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Two-Dimensional Exactly and Completely Integrable Dynamical Systems

Monopoles, Instantons, Dual Models, Relativistic Strings, Lund-Regge Model, Generalized Toda Lattice, etc.

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Abstract. An investigation of two-dimensional exactly and completely integrable dynamical systems associated with the local part of an arbitrary Lie algebra $\mathfrak{g}$ whose grading is consistent with an arbitrary integral embedding of $3d$-subalgebra in $\mathfrak{g}$ has been carried out. We have constructed in an explicit form the corresponding systems of nonlinear partial differential equations of the second order and obtained their general solutions in the sense of a Goursat problem. A method for the construction of a wide class of infinite-dimensional Lie algebras of finite growth has been proposed.

1. Introduction

In papers [1] (see also [2, 3]) we proposed a general scheme for the construction of exactly and completely integrable dynamical systems in two-dimensional space associated with an arbitrary graded Lie algebra or superalgebra $\mathfrak{g}=\sum_{-\infty}^{+\infty}\oplus\mathfrak{g}_a$, and developed a group method to find general solutions to these systems. The method enables us to obtain closed expressions for the solutions. However, due to the absence of a general procedure for the description (finding the structure constants) of Lie algebras of "arbitrary position", it is not always possible to write the equations of the corresponding systems in an explicit form. Furthermore the formulation of the equations is essentially dependent on a choice of gauge constraints.

In view of the aforementioned reasons, the proposed algebraic construction was fully realized [4] [in the sense of explicit formulae for the general solutions and the equations themselves) for the dynamical systems with an abelian invariance subalgebra $\mathfrak{g}_0=\sum_1^r\oplus u(1)$, $r\equiv\operatorname{rank}\mathfrak{g}$, which have no ambiguity related with a choice of gauge constraint. Later on the basis of study of the Lie-Bäcklund group transformation general criteria of exact or complete integrability for the systems of the special form ($\partial^2 x_j/\partial z_+ \partial z_- = f_j(x)$) were worked out [5]. (Here and in what follows, by *exactly integrable systems* we mean conditionally the systems admitting the solutions with the number of arbitrary functions sufficient for the statement of the Goursat problem, which define initial data on the characteristics. The systems with the relevant spectrum of soliton-type solutions are called *completely integrable*]. The final corollary is:

i) a system is exactly integrable if the corresponding internal symmetry group with Lie-Bäcklund algebra is finite-dimensional and

ii) a system is completely integrable if the algebra is infinite-dimensional but possesses finite-dimensional representations. (Note that the presence of the last ones just allows one finally to introduce in a nontrivial way a spectral parameter in Lax-type representation and to obtain soliton solutions by the inverse scattering method (see e.g. [6]).) The class of nonlinear systems considered in Ref. [5] contains the generalized Toda lattice, which is described in one of the equivalent forms by equations [7]

$$\partial^2 x_j/\partial z_+ \partial z_- = \exp(kx)_j. \tag{1.1}$$

Here k is the Cartan matrix of a finite-dimensional simple Lie algebra in the exactly integrable case (finite, nonperiodic problem) and an infinite-dimensional simple Lie algebra of finite growth for completely integrable (periodic) systems. All the other completely and exactly integrable systems of Ref. [5] result from a Toda lattice through some asymptotic transition, which in terms of Lie group representation theory corresponds to the Inönü-Wigner contraction.

In the present paper we investigate nonlinear dynamical systems associated with the definite-type embeddings of three-dimensional ($3d$-) subalgebra A_1 in an arbitrary simple Lie algebra $\mathfrak{g}$[1]. In this a grade of $\mathfrak{g}$ is defined by the Cartan element H of an embedding of $3d$-subalgebra in $\mathfrak{g}$, with respect to which elements of $\mathfrak{g}$ are arranged into multiplets with the fixed values of A_1 irreducible representation weight, in other words, the values ℓ of the angular momentum. We have confined ourselves to consideration of the embedding leading only to an integral spectrum $\{\ell\}$, i.e. when the spinor multiplets are absent. In physical applications such systems are encountered in dual models [9], the problem of relativistic string and minimal surfaces [3, 10], the Lund-Regge model [11], the generalized Toda lattice [4, 12], etc. In particular, the equations of this class are encountered in investigation of cylindrically symmetric self-dual Yang-Mills field configurations in Euclidean space R_4 for an arbitrary embedding of SU(2) in a compact gauge group G [1, 2][2]. Their one-dimensional version describes spherically symmetric monopoles in Minkowski space $R_{3,1}$ with a Higgs scalar field in the adjoint representation of G, dyons and vortices (see e.g. [2, 13–15, 22].

In spite of the seemingly outward difference, all enumerated dynamical systems are joined together due to the presence of non-trivial *internal symmetry groups*. Just this fact allows one to find explicit expressions for the solutions of the

1 As it will be clear from below the requirement of simplicity of the algebra is not too essential

2 Here the requirement of the absence of spinor multiplets in the algebra of G is related to the impossibility of constructing the invariants under a diagonal group of $3d$-rotations in terms of its components and the spatial vector $\mathbf{r}$

corresponding equations in terms of Lie algebra and group representation theory. In this two absolutely different approaches are applied:

i) the method based on a Lax-type representation [1] (which is used in quite another way in the inverse scattering problem) and

ii) the technique of usual perturbation theory [20, 21] (in classical and quantum regions).

In the latter one the role of a coupling constant is played by a small parameter of contraction procedure, which "straightens" the internal symmetry algebra of the system and switches the dynamical system into an asymptotic region with non-interacting fields.

