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H-Surfaces in Lorentzian Manifolds*

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Abstract. We consider surfaces of prescribed mean curvature in a Lorentzian manifold and show the existence of a foliation by surfaces of constant mean curvature.

0. Introduction

Surfaces of prescribed mean curvature, that is what we mean by H-surfaces, are of great physical importance both in the case of a proper Riemannian manifold as well as in a Lorentzian manifold. While H-surfaces in proper Riemannian manifolds, especially in the Euclidean space $\mathbb{R}^n$, have been studied extensively, little is known in the Lorentzian case, except when the manifold is the Minkowski space. Then, there are the papers of Calabi [CA] and Cheng and Yau [CY] on the Bernstein theorem for entire maximal surfaces, the result of Treibergs [TA] on entire surfaces of constant mean curvature, and the paper of Bartnik and Simon [BS] on the Dirichlet problem for surfaces with bounded mean curvature.

For non-flat Lorentz manifolds only local existence results via perturbation arguments, or results concerning the uniqueness are known, cf. [BF1, 2; CB; CFM; GO; MT].

In this paper we consider a connected, oriented, and time-oriented, globally hyperbolic Lorentz manifold M of dimension $(n+1)$.

In the first part of this paper, Sects. 1–5, we consider the Dirichlet problem for bounded H-surfaces. Assuming in this case that M is topologically a product,

$$M=N\times I, \tag{0.1}$$

where I is an interval and N an n-dimensional complete Riemannian manifold, such that the metric in M is given as

$$ds^2=\psi(-dt^2+g_{ij}(x)dx^idx^j) \tag{0.2}$$

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with some positive conformal factor ψ, we prove the existence of a smooth surface $\mathcal{S}$ of prescribed bounded mean curvature H and given boundary $\partial\mathcal{S}$, where the boundary is assumed to be *acausal* and representable as a graph

$$\partial\mathcal{S}=\{(x,\varphi(x):x\in\partial\Omega\}, \tag{0.3}$$

where $\Omega\subset N$ is a relatively compact open set with C^2-boundary, and $\varphi\in C^2(\bar{\Omega})$ is space-like. The solution $\mathcal{S}$ is then also given as the graph of a function u.

In the second part, Sects. 6 and 7, we drop the restriction (0.2) on the metric and assume merely that M has a compact Cauchy surface. Imposing the hypotheses of a *big bang* and a *big crunch,* i.e. assuming the existence of global barriers, we prove the existence of smooth slices of prescribed bounded mean curvature.

Supposing, furthermore, that M satisfies the *time-like convergence* condition, we can show the existence of a *foliation* of M by slices of constant mean curvature. If there are two different maximal slices, then we prove that they are totally geodesic and strictly separated, and that there is a whole continuum of totally geodesic slices in between. If $\mathscr{C}_0$ denotes this continuum, then $\mathscr{C}_0$ can be described as consisting of level surfaces to the "first" totally geodesic slice $\mathcal{S}_0$

$$\mathscr{C}_0=\{\mathcal{S}_t:0\leqq t\leqq\varepsilon_0\}, \qquad d(\mathcal{S}_0,\mathcal{S}_t)=t. \tag{0.4}$$

The tubular neighbourhood of $\mathcal{S}_0$ contains $\mathscr{C}_0$, and the metric is static in $\mathscr{C}_0$

$$ds^2=-dt^2+g_{ij}(x)dx^idx^j \tag{0.5}$$

for $(x,t)\in\mathscr{C}_0$.

The paper is organized as follows:

In Sect. 1, we derive the Euler-Lagrange equation governing surfaces of prescribed mean curvature.

In Sect. 2 we prove boundary estimates, while in Sect. 3 we deal with the so-called *segment condition*, saying, that if the uniform limit of surfaces of uniformly bounded mean curvatures contains a segment of a null geodesic, then this segment has to extend to the boundary.

In Sect. 4 we prove global gradient estimates valid for general metrics. This estimate enables us to show the existence of solutions to the Dirichlet problem in Sect. 5, and of global slices in Sect. 6.

In Sect. 7 we treat the problem of the foliation of M by slices of constant mean curvature.

1. The Euler Equations

In this section we consider a general time-oriented $(n+1)$-dimensional Lorentzian manifold M with metric

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta, \qquad \alpha,\beta=0,1,\dots,n, \tag{1.1}$$

and signature $(-,+,\dots,+)$. In local coordinates the coordinate $x^0=t$ is time-like, while the space-like coordinates x^i are labelled with Roman letters i, $1\leqq i\leqq n$.

A hypersurface $\mathscr{S}\subset M$ is said to have prescribed mean curvature $\frac{1}{n}\cdot H$, if it is space-like, i.e. if

$$g_{\alpha\beta}\nu^\alpha\nu^\beta=-1\,, \tag{1.2}$$

where $\nu=(\nu^\alpha)$ is the future directed unit normal vector to $\mathscr{S}$, and if

$$-\operatorname{div}\nu=-D_\alpha\nu^\alpha=H\,, \tag{1.3}$$

where H is a given function on M.

Let us look locally at such a hypersurface. Choose in the neighbourhood of a point $(x_0,t_0)\in\mathscr{S}$ Gaussian coordinates, i.e. choose a space-like hypersurface N, e.g. $t=t_0$, and take as the new time coordinate the arc length of the time-like geodesics perpendicular to N and as space-coordinates the base-point of the geodesics in N. In a Gaussian coordinate system the metric has the form

$$ds^2=-dt^2+g_{ij}(x,t)dx^idx^j\,, \tag{1.4}$$

where the g_{ij} are positive definite, and any space-like hypersurface $\mathscr{S}$ is locally given as the graph of a function u

$$\mathscr{S}=\{(x,u(x)):x\in\Omega\subset N\}\,, \tag{1.5}$$

the unit normal vector ν is

$$\nu=(\nu_0,\nu_1,\dots,\nu_n)=v\cdot(-1,Du)\,, \tag{1.6}$$

where

$$v=(1-g^{ij}(x,u(x)\cdot D_iu\cdot D_ju)^{-1/2}\,, \tag{1.7}$$

and where as usual we set

$$(g^{ij})=(g_{ij})^{-1}\,. \tag{1.8}$$

If we insert ν in the mean curvature equation, we would get a second order partial differential equation for u. Another more elegant way to derive this equation is to obtain it as the Euler-Lagrange equation of a variational problem, namely, maximize the functional

$$J(\eta)=\int_\Omega(1-|D\eta|^2)^{1/2}\cdot g(x,\eta)^{1/2}+\int_\Omega\int_0^\eta H(x,t)g(x,t)^{1/2} \tag{1.9}$$

in an appropriate function class, e.g. in

$$K=\{\eta\in H^{1,\infty}(\Omega):|D\eta|\leqq 1,\eta|_{\partial\Omega}=\varphi\}\,, \tag{1.10}$$

if we are considering a variational problem of Dirichlet type, where it is to be understood that

$$|D\eta|^2=g^{ij}(x,\eta)\cdot D_i\eta\cdot D_j\eta\,, \tag{1.11}$$

and

$$g=g(x,\eta)=\det(g_{ij}(x,\eta))\,. \tag{1.12}$$

The corresponding Euler-Lagrange equation for a solution u with

$$|Du|<1 \tag{1.13}$$

looks like

$$-\frac{1}{g^{1/2}}\frac{\partial}{\partial x^i}(g^{1/2}\cdot v\cdot g^{ij}D_ju)+\frac{1}{2}v\cdot\frac{\partial g^{ij}}{\partial x^0}\cdot D_iu\cdot D_ju-\frac{1}{2}v^{-1}\cdot g^{ij}\cdot\frac{\partial g_{ij}}{\partial x^0}=H=H(x,u)\,. \tag{1.14}$$

The first term is the divergence of the vector field

$$a^i=v\cdot g^{ij}\cdot D_ju \tag{1.15}$$

with respect to the metric

$$g_{ij}(x,u(x))\,. \tag{1.16}$$

The other terms of the left-hand side are of the form $a\cdot v$, where

$$a=a(x,u,Du)\,; \tag{1.17}$$

i.e. we can rewrite Eq. (1.14) as

$$Au+a\cdot v\equiv -D_i(a^i(Du))+a\cdot v=H(x,u)\,, \tag{1.18}$$

where the symbol "D_i" denotes covariant differentiation with respect to the implicity defined metric (1.16).

This is a quasilinear elliptic differential equation for u, where in contrast to the usually given problems we know in advance that u is already Lipschitz continuous, but where the equation only makes sense if $|Du|$ is strictly less than one.

Hence, if we want to solve a Dirichlet problem

$$\begin{aligned} Au+a\cdot v&=H(x,u) \quad \text{in} \quad \Omega\,,\\ u&=\varphi \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{1.19}$$

we should first prove *a priori* estimates of the kind

$$|Du|\leqq 1-\theta\,, \qquad \theta>0\,, \tag{1.20}$$

and then use some Leray-Schauder-type argument to prove the existence of a solution.

In the case when M is equal to the Minkowski space this has recently been achieved by Leon Simon and Robert Bartnik. For the Minkowski metric the equation simplifies considerably:

$$-D_i(v\cdot D^iu)=H\,, \tag{1.21}$$

where the metric g_{ij} is now the Euclidean metric in $\mathbb{R}^n$.

In the general case, the presence of the v term causes some trouble, though on the other hand, it has the advantage that the *structure* of the equation is invariant under conformal transformations of the metric. Indeed, let

$$d\bar{s}^2=\psi ds^2\,, \qquad ds^2=-dt^2+g_{ij}(x,t)dx^idx^j \tag{1.22}$$

be a conformal metric with some positive C^∞-function ψ. Then the equation for a surface of prescribed mean curvature is

$$Au+\tilde{a}\cdot v=H\,, \tag{1.23}$$

where

$$\tilde{a} = a - \frac{n}{2}(1 - |Du|^2)\frac{\partial}{\partial t}\log\psi - \frac{n}{2}\cdot D_i \log\psi \cdot D^i u, \tag{1.24}$$

and there the operator A is defined with respect to the metric $(g_{ij}(x, u))$.

We shall often exploit this fact even without mentioning it explicitly. Especially we shall always stick to the notation $a(x, u, Du)$ instead of $\tilde{a}(x, u, Du)$.

Finally, let us give some definitions.

Definition 1.1. A hypersurface $\mathscr{S}$ is said to be *space-like* if its normal vector is time-like. If $\mathscr{S}$ is represented as a graph of a function u, then we also say u is space-like.

A subset $A \subset M$ is said to be *acausal*, if any time-like curve or null curve intersects A at most once.

A *slice* $\mathscr{S} \subset M$ is a space-like hypersurface which is also a closed and connected submanifold of M.

We also remark that in the following sections we deal with bounded mean curvature functions H, where we often have to consider compositions of the form $H(x, u)$ with continuous functions u. In order that these composite functions are measurable in x, we therefore have to assume that H is a Borel function. Thus, H bounded always means that we pick a Borel function in the equivalence class defined by H.

2. Boundary Estimates

Suppose $M = N \times I$ with metric ds^2 given by

$$ds^2 = \psi(-dt^2 + g_{ij}(x)dx^i dx^j), \tag{2.1}$$

and let u be a solution to the Dirichlet problem (1.19), (1.23).

