$

Here, as in the first five chapters, all structures are required to be "sufficiently smooth"; only in Chap. 6 more detailed smoothness conditions will be formulated. In the preceding definition only those consequences of Penrose's definition of "asymptotic empty and simple space-times" have been required, which will be needed in the following. The surface $\mathscr{I}^-$ represents "past null infinity" of the space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$. Questions concerning the existence of a smooth structure at future null infinity will not be considered. Condition (iv) in fact follows from (v) and smoothness requirements. However, (iv) has been added to the list, since later $(M, g_{\mu\nu}, \Omega)$ will be constructed by solving (2.1), while initial data on some surface which carry information on (iv) will be given. It can be shown that the conformal Weyl tensor $C^{\mu}_{\lambda\rho\delta}$ of $g_{\mu\nu}$, which is on $\tilde{M}$ identical with the Weyl tensor of $\tilde{g}_{\mu\nu}$, must vanish on $\mathscr{I}^-$ [1]. Hence it may be assumed that the rescaled Weyl tensor $d^{\mu}{}_{\lambda\delta\rho} = \Omega^{-1} C^{\mu}{}_{\lambda\delta\rho}$ is smooth everywhere on M. This is important for the regularity of the solution of the conformal vacuum field equations in their representation discussed in Chap. 3.

To provide the type of initial surface we will be interested in, in addition to (i–v) the space $(M, g_{\mu\nu})$ is required to satisfy:

$$\left.\begin{array}{l}\text{There is a spacelike hypersurface } S \text{ of } (M, g_{\mu\nu}) \\ \text{with two-dimensional boundary } Z\text{, which inter-} \\ \text{sects } \mathscr{I}^- \text{ at } Z\text{. The hypersurface } S \text{ is diffeomorphic} \\ \text{to the closed unit ball in } \mathbb{R}^3\text{, whence } Z \text{ is diffeo-} \\ \text{morphic to the sphere } \mathbb{S}^2. \end{array}\right\} \tag{2.3}$$

By (v) Eq. (2.1) is required to hold everywhere on S. Though this situation is of interest in itself, in many applications one will rather have sources present with compact support in $S \backslash Z$, and (2.1) will only be required to hold in a certain neighbourhood of $\mathscr{I}^-$. Besides the difficulty of providing suitable initial data, the crucial problem will be to solve the propagation equations implied by (2.1) near $\mathscr{I}^-$. Away from $\mathscr{I}^-$ the field equations can be solved by the standard method of employing a harmonic gauge condition. On $\tilde{S} = S \backslash Z$ the metric $\tilde{g}_{\mu\nu}$ implies the first fundamental form $\tilde{h}_{\alpha\beta}$ and the second fundamental form $\tilde{\chi}_{\alpha\beta}$, i.e. an "initial data set" [15] $(\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta})$ for Eq. (2.1) (here and in the following indices $\mu, \nu, \lambda, \ldots$, referring to some coordinate system on M, take values $0, 1, 2, 3$, while indices $\alpha, \beta, \gamma, \ldots$, referring to some coordinate system on S, take values $1, 2, 3$; the summation convention is assumed for both sets of indices). These initial data will determine a solution of (2.1) uniquely only in the domain of dependence $D(S)$ of S in M with respect to $g_{\mu\nu}$ (see [26] for this and related causal notions). In fact, any spacelike hypersurface obtained by a continuous deformation of S, which intersects $\mathscr{I}^-$ at Z will determine the same domain of dependence as S. Replacing M if necessary by a suitable neighbourhood of S in M, the future Cauchy horizon N of S will be a smooth null hypersurface diffeomorphic to $Z \times \mathbb{R}^+$ starting at Z orthogonally to Z into the spacetime $\tilde{M}$, while the past Cauchy horizon I of S will be the past of Z in $\mathscr{I}^-$ and again be diffeomorphic to $Z \times \mathbb{R}^+$. The surfaces intersect each other at their common edge Z. The manifold with boundary and edge $D(S)$ thus looks like two truncated cones $D^+(S), D^-(S)$ (the future respectively past domain of dependence of S) glued together at their common base S. The situation may be illustrated by the following example.

Written with respect to polar coordinates the Minkowskian line element takes the form

$$d\tilde{s}^2 = -dt^2 + dr^2 + r^2 d\omega^2,$$

where $d\omega^2$ denotes the standard line element on the two-dimensional unit sphere. Denote by $\tilde{H}$ the spacelike hyperboloid $\{-t^2 + r^2 = -1,\ t < 0\}$. For the first respectively second fundamental form $\tilde{h}_{\alpha\beta}$ respectively $\tilde{\chi}_{\alpha\beta}$ on $\tilde{H}$ one has $\tilde{\chi}_{\alpha\beta} = -\tilde{h}_{\alpha\beta}$. The domain of dependence of $\tilde{H}$ in Minkowski space is given by the interior of the past light cone at the origin. By performing the coordinate transformation

$$t + r = \tan\left(\frac{t' + r'}{2}\right), \quad t - r = \tan\left(\frac{t' - r'}{2}\right),$$

and rescaling the line element with the conformal factor

$$\Omega(t',r') = -\sin t' \cdot 2\cos\left(\frac{t'+r'}{2}\right)\cos\left(\frac{t'-r'}{2}\right), \tag{2.4}$$

one obtains the "non-physical" metric

$$ds^2 = (\sin t')^2(-dt'^2 + dr'^2 + \sin^2 r' d\omega^2), \tag{2.5}$$

which is regular on

$$-\pi' < t' < 0,\ 0 \leqq r',\ |t'+r'| \leqq \pi,\ |t'-r'| \leqq \pi. \tag{2.6}$$

The metric (2.5) satisfies $ds^2 = \Omega^2 d\tilde{s}^2$ on the interior of (2.6), which corresponds to the part $\{t<0\}$ of Minkowski space. The boundary $\mathscr{I}^- = \{-t'+r'=\pi,\ -\pi<t'<0\}$ of (2.6) represents past null infinity of Minkowski space. The somewhat peculiar conformal factor (2.4) has been chosen (exploiting (3.4)) such that the Ricci scalar of the resulting metric (2.5) vanishes on the domain given by (2.6) (condition (3.5)). In the new coordinate system the conformal closure of $\tilde{H}$ is given by $H = \{t' = -\pi/2,\ 0 \leqq r' \leqq \pi/2\}$. It intersects $\mathscr{I}^-$ at $Z = \{t' = -\pi/2,\ r' = \pi/2\}$, and thus is an example of the type of surface the existence of which is stipulated in (2.3). The domain of dependence of H on the domain (2.6) is

$$D(H) = \left\{\left|t' + \frac{\pi}{2} + r'\right| \leqq \frac{\pi}{2}, \left|t' + \frac{\pi}{2} - r'\right| \leqq \frac{\pi}{2}, -\pi < t' < 0\right\}. \tag{2.7}$$

The future Cauchy horizon of H is the past null cone of the origin of Minkowski space extended to $\mathscr{I}^-$ with the origin removed, while the past Cauchy horizon is given by $\{-t'+r'=\pi,\ -\pi<t'\leqq -\pi/2\} \subset \mathscr{I}^-$. The Cauchy horizons intersect each other and H at Z.

The properties of the surfaces required in (2.3) and the preceding example give rise to

Definition (2.2) A triple $(\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta})$ is called a *hyperboloidal initial data set* if $\tilde{S}$ is a manifold diffeomorphic to the open until ball in $\mathbb{R}^3$, $\tilde{h}_{\alpha\beta}$ is a Riemannian metric, $\tilde{\chi}_{\alpha\beta}$ is a symmetric covariant tensorfield on $\tilde{S}$, such that

i) the vacuum constraints hold on $\tilde{S}$:

$$^3\tilde{r} + (\tilde{\chi}_\alpha{}^\alpha)^2 - (\tilde{\chi}_{\alpha\beta}\tilde{\chi}^{\alpha\beta}) = 0, \tag{2.8}$$

$$\tilde{D}_\alpha(\tilde{\chi}^\alpha{}_\beta - \tilde{h}^\alpha{}_\beta\tilde{\chi}^\gamma{}_\gamma) = 0. \tag{2.9}$$

Here $\tilde{D}_\alpha$ denotes the covariant derivative operator and $^3\tilde{r}$ the Ricci scalar defined by $\tilde{h}_{\alpha\beta}$.

ii) There exists a smooth conformal closure of this initial data set, i.e. $\tilde{S}$ may be diffeomorphically identified with the interior of a manifold S with boundary Z, where S is diffeomorphic to the closed unit ball in $\mathbb{R}^3$ (whence Z is diffeomorphic to the sphere $\mathbb{S}^2$), and on S there exist functions Ω, Σ such that:

$$\Omega > 0 \text{ on } \tilde{S};\ \Omega \equiv 0,\ \Sigma > 0 \text{ on } Z, \tag{2.10}$$

$$h_{\alpha\beta} = \Omega^2\tilde{h}_{\alpha\beta} \text{ extends to a smooth Riemannian metric on } S, \tag{2.11}$$

$$h^{\alpha\beta}\Omega_{,\alpha}\Omega_{,\beta} = \Sigma^2 \text{ on } Z\ (h^{\alpha\beta}h_{\beta\gamma} = \delta^\alpha{}_\gamma \text{ on } S), \tag{2.12}$$

$$\chi_{\alpha\beta} = \Omega(\tilde{\chi}_{\alpha\beta} + \Sigma \tilde{h}_{\alpha\beta}) \text{ extends to a smooth tensor-field on } S. \tag{2.13}$$

iii) $\tilde{\chi}_{\alpha\beta}, \tilde{h}_{\alpha\beta}$ satisfy further "fall-off" requirements near Z which are formulated in (4.10).