The paper is organized as follows: In Sect. 2 we give the necessary information about the embeddings of $3d$-subalgebra in an arbitrary Lie algebra and describe its multiplet structure in accordance with A_1. Section 3 presents the construction of infinite-dimensional Lie algebras related to various embeddings of A_1 in finite-dimensional Lie algebras. In Sects. 4 and 5 an explicit form for the exactly and completely integrable systems under consideration is given. In Sect. 6 we present the Hamilton formalism for these systems. In Sect. 7 exact solutions of the corresponding systems have been constructed. Concrete dynamical systems arising in some physical applications are exposed in the appendix.

Let us complete the Introduction by giving a summary of the main notations used in the text and some definitions: G is an arbitrary simple Lie group of rank $\mathfrak{r}$ with Lie algebra $\mathfrak{g}$ (the same symbols will be taken also for "arbitrary position" Lie algebras and groups); $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$; $R_+(R_-)$ is the system of positive (negative) roots with respect to $\mathfrak{h}$; k is the Cartan matrix of $\mathfrak{g}$; $M(-M)$ is the maximal (minimal) root, $\alpha+M\notin R_+$, $\forall\alpha\in R_+$; $X_{\pm\alpha}$ are the elements of the root space of the root α, $\pm\alpha\in R_\pm$; $h_j\,(\equiv h_{\pi_j})$ are the generators of $\mathfrak{h}$, corresponding to the simple roots $\alpha=\pi_j$, $1\leqq j\leqq\mathfrak{r}$. The elements $X_{\pm j}\,(\equiv X_{\pm\pi_j})$ and h_j satisfy the canonical commutation relations in the Cartan-Weyl form

$$[h_i,h_j]=0\,,\qquad [h_i,X_{\pm j}]=\pm k_{ji}X_{\pm j}\,,\qquad [X_i,X_{-j}]=\delta_{ij}h_j\,. \tag{1.2}$$

By grading of the Lie algebra $\mathfrak{g}$ we mean a decomposition of $\mathfrak{g}$ (in the sense of linear space) into a direct sum of finite-dimensional subspaces

$$\mathfrak{g}=\sum_{a=-\infty}^{\infty}\oplus\mathfrak{g}_a\,,\qquad \dim\mathfrak{g}_a\equiv \mathrm{d}_a<\infty\,, \tag{1.3}$$

for which

$$[\mathfrak{g}_a,\mathfrak{g}_b]\subset\mathfrak{g}_{a+b}\,. \tag{1.4}$$

Growth of $\mathfrak{g}$ is defined as a limit of the following ratio: $\overline{\lim\limits_{a\to\infty}}\ln\sum\limits_{b=-a}^{a}d_b/\ln a$. The subspace $\hat{\mathfrak{g}}\equiv\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is called a local part of the Lie algebra $\mathfrak{g}$; its generators X_a^0 and $X_\alpha^\pm$, $1\leqq a\leqq d_0$, $1\leqq\alpha\leqq d_{\pm1}$, satisfy the commutation relations

$$[X_a^0X_b^0]=\sum_d B_{ab}^dX_d^0\,,\qquad [X_a^0,X_\alpha^\pm]=\sum_\beta C_{a\alpha}^{\pm\beta}X_\beta^\pm\,,\qquad [X_\alpha^+,X_\beta^-]=\sum_a A_{\alpha\beta}^aX_a^0\,. \tag{1.5$_1$}$$

In this the structure constants A, B, and C completely define the structure of the algebra $\mathfrak{g}$ as a whole and satisfy the relations following from the Jacobi identity

$$[B_a, B_b] = -\sum_d B^d_{ab} B_d, \quad [C^\pm_a, C^\pm_b] = -\sum_d B^d_{ab} C^\pm_d, \quad C^+_a A_b + A_b C^-_a = \sum_d B^b_{ad} A_d. \tag{1.5$_2$}$$

Here A_a, B_a, and $C^\pm_a$ are defined by their matrix elements, $(A_a)_{\alpha\beta} \equiv A^a_{\alpha\beta}$, $(B_a)_{cd} \equiv B^d_{ac}$, $(C^\pm_a)_{\alpha\beta} \equiv C^{\pm\beta}_{a\alpha}$. It is obvious that in the particular case of a simple Lie algebra supplied by the canonical grading when $\mathfrak{g}_0 \equiv \{h_1, \dots, h_r\}$, $\mathfrak{g}_{\pm 1} \equiv \{X_{\pm 1}, \dots, X_{\pm r}\}$ one has $B^d_{ab} \equiv 0$, $C^{\pm\beta}_{a\alpha} \equiv \pm\delta_{\alpha\beta} k_{\alpha a}$, $A^a_{\alpha\beta} \equiv \delta_{\alpha\beta}\delta_{\alpha a}$, $1 \leqq \alpha, \beta, a, b \leqq r$, and Eqs. (1.5) reduce identically to (1.2). Note that the contragredient graded Lie algebras characterized by Eqs. (1.2) with some definite conditions on the matrix k lead to simple Lie algebras of finite (including zero) growth [17]. The problem of their classification is equivalent to the description of all possible forms of the relevant Cartan matrices.