For simplicity, we shall assume that $I = \mathbb{R}$, and that ψ remains smooth and positive on compact subsets of M. This has the advantage that a space-like surface $\mathscr{S}$ is *a priori* bounded, if $\partial\mathscr{S}$ is compact. If we would allow I to be a general interval, then we would have to impose further conditions to assume this. Our aim is to prove *a priori* estimates for $|Du|$ at the boundary.

Theorem 2.1. *Let $\Omega \subset N$ be relatively compact with $\partial\Omega \in C^2$, and let $\varphi \in C^2(\bar{\Omega})$ be space-like with*

$$|D\varphi| \leqq 1 - \theta, \quad \theta > 0, \tag{2.2}$$

uniformly in Ω, such that graph $\varphi|_{\partial\Omega}$ is acausal. Let $u \in H^{2,p}(\Omega)$, $p > n$, be a solution of the boundary value problem (1.19), (1.23) *with bounded H. Then*

$$|Du| \leqq 1 - \Theta_0 \tag{2.3}$$

on $\partial\Omega$, where Θ_0 depends on Θ, $\partial\Omega$, $\|\varphi\|_{2,\infty}$, $\|H\|_\infty$, and on the metric.

Proof. We first observe that according to the remarks at the end of Sect. 1 we may assume that the quasi-linear differential operator A is defined with respect to the metric $(g_{ij}(x))$. The lower order terms then look different, but we do not change the notation. We also note that in view of the assumptions u, Du and hence $a(x, u, Du)$ are uniformly bounded in Ω.

Let $x_0 \in \partial\Omega$ be an arbitrary but fixed boundary point. We shall show that there is a neighbourhood U of x_0 and two functions δ^+, $\delta^- \in C^2(\overline{\Omega \cap U})$, such that

$$A\delta^- + a\cdot v(\delta^-) \leqq H \leqq A\delta^+ + a\cdot v(\delta^+) \tag{2.4}$$

in $\Omega \cap U$, and

$$\delta^- \leqq u \leqq \delta^+ \quad \text{in} \quad \partial(\Omega \cap U), \tag{2.5}$$

$$\delta^-(x_0) = u(x_0) = \delta^+(x_0), \tag{2.6}$$

and

$$|D\delta^-|, \qquad |D\delta^+| \leqq 1 - \Theta_0. \tag{2.7}$$

Here, the factor a in (2.4) is evaluated at (x, u, Du). The maximum principle will then yield that (2.5) holds throughout $\Omega \cap U$, and therefore we shall get

$$|Du(x_0)| \leqq 1 - \Theta_0. \tag{2.8}$$

To define δ^+, let $\xi \in N$ be a point outside $\bar{\Omega}$ but near x_0, and label the coordinates so that $\xi = 0$. Let $|x|$ be the geodesic distance, and choose ξ so that the ball $B_R(0)$ is geodesically convex for some $R > |x_0|$. We then define δ^+ through

$$\delta^+(x) = \varphi(x_0) + \int_{|x_0|}^{|x|} (1+\gamma)^{-1/2}, \tag{2.9}$$

where

$$\gamma(t) = \alpha \cdot e^{\lambda t} \tag{2.10}$$

with positive constants α, λ to be determined later: λ is considered to be large depending on $|x_0|$ and H, and α is chosen to be small depending on φ, $\partial\Omega$, and λ.

If α tends to zero, then δ^+ represents the upper light cone with base point $(x_0, \varphi(x_0))$. For positive α and $x \neq 0$ we have

$$|D\delta^+| = (1+\gamma)^{-1/2}, \tag{2.11}$$

and

$$v(\delta^+) = \gamma^{-1/2}(1+\gamma)^{1/2}. \tag{2.12}$$

Furthermore, $\delta^+ \in C^2(\overline{B_R(0)} \setminus \{0\})$, and

$$D_i\delta^+ = (1-\gamma)^{-1/2} \cdot D_i|x|, \tag{2.13}$$

$$D_iD_j\delta^+ = (1+\gamma)^{-1/2} D_iD_j|x| - \tfrac{1}{2}(1+\gamma)^{-3/2} \cdot \lambda \cdot \gamma \cdot D_i|x| D_j|x|. \tag{2.14}$$

Taking into account that $|D|x|| = 1$, and

$$A\delta^+ = -v(\delta^+)\Delta\delta^+ - v^3(\delta^+) \cdot D_i\delta^+ D_j\delta^+ \cdot D^iD^j\delta^+, \tag{2.15}$$

we conclude

$$A\delta^+ = (1+\gamma)^{-1/2} \cdot (\lambda/2 - \Delta|x|) \cdot v, \tag{2.16}$$

where

$$-\Delta|x| = -\frac{n-1}{|x|} + r, \tag{2.17}$$

and where r stands for bounded curvature terms: in Riemannian normal coordinates with center in $\xi=0$,

$$r=-g^{ij}\Gamma_{ij}^{k}D_k|x|\,. \tag{2.18}$$

Thus, we derive

$$A\delta^+ + a\cdot v(\delta^+)\geqq H \quad \text{in} \quad \Omega\cap B_R(0)\,, \tag{2.19}$$

if λ is chosen appropriately and α is small enough, $\alpha\leqq\alpha_0(\lambda)$; we note that this estimate is uniform in α for such α.

Clearly (2.6) is valid for δ^+, so that we merely have to check (2.5) for $U=B_R(0)$. In Lemma 2.3 below we shall show that in *any* neighbourhood of x_0 we can find ξ such that

$$\varphi(x)\leqq\delta^+(x)\,, \qquad \forall x\in\partial\Omega\cap B_R(0)\,, \tag{2.20}$$

if we choose α appropriately, always improving the estimates by choosing α small.

Taking (2.20) for granted for the moment, the final estimate

$$u\leqq\delta^+ \quad \text{in} \quad \partial(\Omega\cap B_R(0)) \tag{2.21}$$

with follow from

Lemma 2.2. *Let $\mathcal{S}=\operatorname{graph} u$ be a surface of bounded mean curvature H, and let $\partial\mathcal{S}$ be acausal. Let $(\bar{x},u(\bar{x}))\in\bar{\mathcal{S}}$, and let $|x|$ be the distance function with respect to the metric $(g_{ij}(x))$ and with base point $\bar{x}$. Then, to any number $R>0$, there exists $\varepsilon=\varepsilon(R,\|H\|_\infty,\partial\mathcal{S})$ such that*

$$u(x)+\varepsilon\leqq u(\bar{x})+|x|\,, \qquad \forall x\in\Omega\cap\partial B_R(\bar{x})\,. \tag{2.22}$$

Proof. Suppose that the lemma were not true. Then we would conclude that $\mathcal{S}$ would contain a segment of a null geodesic. By the results of Sect. 3 below, we would then deduce that this null geodesic segment is maximal, i.e. it would extend to the boundary $\partial\mathcal{S}$, which is impossible since $\partial\mathcal{S}$ is supposed to be acausal.

The fact that ε only depends on R and $\|H\|_\infty$ is due to the observation that the results in Sect. 3 also apply to *uniform limits* of surfaces of uniformly bounded mean curvature, i.e. to surfaces which are not necessarily space-like.

It remains to define the lower barrier δ^-. We set

$$\delta^-(x)=\varphi(x_0)-\int_{|x_0|}^{|x|}(1+\gamma)^{-1/2}\,, \tag{2.23}$$

while choosing ξ, λ, and α as before, and it turns out that the estimates are identical with the appropriate change in sign.

To complete the proof of Theorem 2.1 we claim

Lemma 2.3. *Let $x_0\in\partial\Omega$. Then in any neighbourhood of x_0 we can find ξ not belonging to $\bar{\Omega}$, such that*

$$\delta^-\leqq\varphi\leqq\delta^+ \quad \textit{in} \quad \partial\Omega\cap B_R(\xi)\,, \tag{2.24}$$

if α is chosen sufficiently small.

Proof. We only prove the estimate for δ^+. The proof is similar to the proof of [BS; Proposition 3.1]. Let $x_0 \in \partial\Omega$, and choose a Riemannian normal coordinate system in N around x_0 such that the tangent plane at $\partial\Omega$ in $x_0=0$ is given by $x^n=0$, that the inward unit normal vector of $\partial\Omega$ in x_0 is equal to $(v_i)=(0,\dots,0,1)$, and that the tangential derivative of φ in x_0 is given by

$$D_i\varphi(0)=r\cdot\delta_{1i}, \qquad 1\leqq i\leqq n-1, \tag{2.25}$$

where

$$0\leqq r\leqq 1-\Theta. \tag{2.26}$$

We now want to find a sequence $\xi_\varepsilon\notin\bar{\Omega}$, converging to $x_0=0$, such that, if we define δ^+ with base point in ξ_ε,

$$D_i\delta^+(0)=D_i\varphi(0), \qquad 1\leqq i\leqq n-1, \tag{2.27}$$

holds.

From the definition of δ^+ we deduce that

$$D\delta^+(0)=(1+\alpha e^{\lambda|\xi_\varepsilon|})^{-1/2}\cdot D|\xi_\varepsilon|, \tag{2.28}$$

where

$$D|\xi_\varepsilon|=-\frac{\xi_\varepsilon}{|\xi_\varepsilon|}, \tag{2.29}$$

and hence (2.27) says

$$-(1+\alpha\cdot e^{\lambda|\xi_\varepsilon|})^{-1/2}\cdot\frac{\xi_\varepsilon^i}{|\xi_\varepsilon|}=D^i\varphi(0), \tag{2.30}$$

where $|\xi_\varepsilon|$ is the usual Euclidean norm since the coordinate system is normal.

In view of (2.25) the following definition for ξ_ε seems appropriate:

$$\xi_\varepsilon=\varepsilon(b,0,\dots,0,-1), \tag{2.31}$$

where b is such that

$$-(1+\alpha e^{\lambda|\xi_\varepsilon|})^{-1/2}\cdot b\cdot(1+b^2)^{-1/2}=r. \tag{2.32}$$

The set of the possible b's is uniformly bounded if we choose α so small that

$$r\cdot(1+\alpha e^{\lambda|\xi_\varepsilon|})^{+1/2}\leqq 1-\frac{\Theta}{2}. \tag{2.33}$$

The (ξ_ε) will therefore converge to $x_0=0$ and will lie outside $\bar{\Omega}$ if ε is tending to zero.

Consider now some fixed $\xi=\xi_\varepsilon$ and choose Riemannian normal coordinates in ξ. Then

$$D^iD^j\delta^+=(1+\gamma)^{-1/2}\cdot\left\{\frac{\delta^{ij}}{|x|}-\frac{x^ix^j}{|x|^3}\right\}+c^{ij}, \tag{2.34}$$

where c^{ij} is a bounded tensor if α is small, and hence

$$D_i D_j \delta^+ \cdot \eta^i \eta^j \geqq (1+\gamma)^{-1/2} \cdot \left\{ \frac{|\eta|^2}{|x|} - \frac{|\langle x, \eta \rangle|^2}{|x|^3} \right\} - c \cdot |\eta|^2 \tag{2.35}$$

for any vector field (η^i).