The fields $\Omega, \Sigma, h_{\alpha\beta}$ respectively $\chi_{\alpha\beta}$ correspond to the conformal factor, the derivative of the conformal factor in the direction of the future-directed unit normal (with respect to $g_{\mu\nu}$) on S, the first respectively the second fundamental form implied by $g_{\mu\nu}$ on S. Note that the conditions of Def. (2.2) are still satisfied after the rescalings

$$\Omega \to \Theta\Omega, \Sigma \to \Omega\Lambda + \Sigma,$$
$$h_{\alpha\beta} \to \Theta^2 h_{\alpha\beta}, \ \chi_{\alpha\beta} \to \Theta(\chi_{\alpha\beta} + \Lambda h_{\alpha\beta}), \tag{2.14}$$

with functions Θ, Λ on S such that $\Theta > 0$ on S. However, the conformal closure is uniquely determined by $\tilde{S}, \tilde{h}_{\alpha\beta}, \chi_{\alpha\beta}$ [23].

Since space-time will be constructed by solving the conformal vacuum field equations, the notion of a hyperboloidal initial data set has to be reformulated in terms of the conformally related structures. The analysis of the conformal vacuum field equations in Chaps. 3, 4 will give rise to the following definition, which is equivalent to Definition (2.2):

Definition (*2.3*). A pair (S, u_0) is called a *conformal hyperboloidal initial data set* if S is a manifold diffeomorphic to the closed unit ball in $\mathbb{R}^3$ with boundary Z, and if u_0 is a collection of smooth fields on S

$$u_0 = (\delta^{\mu}{}_0, e^{\alpha}{}_a, \gamma^a_{bc}, \chi_{ab}, \Omega, \Sigma, \Sigma_a, s, \sigma_a, \sigma_{ab}, d_{ab}, d_{abc}), \tag{2.15}$$

such that

i) $\Omega > 0$ on $\tilde{S} = S \backslash Z; \Omega \equiv 0, \ -(\Sigma)^2 + \Sigma_a \Sigma^a = 0, \ \Sigma > 0$ on Z,

ii) u_0 satisfies the conformal vacuum constraints (4.3) on S.

The notation used here and the meaning of the various quantities collected in u_0 will be explained in Chaps. 3, 4. The properties of asymptotically flat solutions listed in Definition (2.1) and the discussion of the domain of dependence motivate

Definition (*2.4*). A triple $\{D(S), \Omega, g_{\mu\nu}\}$ is called a *past asymptotically flat Cauchy development* of the hyperboloidal initial data set $\{\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta}\}$ if

i) $D(S)$ is a four-dimensional manifold which may be mapped by a homeomorphism which together with its inverse is "sufficiently smooth" onto some neighbourhood of H in $D(H)$ (2.7). Under this homeomorphism H corresponds to some hypersurface S of $D(S)$ with boundary, which will again be denoted by Z. The two hypersurfaces with common edge Z, which form the boundary of $D(S)$ will again be denoted by N and I.

ii) Ω is a function, $g_{\mu\nu}$ a Lorentz metric on $D(S)$ such that $(D(S), g_{\mu\nu})$ is an oriented, time-oriented causal spacetime, S is spacelike, N respectively I a null hypersurface in the future respectively past of S. It holds

$$\Omega > 0 \text{ on } D(S) \backslash I; \quad \Omega \equiv 0, \quad d\Omega \neq 0 \text{ on } I.$$

iii) The interior $S \backslash Z$ of S may be identified with $\tilde{S}$ by a homeomorphism, which together with its inverse is "sufficiently smooth", under which $\tilde{h}_{\alpha\beta}$ respectively $\tilde{\chi}_{\alpha\beta}$

is mapped onto the first respectively second fundamental form implied on $S\backslash Z$ by the metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ on $D(S)\backslash I$. By translating this into the conformally related situation one will obtain the equivalent

Definition (2.5). The pair $\{D(S), u\}$ is called a past asymptotically flat Cauchy development of the conformal hyperboloidal initial data set $\{S_0, u_0\}$ if condition (i) of Definition (2.4) holds and if u denotes a collection of fields as described in (3.13) such that the function Ω and the Lorentz metric $g_{\mu\nu}$ provided by u satisfy condition (ii) of Definition (2.4). Furthermore S_0 may be identified by a homeomorphism, which is smooth in both directions, with S such that u_0 is mapped on the quantity induced by u on S.

Now a solution of the *hyperboloidal initial data problem* in relativity for a given hyperboloidal initial data set $\{\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta}\}$ may be described as a past asymptotically flat Cauchy development $\{D(S), \Omega, g_{\alpha\beta}\}$ of this initial data set, such that the metric $\tilde{g}_{\mu\nu}$ solves Einstein's field equations (2.1) on $D(S)\backslash I$. Equivalently a solution of the conformal hyperboloidal initial value problem for a given conformal hyperboloidal initial data set $\{S_0, u_0\}$ is a past asymptotically flat Cauchy development $\{D(S), u\}$ of $\{S_0, u_0\}$ such that u satisfies the conformal vacuum field Eqs. (3.17) on $D(S)$.

The hyperboloidal initial value problem differs from the standard Cauchy problem essentially in two ways. The fall-off requirements for hyperboloidal initial data are quite different from the fall-off conditions for initial data which are euclidean at infinity. Furthermore the solution of the hyperboloidal initial value problem is required to contain past complete null geodesics.

3. The Conformal Vacuum Field Equations

Let $g_{\mu\nu}, \tilde{g}_{\mu\nu}$ be Lorentz metrics on a four-dimensional manifold which are related by a conformal rescaling with a conformal factor Ω

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}. \tag{3.1}$$

Expressing the vacuum field Eqs. (2.1) for $\tilde{g}_{\mu\nu}$ in terms of $g_{\mu\nu}$ and Ω and splitting them into the trace-free part and the trace, one arrives at the equivalent equations

$$\Omega \sigma_{\mu\nu} = -\nabla_\mu \nabla_\nu \Omega + g_{\mu\nu}(\tfrac{1}{4} \nabla_\lambda \nabla^\lambda \Omega), \tag{3.2}$$

$$\Omega^2 R = -6(\Omega \nabla_\lambda \nabla^\lambda \Omega - 2\nabla_\lambda \Omega \nabla^\lambda \Omega), \tag{3.3}$$

where R is the Ricci scalar, $2\sigma_{\mu\nu}$ the trace-free part of the Ricci tensor, and ∇ denotes the Levi-Civita covariant derivative with respect to $g_{\mu\nu}$. Considering these equations as differential equations for $g_{\mu\nu}$, with Ω thought of as being given, the principal part, i.e. the terms involving second order derivatives of $g_{\mu\nu}$, is contained in the left members. Consequently, assuming that $g_{\mu\nu}$ and therefore R and $\sigma_{\mu\nu}$ remain regular, the differential system (3.2), (3.3) becomes singular where Ω vanishes. The conditions given in Definition (2.1) are compatible with rescalings

$$(g, \Omega) \to (\Theta^2 g, \Theta \Omega) \tag{3.4}$$

with functions Θ which are positive everywhere. The Ricci scalars with respect to

$g_{\mu\nu}$ and with respect to $\Theta^2 g_{\mu\nu}$ are related by

$$\Theta R[g_{\mu\nu}] - \Theta^3 R[\Theta^2 g_{\mu\nu}] = 6\nabla_\lambda \nabla^\lambda \Theta.$$

Thus, given Cauchy data for Θ on an initial surface S with Θ positive on S, one may determine Θ in a neighbourhood of S such that the Ricci scalar of the rescaled metric $\Theta^2 g_{\mu\nu}$ vanishes. Whence the conformal factor Ω may be assumed to be such that

$$R = 0, \tag{3.5}$$

and by (3.3)

$$\Omega \nabla_\lambda \nabla^\lambda \Omega = 2\nabla_\lambda \Omega \nabla^\lambda \Omega. \tag{3.6}$$

These conditions are preserved under (3.4), if the function Θ is chosen such that the wave equation

$$\nabla_\lambda \nabla^\lambda \Theta = 0 \tag{3.7}$$

is satisfied. There remains the freedom to prescribe Cauchy data with Θ positive on S. The differential equation (3.6) for Ω obtained by the condition (3.5) still degenerates where Ω vanishes. To avoid this difficulty,Eqs. (3.2), (3.5), (3.6) will be incorporated into a larger system, which is in fact implied by (3.2), (3.5), (3.6), and be given a slightly different interpretation such that the new system remains regular even where Ω vanishes.

Let $g_{\mu\nu}$ be again a Lorentz metric on a four-dimensional manifold $M, (x^\mu)_{\mu=0,1,2\,3}$ some coordinate system, and let now ∇ denote a covariant derivative operator on M which will for the time being only be assumed to be metric with respect to $g_{\mu\nu}$

$$\nabla g_{\mu\nu} = 0. \tag{3.8}$$

The following unknowns will appear in the system we want to consider:

i) The components of a frame field $e_k = e_k{}^\mu(\partial/\partial x^\mu)$, $k = 0, 1, 2, 3$, with respect to x^μ. All tensorfields will be expressed with respect to this frame field. In particular

$$g_{kj} = g_{\mu\nu} e^\mu{}_k e^\nu{}_j, \tag{3.9}$$

and as usual one has

$$g^{ik} g_{kl} = \delta^i{}_l, \; g^{\mu\nu} g_{\lambda\nu} = \delta^\mu{}_\lambda, \; g^{\mu\nu} = g^{ik} e_i{}^\mu e_k{}^\nu \tag{3.10}$$

(frame indices i, j, k, l will take values $0, 1, 2, 3$ everywhere and the summation convention is assumed).

ii) The connection coefficients $\gamma^i{}_{jk}$ of ∇ with respect to e_k

$$\nabla_k e_j \equiv \nabla_{e_k} e_j = \gamma^i{}_{kj} e_i. \tag{3.11}$$

Equation (3.8) is then expressed equivalently by

$$g_{kj,\mu} e^\mu{}_l - g_{ij} \gamma^i{}_{lk} - g_{ki} \gamma^i{}_{lj} = 0. \tag{3.12}$$

iii) The "conformal factor" Ω and a 1-form Σ_i and a function s, the meaning of which will be explained later.

iv) A symmetric traceless (with respect to g_{ik}) tensorfield σ_{ij}.

v) A traceless tensorfield $d^i{}_{jkl}$ possessing all symmetries of the conformal Weyl tensor of the Lorentz metric g_{ik}.