2. Embeddings of the 3*d*-Subalgebra in Lie Algebras

Consider an arbitrary Lie algebra $\mathfrak{g}$ containing a subalgebra A_1 with generators $J_\pm$, H, $[H, J_\pm] = \pm 2J_\pm$, $[J_+, J_-] = H$, embedded in $\mathfrak{g}$ in some way. Then all the elements of $\mathfrak{g}$ are arranged into multiplets in accordance with finite-dimensional irreducible representations of the subalgebra A_1. In this the components F^{ℓ, ν_ℓ}_m of the multiplets are labelled by the values ℓ of the angular momentum (representation weight), its projections $m \in [-\ell, \ell]$ and the index ν_ℓ, which characterizes multiplicity of the given ℓ in all its spectrum. This spectrum $\{\ell\} \equiv \{\ell_1, \dots, \ell_r\}$ is fixed for each embedding. Denote by $\mathfrak{g}_0$ a subalgebra of $\mathfrak{g}$ commuting with H [3] and pick out in it a subalgebra $\mathfrak{g}^0_0$ invariant with respect to A_1, $[J_\pm, \mathfrak{g}^0_0] = 0$, whose elements are scalar under A_1. Denote by $\mathfrak{g}^f_0$ a factor-algebra $\mathfrak{g}_0/\mathfrak{g}^0_0$ and classify its elements F^{ℓ, ν_ℓ}_0 $(\ell > 0)$ in accordance with irreducible representations of A_1 with weight ℓ,

$$[J_+, [J_-, F^{\ell, \nu_\ell}_0]] = \ell(\ell+1) F^{\ell, \nu_\ell}_0 .$$

This relation is invariant under $\mathfrak{g}^0_0$, so the identification of the elements of algebra $\mathfrak{g}$ over index ν_ℓ is carried out according to irreducible representations of $\mathfrak{g}^0_0$. Evidently for the complete description of the elements of an algebra $\mathfrak{g}$ for a definite embedding of the 3*d*-subalgebra in $\mathfrak{g}$ it is sufficient to define its generators corresponding to zero values of m, i.e. the elements of $\mathfrak{g}_0$. Then all other elements of $\mathfrak{g}$ can be obtained by applying the raising and lowering operators constructed with $J_\pm$ to F^{ℓ, ν_ℓ}_0, $\ell > 0$, i.e. by m times commuting with $J_\pm$.

Therefore grading (1.3) of algebra $\mathfrak{g}$ can be realized with respect to the eigenvalues m of the Cartan element H. In this all the elements of $\mathfrak{g}$ with the same indices m are collected in the m^{th} subspace of $\mathfrak{g}$ and it is suitable to take $m/2$ as a grading index for the integral embeddings. In particular the subspaces $\mathfrak{g}_{\pm 1}$ of the

3 Subalgebra $\mathfrak{g}_0$ obviously contains all the components of multiplets with zero value of index m, and its dimensionality d_0 is equal to $\sum_i \nu_{\ell_i}, \ell_i \in \{\ell\}$

local part $\hat{\mathfrak{g}}$ of the algebra $\mathfrak{g}$ are generated by the elements $[J_{\pm}, F_0^{\ell,\nu_\ell}]$. It is important to note that in the framework of this construction its dimensionalities $d_{\pm 1}$ coincide with the dimensionality d_0^f of factor-algebra $\mathfrak{g}_0^f$. (The latter one is a reflection of the fact that all integral multiplets with $\ell > 0$ contain the components both with $m=0$ and $m=\pm 1$.)

This approach applied to the simple Lie algebras allows us to describe all possible embeddings of the $3d$-subalgebra in $\mathfrak{g}$; the problem of their complete classification for the finite-dimensional case was solved in [16]. In this each embedding is defined uniquely (up to equivalence) by the decomposition structure of the Cartan element H of the subalgebra A_1 over the generators $\mathfrak{h}$ of the algebra $\mathfrak{g}$, $H = \sum_{j=1}^{\mathfrak{r}} c_j h_j$. In turn the wanted structure constants c_j of the embedding are defined by the following scheme[4]. The system of simple roots (π-system) of the finite-dimensional simple Lie algebra $\mathfrak{g}$ is supplied by the minimal root. As a result there arises an extended π-system with the canonical generators satisfying the commutation relations

$$[h_I, h_J] = 0\,, \quad [h_I, X_{\pm J}] = \pm \tilde{k}_{JI} X_{\pm J}\,, \quad [X_I, X_{-J}] = \delta_{IJ} h_J\,, \quad 1 \leqq I, J \leqq \mathfrak{r}+1\,. \tag{2.1}$$

Here $\tilde{k}$ is a generalized Cartan matrix, $\tilde{k}_{ij} \equiv k_{ij}$, $1 \leqq i, j \leqq \mathfrak{r}$, with one zero eigenvalue, $X_{\pm(\mathfrak{r}+1)} \equiv X_{\mp M}$. Further one extracts from the extended π-system a set $P^{(s)} \equiv \{p_1, \ldots, p_s, s \leqq \mathfrak{r}\}$ of any roots corresponding to some semisimple Lie subalgebra $\mathfrak{g}^{(s)}$ of rank $\mathfrak{r}_s \leqq \mathfrak{r}$ of the initial algebra $\mathfrak{g}$. Then the unknown element $H = \sum_{i=1}^{s} c_i h_{p_i}$ is found using the condition $[H, X_{p_i}] = 2X_{p_i}$. Therefore the embedding of the $3d$-subalgebra in the simple Lie algebra $\mathfrak{g}$, for which the Cartan element takes the same value on the root vectors of all simple roots of $\mathfrak{g}$, $[H, X_i] = 2X_i$, $1 \leqq i \leqq \mathfrak{r}$[5], plays a peculiar role. The corresponding $3d$-subalgebra is called the principal $3d$-subalgebra. Really, the description of all embeddings of A_1 in the simple Lie algebra $\mathfrak{g}$ is reduced to consideration of the principal $3d$-subalgebras in all algebras $\mathfrak{g}^{(s)}$. (The exceptions are $\left[\frac{r-2}{2}\right]$ embeddings for the series $D_{\mathfrak{r}}$ and $\left[\frac{\mathfrak{r}'-3}{2}\right]$ for $E_{\mathfrak{r}'}$, $\mathfrak{r}' = 6, 7, 8$; we will not deal with them here, referring to [16].) In this the structure constants c_j are expressed in an evident way via the elements of the matrix $\tilde{k}$ with the help of relations (2.1).