Let

$$\beta = 4(1+b^2)^{1/2} \cdot \Theta^{-1}. \tag{2.36}$$

Then it follows from the assumption $\partial\Omega \in C^2$ that

$$\left| \left\langle \frac{x}{|x|}, \frac{x-x_0}{|x-x_0|} \right\rangle \right|^2 \leqq 1 - \frac{\Theta}{2} \tag{2.37}$$

for all $x \in \partial\Omega$ with

$$|x - x_0| \leqq \varepsilon \cdot \beta, \tag{2.38}$$

if ε is small. For such x, we deduce from (2.35)

$$D_i D_j \delta^+ \cdot (x^i - x_0^i)(x^j - x_0^j) \geqq \left\{ \frac{\Theta}{2(1+\gamma)^{1/2}} \cdot \frac{1}{|x|} - c \right\} \cdot |x - x_0|^2. \tag{2.39}$$

For small ε this quadratic form is therefore larger as the corresponding quadratic form derived from φ, hence

$$\varphi(x) \leqq \delta^+(x), \qquad \forall x \in \partial\Omega, \qquad |x - x_0| \leqq \varepsilon \cdot \beta. \tag{2.40}$$

For $|x - x_0| > \varepsilon \cdot \beta$ we argue as follows: for small α we obtain

$$\delta^+(x) - \delta^+(x_0) \geqq \left(1 - \frac{\Theta}{2}\right) \cdot (|x| - |x_0|) \geqq (1-\Theta)|x - x_0|$$

$$+ \frac{\Theta}{2} \cdot |x - x_0| - 2 \cdot |x_0| \geqq (1-\Theta)|x - x_0| \geqq \varphi(x) - \varphi(x_0), \tag{2.41}$$

in view of the definition of β, where we assume R to be small enough so that φ has an extension into $B_R(\xi)$ satisfying the same conditions. The last inequality in (2.41) is then justified.

3. The Segment Condition

Let $\mathcal{S}_\varepsilon = \operatorname{graph} u_\varepsilon$ over a domain Ω be a sequence of surfaces of uniformly bounded mean curvatures H_ε converging locally to a surface $\mathcal{S} = \operatorname{graph} u$, i.e.

$$u_\varepsilon \rightrightarrows u \tag{3.1}$$

on compact subsets, then we have

Theorem 3.1. *If $\mathcal{S}$ contains a segment of a null geodesic, then this segment has to be maximal,* i.e. *it extends to the boundary of $\mathcal{S}$.*

Proof. The proof is a modification of the arguments given in [BS; Theorem 3.2]. Let $\mathcal{S}=\{(x,u(x):x\in\Omega\}$ and suppose the statement were false. Then, we could find $x_0\in\Omega$, $R>0$, and $x_1\in B_R(x_0)$ such that $B_R(x_0)$ would be geodesically convex, $\overline{B_R}(x_0)\subset\Omega$, (3.1) would hold in $B_R(x_0)$, and if we would introduce a Riemannian normal coordinate system in x_0 and set

$$x_t:=x_0+t\cdot(x_1-x_0),\quad -1\leqq t\leqq 1\,, \tag{3.2}$$

then we could arrange that

$$u(x_t)=u(x_0)+t\cdot|x_0-x_1|,\quad -\tfrac{1}{2}\leqq t\leqq 1 \tag{3.3}$$

and

$$u(x_t)>u(x_0)+t\cdot|x_0-x_1|,\quad t=-1\,, \tag{3.4}$$

where $|\cdot|$ denotes the geodesic distance function, and where we point out that because of (3.1)

$$|Du|\leqq 1 \text{ in } \Omega\,. \tag{3.5}$$

Let χ be defined through

$$\chi(x)=u(x_0)-|x-x_0|\,. \tag{3.6}$$

Then $\chi\in C^2(B_R(x_0)\setminus\{x_0\})$ and

$$\chi(x)<u(x)\quad\text{for}\quad |x-x_0|=|x_0-x_1|\,. \tag{3.7}$$

Indeed, if equality would hold in (3.7) for some x, then

$$u(x)=u(x_0)-|x-x_0|=u(x_1)-|x_1-x_0|-|x-x_0| \tag{3.8}$$

in view of (3.3), and hence

$$|x-x_0|+|x_0-x_1|\leqq|x-x_1|\,, \tag{3.9}$$

i.e. x_0, x and x_1 would lie on a common geodesic, in other words

$$x=x_1,\quad\text{or}\quad x=x_{-1}\,, \tag{3.10}$$

but both cases are excluded by (3.3) and (3.4).

Let B_0 be the geodesic ball with center in x_0 and radius $|x_0-x_1|$. Since ∂B_0 is compact, we conclude from (3.1) and (3.7)

$$\chi(x)<u_\varepsilon(x)\,,\qquad\forall x\in\partial B_0\,, \tag{3.11}$$

if ε is small, and hence that

$$\delta^-(x)<u_\varepsilon(x)\,,\qquad\forall x\in\partial B_0\,,$$

where

$$\delta^-(x)=u_\varepsilon(x_0)-\int_0^{|x-x_0|}(1+\gamma)^{-1/2}+\int_0^{\varrho}(1+\gamma)^{-1/2}-\varrho\,, \tag{3.12}$$

provided the constant α involved with γ is small enough; here, we have set

$$\varrho=\tfrac{1}{4}|x_0-x_1|\,. \tag{3.13}$$

Moreover, the estimate (3.12) holds trivially on $\partial B_\varrho(x_0)$, since u_ε is space-like, so δ^- is a good candidate for a lower barrier in $G = B_0 \setminus \bar{B}_\varrho(x_0)$. Indeed, from the results in Sect. 2 we deduce that

$$\delta^-(x) \leqq u_\varepsilon(x), \qquad \forall x \in G, \tag{3.14}$$

if α and λ are chosen appropriately, independent of ε; hence

$$\delta^-(x) \leqq u(x), \qquad \forall x \in G. \tag{3.15}$$

Specifying $x = x_{-1/2}$, we deduce from (3.3)

$$\int_0^\varrho (1+\gamma)^{-1/2} - \varrho \leqq \int_0^{2\varrho} (1+\gamma)^{-1/2} - 2\varrho, \tag{3.16}$$

a contradiction.

4. Global Estimates

In this section we consider a surface $\mathscr{S}$ of prescribed mean curvature H given as a graph of a function u defined in an open, relatively compact set $\Omega \subset N$, where $M = N \times I$, and the metric ds^2 is given in the general form

$$ds^2 = \psi(-dt^2 + g_{ij}(x,t)dx^i dx^j). \tag{4.1}$$

We assume that u is bounded

$$m_1 \leqq u \leqq m_2, \tag{4.2}$$

that the metric (g_{ij}) is uniformly elliptic and of class C^2 in $\bar{\Omega} \times [-m_1, m_2]$, that ψ is of class C^2, and that H is uniformly bounded.

Let v be defined as in Sect. 1 through

$$v = (1 - |Du|^2)^{-1/2}. \tag{4.3}$$

We are going to prove that v is uniformly bounded in Ω with a fixed *a priori* estimate, provided $v|_{\partial\Omega}$ is bounded, including the case $\partial\Omega = \emptyset$.

Theorem 4.1. *Let u be a solution of Eq.* (1.23) *with bounded H, and suppose that v is bounded on $\partial\Omega$ by a constant k_0. Then, under the assumptions stated above, we have*

$$\sup_\Omega v \leqq k = k(k_0, \|H\|_\infty, m_1, m_2, |\Omega|, \psi, (g_{ij})), \tag{4.4}$$

where

$$|\Omega| = \int_\Omega \sqrt{g(x,u)}. \tag{4.5}$$

Proof. We prove *a priori* estimates, so we assume that $v \in L^\infty(\Omega)$ and that u is thus a solution of a uniformly elliptic equation. In view of our assumptions of the metric and of H we conclude

$$u \in H^{2,p}(\Omega),\ \forall 1 \leqq p < \infty, \tag{4.6}$$

i.e. v is of class $H^{1,p}(\Omega)$ for any finite p.

To obtain a differential equation satisfied by v, we differentiate Eq. (1.23) covariantly with respect to the differential operator

$$vD^k u \cdot D_k \tag{4.7}$$

to conclude

$$\begin{aligned} &-D_i(v^{-2}a^{ij}D_j v)+v^{-2}|Dv|^2+|\langle Du, Dv\rangle|^2 \\ &+v\cdot a^{ij}D_jD^k u\cdot D_iD_k u+v^2\cdot R_{ij}D^i uD^j u+a\cdot v\cdot\langle Du, Dv\rangle+v^2\cdot\frac{\partial a}{\partial x^k}\cdot D^k u \\ &+v^2\cdot\frac{\partial a}{\partial u}\cdot|Du|^2+v^2\cdot\frac{\partial a}{\partial p^i}D_iD_k u\cdot D^k u=vD^k u\cdot D_k H, \end{aligned} \tag{4.8}$$

where

$$a^{ij}=v\cdot g^{ij}+v^3\cdot D^i uD^j u. \tag{4.9}$$

The Ricci tensor and the covariant differentiations are calculated with respect to the implicitly defined metric $(g_{ij}(x,u))$, and the right-hand side of (4.8) is to be understood as a weak derivative of $H=H(x,u)$.

We note for the subsequent considerations the estimates

$$a^{ij}D_i vD_j v\geqq v\cdot|Dv|^2, \tag{4.10}$$

$$a^{ij}D_jD^k u\cdot D_iD_k u\geqq v\cdot D^iD^k u\cdot D_iD_k u\equiv v\cdot|D^2u|^2, \tag{4.11}$$

and

$$|R_{ij}D^i u\cdot D^j u|\leqq c(1+|D^2u|), \tag{4.12}$$

where c depends on the C^2-norm of the metric $(g_{ij}(x,t))$.

Estimating all non-positive terms on the left-hand side of (4.8) in the coarsest way, and using (4.11) and (4.12), we deduce

$$-D_i(v^{-2}a^{ij}D_j v)+|\langle Du, Dv\rangle|^2+\tfrac{1}{2}v^2\cdot|D^2u|\leqq c\cdot v^2+vD^k uD_k H, \tag{4.13}$$

where c depends on the first derivatives of a and on the second derivatives of $(g_{ij}(x,t))$.

Moreover, looking at the differentiated form of (1.23)

$$-v\cdot\Delta u-\langle Du, Dv\rangle+a\cdot v=H, \tag{4.14}$$

and using

$$|D^2u|^2\geqq\frac{1}{n}|\Delta u|^2, \tag{4.15}$$

we finally obtain from (4.13) the crucial inequality

$$-D_i(v^{-2}a^{ij}D_j v)+(1+2\varepsilon)\cdot|\langle Du, Dv\rangle|^2+\varepsilon\cdot v^2|D^2u|^2\leqq c\cdot v^2+H^2+vD^k u\cdot D_k H \tag{4.16}$$

for some positive constant ε.

Our first observation is that this relation immediately yields an estimate for the L^∞-norm of v in terms of k_0 and a smaller L^p-norm of v, e.g. we may use the Moser iteration technique or Stampacchia's method.