The unknowns will be collected in the quantity

$$u = (e^\mu{}_k, \gamma^i{}_{jk}, \Omega, \Sigma_i, s, \sigma_{ij}, d^i{}_{jkl}), \tag{3.13}$$

g_{ij} being omitted here, because it will be given a fixed known value later by specifying a gauge of the framefield. In the later discussion the symmetries of the various objects will play a decisive role.
Given u and g_{ik}, one can define the "zero-quantity"

$$z = (O_j, P_j, Q_{jl}, T_j{}^k{}_l, K^i{}_{jkl}, L_{jkl}, H_{jkl}), \tag{3.14a}$$

where the various tensorfields constituting z are given by

$$\left.\begin{aligned} O_j &\equiv \nabla_j \Omega - \Sigma_j, \\ P_j &\equiv \nabla_j s + \sigma_j{}^k \Sigma_k, \\ Q_{jk} &\equiv \nabla_j \Sigma_k + \Omega \sigma_{jk} - s g_{jk}, \\ T^i{}_{jk} e_i{}^\mu &\equiv (\gamma^i{}_{jk} - \gamma^i{}_{kj}) e_i{}^\mu - (e_k{}^\mu{}_{,\nu} e^\nu{}_j - e^\mu{}_{j,\nu} e^\nu{}_k) \\ K^i{}_{jkl} &\equiv r^i{}_{jkl} - R^i{}_{jkl}, \\ L_{jkl} &\equiv \nabla_k \sigma_{lj} - \nabla_l \sigma_{kj} - d^i{}_{jkl} \Sigma_i, \\ H_{jkl} &\equiv \nabla_i d^i{}_{jkl}, \end{aligned}\right\} \tag{3.14b}$$

with $T_j{}^i{}_k$ being the torsion tensor of ∇ while

$$r^i{}_{jkl} \equiv e_k(\gamma^i{}_{lj}) - e_l(\gamma^i{}_{kj}) + \gamma^i{}_{km}\gamma^m{}_{lj} - \gamma^i{}_{lm}\gamma^m{}_{kj} - \gamma^i{}_{mj}(\gamma^m{}_{kl} - \gamma^m{}_{lk} - T^m{}_{kl}) \tag{3.15}$$

is the curvature tensor of ∇ and

$$R^i{}_{jkl} \equiv \Omega d^i{}_{jkl} + g^i{}_k \sigma_{lj} - g^i{}_l \sigma_{kj} + g_{jl}\sigma^i{}_k - g_{jk}\sigma^i{}_l. \tag{3.16}$$

The purpose of introducing the quantities collected by z is to give labels to the *conformal vacuum field equations* which will allow a convenient discussion of those equations. One has

Theorem (3.1). *Suppose g_{ik} and u as defined by (3.13) are such that (3.12) holds and the conformal vacuum field equations*

$$z(u) = 0 \tag{3.17}$$

are satisfied on M. Then, if Eq. (3.6) holds at one point of M, this equation in fact is satisfied everywhere on M. Furthermore the metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$, defined where Ω does not vanish, is a solution of Einstein's vacuum field equations (2.1).

Since this has been shown in previous papers [5, 9] only a short discussion of the meaning of Eq. (3.17) will be given here. The equation $T^i{}_{jk} = 0$ together with (3.12) implies that ∇ is in fact the Levi-Civita covariant derivative operator with respect to the metric g_{ik} and consequently $r^i{}_{jkl}$ as defined by (3.15) is the curvature tensor of g_{ik}. Because of $K^i{}_{jkl} = 0$, the representation of the curvature tensor by its irreducible parts is given by $R^i{}_{jkl}$. Thus $\Omega d^i{}_{jkl}$ represents the Weyl tensor, $2\sigma_{ik}$ the Ricci tensor of g_{ik}. Since σ_{ik} is traceless the Ricci scalar of g_{ik} vanishes, i.e. (3.5) is satisfied. From $O_j = 0$ it follows that Σ_j is the differential of Ω and $Q_j{}^j = 0$ implies

$s = \frac{1}{4}\nabla_k\nabla^k\Omega$. The equation $Q_{jk} = 0$ is now seen to be just (3.2). Furthermore one concludes now from (3.17)

$$\nabla_i(\Omega\nabla_j\Omega\nabla^j\Omega - 2\nabla_j\Omega\nabla^j\Omega) = 4(\Omega P_i - \nabla^j\Omega Q_{ij}) = 0. \tag{3.18}$$

Hence (3.6) must hold everywhere on M if it is satisfied at one point. In fact the surprising identity (3.18) is the motivation for the introduction of the equation $P_j = 0$, which itself is a consequence of the other equations implied by (3.17). The equation $H_{jkl} = 0$ is the vacuum Bianchi identity $\tilde{\nabla}_\mu \tilde{C}^\mu{}_{\nu\lambda\rho} = 0$ for the Weyl tensor of $\tilde{g}_{\mu\nu}$ expressed in terms of the rescaled Weyl tensor $d^\mu{}_{\nu\lambda\rho} = \Omega^{-1}\tilde{C}^\mu{}_{\nu\lambda\rho}$ and the connection ∇ with respect to $g_{\mu\nu}$. Using this equation and (3.5) one can derive $L_{jkl} = 0$ from the Bianchi identities for $r^i{}_{jkl}$.

Equation (3.17) together with (3.12) forms an overdetermined quasilinear first order differential system for g_{ik} and u. As a special case one may have

$$\Omega \equiv 1, \Sigma_i \equiv 0, s \equiv 0, \sigma_{ik} \equiv 0, \tag{3.19}$$

which will be referred to in the following as the "vacuum field equations." In this case (3.17) simplifies to

$$T^i{}_{jk} = 0, K^i{}_{jkl} = 0, H_{jkl} = 0. \tag{3.20}$$

Gauge Conditions. To integrate the conformal field equations (3.17) one has to specify an appropriate coordinate system (x^μ) and a frame field (e_μ). Both will depend on the choice of the conformal factor. While the conformal factor is fixed off the initial surface by (3.6) respectively (3.7), there is still the freedom to prescribe Cauchy data for (3.6) respectively (3.7) on an initial surface. Here the choice of Cauchy data for the conformal factor will be left open and will be dealt with in the same way as the Cauchy data for the other quantities figuring in (3.13).

The following coordinate system and frame field can be constructed for any choice of the conformal factor. Their choice is suggested by the resulting form of the "reduced conformal vacuum field equations" (5.1). For other purposes a different choice of coordinates and frame fields may be more convenient. Then one would use (3.12) to extract propagation equations for g_{ik}.

Let $(x^\alpha)_{\alpha=1,2,3}$ be a coordinate system and $(e_a)_{a=1,2,3}$ an orthonormal frame field on S (here as in the following indices $\alpha, \beta, \ldots, a, b, \ldots$ will take values $1, 2, 3$ and the summation convention will be assumed for these indices). A (Gauss) coordinate system $(x^\mu)_{\mu=0,1,2,3}$ and a frame field $(e_k)_{k=0,1,2,3}$, which extend the coordinates and frame field already given on S, will be fixed on a neighbourhood of S in $D(S)$ by the conditions

e_0 is the future-directed unit normal of S, the frame e_k is propagated parallel in the direction of e_0 on $D(S)$ (3.21)

x^0 is the parameter of the integral curves of e_0 which vanishes at S, $x^\alpha = \text{const}$ on the integral curves of e_0. (3.22)

In the rest of the paper—unless stated otherwise—this choice of coordinates and frame fields will be assumed. With respect to $(x^\mu), (e_k)$, one then finds

$$g_{ik} = \text{diag}(-1, 1, 1, 1), \tag{3.23}$$

$$e_0 = \frac{\partial}{\partial x^0}, \; e_a = e_a^\alpha \frac{\partial}{\partial x^\alpha}, \tag{3.24}$$

and the e_a are tangent to the surfaces $S_t = \{x^0 = t\}$. Furthermore

$$\gamma_0{}^i{}_k = 0. \tag{3.25}$$

Because of (3.23), Eq. (3.12) reduces to the algebraic condition

$$g_{ij}\gamma^i{}_{lk} + g_{ki}\gamma^i{}_{lj} = 0. \tag{3.26}$$

Let D denote the covariant Levi–Civita derivative-operator implied on the surface $S = S_0$ (respectively on S_t) by the interior metric induced by g_{ik}. Then one has

$$D_a e_b \equiv D_{e_a} e_b = \gamma_{ab}{}^c e_c. \tag{3.27}$$

For the remaining connection coefficients which do not necessarily vanish because of (3.25), (3.26) one has

$\chi_{ab} \equiv \gamma_{ab}{}^0 = \gamma_a{}^b{}_0$ is the second fundamental form on S(respectively on S_t), expressed with respect to e_a. (3.28)

There is still the freedom to choose the coordinates x^α, the frame field (e_a), the conformal factor Ω and its normal derivative $\Sigma \equiv \Sigma_0$ on S. In the following only near Z a more detailed discussion of a possible gauge will be given to illustrate the specific properties of the intersection of S with null infinity. For computational reasons, for giving geometrical interpretations and for the comparison of the general case with the standard situation in Minkowski space it is convenient to choose near Z the field e_a on S such that

e_1 is the outward pointing unit normal of Z in S(hence e_1 is tangent to S) and $D_1 e_a = 0$ near Z. (3.29)