Let us stress that the scheme given above allows us to describe integral embeddings as well as embeddings leading to spinor multiplets.

4 Note that the analogous scheme is used below in construction of infinite-dimensional simple Lie algebras of finite growth starting from the finite-dimensional ones

5 Note that for this embedding, which naturally corresponds to the canonical grading of $\mathfrak{g}$, the spectrum $\{\ell\}$ coincides with the values of the indices of the algebra [18]. In this the multiplicity ν_ℓ of a multiplet is always equal to unity except the series $D_{\mathfrak{r}}$ with even $\mathfrak{r}$ when $\ell_{\mathfrak{r}/2} = \ell_{\mathfrak{r}} = \mathfrak{r} - 1$

3. Infinite-Dimensional Graded Lie Algebras Associated with Embeddings of the 3d-Subalgebra in Finite-Dimensional Lie Algebras

To each embedding of A_1 in an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ one can put in correspondence an infinite-dimensional graded Lie algebra of finite growth as well. The approach described below represents a special case of some wider scheme for the construction of Lie algebras of finite (nonzero) growth associated with an arbitrary finite-dimensional Lie algebra supplied with a grading and completely defined by its local part $\hat{\mathfrak{g}}$.

The solution of the problem lies in extension of subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_{\pm 1}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ with the elements $X^0, X^{\pm}$ [see Eq. (1.5)] through inclusion of some number of additional generators Y^0 and $Y^{\pm}$. In this, the domains of finite-dimensional representations of the initial algebra $\mathfrak{g}$ and an infinite-dimensional algebra $\tilde{\mathfrak{g}}$ constructed with its help should coincide. For this goal consider the highest and lowest components $F^{\ell, \nu_\ell}_{\pm \ell}$ of multiplets of the elements of $\mathfrak{g}$ under A_1 (for a fixed embedding) with a maximal value of the momentum $\ell_{\max} \equiv L$ belonging to the spectrum $\{\ell\}$, i.e. $F^{L, \nu_L}_{\pm L} \equiv F^{\pm}_{\nu}$, $1 \leqq \nu \leqq \nu_L$. They satisfy the commutation relations

$$[F^{-}_{\nu}, F^{+}_{\mu}] = \sum_i R^i_{\nu\mu} V^0_i, \qquad [X^0_a, F^{\pm}_{\nu}] = \sum_\mu Q^{\pm\mu}_{a\nu} F^{\pm}_{\mu}, \tag{3.1}$$

where V^0_i are some linear independent combinations of the elements of $\mathfrak{g}_0$, $V^0_i = \sum_a b_{ia} X^0_a$. Now let us extend the subspaces $\mathfrak{g}_{\pm 1}$ to $\tilde{\mathfrak{g}}_{\pm 1}$ by adding to the generators $X^{\pm}_{\alpha}$, $1 \leqq \alpha \leqq d_{\pm 1}$, exactly ν_ℓ elements $Y^{\pm}_{\nu}$, $1 \leqq \nu \leqq \nu_L$, i.e. $\tilde{\mathfrak{g}}_{\pm 1} = \{X^{\pm}_{\alpha}, Y^{\pm}_{\nu}\}$ with $\dim \tilde{\mathfrak{g}}_{\pm 1} \equiv \tilde{d}_{\pm 1} = d_{\pm 1} + \nu_L$, submitting them to the relations

$$[X^{+}_{\alpha}, Y^{-}_{\nu}] = 0, \qquad [X^{-}_{\alpha}, Y^{+}_{\nu}] = 0, \qquad [Y^{+}_{\nu}, Y^{-}_{\mu}] = \sum_i R^i_{\nu\mu}(V^0_i + Y^0_i). \tag{3.2}$$

In this we shall commit additionally introduced elements Y^0_i to subspace $\mathfrak{g}_0$. Jacobi identities similar to (1.5_2) single out nonzero elements Y^0_i, i.e. the consistency requirements automatically restrict the set of these nonzero elements of $\tilde{\mathfrak{g}}_0/\mathfrak{g}_0$. The constants $R^i_{\nu\mu}$ are defined in accordance with (3.1). Then define

$$[X^0_a, Y^{\pm}_{\nu}] = \sum_\mu Q^{\mp\mu}_{a\nu} Y^{\pm}_{\mu}, \tag{3.3}$$

where the constants $Q^{\pm\mu}_{a\nu}$ are fixed by relations (3.1). Note that (3.3) is a direct modification of (3.1). It follows from (1.5_2) that Y^0_i $(\neq 0)$ form the centre of $\hat{\tilde{\mathfrak{g}}}$, i.e.