Using Stampacchia's truncation method we multiply (4.16) with

$$\eta = v \cdot \max(v-k, 0), \qquad k \geqq k_0, \tag{4.17}$$

and integrate by parts to conclude after some further steps

$$\sup v \leqq k_0 + c \cdot (1 + |v|_{2n}^3), \tag{4.18}$$

where c depends on n, $|\Omega|$, the ellipticity constants of $(g_{ij}(x,u))$, and on $\|H\|_\infty$, cf. the appendix for details.

Thus we have to prove *a priori* estimates for finite L^p-norms of v. Let $p \geqq 2$ be an arbitrary but fixed number, and let λ be a large positive constant to be determined later. Then, we multiply (4.16) with

$$v_k^p e^{\lambda u}, \qquad k \geqq k_0, \tag{4.19}$$

where $v_k = \max(v-k, 0)$ and k is fixed. We could have chosen $k = k_0$; it is only for notational convenience that we prefer to use the subscript k.

We note that the function in (4.19) is of class $H_0^{1,q}(\Omega)$ for any finite q, so we can integrate by parts to obtain

$$\begin{aligned} &p \cdot \int_\Omega v^{-2} \cdot a^{ij} D_j v \cdot D_i v v_k^{p-1} e^{\lambda u} + \lambda \int_\Omega v^{-2} a^{ij} D_j v D_i u \cdot v_k^p e^{\lambda u} \\ &+ (1+2\varepsilon) \int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u} + \varepsilon \int_\Omega v^2 |D^2 u|^2 v_k^p e^{\lambda u} \\ &\leqq c \cdot \int_\Omega v^2 v_k^p e^{\lambda u} + c(p+1) \int_\Omega v \cdot v_k^{p-1} |\langle Du, Dv\rangle| e^{\lambda u} + c\lambda \int_\Omega v v_k^p |D^2 u| e^{\lambda u}, \end{aligned} \tag{4.20}$$

where we have simplified the expressions occurring on the right-hand side already a little bit; c depends also on $\|H\|_\infty$, and we should point out that integration is taking place with respect to the volume element $\sqrt{g}\, dx$.

We have in mind to compare the terms

$$\int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u}, \tag{4.21}$$

and

$$\int_\Omega v^2 v_k^p e^{\lambda u}, \tag{4.22}$$

carefully keeping track of the constants in front of these integrals. Lower order terms involving only powers of v up to the order $(p+1)$ are negligible and we shall use the common abbreviation B for them.

Now, using the relation

$$a^{ij} D_j v \cdot D_i v \geqq v^3 |\langle Du, Dv\rangle|^2, \tag{4.23}$$

we deduce from (4.20)

$$(p+1+\varepsilon) \int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u} \leqq \lambda \int_\Omega v_k^p |\langle Du, Dv\rangle| \cdot v e^{\lambda u} + c \cdot \int_\Omega v^2 v_k^p e^{\lambda u} + B. \tag{4.24}$$

Dividing by $(p+1+\varepsilon)$ and using the inequality

$$ab \leqq \tfrac{1}{2} a^2 + \tfrac{1}{2} b^2, \tag{4.25}$$

we conclude

$$\int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u} \leqq \frac{\lambda^2}{(p+1+\varepsilon)^2} \cdot \int_\Omega v^2 v_k^p e^{\lambda u} + \frac{2c}{p+1+\varepsilon} \int_\Omega v^2 v_k^p e^{\lambda u} + B. \quad (4.26)$$

In order to obtain an opposite inequality matching the leading terms, we multiply the Euler-Lagrange-equation (1.23) with

$$v v_k^p e^{\lambda u}, \quad (4.27)$$

and conclude

$$\lambda \cdot \int_\Omega v^2 v_k^p |Du|^2 e^{\lambda u} \leqq c \cdot \int_\Omega v^2 v_k^p e^{\lambda u} + (p+1) \int_\Omega v^2 v_k^{p-1} |\langle Du, Dv\rangle| e^{\lambda u} + B, \quad (4.28)$$

where we used the fact that $p \geqq 2$. Choosing λ larger than c and having in mind that

$$v^{-2} = 1 - |Du|^2, \quad (4.29)$$

we conclude

$$\int_\Omega v^2 v_k^p e^{\lambda u} \leqq \frac{(p+1)^2}{(\lambda - c)^2} \int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u} + B. \quad (4.30)$$

Combining (4.26) and (4.30), we deduce that for large λ, depending on p and known quantities, we have the estimate

$$\int_\Omega v_k^p |\langle Du, Dv\rangle|^2 e^{\lambda u} \leqq B, \quad (4.31)$$

which in turn yields

$$\int_\Omega v^2 v_k^p e^{\lambda u} \leqq B, \quad (4.32)$$

i.e. an estimate for $\|v\|_{p+2}$. Theorem 4.1 is thus proved.

Remark 4.2. The same estimate with the same proof is also valid if we consider solutions u of variational inequalities of the form

$$u \in K; \langle Au + a \cdot v - H, \eta - u\rangle \geqq 0, \quad \forall \eta \in K, \quad (4.33)$$

where K is defined through

$$K = \{\eta \in H^{1,\infty}(\Omega) : \psi_1 \leqq \eta \leqq \psi_2, \eta|_{\partial\Omega} = \varphi\}, \quad (4.34)$$

and where the obstacles ψ_i, $i = 1, 2$, are of class $H^{2,\infty}(\Omega)$ and space-like, i.e.

$$|D\psi_i| \leqq 1 - \theta, \qquad i = 1, 2. \quad (4.35)$$

Here, we suppose u to satisfy the same assumptions as before, namely, to be of class $H^{2,p}(\Omega)$ for any finite p, and to be space-like with $v|_{\partial\Omega} \leqq k_0$. If we now choose in (4.19) and (4.27)

$$k > \max\{k_0, \sup v(\psi_1), \sup v(\psi_2)\}, \quad (4.36)$$

then these functions have support in

$$E = \{x \in \Omega : \psi_1 < u < \psi_2\}, \quad (4.37)$$

so that we can exploit the fact that u is a solution to the equation

$$Au + av = H \text{ in } E. \tag{4.38}$$

The considerations yielding an estimate for $\sup v$ are therefore still applicable, see [GE1; Appendix], where a similar situation has been treated.

5. Existence of a Solution

Let us first consider solutions of the Dirichlet problem (1.19), (1.23), where we suppose the assumptions of Theorem 2.1 to be satisfied; this means especially that the differential equation looks as in (1.23), though by abuse of notation we still write $a(x, u, Du)$ instead of $\tilde{a}(x, u, Du)$.

Theorem 5.1. *Under the assumptions of Theorem 2.1 the Dirichlet problem* (1.19), (1.23) *has a solution* $u \in H^{2,p}(\Omega)$, *for any* $1 \leqq p < \infty$.

Proof. Consider in $C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, the closed and convex set

$$\mathscr{C} = \{w \in C^{1,\alpha}(\bar{\Omega}) : m_1 - \varepsilon \leqq w \leqq m_2 + \varepsilon, |Dw| \leqq 1 - \Theta^* + \varepsilon\}, \tag{5.1}$$

where m_1, m_2, and Θ^* are the constants that can be deduced from the *a priori* estimates in Theorems 2.1 and 4.1, i.e. any solution u of the Dirichlet problem satisfies

$$|Du| \leqq 1 - \Theta^*, \; m_1 \leqq u \leqq m_2, \tag{5.2}$$

and where ε is so small that

$$1 - \Theta^* + \varepsilon < 1 \quad \text{and} \quad [m_1 - \varepsilon, m_2 + \varepsilon] \subset I. \tag{5.3}$$

The interior of $\mathscr{C}$ is certainly not empty since $0 \in \mathring{\mathscr{C}}$, where, to be absolutely precise, we assume that $0 \in I$ and m_1 is to be chosen nonpositive.

For $w \in \mathscr{C}$, consider the differential operator

$$-a^{ij} D_i D_j u + av = H \tag{5.4}$$

with coefficients

$$\begin{aligned} a^{ij} &= v g^{ij} + v^3 D^i w D^j w, \\ v &= (1 - |Dw|^2), \\ a = a(x, w, Dw), &\qquad H = H(x, w), \end{aligned} \tag{5.5}$$

and let

$$T : \mathscr{C} \to \mathscr{C}^{1,\alpha}(\bar{\Omega}), \quad u = Tw \tag{5.6}$$

be defined through the requirement that u is a solution of (5.4) subject to the boundary condition $u = \varphi$ on $\partial\Omega$.

Since the coefficients (a^{ij}) are Hölder continuous and uniformly elliptic and the lower order terms bounded, this Dirichlet problem has a unique solution

$$u \in H^{2,p}(\Omega) \quad \text{for any} \quad 1 \leqq p < \infty, \tag{5.7}$$

with *a priori* bounds for the $H^{2,p}$-norm depending only on p, $\|w\|_{1,\alpha}$, and fixed quantities. Hence, the operator T is continuous and compact.

In order to find a fixed point of T, which would necessarily be a solution of the Dirichlet problem (1.19), (1.23), we apply the following sufficient criterion: any quasi fixed point u of T, i.e. any u satisfying

$$Tu=\lambda u\,,\qquad \lambda>1\,, \tag{5.8}$$

has to lie in the interior of $\mathscr{C}$, cf. [LL; Theorem 4.4.3].

Thus, let u be a quasi fixed point, then u is a solution to the Dirichlet problem

$$\begin{aligned} Au+\lambda^{-1}av&=\lambda^{-1}H \quad \text{in} \quad \Omega\,,\\ u&=\lambda^{-1}\varphi \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{5.9}$$

Due to the fact that λ is larger than one, the *a priori* estimates, applied to the present situation, then yield

$$m_1\leqq u\leqq m_2\,,\qquad |Du|\leqq 1-\Theta^*\,, \tag{5.10}$$

i.e. $u\in\mathring{\mathscr{C}}$.

Finally, let us show the existence of a solution to the variational inequality (4.33), in the case when Ω is a compact, connected n-dimensional manifold, $M=\Omega\times I$, where the metric in M is given by

$$ds^2=\psi(-dt^2+g_{ij}(x,t)dx^idx^j)\,. \tag{5.11}$$

Theorem 5.2. *Let $\psi,(g_{ij}(x,t))$ be of class C^2 in M, and assume that the matrix (g_{ij}) is uniformly elliptic on compact subsets of M. Then the variational inequality* (4.33) *has a solution $u\in H^{2,p}(\Omega)$, $1\leqq p<\infty$, for any bounded H. Here, $\partial\Omega$ is assumed to be empty.*

Proof. First, we assume the obstacles are constant

$$\psi_i=m_i\,,\qquad i=1,2\,. \tag{5.12}$$

Then, we define $\mathscr{C}$ and T similarly as before, where we note that now in the definitions (5.4), (5.5) of the linearized operator the metric is evaluated at the points (x,w), and where the covariant derivatives are also taken with respect to this metric. Of course, $u=Tw$ is now defined through

$$u\in K\,;\langle -a^{ij}D_iD_ju+av-H,\eta-u\rangle\geqq 0,\quad \forall\eta\in K\,. \tag{5.13}$$

It is very easy to see that this variational inequality has a unique solution

$$u\in H^{2,p}(\Omega)\quad \text{for any}\quad 1\leqq p<\infty\,, \tag{5.14}$$

e.g. by using the penalization method, cf. [KS] or the subsequent considerations in the case of general obstacles.