(In the following indices $A, B, \ldots$ will take the values 3, 4, will be lowered or raised by $g_{AB} = g^{AB} = \mathrm{diag}(1.1)$, and the summation convention is assumed.) Then $\gamma_{AB}{}^1$ is the second fundamental form on Z with respect to S.
It holds

$$\Omega = 0, \Sigma_i \Sigma^i = 0 \text{ on } Z, \tag{3.30}$$

$$\Sigma_A = 0, \Sigma = -\Sigma_1 > 0 \text{ on } Z. \tag{3.31}$$

Equation $Q_{AB} = 0$ implies on Z

$$-(\chi_{AB} - \gamma_{AB}{}^1) = s g_{AB} \tag{3.32}$$

from which follows in particular

$$\chi_{23} = \gamma_{23}{}^1, \chi_{22} - \chi_{33} = \gamma_{22}{}^1 - \gamma_{33}{}^1. \tag{3.33}$$

These relations show that the shear of null infinity vanishes [1]. The equation $K^0{}_{1AB} = 0$ together with (3.33) gives on Z an integrability condition by which one concludes that

$$\chi_{A1} = e_A(\chi) \text{ on } Z \tag{3.34}$$

with some function χ on Z. From this it follows that by a boost in the plane orthogonal to Z, i.e. by tilting the tangent plane of S at Z appropriately, one can obtain

$$\chi_{A1} = 0 \text{ on } Z. \tag{3.35}$$

The freedom in (3.4), (3.7) to prescribe $\Theta, e_1(\Theta), e_0(\Theta), e_1(e_1(\Theta)), e_1(e_0(\Theta))$ on Z can be used to obtain (using (3.33) and the equations $Q_{jk} = 0, K^A{}_{BCD} = 0$) on Z:

$$\left.\begin{array}{l} g_{\mu\nu} \text{ implies the standard unit metric of the} \\ \text{two-sphere on } Z, \chi_{22} + \chi_{33} = 0, \gamma_{22}{}^1 + \gamma_{33}{}^1 = 0, \\ \chi_{AB} = \gamma_{AB}{}^1, s = 0, \ -\sigma_{1B} = \sigma_{0B} = g^{AC} \Delta_C \chi_{AB}, \\ \sigma_{10} = 0, 2\sigma_{11} = 1, g^{AB}\sigma_{AB} = 1, \end{array}\right\} \tag{3.36}$$

where Δ_A denotes the covariant derivative operator on Z in the direction of e_A. The surface S may furthermore be chosen such that

$$\chi_{11} = \chi^a{}_a = 0 \text{ on } Z. \tag{3.37}$$

The two unknown functions in $\chi_{ab} = \gamma_{ab}^1$ contain the information on the shear of the null hypersurface N, while the two unknown functions in σ_{ab} contain the information on the Bondi–Sachs news function [2, 3, 27]. These functions constitute free initial data in the asymptotic characteristic initial value problem.

With the choice (3.30–37) one has

$$\nabla_i \nabla_k \Omega = 0, e_j(e_k(\Omega)) = 0 \text{ on } Z. \tag{3.38}$$

4. The Conformal Constraint Equations

With the choice of frame above the conformal vacuum field Eqs. (3.17) can easily be split into the propagation equations and into the constraint equations implied on the surfaces S_t, in particular on S. The quantity u defines various tensorfields on S which will be expressed with respect to (e_a). To obtain convenient expressions for the constraint equations the following notation is used:

$$\Sigma = \Sigma_0, \sigma_a = \sigma_{a0}, d_{ab} = d_{a0b0}, d_{abc} = d_{a0bc}.$$

These tensors on S have the algebraic properties

$$\sigma^a{}_a = \sigma_{00}, d_{ab} = d_{ba}, d^a{}_a = 0, d_{acb} = -d_{acb}, d_{[abc]} = 0, d^a{}_{ac} = 0, \tag{4.1}$$

$$d_{eabc} = g_{eb} d_{ac} - g_{ec} d_{ab} + g_{ac} d_{be} - g_{ab} d_{ce}. \tag{4.2}$$

The fields d_{ab}, d_{abc} contain all the information on the rescaled Weyl tensor (essentially the "electric" and "magnetic" part).

The constraint equations implied by (3.17) on S are

$$\left.\begin{array}{l} 0 = T_a{}^b{}_c, \\ 0 = O_a = D_a\Omega - \Sigma_a, \\ 0 = P_a = D_a s + \sigma_{ab}\Sigma^b - \sigma_a \Sigma, \\ 0 = Q_a \equiv Q_{a0} = D_a \Sigma - \chi_{ac}\Sigma^c + \Omega \sigma_a, \\ 0 = Q_{ab} = D_a \Sigma_b - \chi_{ab}\Sigma + \sigma_{ab}\Omega - g_{ab} s, \\ 0 = K^e{}_{abc} = {}^3r^e{}_{abc} + \chi^e{}_b \chi_{ac} - \chi^e{}_c \chi_{ab} - \Omega d^e{}_{abc} - g^e{}_b \sigma_{ca} \\ \quad + g^e{}_c \sigma_{ba} - g_{ac}\sigma_b{}^e + g_{ab}\sigma_c{}^e, \\ \quad \text{(Gauss' equation)} \end{array}\right\} \tag{4.3}$$

$$\left.\begin{aligned}
&0 = K_{abc} \equiv K^0{}_{abc} = D_b\chi_{ca} - D_c\chi_{ba} - \Omega d_{abc} - g_{ab}\sigma_c + g_{ac}\sigma_b,\\
&\quad\text{(Codacci's equation)}\\
&0 = L_{cab} = D_a\sigma_{bc} - D_b\sigma_{ac} - \chi_{ac}\sigma_b + \chi_{bc}\sigma_a - d_{cab}\Sigma - d_{ecab}\Sigma^e,\\
&0 = L_{ab} \equiv L_{0ab} = D_a\sigma_b - D_b\sigma_a - \chi_a{}^c\sigma_{bc} + \chi_b{}^c\sigma_{ac} - d_{cab}\Sigma^c,\\
&0 = H_{bc} \equiv H_{0bc} = D_a d^a{}_{bc} - \chi^e{}_c d_{eb} + \chi^e{}_b d_{ec},\\
&0 = H_b \equiv H_{0b0} = D_a d^a{}_b - \chi^{ef} d_{ebf}.
\end{aligned}\right\}$$

By taking linear combinations of some of those equations (3.17) which do contain the operator e_0, interior equations are obtained on S which, however, are also implied by (4.3). Contracting indices in Gauss' and in Codacci's equations gives in particular:

$$0 = K_{ac} \equiv {}^3r_{ac} + \chi^b{}_b\chi_{ac} - \chi^b{}_c\chi_{ab} - \Omega d_{ac} - \sigma_{ac} - g_{ac}\sigma^b{}_b, \tag{4.4}$$

$$0 = K \equiv {}^3r + (\chi^b{}_b)^2 - \chi^{bc}\chi_{bc} - 4\sigma^b{}_b, \tag{4.5}$$

$$0 = K_c \equiv D_b(\chi_c{}^b - g_c{}^b\chi^d{}_d) - 2\sigma_c. \tag{4.6}$$

In the equations above ${}^3r^a{}_{bcd}$ denotes the curvature of the metric implied on S, ${}^3r_{ac}$ its Ricci Tensor, ${}^3r$ its Ricci scalar. The constraint equations (4.3) are considerably more complicated than the vacuum constraints (2.8), (2.9). However, the equations (4.3) are not independent, since integrability conditions have been used to build up the system (3.17). The following relations are obtained by straightforward calculations if $T_a{}^b{}_c = 0$ is assumed to hold:

$$\begin{aligned}
&\Omega L_{ab} + \Sigma^d K_{dba} + O_a\sigma_b - O_b\sigma_a + Q_{bd}\chi^d{}_a\\
&\quad - \Omega_{ad}\chi^d{}_b + D_bQ_a - D_aQ_b = 0,\\
&\Omega L_{cba} + \Sigma^d K_{dabc} + \Sigma K_{abc} + O_c\sigma_{ba} - O_b\sigma_{ca} + Q_b\chi_{ca} - Q_c\chi_{ba}\\
&\quad + g_{ac}P_b - g_{ab}P_c + D_bQ_{ca} - D_cQ_{ba} = 0,\\
&\Omega H_b + L_{ab}{}^a + L_{eab}\chi^{ea} + K_b\chi^e{}_e - K_f\chi^f{}_b + O_c d^c{}_b\\
&\quad + D_fK^f{}_b - \tfrac{1}{2}D_bK = 0,\\
&\Omega H_{bc} + L_{cb} + K_{fc}\chi^f{}_b - K_{fb}\chi^f{}_c + O_e d^e{}_{bc} + D_aK^a{}_{bc}\\
&\quad + D_cK_b - D_bK_c = 0,\\
&\Sigma L_{ab} + \Sigma^d L_{abd} + Q_a\sigma_b - Q_b\sigma_a + Q_{bd}\sigma_a{}^d - Q_{ad}\sigma_b{}^d\\
&\quad + D_aP_b - D_bP_a = 0.
\end{aligned} \tag{4.7}$$

$$D_a(2\Omega s + (\Sigma)^2 - \Sigma_b\Sigma^b) = 2(sO_a + \Omega P_a + \Sigma Q_a - \Sigma^c Q_{ca}). \tag{4.8}$$

From (4.8) follows immediately

Lemma (4.1). *If the quantity u is such that (3.30) holds and if u satisfies the constraints (4.3) on S then*

$$2\Omega s = \Sigma_i\Sigma^i \tag{4.9}$$

holds on S.