$$[Y^0_i, X^{\pm}_{\alpha}] = [Y^0_i, Y^{\pm}_{\nu}] = [Y^0_i, X^0_a] = 0, \qquad [Y^0_i, Y^0_j] = 0, \tag{3.4}$$

as well as of the whole algebra $\tilde{\mathfrak{g}}$. The reconstruction procedure of algebra $\tilde{\mathfrak{g}}$ as a whole is the same as in [17], which is based on introduction of a bilinear symmetric invariant form on $\hat{\tilde{\mathfrak{g}}}$ with its subsequent extension on algebra $\tilde{\mathfrak{g}}$; $\sum_b \tilde{A}^b_{\alpha\beta}(\tilde{X}^0_a, \tilde{X}^0_b) = -\sum_\gamma \tilde{C}^{-\gamma}_{\alpha\beta}(\tilde{X}^{+}_{\alpha}, \tilde{X}^{-}_{\gamma})$, $\tilde{X}(\tilde{A}, \tilde{B}, \tilde{C}) \in \hat{\tilde{\mathfrak{g}}}$. Therefore commutation relations (1.5), (3.1)–(3.4) define the structure constants of the infinite-dimensional Lie algebra with the local part $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ and play the same role as relations

(1.5) for the finite-dimensional algebra $\mathfrak{g}$, whose extension is $\tilde{\mathfrak{g}}$. By the construction the obtained algebra $\tilde{\mathfrak{g}}$ is known to possess a series of representations of the initial algebra $\mathfrak{g}$ (finite-dimensional, in particular). Thus the question of consistency of (3.1)–(3.4) is taken off automatically.

For the case of the principal (minimal) embedding of A_1 in an arbitrary simple Lie algebra $\mathfrak{g}$ its local part (1.2) is added in accordance with the general scheme given above by the elements Y^-, $Y^0 \equiv [Y^+, Y^-]$, and Y^+. The generators of the local part $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ of the extended algebra $\tilde{\mathfrak{g}}$ satisfy relations (2.1), and also $\sum_{I=1}^{r+1} \lambda_I h_I$ belongs to the centre of $\tilde{\mathfrak{g}}$. Here $\lambda \equiv \{\lambda_I\}$ is the eigenvector of the Cartan matrix $\tilde{k}$ with zero eigenvalue, i.e. $\sum_{J=1}^{r+1} \tilde{k}_{IJ} \lambda_J = 0$. In this the construction leads to one of the possible versions of simple Lie algebras of finite growth described in [17].

Note that an algebra $\tilde{\mathfrak{g}} = \sum_{-\infty}^{\infty} \oplus \tilde{\mathfrak{g}}_a$ constructed by the indicated method seems always to have a finite growth because each subspace $\tilde{\mathfrak{g}}_a$ for finite-dimensional (degenerated) representations of $\tilde{\mathfrak{g}}$[6] cannot contain more elements than the dimensionality of the initial Lie algebra $\mathfrak{g}$ as a whole. In any case the Lax pair operators taking the values in these algebras ($\hat{\tilde{\mathfrak{g}}}$) can be realized by finite-dimensional matrices, which is known to be enough to find soliton-type solutions of the corresponding dynamical systems by the inverse scattering method.

4. Exactly Integrable Systems

In accordance with general method [1] the nonlinear equations describing exactly and completely integrable dynamical systems in $2d$-space with coordinates (z_+, z_-) arise from the relation of Lax-type representation $[\partial/\partial z_+ + A_+, \partial/\partial z_- + A_-] = 0$ with operators $A_\pm$, which take the values in subspaces $\mathfrak{g}_{\pm a}$, $a \geqq 0$, of an arbitrary graded Lie algebra or superalgebra $\mathfrak{g}$. The systems considered in the present paper are connected with the local part $\hat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$, whose grading is consistent with integral embeddings of the $3d$-subalgebra in $\mathfrak{g}$. In this the starting point is the representation in the form

$$[\partial/\partial z_+ + E_0^{0+} + E_0^{f+} + E_1^+, \partial/\partial z_- + E_0^{0-} + E_0^{f-} + E_1^-] = 0, \tag{4.1}$$

where $E_0^{0\pm}$, $E_0^{f\pm}$, and $E_1^\pm$ $(\equiv E_{\pm 1})$ take the values in subspaces $\mathfrak{g}_0^0$, $\mathfrak{g}_0^f$, and $\mathfrak{g}_{\pm 1}$ of the Lie algebra $\mathfrak{g}$, respectively; $E_a^\pm = \sum_{\alpha=1}^{d_a} \varphi_{\pm\alpha}^a(z_+, z_-) X_\alpha^a$, $X_\alpha^a \in \mathfrak{g}_a$. The decomposition of the local part $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus (\mathfrak{g}_0^0 \oplus \mathfrak{g}_0^f) \oplus \mathfrak{g}_1$, taking account of property (1.4), induces the subdivision of (4.1) into subsystems

$$\begin{gathered}[\partial/\partial z_- + E_0^{0-} + E_0^{f-}, E_1] = 0, \quad [\partial/\partial z_+ + E_0^{0+} + E_0^{f+}, E_{-1}] = 0, \\ [\partial/\partial z_+ + E_0^{0+} + E_0^{f+}, \partial/\partial z_- + E_0^{0-} + E_0^{f-}] + [E_1, E_{-1}] = 0.\end{gathered} \tag{4.2}$$

Due to the properties of embedding of A_1 in $\mathfrak{g}$ mentioned in Sect. 2 the dimensionalities of $\mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_0^f$ coincide ($d_{\pm 1} = d_0^f$) and some representation of $\mathfrak{g}_0^f$ is