Hence, T is continuous and compact, and exactly the same conclusions as before yield the existence of a fixed point, since the obstacles are constants. This assumption is used when the *a priori* estimates are applied to a quasi fixed point u of the form (5.8); u is then a solution to a variational inequality with obstacles

$$\lambda^{-1}\psi_i\,,\qquad i=1,2\,, \tag{5.15}$$

instead of ψ_i, and in general it does not hold that the gradient of the new obstacles can be bounded uniformly strict from one, because of the time-dependence of the metric coefficients.

In the general case, we choose constants m_i, $i=1,2$, such that

$$m_1<\psi_1 \leqq \psi_2 < m_2, \tag{5.16}$$

and penalization functions $\beta_{i,\varepsilon}$ defined through

$$\beta_{1,\varepsilon}(t)=\begin{cases} -1, & t\leqq -\varepsilon \\ \text{linear}, & -\varepsilon\leqq t\leqq 0 \\ 0, & t\geqq 0 \end{cases} \tag{5.17}$$

and

$$\beta_{2,\varepsilon}(t)=\begin{cases} 0, & t\leqq 0 \\ \text{linear}, & 0\leqq t\leqq \varepsilon \\ 1, & t\geqq \varepsilon \end{cases} \tag{5.18}$$

for positive values of ε, and consider the variational inequality

$$u\in K; \langle Au+av-H+\mu\beta_{1,\varepsilon}(u-\psi_1)+\mu\beta_{2,\varepsilon}(u-\psi_2), \eta-u\rangle \geqq 0,\ \forall \eta\in K, \tag{5.19}$$

where K is defined by the obstacles m_1, m_2 and where μ is a large positive constant.

According to the first part of the proof, there is a solution to this variational inequality, since the penalization functions are Lipschitz continuous. Moreover, if ε is small such that

$$m_1+\varepsilon \leqq \psi_1 \leqq \psi_2 \leqq m_2-\varepsilon, \tag{5.20}$$

and μ large such that

$$|a(x,u,0)-H|\leqq \mu, \tag{5.21}$$

then

$$m_1<u<m_2, \tag{5.22}$$

so that u is actually a solution of

$$Au+av-H+\mu\beta_{1,\varepsilon}(u-\psi_1)+\mu\beta_{2,\varepsilon}(u-\psi_2)=0. \tag{5.23}$$

For a verification of (5.22), see the proof of Theorem 6.1 below, where a more general situation is treated; we note that in the places where u touches an obstacle the sum of the penalization functions is equal to -1 or $+1$.

Moreover, for large μ, depending on the C^2-norm of the obstacles, we have

$$\psi_1-\varepsilon \leqq u \leqq \psi_2+\varepsilon. \tag{5.24}$$

Indeed, suppose e.g. that the second inequality is violated, and denote the graph of $\psi_2+\varepsilon$ with $\mathscr{S}_1$ and the graph of u with $\mathscr{S}_2$ and their respective mean curvatures with H_1 and H_2. As in the proof of Lemma 7.2 below we would then find a time-like future directed geodesic $\gamma=(\gamma^\alpha)$ from $\mathscr{S}_1$ to $\mathscr{S}_2$ maximizing the distance between the two surfaces. Furthermore, we would derive the inequality

$$H_1(\xi_1)-H_2(\xi_2)+\int_0^{d_0} R_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta \leqq 0, \tag{5.25}$$

where $\xi_i \in \mathcal{S}_i$ are the endpoints of γ, d_0 is the distance, and $(R_{\alpha\beta})$ is the Ricci tensor of M, cf. formula (7.15) below.

We remark that $\mathcal{S}_1$ and $\mathcal{S}_2$ are staying in a compact subset of M in view of (5.22) for all values of μ and ε, and so do the corresponding geodesics γ, cf. [HE; the corollary after Proposition 6.6.1]. Hence, the integral in (5.25) is uniformly bounded, cf. Lemma 5.3 below.

The endpoints ξ_i can be expressed as

$$\xi_1 = (x_1, \psi_2(x) + \varepsilon), \qquad \xi_2 = (x_2, u(x_2)), \tag{5.26}$$

where

$$u(x_2) > \psi(x_2) + \varepsilon, \tag{5.27}$$

otherwise we could find a future directed time-like path intersecting $\mathcal{S}_1$ twice, which is impossible since $\mathcal{S}_1$ is a Cauchy surface, cf. [BU].

Thus, we finally deduce from (5.25)

$$H_1(\xi_1) - H + \mu + \int_0^{d_0} R_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta \leqq 0, \tag{5.28}$$

since

$$H_2(\xi_2) = H - \mu, \tag{5.29}$$

in view of (5.27).

We conclude that the estimate (5.24) holds for large μ, independent of ε. In the limit, when ε goes to zero, we then obtain a solution of the variational inequality (4.33).

It remains to prove that the integral in (5.25) is uniformly bounded. In view of the boundedness of the components of the Ricci tensor, this is tantamount to prove the boundedness of the components of the tangent vectors $(\dot\gamma^\alpha)$. Since the geodesics emanate from compact space-like surfaces, the result will follow from

Lemma 5.3. *Let $\gamma = (\gamma^\alpha)$ be a time-like geodesic contained in a compact subset K of a globally hyperbolic manifold M, the metric of which can be expressed in the form (4.1). Let $\xi_0 = (\gamma^0(0), \ldots, \gamma^n(0))$ be an endpoint of the geodesic and assume that the time-like curve*

$$\sigma(\tau) = (\gamma^0(\tau), \gamma^1(0), \ldots, \gamma^n(0)) \tag{5.30}$$

also stays in K. Then, if γ is parametrized by arclength τ, $0 \leqq \tau \leqq d_0$, the estimate

$$|\dot\gamma^0(\tau)| \leqq c \cdot |\dot\gamma^0(0)|, \qquad \forall 0 \leqq \tau \leqq d_0 \tag{5.31}$$

holds with a constant c depending only on K and the metric.

Proof. We first note that the space-like components of $\dot\gamma$ can be estimated by $\dot\gamma^0$ since

$$-1 = \psi(-|\gamma^0|^2 + g_{ij}\dot\gamma^i\dot\gamma^j). \tag{5.32}$$

The estimate (5.31) now follows from the geodesic equation

$$\dot\gamma^0 + \Gamma^0_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = 0, \tag{5.33}$$

and the fact that the length of the time-like path σ is bounded by a constant depending only on K, i.e.

$$\int_0^{d_0} |\dot{\gamma}^0| \leqq c. \tag{5.34}$$

Multiplying now (5.33) with $\dot{\gamma}^0$ we obtain

$$\frac{d}{d\tau}|\dot{\gamma}^0|^2 \leqq c\cdot|\dot{\gamma}^0|\cdot|\dot{\gamma}^0|^2 \tag{5.35}$$

in view of the boundedness of the Christoffel symbols. The desired estimate then follows from (5.34).

6. Slices of Prescribed Mean Curvature

In this section we suppose M to be globally hyperbolic and connected, having a *compact Cauchy surface.* This implies especially that there exists a global time-function f

$$f: M \to \mathbb{R},\ Df \neq 0 \text{ everywhere}, \tag{6.1}$$

such that Df is a time-like gradient field.

The level surfaces

$$\mathcal{S} = \{f = \text{const}\} \tag{6.2}$$

are all Cauchy surfaces and hence compact and connected, since Cauchy surfaces are all homeomorphic, see [GR], [HE; p. 212], and [GRH; p. 252]. The f is usually supposed to be of class C^∞, which we shall assume, too, though C^3 would be sufficient for our purposes.

Let ds^2 be the original metric in M and let

$$d\sigma^2 = \psi^{-1} ds^2,\ \psi = -\|Df\|^2. \tag{6.3}$$

In this metric Df is a *unit gradient field* and its integral curves are therefore geodesics.

Let

$$I = (T_0, T_1) = f(M),\ -\infty \leqq T_0 < T_1 \leqq \infty, \tag{6.4}$$

and assume for simplicity that $0 \in I$, then M is homeomorphic to $\Omega \times I$, where

$$\Omega = f^{-1}(0),\ \partial\Omega = \emptyset \tag{6.5}$$

is a compact Cauchy surface and the metric (6.3) can be represented in Gaussian normal coordinates relative to Ω

$$d\sigma^2 = -dt^2 + g_{ij}(x,t)\,dx^i dx^j, \tag{6.6}$$

because the time-like geodesics orthogonal to Ω are integral curves of Df, and

$$t = f(x,t). \tag{6.7}$$

Thus, the original metric can be recovered in the familiar and convenient form

$$ds^2 = \psi(-dt^2 + g_{ij}(x,t)\,dx^i dx^j). \tag{6.8}$$

We want to find compact slices of prescribed mean curvature H in M, which are necessarily representable as graphs over Ω. In view of the *a priori* estimates in Sect. 4 the only difficulty in this achievement stems from preventing the slices to run into the singularities T_0, T_1, i.e. we must be able to obtain *a priori* estimates for the height of the graphs. For this reason we postulate the *big bang* and the *big crunch* hypotheses:[1] there exist sequences (ψ_k), $(\tilde{\psi}_k)$ of C^2-functions over Ω the graphs of which have mean curvatures $H(\psi_k)$, $H(\tilde{\psi}_k)$ respectively, such that

$$(\psi_k) \searrow, \quad \sup \psi_k \to T_0, \quad H(\psi_k) \to H_-, \tag{6.9}$$

and

$$(\tilde{\psi}_k) \nearrow, \quad \inf \tilde{\psi}_k \to T_1, \quad H(\tilde{\psi}_k) \to H_+, \tag{6.10}$$

where we furthermore assume from the start

$$\psi_k < \tilde{\psi}_k. \tag{6.11}$$

Here, H_- and H_+ are extended real numbers, satisfying

$$-\infty \leqq H_- < H_+ \leqq \infty. \tag{6.12}$$

We also ought to explain the notation $H(u)$ to represent the mean curvature of a graph evaluated at $(x, u(x))$.

We further remark that not all the assumptions stated in (6.9) and (6.10) are really necessary to prove the existence of a slice with mean curvature H. The proof of Theorem 6.1 below indicates what is really essential; the full hypotheses are only needed when we want to prove the existence of a *foliation* of M by surfaces of constant mean curvature.

Theorem 6.1. *Let H be a bounded function on M satisfying*

$$H_- < \inf H \leqq \sup H < H_+. \tag{6.13}$$

Then, there exists a space-like function $u \in H^{2,p}(\Omega)$, $1 \leqq p < \infty$, such that

$$Au + av = H \quad in \quad \Omega, \tag{6.14}$$

i.e. graph u has mean curvature H.

Proof. Choose two barriers ψ, $\tilde{\psi}$ such that

$$H(\psi) < \inf H \leqq \sup H < H(\tilde{\psi}), \qquad \psi < \tilde{\psi}, \tag{6.15}$$

and let u be a solution to the variational inequality

$$\begin{gathered} \langle Au + av - H, \eta - u \rangle \geqq 0, \qquad \forall \eta \in K, \\ K = \{\eta \in H^{1,\infty}(\Omega) : \psi \leqq \eta \leqq \tilde{\psi}\}, \end{gathered} \tag{6.16}$$

which exists and is of class $H^{2,p}(\Omega)$, $1 \leqq p < \infty$, according to Theorem 5.2.