If a hyperboloidal initial data set (Definition (2.2)) is given, then by (2.10)–(2.13) one may assume that the conformal closure S and on it the fields Ω, Σ, $\Sigma_a = e_a(\Omega)$,

e_a, χ_{ab} are known. Then s can be calculated from (4.9) and consequently σ_a and σ_{ab} follow from $Q_a = 0$, $Q_{ab} = 0$. Since the vacuum constraints are satisfied by the physical quantities, one concludes from the transformation behaviour of the various fields under conformal rescalings that in fact Eq. (4.5), (4.6) are satisfied by the fields known so far. Because of this the fields d_{ab}, d_{abc}, obtained by requiring $K_{ac} = 0$ (or, equivalently, $K^e{}_{abc} = 0$) and $K_{abc} = 0$ are in fact tracefree. Now all the fields constituting the unknown u are given and it follows from the identities (4.7) that in fact all the constraint equations (4.3) are satisfied. The fall-off conditions mentioned in Definition (2.2) can now be formulated in terms of smoothness conditions on (S, u).

$$\left.\begin{array}{l}\text{The fields } s,\ \sigma_a,\ \sigma_{ab},\ d_{ab},\ d_{abc} \text{ obtained by}\\ \text{solving (4.9), } Q_a = 0,\ Q_{ab} = 0,\ K_{ac} = 0,\\ K_{abc} = 0 \text{ for these quantities are required to}\\ \text{be ``sufficiently smooth.''}\end{array}\right\} \tag{4.10}$$

More precise smoothness conditions will be formulated in Chap. 6.

If hyperboloidal initial data sets are to be obtained by means of York's approach [18, 19, 15] one has to express (4.10) in terms of the physical quantities $\tilde{h}_{\alpha\beta}$, $\tilde{\chi}_{\alpha\beta}$. To avoid this problem and to make it easier to formulate the appropriate smoothness conditions near Z, it is desirable to have an analogue of York's procedure which applies immediately to Eqs. (4.3). For this it is of interest that by the identities (4.7) it suffices to solve only part of Eq. (4.3) while the rest follows automatically.

To illustrate the problems which occur by working in terms of the nonphysical quantities and at the same time to give examples of the type of surfaces one will have to deal with, one may consider the case where the second fundamental form χ_{ab} vanishes everywhere on S. By (4.3) one must have $\Sigma = \text{const}$, $\sigma_a \equiv 0$, $d_{abc} \equiv 0$ and from $Q_a{}^a = 0$, (4.5), (4.9)

$$4\Omega D_a D^a \Omega + {}^3r\Omega^2 - 6D_a\Omega D^a\Omega = -6(\Sigma)^2. \tag{4.11}$$

This equation is the analogue of Lichnerowicz's equation (which is in fact the formally regular equation obtained from (4.11) by expressing it in terms of the singular quantity $\Phi = \Omega^{-1/2}$) if the nonphysical ("test") metric is thought as being given. The equation is singular on Z. To escape the problems posed by this degeneracy one may try to solve (4.3) which represents a regular system. However, then $Q_a{}^a = 0$ couples via s in a complicated way to the other equations and it is not easy to see whether (4.3) can be broken up into a hierarchy of "manageable" equations.

Considering (4.11) for a moment not as a differential equation for Ω, one may assume $\Omega \equiv 1$. Then (4.11) reduces to the vacuum constraint ${}^3\tilde{r} = -6(\Sigma)^2$, which shows that the initial surface has a constant negative Ricci scalar. From (2.13) follows in fact that $\tilde{\chi}_{\mu\nu} = -\Sigma\tilde{h}_{\mu\nu}$ on $\tilde{S}$. Surfaces of this type are provided by the hyperboloids in Minkowski space.

Just to see whether the set of initial data with the appropriate behaviour near

Z is sufficiently rich, one may take refuge in the analytic case. For analytic data for the asymptotic characteristic initial problem, which are essentially given by 4 analytic functions of three variables, there exists a unique solution of the conformal vacuum field equations [13]. This, of course, provides solutions of (4.3) on spacelike surfaces intersecting null infinity. However, it is not clear how many of these local solutions near Z extend to a solution on a simply connected surface.

5. The Reduced Conformal Vacuum Field Equations

The complete information about the propagation equations contained in the conformal vacuum field equations (3.17) is reflected by the following system (5.1) of "reduced conformal vacuum field equations." It is obtained by taking linear combinations of Eqs. (3.17). Here the gauge (3.21), (3.22) is assumed by which ∇_0 is reduced to $\partial/\partial x^0$.

$$\left.\begin{aligned}
&0 = O_0 = \Omega_{,0} - \Sigma_0,\\
&0 = P_0 = s_{,0} + \sigma_{0j}\Sigma^j,\\
&0 = Q_{0k} = \Sigma_{k,0} + \Omega\sigma_{0k} - s g_{0k},\\
&0 = T_j{}^i{}_0 e_i{}^\mu = e^\mu{}_{j,0} + \gamma_j{}^i{}_0 e_i{}^\mu,\\
&0 = K^i{}_{j01} = \gamma^i{}_{1j,0} + \gamma^i_{mj}\gamma^m_{10} - \Omega d^i{}_{j01} - g^i{}_0\sigma_{1j}\\
&\qquad + g^i{}_1\sigma_{0j} - g_{j1}\sigma^i{}_0 + g_{j0}\sigma^i{}_1,
\end{aligned}\right\} \tag{5.1a}$$

$$\left.\begin{aligned}
&0 = -\eta^{jk}L_{jka} = \nabla_0\sigma_{0a} - \nabla_1\sigma_{1a} - \nabla_2\sigma_{2a} - \nabla_3\sigma_{3a}, \quad a = 1, 2, 3,\\
&0 = L_{a0a} = \nabla_0\sigma_{aa} - \nabla_a\sigma_{0a} - \Sigma_i d^i{}_{a0a}, \quad a = 1, 2, 3,\\
&0 = L_{a0b} - L_{b0a} = 2\nabla_0\sigma_{ab} - \nabla_a\sigma_{0b} - \nabla_b\sigma_{0a}, \quad (a,b) = (1,2), (1,3), (2,3),
\end{aligned}\right\} \tag{5.1b}$$

$$\left.\begin{aligned}
0 &= H_{212} - H_{313} + H_{202} - H_{303} = \nabla_i(d^i{}_{212} - d^i{}_{313} + d^i{}_{202} - d^i{}_{303}),\\
0 &= -H_{102} + H_{121} = \nabla_i(-d^i{}_{102} + d^i{}_{121}),\\
0 &= H_{101} = \nabla_i(d^i{}_{101}),\\
0 &= H_{102} + H_{121} = \nabla_i(d^i{}_{102} + d^i{}_{121}),\\
0 &= H_{313} - H_{212} + H_{202} - H_{303} = \nabla_i(d^i{}_{313} - d^i{}_{212} + d^i{}_{202} - d^i{}_{303}),\\
0 &= H_{213} + H_{312} + H_{203} + H_{302} = \nabla_i(d^i{}_{213} + d^i{}_{312} + d^i{}_{203} + d^i{}_{302}),\\
0 &= -H_{103} + H_{131} = \nabla_i(-d^i{}_{103} + d^i{}_{131}),\\
0 &= -H_{123} = \nabla_i(-d^i{}_{103}),\\
0 &= -H_{103} - H_{131} = \nabla_i(-d^i{}_{103} - d^i{}_{131}),\\
0 &= H_{213} + H_{312} - H_{203} - H_{302} = \nabla_i(d^i{}_{213} + d^i{}_{312} - d^i{}_{203} - d^i{}_{302}).
\end{aligned}\right\} \tag{5.1c}$$

This is a first order quasilinear system of the form

$$A^\mu u_{,\mu} + B\cdot u = 0, \tag{5.2}$$

where the unknown $d^i{}_{jkl}$ is understood as being represented by the ten unknowns

$$\begin{aligned}
&d_1 = d^0{}_{212} - d^0{}_{313} + d^0{}_{202} - d^0{}_{303},\\
&d_2 = -d^0{}_{102} + d^0{}_{121}, d_3 = d^0{}_{101}, d_4 = d^0{}_{102} + d^0{}_{121},\\
&d_5 = d^0{}_{313} - d^0{}_{212} + d^0{}_{202} - d^0{}_{303},\\
&d_6 = d^0{}_{213} + d^0{}_{312} + d^0{}_{203} + d^0{}_{302},\\
&d_7 = -d^0{}_{103} + d^0{}_{131}, d_8 = -d^0{}_{123}, d_9 = -d^0{}_{103} - d^0{}_{131},\\
&d_{10} = d^0{}_{213} + d^0{}_{312} - d^0{}_{203} - d^0{}_{302}.
\end{aligned} \tag{5.3}$$

With this proviso, the system (5.1) written in the order as given above, constitutes a quasilinear "symmetric hyperbolic system" [11,28].

The square matrices A^μ, B are of the form

$A^\mu = A^k e_k{}^\mu$, $B = B_1 + B_2(u)$
with constant matrices A^k, B_1 and B_2 being a linear function of u. Here $e_k{}^\mu$ is the frame provided by u. (5.4)

The matrices A^μ are symmetric and A^0, being a constant matrix with only 1's and 2's on the diagonal and zeros in all the other entries, is positive definite for any frame e_k satisfying (2.34). (5.5)

The subsystem (5.1c) and the combinations (5.3) look somewhat complicated here, however, they come out quite naturally in the spin frame formalism where it is much easier to keep track of the symmetries of the Weyl tensor and of the Bianchi identities [5].

If u is a solution of (5.1) respectively (5.2) the quantity $z(u)$ determined by (3.14) from u need not vanish necessarily since not all equations contained in (3.17) are given by (5.1). However, one has

Theorem (5.1). *If u is a solution of (5.1), the quantity z defined from u by (3.14) satisfies a "subsidary system" of the form*

$$F^\mu z_{,\mu} + Gz = 0 \tag{5.6}$$

(constituted by the subsystems (5.7), (5.8), (5.10), (5.12), (5.14)). Here the F^μ are symmetric matrices depending on the frame $e^\mu{}_k$ supplied by u, F^0 is a constant matrix with 1's and 2's on the diagonal and zeros in the other entries, hence positive definite, and G is a square matrix which is a function of u and its first order derivatives.

The properties of F^μ, G ensure in fact that (5.6) is a linear symmetric hyperbolic system of differential equations for z.