6 For these representations $Y_\nu^\pm = F_\nu^\mp$ and $Y_i^0 = 0$

realized on subspaces $\mathfrak{g}_{\pm 1}$, $[\mathfrak{g}_0^f, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}$. All space $\mathfrak{g}_1$ ($\mathfrak{g}_{-1}$) can be recovered from a single element J_+ (J_-) known to be contained in it by means of transformations of group G_0 with Lie algebra $\mathfrak{g}_0$, i.e. $\mathfrak{g}_{\pm 1} = \{g_0^{-1} J_\pm g_0\}$, $g_0 \in G_0$. In this due to commutativity of $J_\pm$ with transformations of group G_0^0 with Lie algebra $\mathfrak{g}_0^0$, the parameters of g_0 connected with $\mathfrak{g}_0^0$ are absent in the last reconstruction formula. The equality $d_{\pm 1} = d_0^f$ is just ensured by this fact. In other words the elements $q \equiv g_0^{-1} g_0^0 g_0$, $g_0^0 \in G_0^0$, generate the stationary subgroup of the "points" of the subspace $\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1} = q \mathfrak{g}_{\pm 1} q^{-1}$, because G_0^0 is the stationary subgroup for $J_\pm$. For this reason using the appropriate gauge transformation the operator E_1 can be reduced to J_+ and then the first equation in (4.2) takes the form $[E_0^{0-} + E_0^{f-}, J_+] = 0$, i.e. $[E_0^{f-}, J_+] = 0$. The last relation gives $E_0^{f-} \equiv 0$ due to the fact that each element of $\mathfrak{g}_1$ can be constructed from the corresponding element of $\mathfrak{g}_0^f$ by the action of the raising operator (J_+). Thus system (4.2) is rewritten as

$$\begin{gathered} \partial E_{-1}/\partial z_+ + [E_0^{0+} + E_0^{f+}, E_{-1}] = 0, \quad \partial E_0^{f+}/\partial z_- + [E_{-1}, J_+] = 0, \\ \partial E_0^{0+}/\partial z_- - \partial E_0^{0-}/\partial z_+ + [E_0^{0-}, E_0^{0+}] \equiv [\partial/\partial z_- + E_0^{0-}, \partial/\partial z_+ + E_0^{0+}] = 0. \end{gathered} \tag{4.3}$$

The last equation in (4.3) means the gradientness of $E_0^{0\pm}$, i.e. $E_0^{0\pm} = (g_0^0)^{-1} \partial g_0^0 / \partial z_\pm$, and these quantities are eliminated from the remaining equations with the help of the appropriate gauge transformation. Therefore we come to the following final form of the initial system (4.2):

$$\partial E_{-1}/\partial z_+ + [E_0^{f+}, E_{-1}] = 0, \quad \partial E_0^{f+}/\partial z_- + [E_{-1}, J_+] = 0. \tag{4.4}$$

Having solved the first equation with respect to E_0^{f+} and substituting its expression in the second one, we obtain the desired system of nonlinear partial differential equations of the second order for the functions $\varphi_{-\alpha}^1(z_+, z_-)$. (Note once more that $d_{-1} = d_0^f$, and consequently the number of functions $\varphi_{-\alpha}^1$ and $\varphi_{+\alpha}^{0f}$ coincides.) For the explicit realization of this procedure one removes the commutator $[E_0^{f+}, E_{-1}] = \sum_{\alpha,\beta,\gamma} \varphi_{+\alpha}^{0f} \varphi_{-\beta}^1 C_{\alpha\beta}^\gamma X_\gamma^-$, where $C_{\alpha\beta}^\gamma$ is a matrix of representation of $\mathfrak{g}_0^f$ in $\mathfrak{g}_{-1}$, $[X_\alpha^{0f}, X_\beta^-] = \sum_\gamma C_{\alpha\beta}^\gamma X_\gamma^-$, and puts this form with the notation of matrix function $R_{\gamma\alpha} \equiv \sum_\beta C_{\alpha\beta}^\gamma \varphi_{-\beta}^1$ in the first equation (4.4). Then after trivial algebraic operations we obtain

$$\varphi_{+\alpha}^{0f} = -\sum_\beta R_{\alpha\beta}^{-1} \partial \varphi_{-\beta}^1 / \partial z_+ . \tag{4.5}$$

In accordance with (4.5) the second equation in (4.4) is brought to a form

$$\sum_\beta \partial (R_{\alpha\beta}^{-1} \partial \varphi_{-\beta}^1 / \partial z_+) / \partial z_- = \sum_\beta A_{\alpha\beta} \varphi_{-\beta}^1 , \tag{4.6$_1$}$$

or

$$\partial^2 \varphi_{-\alpha}^1 / \partial z_+ \partial z_- = \sum_{\beta,\gamma} [\partial R_{\alpha\beta} / \partial z_- R_{\beta\gamma}^{-1} \partial \varphi_{-\gamma}^1 / \partial z_+ + R_{\alpha\beta} A_{\beta\gamma} \varphi_{-\gamma}^1] . \tag{4.6$_2$}$$

Here the matrix $A_{\alpha\beta}$ is defined from the relation $[X_\beta^-, J_+] \equiv \sum_\alpha A_{\alpha\beta} X_\alpha^{0f}$. Let us stress that the matrix R is nondegenerated and thus formulae (4.5) and (4.6) are quite correct.