We shall now show that u cannot touch the obstacles, i.e.

$$\psi < u < \tilde{\psi}, \tag{6.17}$$

and hence that u is a solution of (6.14).

1 Similar hypotheses have been used in D. Eardley and L. Smarr: Time functions in numerical relativity. Phys. Rev. D **19**, 2239 (1979), as the author learnt from the referee

Let us only prove the first inequality in (6.17). Assume that there is $x_0 \in \Omega$ such that

$$u(x_0) = \psi(x_0), \tag{6.18}$$

then

$$Du(x_0) = D\psi(x_0), \qquad g_{ij}(x_0, u) = g_{ij}(x_0, \psi), \tag{6.19}$$

and in a neighbourhood $B_R(x_0) \subset \Omega$ there holds

$$Au + av \geqq H, \tag{6.20}$$

since $\psi < \tilde{\psi}$.

On the other hand, ψ satisfies in $B_R(x_0)$ the equation

$$A\psi + av = H(\psi), \tag{6.21}$$

where, of course, the differential operator is now defined with respect to the metric $(g_{ij}(x, \psi))$. But due to (6.15), (6.19) and the fact that ψ is of class C^2, we can choose R so small that ψ and u satisfy the differential inequality

$$-a^{ij} D_i D_j (u - \psi) > 0, \quad \text{a.e. in} \quad B_R(x_0), \tag{6.22}$$

where (a^{ij}) is a Hölder continuous, symmetric uniformly elliptic matrix, and where the second derivatives are ordinary derivatives.

But from the well-known strong maximum principle for $H^{2,p}$-solutions (see e.g. [TR; Theorem 2]) we deduce that (6.18) and (6.22) exclude each other, hence the result.

We actually proved a little bit more than the mere claim of the theorem, let us state this extra information as a lemma.

Lemma 6.2. *Let $\psi_1, \psi_2 \in C^2(\Omega)$ be given, the graphs of which have mean curvatures $H_1(x, \psi_1)$ and $H_2(x, \psi_2)$, respectively, and assume*

$$\psi_1 < \psi_2 \quad \textit{and} \quad H_1(x, \psi_1) < H_2(x, \psi_2). \tag{6.23}$$

Then, for any continuous function $H = H(x, t)$ satisfying

$$H_1(x, t) < H(x, t) < H_2(x, t), \tag{6.24}$$

we can find a function $u \in H^{2,p}(\Omega)$, $1 \leqq p < \infty$, the graph of which has mean curvature H, and such that

$$\psi_1 < u < \psi_2. \tag{6.25}$$

Of special physical interest are slices of constant mean curvature. With the help of Lemma 6.2 we obtain

Theorem 6.3. *There exists a family $(\mathscr{S}_\tau)$ of slices with constant mean curvature τ, $H_- < \tau < H_+$, which can be represented as graphs of functions u_τ defined over Ω, such that*

$$u_\tau < u_{\tau'}, \quad \textit{for} \quad \tau < \tau'. \tag{6.26}$$

Moreover, there holds

$$\lim_{\tau \to H_-} \sup_\Omega u_\tau = T_0, \tag{6.27}$$

and

$$\lim_{\tau \to H_+} \inf_{\Omega} u_\tau = T_1 . \tag{6.28}$$

Proof. Let us first construct countably many slices $(u_{\tau_k})_{k\in\mathbb{Z}}$, such that

$$\tau_k < \tau_{k+1}, \qquad \forall k \in \mathbb{Z}, \tag{6.29}$$

$$\lim_{k\to -\infty} \tau_k = H_-, \qquad \lim_{k\to\infty} \tau_k = H_+, \tag{6.30}$$

and the relations (6.26), (6.27), and (6.28) are valid for this family.

Choose a sequence of barriers (ψ_k), $k\in\mathbb{Z}$, such that

$$\psi_k < \psi_{k+1}, \qquad \sup_{\Omega} H(\psi_k) < \inf_{\Omega} H(\psi_{k+1}), \tag{6.31}$$

$$\lim_{k\to\infty} \inf_{\Omega} \psi_k = T_1, \qquad \lim_{k\to\infty} H(\psi_k) = H_+, \tag{6.32}$$

and

$$\lim_{k\to -\infty} \sup_{\Omega} \psi_k = T_0, \qquad \lim_{k\to -\infty} H(\psi_k) = H_- . \tag{6.33}$$

Let (τ_k), $k\in\mathbb{Z}$, be defined through

$$H(\psi_k) < \tau_k < H(\psi_{k+1}), \tag{6.34}$$

and let (u_{τ_k}) be slices with mean curvature τ_k satisfying

$$\psi_k < u_{\tau_k} < \psi_{k+1}, \tag{6.35}$$

according to Lemma 6.2.

Next, let

$$I_k = [\tau_k, \tau_{k+1}], \tag{6.36}$$

and consider for fixed k the family $\mathscr{F}$ of sets F defined through the requirement:

$$F \subset \{\eta \in C^{2,\alpha}(\Omega) : u_{\tau_k} \leqq \eta \leqq u_{\tau_{k+1}}\}, \tag{6.37}$$

graph η has constant mean curvature $H(\eta)$ such that

$$\tau_k \leqq H(\eta) \leqq \tau_{k+1} \tag{6.38}$$

and, for $\eta_1, \eta_2 \in F$ there holds

$$\eta_1 \leqq \eta_2, \quad \text{if} \quad H(\eta_1) \leqq H(\eta_2). \tag{6.39}$$

In view of the *a priori* estimates in Sect. 4 the $C^{2,\alpha}$-norms of the functions η are uniformly bounded and the graphs are uniformly space-like. Therefore, Zorn's lemma is applicable to conclude that $\mathscr{F}$ contains a maximal subset F_k, maximal with respect to inclusion. From Lemma 6.2 we then deduce that

$$I_k = \{H(\eta) : \eta \in F_k\}, \tag{6.40}$$

for let $\tau \in I_k$ be such that there is no $\eta \in F_k$ with $H(\eta) = \tau$, and consider the non-empty subsets F_-, F_+ defined through

$$F_- = \{\eta \in F_k : H(\eta) < \tau\}, \tag{6.41}$$

$$F_+ = \{\eta \in F_k : H(\eta) > \tau\}. \tag{6.42}$$

The functions

$$u_- = \sup\{\eta : \eta \in F_-\} \tag{6.43}$$

and

$$u_+ = \inf\{\eta : \eta \in F_+\} \tag{6.44}$$

then have constant mean curvatures

$$H(u_-) \leqq \tau \leqq H(u_+) \tag{6.45}$$

and are members of F_k because of its maximality. Moreover, $H(u_-)$ or $H(u_+)$ have to coincide with τ, because otherwise Lemma 6.2 would yield the existence of a function u with $H(u) = \tau$ and

$$u_- < u < u_+ , \tag{6.46}$$

in contrast to the maximality of F_k.

To complete the proof of the theorem, we take the union of all F_k's to obtain the desired family of slices.

7. Foliation of Space-Time by Surfaces of Constant Mean Curvature

The obvious question, if the family of slices given in Theorem 6.3 is a foliation of M, can be affirmatively answered if we assume furthermore that the manifold satisfies the *time-like convergence* condition, i.e.

$$R_{\alpha\beta} t^\alpha t^\beta \geqq 0 \tag{7.1}$$

for any time-like vector field (t^α). If the Einstein equations hold in M with zero cosmological constant, then (7.1) is equivalent to the *strong energy condition*, cf. [HE; p. 95].

One consequence of this assumption is that slices with a given mean curvature are unique if the mean curvature does not vanish identically, and if there are two different maximal slices, then both have to be totally geodesic. This is well-known in the literature, and has already been used to prove uniqueness and local foliation results, cf. [BF1, CB, CFM, GO, MT]. We use these ideas to give a rigorous and comprehensive proof of the existence of a foliation by surfaces of constant mean curvature, where we are also able to overcome the difficulties arising from the presence of different maximal slices.

Let us first state

Lemma 7.1. *Let M satisfy the assumptions stated above and let $\mathscr{S}$ be a compact slice with constant mean curvature τ which is not totally geodesic. Then, in any neighbourhood of $\mathscr{S}$ there are slices with strictly larger and smaller mean curvatures.*

A proof of the lemma can be found in [MT; Lemma 4].

In the following we shall use definitions and terminology from [HE; Chap. 6], for the definition of the Lorentz distance function d see especially [HE; p. 215]. In a globally hyperbolic manifold d is continuous in $M \times M$.

Lemma 7.2. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be compact slices with mean curvatures H_1, respectively, H_2, and assume*

$$d(\mathcal{S}_1, \mathcal{S}_2) > 0. \tag{7.2}$$

Let $\xi_i \in \mathcal{S}_i$ be points satisfying

$$d(\xi_1, \xi_2) = d(\mathcal{S}_1, \mathcal{S}_2). \tag{7.3}$$

Then there holds

$$H_1(\xi_1) \leqq H_2(\xi_2). \tag{7.4}$$

Proof. This lemma is also well-known, see e.g. [BF1; p. 161] and [MT; p. 119]. First, we observe that the points $\xi_i \in \mathcal{S}_i$, $i=1,2$, exist since d is continuous and the slices are compact. Moreover, the points are joined by a time-like, future directed geodesic $\gamma = (\gamma^\alpha)$ orthogonal to both $\mathcal{S}_1$ and $\mathcal{S}_2$, having maximal length among all non-space-like curves connecting $\mathcal{S}_1$ and $\mathcal{S}_2$. The second variation of its curve length is therefore non-negative. Choosing the variation appropriately it turns out that

$$H_1(\xi_1) - H_2(\xi_2) + \int_0^{d_0} R_{\alpha\beta} \dot{\gamma}^\alpha \dot{\gamma}^\beta \leqq 0, \tag{7.5}$$

where $d_0 = d(\xi_1, \xi_2)$ and γ is parametrized by arc length, cf. [MT; p. 119], [GKM; p. 126], and [HE; Lemma 4.5.7]. We point out that we use the opposite sign convention in the definition of the mean curvature. The time-like convergence condition then yields the result. Combining the lemmata we can deduce

Theorem 7.3. *Let $\mathcal{S}_i$, $i=1,2$, be compact slices with constant mean curvatures τ_i. Assume $\tau_1 \leqq \tau_2$, then there holds:*

(i) *if the τ_i are not identically zero, then*

$$d(\mathcal{S}_2, \mathcal{S}_1) = 0, \tag{7.6}$$

and

(ii) *if $\tau_1 = \tau_2 = 0$ and if $\mathcal{S}_1 \neq \mathcal{S}_2$, then both surfaces are totally geodesic.*

Proof. We only prove (i), since the proof of (ii) will be identical. For definiteness, suppose that $\tau_1 \neq 0$. This means, that $\mathcal{S}_1$ is not totally geodesic, since

$$\frac{1}{n} \tau_1^2 \leqq \omega^2 \tag{7.7}$$

Hence, in any neighbourhood of $\mathcal{S}_1$, there are slices with smaller or larger mean curvatures. Let

$$d(\mathcal{S}_2, \mathcal{S}_1) > 0. \tag{7.8}$$

Then, because of the continuity of d, there exists a compact slice $\mathcal{S}_1^*$, such that

$$d(\mathcal{S}_2, \mathcal{S}_1^*) > 0, \tag{7.9}$$

and

$$H(\mathcal{S}_1^*) < \tau_1, \tag{7.10}$$

in contradiction with the result of Lemma 7.2. Theorem 7.3 is therefore proved.