An analogue of Theorem (5.1) has been shown in the case of the characteristic and asymptotic characteristic initial value problems [9] where, however, somewhat different reduced equations have been used. The basic argument, which has been explained there in detail, is essentially the same: The Bianchi identities for the (metric) connection ∇ which is defined by the connection coefficients (satisfying (3.25), (3.26)) supplied by the solution of (5.1), the symmetries of the various fields which constitute u, and the fact that by (5.1) some components of z vanish are exploited to derive (5.6). To display clearly the principal part of this system it is convenient to introduce the notation

$$\nabla_j - e_j = \Gamma_j.$$

Then e.g. for a tensor L_j with $L_0 = 0$, one has

$$\nabla_j L_0 = \Gamma_j{}^i{}_0 L_i = \Gamma_j L_0,$$

and no derivative operator is involved.

The equations

$$\begin{aligned} O_{j,0} &= \Gamma_j O_0 + Q_{j0}, \\ P_{j,0} &= \Gamma_j P_0 + L_{l0j}\Sigma^l - \sigma_0{}^l Q_{jl}, \\ Q_{lj,0} &= \Gamma_l Q_{0j} - O_l \sigma_{0j} + 2P_l \sigma_{0j} + \Omega L_{j0l}, \end{aligned} \tag{5.7}$$

are obtained by using the definition of z to express

$$\nabla_l O_j - \nabla_j O_l, \nabla_k P_j - \nabla_j P_k, \nabla_k Q_{lj} - \nabla_l Q_{kj}$$

in terms of the components of z and of the commutator of ∇_l, ∇_j which is given by the curvature tensor $K^i{}_{jkl} + R^i{}_{jkl}$ of ∇ and the torsion $T^i{}_{jk}$, and by evaluating the expressions so obtained for $k = 0$, taking into account (5.1).

The Bianchi identity

$$\sum_{(jkl)} \nabla_j T_{kl}{}^h = \sum_{(jkl)} (r^h{}_{ljk} + T_{jl}{}^m T_{mk}{}^h),$$

(where $\sum_{(jkl)}$ denotes the sum over the cyclic permutation of the indices j,k,l) evaluated for $j = 0$ gives in view of (5.1) and the symmetries of the tensor fields involved:

$$T^h_{kl,0} = -\Gamma_k T^h_{l0} - \Gamma_l T_0{}^h{}_k + K^h{}_{0kl}. \tag{5.8}$$

The identity

$$\begin{aligned} 2\sum_{(mij)} \nabla_m d_{klij} &= -\varepsilon_{mij}{}^r \nabla_t d^t{}_{rpq} \varepsilon_{kl}{}^{pq} \\ &= -\varepsilon_{mij}{}^r H_{rpq} \varepsilon_{kl}{}^{pq}, \end{aligned} \tag{5.9}$$

which follows from the behaviour of tensors with the algebraic properties of conformal Weyl tensors under duality operations, allows one to derive

$$\begin{aligned} \sum_{(jkl)} \nabla_j R^m{}_{ikl} = \sum_{(jkl)} & (O_j d^m{}_{ikl} + g^m{}_l L_{ikj} + g_{il} L^m{}_{jk}) \\ & + \tfrac{1}{2}\Omega \varepsilon_{jkl}{}^r H_{rpq} \varepsilon_i{}^{mpq}. \end{aligned}$$

Evaluating this and the Bianchi identity

$$\sum_{(jkl)} \nabla_i r^m{}_{ikl} = \sum_{(jkl)} T^h{}_{kj} r^m{}_{ihl}$$

for $j = 0$ gives

$$\begin{aligned} K^m{}_{ikl,0} = & -\Gamma_k K^m{}_{il0} - \Gamma_l K^m{}_{i0k} + \tfrac{1}{2}\Omega \varepsilon_{0kl}{}^r H_{rpq} \varepsilon_i{}^{mpq} \\ & + 3(r^m{}_{ih[} T_{k0]}{}^h + d^m{}_{i[kl} O_{0]} + L_{i[k0} g_{l]}{}^m + L^m{}_{[0k} g_{l]i}). \end{aligned} \tag{5.10}$$

Taking into account the definition (3.14b) of L_{ikl}, expressing again commutators of covariant derivatives by the curvature and torsion tensors of ∇, one finds for the quantity

$$L_{ijkl} = \sum_{(jkl)} \nabla_j L_{ikl}, \tag{5.11}$$

the following expression, which is linear in the components of z,

$$L_{ijkl} = \sum_{(jkl)} (\sigma_{it} K^t{}_{lkj} + \sigma_{lt} K^t{}_{ikj} - Q_{jt} d^t{}_{ikl}) + \tfrac{1}{2} \varepsilon_{jkl}{}^r H_{rpq} \varepsilon_{ti}{}^{pq} \Sigma^t. \tag{5.12a}$$

Here again (5.9) has been used. Interpreting now the right member of (5.11) as the result of the action of a differential operator on L_{ikl}, the following three subsystems (for $(a, b) = (1, 2), (1, 3), (2, 3)$) of differential equations for those components of L_{ijk}, which do not vanish necessarily because of (5.1), are obtained

$$\begin{aligned} &2\nabla_0 L_{cab} - \nabla_c L_{0ab} = \Gamma_a(L_{b0c} + L_{c0b}) - \Gamma_b(L_{c0a} + L_{a0c}) + 2L_{c0ab} - L_{0cab}, c = 1, 2, 3, \\ &\nabla_0 L_{0ab} - g^{cd} \nabla_c L_{dab} = L_{00ab} + L_{a0b0} + L_{b00a} - g^{cd} L_{cdab}. \end{aligned} \tag{5.12b}$$

Here on the right hand side L_{ijkl} is understood as being given by (5.12a).

Similarly, for

$$H_{pq} = \nabla_r H^r{}_{pq} \tag{5.13}$$

one derives on the one hand, using the definition of H_{rpq} and (5.9), the identity

$$\begin{aligned} H_{pq} = &- \tfrac{1}{24} \varepsilon^{mijs} (3 T^t_{n\,s} \nabla_t d_{klij} + 6K^t{}_{kms} d_{tlij} + 2K^t{}_{mjs} d_{klti} \\ &+ 2K^t{}_{ims} d_{kltj} + 2K^t{}_{jis} d_{kltm}) \varepsilon_{pq}{}^{kl}. \end{aligned} \tag{5.14a}$$

Here again the right member is linear in the components of z. While on the other hand, interpreting the right member of (5.13) again as the result of the action of a differential operator on H_{rpq}, one derives for those components of H_{rpq} which do not vanish necessarily because of (5.1) the differential system:

$$\begin{aligned} \nabla_0 H_{023} + \nabla_2 H_{030} - \nabla_3 H_{020} &= - H_{23} + \Gamma_1 H_{123} + \Gamma_2 (H_{121} + H_{333}) \\ &\quad + \Gamma_3 (H_{121} + H_{222}), \\ 2\nabla_0 H_{013} - \nabla_3 H_{010} &= - 2H_{13} - \Gamma_3 (H_{212} - H_{313}) \\ &\quad - \Gamma_1 H_{131} + \Gamma_2 H_{213} - \Gamma_3 H_{111}, \\ 2\nabla_0 H_{012} - \nabla_2 H_{010} &= - 2H_{12} + \Gamma_2 (H_{212} - H_{313}) \\ &\quad - \Gamma_2 H_{111} - \Gamma_1 H_{121} - \Gamma_3 H_{321}, \\ 2\nabla_0 H_{030} + \nabla_2 H_{023} &= - 2H_{30} - \Gamma_2 (H_{302} + H_{203}) \\ &\quad - 2\Gamma_1 H_{103} - 2\Gamma_3 H_{303}, \\ 2\nabla_0 H_{020} - \nabla_3 H_{023} &= - H_{20} - \Gamma_3 (H_{302} + H_{203}) \\ &\quad - 2\Gamma_1 H_{102} - 2\Gamma_2 H_{202}, \\ \nabla_0 H_{010} - \nabla_2 H_{012} - \nabla_3 H_{013} &= - H_{10} - \Gamma_1 H_{101} + \Gamma_2 H_{120} + \Gamma_3 H_{130}, \end{aligned} \tag{5.14b}$$

where on the right hand side H_{pq} is regarded as being given by (5.14a). The system of equations (5.7), (5.8), (5.10), (5.12), (5.14) (supplemented by equations of the type

$x_{,0} = 0$ for those components of x of z which vanish already because of (5.1), to obtain a system of the form (5.6)) constitute the symmetric hyperbolic system of subsidiary equations.

Although the zero-quantity z defined by (3.14) from a solution u of (5.1) need not vanish everywhere, one has

Lemma (3.2). *If u, given by (3.13), is defined near S and satisfies the conformal constraint Eqs. (4.3) and the reduced conformal vacuum field Eqs. (5.1) on S, the zero quantity $z(u)$ defined by (3.14) from u vanishes on S. Furthermore*

$$s = \tfrac{1}{4}\nabla_i \nabla^i \Omega \tag{5.15}$$

holds on S. If u is such that $\Omega = 0, \Sigma_i \Sigma^i = 0$ on Z, then Eq. (3.6)

$$\Omega \nabla_k \nabla^k \Omega = 2\nabla_k \Omega \nabla^k \Omega$$

is satisfied on S.

The vanishing of $T^0{}_{ab}$ is equivalent to the symmetry of the second fundamental form χ_{ab} on S. The vanishing of all the other components of z follows by taking linear combinations of the constraint equations (4.3) and the propagation equations (5.1) on S. The relation (5.15) then follows from $Q^a{}_a = 0$ and (3.6) is obtained from (5.15) and Lemma (4.1).

It may be noted that the torsion has been introduced here only as a technical device.

6. On the Solution of the Hyperboloidal Initial Value Problem

There exists an extensive literature on linear symmetric hyperbolic systems (see the lists in [29, 16]), which were introduced and studied first by K. O. Friedrichs [11]. Existence and uniqueness proofs for quasilinear symmetric hyperbolic systems have been sketched in [28] and worked out in detail by Fischer & Marsden [16]. These authors applied their results to the standard Cauchy problem in relativity by expressing the field equations with respect to harmonic coordinates and writing them as a first order quasilinear symmetric hyperbolic system. Their treatment of quasilinear systems was generalized by Kato. Taylor [30] used a different approach and obtained weaker results. In the following Kato's results as described in [31] will be employed.