System (4.4) can be represented as a single operator equation of the second order for a group element g_0 of G_0. Really, due to the definition $E_{-1}=g_0 J_- g_0^{-1}$, the first equation (4.4) is rewritten in a form of the commutator $[g_0^{-1}\partial g_0/\partial z_+ + g_0^{-1}E_0^{f+}g_0, J_-]=0$. Hence $g_0^{-1}\partial g_0/\partial z_+ + g_0^{-1}E_0^{f+}g_0 \equiv E_0^0$ takes a value in the subalgebra $\mathfrak{g}_0^0$ and $E_0^{f+}=g_0E_0^0g_0^{-1}-\partial g_0/\partial z_+ g_0^{-1}$. The gauge transformation $g_0\to g_0 g_0^0$ with $g_0^0\in G_0^0$ conserving the form of E_{-1} permits us to make the element E_0^0 zero and gives $E_0^{f+}=-\partial g_0/\partial z_+ g_0^{-1}$. The latter equality means that between the parameters of the elements g_0 and their first derivatives over z_+, there are d_0^0 relations, which play the role of constraints. Substituting it into the second equation (4.4) we have[7]

$$\partial(\partial g_0/\partial z_+ g_0^{-1})/\partial z_- + [g_0 J_- g_0^{-1}, J_+]=0\,, \tag{4.7$_1$}$$

$$\partial g_0/\partial z_+ g_0^{-1} \dot{\in} \mathfrak{g}_0^f \tag{4.7$_2$}$$

(compare with system (4) of [3]). Equation (4.7$_2$) can be considered as an additional (or initial) condition on the characteristic since it follows from (4.7$_1$) that the components of $\partial g_0/\partial z_+ g_0^{-1}$ on $\mathfrak{g}_0^0$ depend only on the argument z_+. Naturally, this condition is trivial when the subspace $\mathfrak{g}_0^0$ is empty ($E_0^{0\pm}\equiv 0$) and then systems (4.4) and (4.7) are completely equivalent. System (4.7) can be rewritten as

$$\begin{aligned} &\partial(g_0^{-1}\partial g_0/\partial z_-)/\partial z_+ + [g_0^{-1}J_+ g_0, J_-]=0\,,\\ &g_0^{-1}\partial g_0/\partial z_- \dot{\in} \mathfrak{g}_0^f\,. \end{aligned} \tag{4.8}$$

5. Systems Connected with Infinite-Dimensional Lie Algebras

In accordance with the construction given in Sect. 3, the local part $\hat{\tilde{\mathfrak{g}}}$ of the infinite-dimensional graded Lie algebra $\tilde{\mathfrak{g}}$ contains in addition to the generators X_a^0, $1\leqq a\leqq d_0$, and $X_\alpha^\pm$, $1\leqq\alpha\leqq d_{\pm 1}$, of subspace $\hat{\mathfrak{g}}$ of the initial finite-dimensional Lie algebra $\mathfrak{g}$, the elements Y_i^0 and $Y_\nu^\pm$ satisfying relations (3.2)–(3.4). In this the reasonings leading to Eqs. (4.4) remain valid under corresponding modification. Now the operators E_0^{f+} and E_{-1} entering Eqs. (4.4) take values in subspaces $\mathfrak{g}_0^f$ and $\tilde{\mathfrak{g}}_{-1}$, respectively. In the case under consideration by analogy with the derivation of Eqs. (4.4) according to commutation relations (1.5), (3.2)–(3.4), one obtains

$$\begin{aligned} \partial E_{-1}/\partial z_+ + [E_0^{f+}, E_{-1}]&=0\,,\\ \partial \tilde{E}_{-1}/\partial z_+ + [E_0^{f+}, \tilde{E}_{-1}]&=0\,,\\ \partial E_0^{f+}/\partial z_- + [E_{-1}, J_+] - [\tilde{E}_1, \tilde{E}_{-1}]'&=0\,,\\ \partial \tilde{E}_0^{0+}/\partial z_- - [\tilde{E}_1, \tilde{E}_{-1}]''&=0\,,\\ \partial \tilde{E}_1/\partial z_- &=0\,, \end{aligned} \tag{5.1}$$

7 Here symbol $A\dot{\in}\mathfrak{g}_0^f$ means that the corresponding operator A takes the values in $\mathfrak{g}_0^f$

where

$$E_{0,\pm 1} \dot{\in} \mathfrak{g}_{0,\pm 1}; \quad \tilde{E}_{0,\pm 1} \dot{\in} \tilde{\mathfrak{g}}_{0,\pm 1}/\mathfrak{g}_{0,\pm 1};$$

$$[\tilde{E}_1, \tilde{E}_{-1}] \equiv [\tilde{E}_1, \tilde{E}_{-1}]'(\dot{\in} \mathfrak{g}_0^f) + [\tilde{E}_1, \tilde{E}_{-1}]''(\dot{\in} \tilde{\mathfrak{g}}_0^0/\mathfrak{g}_0^0),$$

$$E_{0,\pm 1} \equiv \sum_\alpha \varphi_\alpha^{0,\pm 1}(z_+, z_-) X_\alpha^{0,\pm 1},$$

$$\tilde{E}_0^0 \equiv \sum_i \tilde{\varphi}_i^0(z_+, z_-) Y_i^0,$$

$$\tilde{E}_{\pm 1} \equiv \sum_\mu \tilde{\varphi}_{\pm\mu}^1(z_+, z_-) Y_\mu^\pm .$$

A matrix form of system (5.1) is

$$\partial \varphi_{-\alpha}^1/\partial z_+ + \sum_{\beta,\gamma} C_{\beta\gamma}^{-\alpha} \varphi_\beta^0 \varphi_{-\gamma}^1 = 0, \tag{5.2$_1$}$$

$$\partial \tilde{\varphi}_{-\mu}^1/\partial z_+ + \sum_{\beta,\nu} Q_{\beta\nu}^{+\mu} \varphi_\beta^0 \tilde{\varphi}_{-\nu}^1 = 0, \tag{5.2$_2$}$$

$$\partial \varphi_\alpha^0/\partial z_- + \sum_\beta A_{\alpha\beta} \varphi_{-\beta}^1 - \sum_\nu \mathfrak{A}_{\alpha\nu} \tilde{\varphi}_{-\nu}^1 = 0 \quad \Big(\mathfrak{A}_{\alpha\nu} \equiv \sum_{i,\mu} R_{\mu\nu}^i b_{i\alpha} \tilde{\varphi}_{+\mu}^1\Big), \tag{5.2$_3$}$$