Let us now analyze the situation described in the second part of the theorem more closely. Suppose there are two different compact maximal slices $\mathcal{S}_-$ and $\mathcal{S}_+$ which are then necessarily totally geodesic. Since both slices are Cauchy surfaces, cf. [BU], we either have

$$d(\mathcal{S}_-,\mathcal{S}_+)>0 \quad \text{or} \quad d(\mathcal{S}_+,\mathcal{S}_-)>0. \tag{7.11}$$

Assume the first inequality to be valid, and consider a Gaussian normal coordinate system relative to $\mathcal{S}_-$ which is defined in a neighbourhood of $\mathcal{S}_-$, so that

$$ds^2 = -dt^2 + g_{ij}(x,t)\,dx^i dx^j. \tag{7.12}$$

The time-like convergence condition then says that in this coordinate system

$$R_{00} \geqq 0. \tag{7.13}$$

On the other hand, we know (cf. [EI; p. 21] and observe that we use the opposite sign in the definition of the Ricci tensor)

$$R_{00} = -\frac{\partial^2 \log\sqrt{g}}{\partial t^2} - \tfrac{1}{4} g^{mi} g^{kl} \cdot \frac{\partial g_{ik}}{\partial t} \cdot \frac{\partial g_{ml}}{\partial t}. \tag{7.14}$$

The second term on the right-hand side is non-positive, it is exactly "$-\omega^2$" of the surface $t=\text{const}$, thus

$$\frac{\partial^2 \log\sqrt{g}}{\partial t^2} \leqq 0 \tag{7.15}$$

in a neighbourhood of $\mathcal{S}_-$.

Now, look at the surfaces

$$\mathcal{S}_\varepsilon := \{(x,t) : t=\varepsilon\} \tag{7.16}$$

for $\varepsilon \geqq 0$. Their mean curvatures $H(\mathcal{S}_\varepsilon)$ are equal to

$$H(\mathcal{S}_\varepsilon) = -\frac{\partial}{\partial t} \log\sqrt{g} \tag{7.17}$$

evaluated for $t=\varepsilon$. From

$$H(\mathcal{S}_0) \equiv H(\mathcal{S}_-) = 0 \tag{7.18}$$

and from (7.15), we thus deduce

$$H(\mathcal{S}_\varepsilon) \geqq 0 \quad \text{for} \quad \varepsilon \geqq 0. \tag{7.19}$$

But for small ε, we certainly have

$$d(\mathcal{S}_\varepsilon, \mathcal{S}_+) > 0, \tag{7.20}$$

hence, the level surfaces $\mathcal{S}_\varepsilon$ are all totally geodesic for small positive ε.

This information will enable us to prove, that the tubular neighbourhood of $\mathcal{S}_-$ in which the Gaussian coordinate system is defined contains the set

$$\{\xi \in M : d(\mathcal{S}_-, \xi) \leqq d(\mathcal{S}_-, \mathcal{S}_+)\} \tag{7.21}$$

and

$$\mathscr{S}_+ = \{\xi \in M : d(\mathscr{S}_-, \xi) = d(\mathscr{S}_-, \mathscr{S}_+)\,. \tag{7.22}$$

Indeed, let the neighbourhood contain the surfaces

$$\mathscr{S}_\varepsilon = \{\xi \in M : d(\mathscr{S}_-, \xi) = \varepsilon\}, \quad 0 \leqq \varepsilon < \varepsilon_0\,, \tag{7.23}$$

and let ε_0 be maximal subject to the condition $\varepsilon_0 \leqq d(\mathscr{S}_-, \mathscr{S}_+)$. As we have just proved, those surfaces are all compact maximal slices and are therefore representable as graphs of functions u_ε over Ω, satisfying the differential equation

$$Au_\varepsilon + av = 0 \quad \text{in} \quad \Omega\,. \tag{7.24}$$

The *a priori* estimates in Sect. 4 will yield uniform estimates for $v = v_\varepsilon$ and the $C^{2,\alpha}$-norms of u_ε, provided the surfaces $\mathscr{S}_\varepsilon$ remain in a compact set of M, or equivalently, provided the ranges of u_ε are compactly contained in I.

Various conditions can be imposed to force the uniform compactness. We shall consider the one appropriate for our purposes, namely, we shall assume that there exists a compact slice $\mathscr{S}$ in M with $H(\mathscr{S}) > 0$. Then, if $\mathscr{S} = \operatorname{graph} u$, we conclude

$$u_- \leqq u_\varepsilon \leqq u\,, \qquad \forall 0 \leqq \varepsilon < \varepsilon_0\,, \tag{7.25}$$

due to the time-like convergence condition, cf. Lemma 7.2. Thus, the uniform compactness is proved. Let us remark that a similar consideration would be possible if we had assumed the existence of a compact slice $\mathscr{S}$ with strictly negative mean curvature; instead of looking to the future of $\mathscr{S}_-$, we then would look to the past of $\mathscr{S}_+$.

The surfaces $\mathscr{S}_\varepsilon$ are therefore uniformly smooth, and going to the limit we obtain a smooth totally geodesic

$$\mathscr{S}_{\varepsilon 0} = \operatorname{graph} u_{\varepsilon 0} = \{\xi \in M : d(\mathscr{S}_-, \xi) = \varepsilon_0\}\,. \tag{7.26}$$

We shall show that the tubular neighbourhood of $\mathscr{S}_-$ contains $\mathscr{S}_{\varepsilon 0}$, by proving that the geodesics orthogonal to $\mathscr{S}_-$ cannot intersect in a sufficiently small neighbourhood of $\mathscr{S}_{\varepsilon 0}$.

Consider a point ξ in a tubular neighbourhood $U_{\varepsilon 0}$ of $\mathscr{S}_{\varepsilon 0}$, where we may restrict our attention to points ξ lying in $\mathscr{S}_{\varepsilon 0}$ or in its future, and let γ be a time-like geodesic from $\mathscr{S}_-$ to ξ, orthogonal to $\mathscr{S}_-$. Our first observation is that γ has to be maximal, i.e.

$$\operatorname{length} \gamma = d(\mathscr{S}_-, \xi)\,. \tag{7.27}$$

Indeed, let

$$\xi_0 \in \gamma \cap \mathscr{S}_{\varepsilon 0}\,, \tag{7.28}$$

where $\xi_0 = \xi$ if ξ belongs to $\mathscr{S}_{\varepsilon 0}$, and let γ_0 be the corresponding segment of γ. Then γ_0 is a maximal geodesic from ξ_0 to $\mathscr{S}_-$, and also a maximal geodesic from $\mathscr{S}_{\varepsilon 0}$ to $\mathscr{S}_-$ since $\mathscr{S}_{\varepsilon 0}$ is a level surface. Thus, γ is orthogonal both to $\mathscr{S}_-$ and $\mathscr{S}_{\varepsilon 0}$, and hence maximal, since ξ is contained in a tubular neighbourhood of $\mathscr{S}_{\varepsilon 0}$.

Assume now that there were two geodesics γ, γ' orthogonal to $\mathscr{S}_-$ and containing ξ. Both had to be maximal, and therefore their segments from ξ to $\mathscr{S}_{\varepsilon 0}$ had to coincide, which would yield an immediate contradiction if ξ lies in the future of $\mathscr{S}_{\varepsilon 0}$. If ξ belongs to $\mathscr{S}_{\varepsilon 0}$, then γ, γ' would be orthogonal to $\mathscr{S}_{\varepsilon 0}$, yielding a contradiction, too.

Summarizing we conclude that the tubular neighbourhood of $\mathscr{S}_-$ contains the set

$$\{\xi \in M : d(\mathscr{S}_-, \xi) \leqq \varepsilon_0\}. \tag{7.29}$$

Hence, ε_0 has to be equal to $d(\mathscr{S}_-, \mathscr{S}_+)$ because of its maximality.

It remains to prove that $\mathscr{S}_{\varepsilon_0}$ is equal to $\mathscr{S}_+$. Their intersection is certainly not empty; in fact the surfaces touch each other where they intersect.

Let $\xi_0 \in \mathscr{S}_{\varepsilon_0} \cap \mathscr{S}_+$ and choose Gaussian normal coordinates relative to $\mathscr{S}_+$. Let ξ_0 be given as $(x_0, 0)$. Then, in a neighbourhood $B_R(x_0)$, $\mathscr{S}_{\varepsilon_0}$ is represented as a graph of a function u such that

$$u \geqq 0, \qquad u(x_0) = 0, \qquad Du(x_0) = 0, \tag{7.30}$$

and

$$Au + av = 0 \quad \text{in} \quad B_R(x_0). \tag{7.31}$$

The last equation can be written in a more convenient form as

$$-D_i(a^{ij} D_j u) + b = 0 \quad \text{in} \quad B_R(x_0) \tag{7.32}$$

with ordinary derivatives. The lower order term $b = b(x, u, Du)$ is smooth, and because $\mathscr{S}_+$ is maximal we have

$$b(x, 0, 0) = 0 \quad \text{in} \quad B_R(x_0), \tag{7.33}$$

compare the formula (1.14). Thus,

$$b(x, u, Du) = \int_0^1 \frac{\partial}{\partial \tau} b(x, \tau u, \tau Du) = b_0 \cdot u + b^i D_i u. \tag{7.34}$$

We can therefore apply the Harmack inequality for linear operators, cf. [GT; Theorem 8.20], to conclude

$$u \equiv 0 \quad \text{in} \quad B_R(x_0). \tag{7.35}$$

Thus, we have proved that the subset

$$\{\xi \in \mathscr{S}_+ : d(\mathscr{S}_-, \xi) = d(\mathscr{S}_-, \mathscr{S}_+)\} \tag{7.36}$$

is relatively open in $\mathscr{S}_+$; since it is also closed, because of the continuity of d, and since $\mathscr{S}_+$ is connected we obtain:

Theorem 7.4. *Let $\mathscr{S}_-$ and $\mathscr{S}_+$ be compact maximal slices which do not coincide, then they are both totally geodesic. Assume, furthermore, that there exists a compact slice $\mathscr{S}$ whose mean curvature $H(\mathscr{S})$ is either strictly negative or strictly positive. Then by changing the labels if necessary, we have*

$$0 < d(\mathscr{S}_-, \mathscr{S}_+), \qquad \mathscr{S}_+ \subset I^+(\mathscr{S}_-), \tag{7.37}$$

and the slices

$$\mathscr{S}_\varepsilon = \{\xi \in M : d(\mathscr{S}_-, \xi) = \varepsilon\} \tag{7.38}$$

are all totally geodesic for $0 \leqq \varepsilon \leqq d(\mathscr{S}_-, \mathscr{S}_+)$. Especially $\mathscr{S}_+$ is the level surface with distance $d(\mathscr{S}_-, \mathscr{S}_+)$. All slices are contained in a tubular neighbourhood of $\mathscr{S}_-$. The *metric coefficients* (g_{ij}) *in the representation*

$$ds^2 = -dt^2 + g_{ij}(x,t)\,dx^i dx^j \tag{7.39}$$

are such that

$$g_{ij}(x,0) = g_{ij}(x,t), \qquad \forall 0 \leqq t \leqq d(\mathscr{S}_-, \mathscr{S}_+). \tag{7.40}$$

Only the last statement needs some further moment of consideration: from (7.14) and the reasoning thereafter we conclude

$$g^{mi} g^{kl} \frac{\partial g_{ik}}{\partial t} \frac{\partial g_{ml}}{\partial t} = 0, \tag{7.41}$$

which simply means that the surfaces $t = \text{const}$ are totally geodesic. But from [GE2; formula (3.6)] we then deduce

$$\frac{\partial g_{ij}}{\partial t} = 0, \quad \forall i, j. \tag{7.42}$$

Physically, it is not realistic that a space-time would be static for a positive period of time, cf. the corresponding considerations in [CB, MT], but there are of course numerous examples of globally hyperbolic Lorentz manifolds satisfying the time-like convergence condition, and being endowed with a metric ds^2 of the form (7.39), (7.40).