Kato proved for a very general class of symmetric hyperbolic systems existence and uniqueness theorems, local in time, for Cauchy data given on $\mathbb{R}^n$. These apply in particular to the system (5.1). By (3.13) the unknown u takes its values in $\mathbb{R}^N$ for some N. For any positive integer s. let $H^s(\Omega, \mathbb{R}^N)$ (sometimes denoted simply by $H^s(\Omega)$ or H^s if the meaning is clear from the context) denote the L^2-type Sobolev-space of $\mathbb{R}^N$-valued functions defined on a domain Ω of $\mathbb{R}^k$ with respect to the measure implied by the standard euclidean metric on $\mathbb{R}^k$.

Theorem (6.1). *Suppose $s \geqq 3$ and initial data v_0 of the form (3.13) are given such that $e^\mu{}_0 = \delta^\mu{}_0$, $v_0 = (\delta^\mu{}_0, w_0)$ and $w_0 = (e^\mu{}_a, \gamma^i{}_{jk}, \Omega, \Sigma_i, s, \sigma_{ij}, d^i{}_{jkl}) \in H^s(\mathbb{R}^3, \mathbb{R}^N)$.*
Then there exists a unique solution $v(t)$ of (5.1), defined on $[-T, T]$ with some $T > 0$,

which takes the value v_0 *for* $t=0$ *and is of the form* $v(t)=(\delta^\mu{}_0, w(t))$ *with (w defined similarly as above)* $w \in C[-T, T; H^s(\mathbb{R}^3, \mathbb{R}^{N'})] \cap C^1[-T, T; H^{s-1}(\mathbb{R}^3, \mathbb{R}^{N'})]$.
The number T can be chosen common to all initial conditions $v_0' = (\delta^\mu{}_0, w_0')$ *such that* w_0' *is sufficiently close to* w_0.

This follows by straightforward application of Theorem 2 in [31], observing (5.4), (5.5).

Remarks. (i) In his paper Kato formulates a stability property for solutions of systems of the type (5.1). This will not be reproduced here.

(ii) By inspecting the particular structure of (5.1a) one finds that the smoothness result for some components of v can be improved:

$$(s, \Sigma_k, \gamma^i{}_{kj}) \in C^1[-T, T; H^s(\mathbb{R}^3, \mathbb{R}^{N''})],$$
$$(\Omega, e^\mu{}_k) \in C^2[-T, T; H^s(\mathbb{R}^3, \mathbb{R}^{N'''})]. \tag{6.1}$$

(iii) Using the multiplicative properties of Sobolev functions one concludes from (5.1) that the function v, considered as a function of the space and time variables, is such that

$$w \in H^s(]-T, T[\times \mathbb{R}^3, \mathbb{R}^{N'}). \tag{6.2}$$

(iv) Theorem (6.1) has been stated such as to give sufficient results for the following local considerations. If one is interested in existence theorems for the standard Cauchy problem for the vacuum field Eqs. (3.19), (3.20), a somewhat different result has to be extracted from Kato's theorem. One has to keep open the possibility for the frame e_k to approach a standard Minkowskian frame at infinity. If the initial value v_0 is required to be such that $e^\mu{}_k$ is the sum of a constant and a function of class $H^s(\mathbb{R}^3)$, Kato's theorem will ensure the existence of a solution of (5.1) with the same structure.

(v) Notice that v_0 was not assumed to satisfy the constraint equations.

Let now (S_0, u_0) be a conformal hyperboloidal initial data set. The surface S_0 will be assumed to be diffeomorphically identified with the closed unit ball in $\mathbb{R}^3$. Furthermore the coordinaes x^α on S_0 will be those implied on S_0 by the standard euclidean coordinates on $\mathbb{R}^3$. Suppose u_0 is of class $H^s(S_0, \mathbb{R}^N)$ with $s \geqq 4$. It will be assumed that the frame e_a supplied by u_0 exists on the whole of S_0. This assumption is only made for convenience, otherwise one would have to repeat the following argument for sufficiently small subsets of S_0 and patch together the resulting developments. The function u_0 on S_0 can be extended to a function v_0 on $\mathbb{R}^3$ with the properties stated in Theorem (6.1) (with $s \geqq 4$) [32]. Let v denote the solution of (5.1) for initial data v_0, the existence of which is asserted in Theorem (6.1) (again with $s \geqq 4$). There is a neighbourhood U of S_0 in $\mathbb{R}^3$ on which $\det(e^\mu{}_k \neq 0$. Possibly after shrinking U and T one has $\det(e^\mu{}_k) \neq 0$ on $U \times]-T, T[$. By (3.10), (6.2) and the Sobolev embedding theorems the frame defines a metric $g_{\mu\nu}$ of class $H^s(U \times]-T, T[) \cap C^{s-2}(U \times]-T, T[)$. Let $D_{u_0}(S_0)$ denote the domain of dependence of S_0, N respectively I the future respectively past Cauchy horizon of S_0 in $U \times]-T, T[$ with respect to $g_{\mu\nu}$ (the vector field e_0 being future directed by definition), and let u denote the restriction of v to $D_{u_0}(S_0)$. At this stage it is not known, whether the $\gamma^i{}_{jk}$ supplied by u are in fact the coefficients with respect to e_k of the Levi–Civita connection determined by $g_{\mu\nu}$. The Levi–Civita connection coefficients calculated from $g_{\mu\nu}$ are of class $H^{s-1}(U\times]-T,T[)$, hence the surfaces N, I are near Z, where they intersect S_0, null hypersurfaces of class C^{s-3}. Further away from Z these surfaces may develop caustics and selfintersections. To simplify the following argument, T will be restricted such that N and I remain smooth in $U\times]-T,T[$.

Lemma (6.2). *The pair $(D_{u_0}(S_0),u)$ only depends on the conformal hyperboloidal initial data set (S_0,u_0).*

Consider on $U\times]-T,T[$ a differential operator of the form

$$L_u w = A^\mu w_{,\mu} + Cw, \tag{6.3}$$

where $A^\mu = A^k e_k{}^\mu$ (see (5.4)) with $e_k{}^\mu$ being the frame supplied by v and C being some continuous matrix valued function. It will be shown that w must vanish on $D_{u_0}(S_0)$ if it is a solution of $L_u w=0$ of class C^1 and vanishes on S_0. This will entail Lemma (6.2) because if $\tilde{v}_0$ is another extension of u_0 and $\tilde{v}$ the corresponding solution of (5.1), $w=v-\tilde{v}$ satisfies an equation of type $L_u w=0$ as follows from (5.4), (5.5).

From (5.1) one obtains

$$\det(A^\mu\xi_\mu) = c(Q_0(\xi))^{k_0}(Q_1(\xi))^{k_1}(Q_2(\xi))^{k_2}(Q_3(\xi))^{k_3}, \tag{6.4}$$

where $c\neq 0$ is a constant, the k_j are positive integers and

$$\begin{aligned} &Q_0(\xi) = g^{kl}e_k{}^\mu e_l{}^\nu \xi_\mu\xi_\nu = g^{\mu\nu}\xi_\mu\xi_\nu,\ Q_1(\xi) = e_0{}^\mu\xi_\mu,\\ &Q_2(\xi) = (g^{\mu\nu} - e^\mu{}_0 e^\nu{}_0)\xi_\mu\xi_\nu,\\ &Q_3(\xi) = (g^{\mu\nu} - e^\mu{}_0 e^\nu{}_0 - e_1{}^\mu e_1{}^\nu)\xi_\mu\xi_\nu. \end{aligned} \tag{6.5}$$

The characteristics of (6.3) are hypersurfaces of the form $\{\Phi(x^\mu)=\text{const},\ \Phi_{,\mu}\neq 0\}$ with Φ being a function of class C^1 such that $\det(A^\mu\Phi_{,\mu})=0$ on these surfaces. From (6.4), (6.5) one finds that all characteristics of (6.3) are in fact timelike or null hypersurfaces with respect to the metric $g_{\mu\nu}$. Furthermore from (5.5), (6.5) one concludes

$A^\mu\xi_\mu$ is positive definite for all covectors ξ_μ such that $-g^{\mu\nu}\xi_\nu$ is timelike future-directed with respect to $g_{\mu\nu}$. (6.6)

Now the result follows from a standard argument for symmetric hyperbolic systems [28]. For $0\leq t<T$, set

$$M_t=(U\times[0,t])\cap \mathrm{D}_{u_0}(S_0),\ S_t=(U\times\{t\})\cap D_{u_0}(S_0),\ N_t=M_t\cap N,$$

denote by dv the volume element defined by the euclidean metric on $\mathbb{R}^k$ and by $(u|v)_{M_t}$ the L^2 scalar product on M_t with respect to dv. One has

$$2(L_u w|w)_{M_t} = (w,(L_u+L_u{}^*)w)_{M_t} + \int_{M_t} ({}^t w A^\mu w)_{,\mu}\,dv, \tag{6.7}$$

where $L_u{}^*$ is the formal adjoint of L_u. Because of the symmetry of the matrices A^μ the operator $L_u+L_u{}^*$ is in fact of zeroth order. Since A^0 is positive definite one has an estimate

$$|(w,(L_u+L_u{}^*)w)_{M_t}| \leq c'(A^0 w,w)_{M_t} \tag{6.8}$$

with some constant c' depending only on T. Applying the divergence theorem to (6.7) and taking into account (6.8) and $L_u w = 0, w|_s = 0$ one obtains

$$\int_{S_t} ({}^t w A^0 w) dS \leqq - \int_{N_t} ({}^t w A^\mu w) \eta_\mu dS + c' \int_0^t \left(\int_{S_t} ({}^t w A^0 w) dS \right) dt', \tag{6.9}$$

where dS denotes the respective surface elements implied on S_t respectively N_t, and $\eta_\mu dx^\mu$ is a 1-form proportional to the differential of a function with level surface N and gradient (with respect to the euclidean metric) pointing out of $D_{u_0}(S_0)$. By (6.6) the first integral of the right member of (6.9) is positive since $A^\mu \eta^\mu$ is positive (though not definite). Because the integral on the left vanishes for $t = 0$, it vanishes for all t, $0 \leqq t < T$ by Gronwall's lemma [33]. Hence w vanishes in the future of S_0 in $D_{u_0}(S_0)$. Similarly one concludes that w vanishes in the past of S_0 in $D_{u_0}(S_0)$.