Combining now the results of the Theorems 6.3, 7.3, and 7.4 we can prove the existence of a foliation consisting of slices of constant mean curvature.

First, we deduce from Theorems 6.3 and 7.3 that there are uniquely determined slices $\mathscr{S}_\tau = \operatorname{graph} u_\tau$ with mean curvature $H(\mathscr{S}_\tau) = \tau$ for any non-zero $\tau \in (H_-, H_+)$ satisfying (6.26), (6.27), and (6.28). Moreover, if

$$0 \in (H_-, H_+), \tag{7.43}$$

then define

$$u_- = \sup\{u_\tau : \tau < 0\}, \tag{7.44}$$

and

$$u_+ = \inf\{u_\tau : \tau > 0\}. \tag{7.45}$$

Both functions are smooth and their graphs are compact maximal slices. If they are different, then the results of Theorem 7.4 apply, and we obtain in any case a family $\mathscr{C}_0$ of disjoint maximal slices

$$\mathscr{C}_0 = \{\mathscr{S}_\varepsilon : 0 \leqq \varepsilon \leqq d(\mathscr{S}_-, \mathscr{S}_+)\}, \tag{7.46}$$

where $\mathscr{S}_- = \operatorname{graph} u_-$, $\mathscr{S}_+ = \operatorname{graph} u_+$, and

$$\mathscr{S}_\varepsilon = \{\xi \in M : d(\mathscr{S}_-, \xi) = \varepsilon\}. \tag{7.47}$$

If $\mathscr{C}_0$ contains more than one element, than all maximal slices are totally geodesic, and

$$\mathscr{S}_+ = \{\xi \in M : d(\mathscr{S}_-, \xi) = d(\mathscr{S}_-, \mathscr{S}_+)\}. \tag{7.48}$$

Furthermore, we can prove

Theorem 7.5. *The disjoint family of slices*

$$(\mathscr{S}_\tau)_{\tau \neq 0} \cup \mathscr{C}_0 , \tag{7.49}$$

where $\mathscr{C}_0$ *can be empty, is a foliation of* M.

Proof. We have only to prove that this family is a covering of M. Thus, let $\xi_0 \in M$, $\xi_0 = (x_0, t_0)$ with $x_0 \in \Omega$ and $t_0 \in (T_0, T_1)$. In view of (6.27) and (6.28) the families

$$\Lambda_1 = \{u_\tau : u_\tau(x_0) \leqq t_0\} , \tag{7.50}$$

and

$$\Lambda_2 = \{u_\tau : u_\tau(x_0) \geqq t_0\} , \tag{7.51}$$

are non-empty and for arbitrary members $u_1 \in \Lambda_1$, $u_2 \in \Lambda_2$, we have

$$u_1 \leqq u_2 . \tag{7.52}$$

Here, τ is a general label for the members of the family. Let u_1 be the largest element in Λ_1, and u_2 be the smallest element in Λ_2. Both u_1, u_2 exist in view of the *a priori* estimates. Then, we have

$$u_1(x_0) \leqq t_0 \leqq u_2(x_0) , \qquad u_1 \leqq u_2 . \tag{7.53}$$

If the inequalities would hold strictly, then we could conclude that either

$$H(u_1) < H(u_2) , \tag{7.54}$$

or that

$$u_1, u_2 \in \mathscr{C}_0 . \tag{7.55}$$

In both cases, we then would find u_τ such that e.g.

$$u_1(x_0) < u_\tau(x_0) < t_0 , \tag{7.56}$$

contradicting the definition of u_1.

Appendix

For the convenience of the reader we indicate the details of Stampacchia's method to derive an upper bound for subsolutions of an elliptic equation. Our starting point is the inequality (4.16) which we multiply with the function given in (4.17) and integrate by parts. Using the notation

$$v_k = \max(v - k, 0) , \qquad A(k) = \{x \in \Omega : v_k > 0\} , \tag{A 1}$$

and

$$|A(k)| = \int_{A(k)} \sqrt{g} , \tag{A2}$$

we then obtain

$$\int_\Omega |Dv_k|^2 \leqq c \int_{A(k)} v^3 \tag{A3}$$

with some constant c, where we used the fact that from the main part of the left-hand side of (4.16) we also got a dominating term of the form

$$\int_{A(k)} v^2 |\langle Du, Dv\rangle|^2 \tag{A4}$$

in view of (4.9).

Next, we apply the Sobolev inequality

$$\left(\int_\Omega |v_k|^{n/n-1}\right)^{\frac{n-1}{n}} \leqq c \cdot \left\{\int_\Omega |Dv_k| + \int_\Omega |v_k|\right\}, \tag{A5}$$

valid for any compact manifold with or without boundary, to deduce from (A3)

$$\left(\int_\Omega |v_k|^{n/n-1}\right)^{\frac{n-1}{n}} \leqq c \cdot |A(k)|^{1/2} \cdot \left(\int_{A(k)} v^3\right)^{1/2}. \tag{A6}$$

Estimating the integral on the right-hand side with Hölder's inequality we obtain

$$\left(\int_\Omega |v_k|^{n/n-1}\right)^{\frac{n-1}{n}} \leqq c \cdot \|v\|_{2n}^{3/2} \cdot |A(k)|^{1-3/4n}. \tag{A7}$$

Moreover, for $h > k$ we have

$$(h-k)|A(h)| \leqq \int_{A(k)} (v-k) \leqq |A(k)|^{1/n} \cdot \left(\int_\Omega |v_k|^{n/n-1}\right)^{\frac{n-1}{n}}, \tag{A8}$$

and hence

$$(h-k)|A(h)| \leqq c \cdot \|v\|_{2n}^{3/2} \cdot |A(k)|^{1+1/4n}, \tag{A9}$$

valid for all $h > k \geqq k_0$.

From [ST; Lemma 4.1] we then conclude the estimate (4.18).

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References

[AV] Avez, A.: Essais de géometrie riemannienne hyperbolique global-applications à la rélativité générale. Ann. Inst. Fourier **13**, 105–190 (1963)

[BA] Bancel, D.: Problème aux limites pour les hypersurfaces maximales Lorentziennes. Rend. Circolo Mat. Palermo, Ser. II, **28**, 183–190 (1979)

[BS] Bartnik, R., Simon, L.: Space-like hypersurfaces with prescribed boundary values and mean curvature. Commun. Math. Phys. **87**, 131–152 (1982)

[BF1] Brill, D., Flaherty, F.: Isolated maximal surfaces in space-time. Commun. Math. Phys. **50**, 157–165 (1976)

[BF2] Brill, D., Flaherty, F.: Maximizing properties of extremal surfaces in general relativity. Ann. Inst. Henri Poincaré, A, **28**, 335–347 (1978)

[BU] Budic, R. et al.: On the determination of Cauchy surfaces from intrinsic properties. Commun. Math. Phys. **61**, 87–95 (1978)

[CA] Calabi, E.: Examples of Bernstein problems for some nonlinear equations. Proc. Symp. Pure Math. **25**, 223–230 (1970)

[CY] Cheng, S.Y., Yau, S.-T.: Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. Ann. Math. **104**, 407–419 (1976)

[CB] Choquet-Bruhat, Y.: Maximal submanifolds and submanifolds with constant mean extrinsic curvature of a Lorentzian manifold. Ann. Sc. Norm. Sup. Pisa, Sér. IV, **3**, 361–376 (1976)

[CFM] Choquet-Bruhat, Y., Fischer, A.E., Marsden, J.E.: Maximal hypersurfaces and positivity of mass. In: Isolated gravitating systems in general relativity. Ehlers, J. (ed.). Italian Physical Society 1979, pp. 396–456

[EI] Eisenhart, L.P.: Riemannian geometry. Princeton: Princeton University Press 1964

[GE1] Gerhardt, C.: Existence, regularity, and boundary behaviour of generalized surfaces of prescribed mean curvature. Math. Z. **139**, 173–198 (1974)

[GE2] Gerhardt, C.: Evolutionary surfaces of prescribed mean curvature. J. Differential Equations **36**, 139–172 (1980)

[GR] Geroch, R.P.: The domain of dependence. J. Math. Phys. **11**, 437–449 (1970)

[GRH] Geroch, R.P., Horowitz, G.T.: Global structure of space-time. In: General relativity. An Einstein centenary survey. Hawking, S.W., Israel, W. (eds.). Cambridge: Cambridge University Press 1979, pp. 212–293

[GT] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1977

[GO] Goddard, A.J.: Foliations of space-time by space-like hypersurfaces of constant mean curvature. Commun. Math. Phys. **54**, 279–282 (1977)

[GKM] Gromoll, D., Klingenberg, W., Meyer, W.: Riemannsche Geometrie im Großen. In: Lecture Notes in Mathematics, Vol. 55. Berlin, Heidelberg, New York: Springer 1968

[HE] Hawking, S.W., Ellis, G.F.R.: The large scale structure of space-time. Cambridge: Cambridge University Press 1973

[KS] Kinderlehrer, D., Stampacchia, G.: An introduction to variational inequalities and their applications. New York: Academic Press 1980

[LL] Lloyd, N.G.: Degree theory. Cambridge: Cambridge University Press 1978

[MT] Marsden, J.E., Tipler, F.J.: Maximal hypersurfaces and foliations of constant mean curvature in general relativity. Phys. Rep. **66**, 109–139 (1980)

[ST] Stampacchia, G.: Equations elliptique du second ordre à coefficients discontinus. Montréal: Les Presses de l'Université 1966

[TA] Treibergs, A.E.: Entire space-like hypersurfaces of constant mean curvature in Minkowski space. Invent. Math. **66**, 39–56 (1982)

[TR] Trudinger, N.S.: Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations. Invent. Math. **61**, 67–79 (1980)

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