Lemma (6.3). *The solution u of (5.1) on $D_{u_0}(S_0)$ satisfies in fact the conformal vacuum field equations (3.17) and Eq. (3.6)*

$$\Omega \nabla_k \nabla_k \Omega = 2 \nabla_k \Omega \nabla^k \Omega$$

on $D_{u_0}(S_0)$. Moreover one has, possibly after restricting T further, $\Omega > 0$ on $D_{u_0}(S) \backslash I$, $\Omega \equiv 0$ on I, $\nabla_i \Omega \neq 0$ on I. The hypersurfaces N and I are given as level surfaces of functions of class $H^s(U \times]-T, T[) \cap C^{s-2}(U \times]-T, T[)$.

The zero-quantity z obtained from v on $U \times]-T, T[$ by (3.14) is of class $H^{s-1}(U \times]-T, T[)$ and satisfies the subsidiary equation (5.6). By our assumptions on u_0 and by Lemma (3.2) z vanishes on S_0. For the subsidiary equations (5.6) one has with the notation (6.5)

$$\det(F^\mu \xi_\mu) = c''(Q_1(\xi))^{j_1} (Q_2(\xi))^{j_2} (Q_3(\xi))^{j_3}.$$

where again $c'' \neq 0$ is a constant and the j_a are positive integers. It follows that the characteristics of (5.6) are timelike or null hypersurfaces with respect to the metric $e_{\mu\nu}$, which is defined by $e_{\mu\nu} e^{\nu\lambda} = \delta_\mu{}^\lambda$, where $e^{\nu\lambda} = g^{\nu\lambda} - e_0{}^\nu e_0{}^\lambda$. Hence the characteristics of (5.6) are in particular all timelike with respect to $g_{\mu\nu}$. Furthermore, $F^\mu \xi_\mu$ is positive definite for all ξ_μ such that $-\xi_\mu e^{\mu\nu}$ is future-directed and timelike with respect to $e_{\mu\nu}$. This is the case in particular for η_μ as discussed before. By repeating for the subsidiary equations the arguments used above one concludes that the zero quantity vanishes on $D_{u_0}(S_0)$ (in fact, since N and I are spacelike with respect to $e_{\mu\nu}$, the zero-quantity vanishes even on a neighbourhood of $D_{u_0}(S_0) \backslash Z$ in $(U \times]-T, T[) \backslash Z)$. Equation (3.6) holds because of Theorem (3.1) and Lemma (3.2). Let n^i be a past-directed parallel propagated non-vanishing null vector field on I. From $O_j = 0, Q_{jk} = 0$ is obtained the system of ordinary linear homogeneous differential equations

$$n^i \nabla_i \Omega = (n^i \Sigma_i),$$
$$n^i \nabla_i (n^j \Sigma_j) = -\Omega \sigma_{ij} n^i n^j,$$

for Ω, $n^i \Sigma_i$ along the null generators of I. Since by choice of u_0 Ω and $n^i \Sigma_i$ vanish on Z, these quantities vanish everywhere on I. Because $\Sigma_i \neq 0$ on Z, $\Omega > 0$ on S

one has $\Sigma_i \neq 0$ on I and $\Omega > 0$ on $D_{u_0}(S_0)\backslash I$ for sufficiently small T. Finally the geodesic flow is of class H^s since it is determined by the system

$$\dot{x}^\mu = z^k e_k{}^\mu,$$
$$\dot{z}^j = -z^k z^i \gamma_{k}{}^j{}_i, \tag{6.10}$$

which is defined by the functions $e_k{}^\mu$, $\gamma_k{}^j{}_i$ of class H^s [29].

Going from any coordinate system to Gauss-coordinates formally implies a loss of differentiability. However, one has

Lemma (6.4). *Suppose $s \geqq 4$, $x^{\mu'}$ is a coordinate system of class H^{s+1} on $D_{\mu_0}(S_0)$, u' a collection of quantities as in (3.13) given with respect to $x^{\mu'}$ and an orthonormal frame $e_{k'}$. If u' is of class H^s, satisfies the conformal vacuum field equations (3.17) on $D_{u_0}(S_0)$ and implies on S_0 the data u_0, then, near S, there exists a coordinate transformation $x^\mu(x^{\mu'})$ of class H^{s+1} and a field of Lorentz transformations $A^{l'}{}_k(x^\mu)$ of class H^s such that u', if expressed with respect to the coordinates x^μ and the frame $e_k = A^{l'}{}_k e_{l'}$ is of class H^s and satisfies the gauge (3.21), (3.22).*

Solving the geodesic equation in the form (6.10) and similarly the parallel transport equations gives $x^{\mu'}(x^\mu)$, $e^{\mu'}{}_k(x^\mu)$ of class H^s, and $\det(x^{\mu'}/x^\mu) \neq 0$ near S_0. Then $A_k^{j'}(x^\mu)$, obtained from $e^{\mu'}{}_k = A_k{}^{j'} e_{j'}{}^{\mu'}$ is of class H^s and consequently Ω, $\Sigma_{k'}$, $\sigma_{k'j'}$, $d^{i'}{}_{j'k'l'}$ and $R^{i'}{}_{j'k'l'}$ if expressed with respect to x^μ, e_k are of class H^s. But formally $e^\mu{}_k = (\partial x^\mu / \partial x^{\mu'}) e^{\mu'}{}_k$ and $\gamma^i_{jk} = A^{l'}_j (A^{i'}{}_{k,\mu'} e^{\mu'}{}_{l'} + A^{n'}{}_k \gamma_{l'}{}^{i'}{}_{n'}) A^{-1i}{}_{i'}$ are of class H^{s-1}, though the corresponding quantities implied on S_0 are of class $H^s(S_0)$. Then $r^i{}_{jkl}$ as given in (3.15) is of class H^{s-2}. However, since u' satisfies the conformal vacuum field equations, $r^i{}_{jkl} = R^i{}_{jkl}$ is in fact of class H^s. Now from $T^i{}_{j0} = 0$, $K^i{}_{j0l} = 0$ (see 5.1a) follows that $e^\mu{}_k$, $\gamma^i{}_{jk}$ are in fact of class H^s. Since $\sigma^k{}_{\nu'}$ defined by $\sigma^k{}_{\mu'} e_k{}^{\nu'} = \delta_{\mu'}{}^{\nu'}$ is of class H^s the same is true for $\partial x^\mu / \partial x^{\mu'} = e^\mu{}_k \sigma^k{}_{\mu'}$. Collecting results one obtains

Theorem (6.5). *Suppose $s \geqq 4$ and (S_0, u_0) is a conformal hyperboloidal initial data set such that u_0 is of class $H^s(S_0)$. Then there exists a unique (up to questions of extensibility) solution $(D(S), u)$ of the conformal hyperboloidal initial value problem for these initial data, such that:*
$D(S)$ may be considered as a submanifold with boundary of a manifold of class H^{s+1}. The boundary surfaces of $D(S)$ may be given as level surfaces of functions of class H^s. The solution u is of class $H^s(D(S))$. In particular the conformal factor Ω and the metric $g_{\mu\nu}$ supplied by u are of class H^s, $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ solves Einstein's vacuum field equations on $D(S)\backslash I$ and I represents past null infinity with respect to $\tilde{g}_{\mu\nu}$.

Remarks. (i) From the Sobolev imbedding theorems it follows in particular that the solution $(D(S), u)$ is of class C^∞ if the data set (S_0, u_0) is of class C^∞.

(ii) The integration of Eq. (5.1) may stop for various reasons. There may build up curvature singularities which should be indicated by the tendency of some components of $d^i{}_{jkl}$ to blow up. There may also develop caustics of the Gauss coordinates used in (5.1). However, since one is dealing with the "non-physical" geodesics, it is difficult to estimate when this may happen. Furthermore the conformal factor may go to zero in regions not belonging to null infinity. In the example in Minkowski-space discussed in Chap. 2 this happens at the origin of Minkowski space, which represents the "top" of $D(H)$.

(iii) Similarly as the propagation equations (5.1) imply improved smoothness results (6.1) for the lower order structures, one may expect the constraint equations (4.3), which are satisfied on the surfaces S_t, to imply an increase in smoothness for the lower order structure on these surfaces. However, this requires a more detailed analysis of the system (4.3).

Concluding Remarks

The conformal vacuum field equations in the regular representation (3.17) and the technique of reducing initial value problems of different types for these equations as developed in [5, 9] and the present paper are seen to be sufficient to deal with questions concerning the propagation of the field in regions comprising part of past or future null infinity. Whereas in the asymptotic characteristic initial value problem the equations can completely be solved in terms of the nonphysical quantities, because there the constraints essentially reduce to ordinary differential equations, in the Cauchy problems one possibly has to solve the constraint equations in terms of the vacuum field. In York's approach to the constraint equations these are solved by using conformal techniques. Notice, however, the difference between the type of rescalings used there for the various parts of the second fundamental form and the transformation (2.13) of the second fundamental form implied by conformal rescalings of the space-time metric. The investigation of the problem of finding hyperboloidal initial data should also lead to a formulation, in terms of the physical fields, of the differentiability conditions, which are specified here completely in terms of the non-physical quantities. Correspondingly should the differentiability properties of the solution of the propagation equations be translated into physical terms. At first sight one may expect the conformal factor and its derivatives to come in as weight factors in the definition of the appropriate function spaces, however, the conformal factor itself is in the present treatment of the problem provided by the solution of the differential equations.

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