$$\partial \tilde{\varphi}_i^0/\partial z_- - \sum_\nu S_\nu^i \tilde{\varphi}_{-\nu}^1 = 0 \quad \Big(S_\nu^i \equiv \sum_\mu R_{\mu\nu}^i \tilde{\varphi}_{+\mu}^1\Big), \tag{5.2$_4$}$$

$$\partial \tilde{\varphi}_{+\mu}^1/\partial z_- = 0. \tag{5.2$_5$}$$

(In what follows consider that $1 \leqq \alpha$, $\beta \leqq d_{-1}$, $1 \leqq \mu$, $\nu \leqq d_L$.) Subsystem (5.2$_1$) coincides with those of Sect. 4 and leads to formula (4.5). It follows from Eq. (5.2$_5$) that $\tilde{\varphi}_{+\mu}^1$ are arbitrary functions depending only on argument z_+, which can be eliminated from the other equations using an appropriate transformation.

Therefore substituting Eqs. (4.5) in (5.2$_3$) we come to the quested equations of the second order, which generalize Eqs. (4.6) for the case of infinite-dimensional Lie algebras

$$\begin{aligned} \partial^2 \varphi_{-\alpha}^1/\partial z_+ \partial z_- = & \sum_{\beta,\gamma} [\partial R_{\alpha\beta}/\partial z_- R_{\beta\gamma}^{-1} \partial \varphi_{-\gamma}^1/\partial z_+ + R_{\alpha\beta} A_{\beta\gamma} \varphi_{-\gamma}^1] \\ & - \sum_{\beta,\nu} R_{\alpha\beta} \mathfrak{A}_{\beta\nu} \tilde{\varphi}_{-\nu}^1, \end{aligned} \tag{5.3}$$

where the functions $\tilde{\varphi}_{-\mu}^1$ are related with $\varphi_{-\beta}^1$ by equations

$$\partial \tilde{\varphi}_{-\mu}^1/\partial z_+ = \sum_\alpha \Big(\sum_\beta \mathcal{R}_{\mu\beta} R_{\beta\alpha}^{-1}\Big) \partial \varphi_{-\alpha}^1/\partial z_+, \quad \mathcal{R}_{\mu\beta} \equiv \sum_\nu Q_{\beta\nu}^{+\mu} \tilde{\varphi}_{-\nu}^1 .$$

6. Hamilton Formalism

In this section we give the Hamilton formulation of the fundamental equations (4.8) or (4.7) constructed above for the one-dimensional case, when all the unknown functions depend on the unique argument $t \equiv z_+ + z_-$ and these equations take the form

$$d(g_0^{-1} dg_0/dt)/dt + [g_0^{-1} J_+ g_0, J_-] = 0, \tag{6.1$_1$}$$

with the supplementary condition

$$g_0^{-1}dg_0/dt\dot{\in}\mathfrak{g}_0^f. \tag{6.1$_2$}$$

Consider an operator[8]

$$\mathfrak{H}=\tfrac{1}{2}K_2+\mathrm{Sp}(g_0^{-1}J_+g_0J_-), \tag{6.2}$$

where K_2 is the quadratic Casimir operator constructed with infinitesimal left shifts L_a, $1\leqq a\leqq d_0$, on G_0, $K_2\equiv\sum_a L_a^2$. A Poisson bracket of $\mathfrak{H}$ with a group element g_0 in some representation of G_0 with generators M_a has the form $\{\mathfrak{H},g_0\}=\sum_a L_aM_ag_0$, or $\{\mathfrak{H},g_0\}g_0^{-1}=\sum_a L_aM_a$. Now composing a Poisson bracket of $\mathfrak{H}$ with the latter element one obtains

$$\begin{aligned}\left\{\mathfrak{H},\sum_a L_aM_a\right\}&=\mathrm{Sp}\left\{g_0^{-1}J_+g_0J_-,\sum_a L_aM_a\right\}\\&=-\sum_a M_a\,\mathrm{Sp}\{M_a,[g_0^{-1}J_+g_0,J_-]\}=-[g_0^{-1}J_+g_0,J_-].\end{aligned}$$

Here for simplicity we make use of the orthonormal basis in representation space of G_0, $\mathrm{Sp}M_aM_b=\delta_{ab}$, and take into account the permutability of the Casimir operator with all generators. Therefore if one identifies operator $\mathfrak{H}$ with the Hamiltonian of a dynamical system, then the equations describing it coincide with (6.1$_1$). In this condition (6.1$_2$) is fulfilled when one imposes the requirement for vanishing of generalized momenta M_a (right shifts) for the elements $g_0^{-1}dg_0/dt=\sum_a L_aM_a$, taking values in $\mathfrak{g}_0^0$.

The presence of the Hamilton formalism for the considered dynamical systems (4.8) allows us to apply standard perturbation methods of classical and quantum mechanics for its investigation. In this the first term in expression (6.2) plays the role of the free Hamiltonian, while the second one describes an interaction in the system with some coupling constant λ, with which we supply this term. Perturbation series in one- and two-dimensional cases turn out to be finite polynomials over λ and reproduce exact solutions of corresponding systems, giving another form completely equivalent to the expressions in Sect. 7. It should be stressed once more that *here the perturbation method is used not for establishing approximate results but for the direct construction of the explicit expressions of exact solutions.* This approach has been developed in [20] for a particular case of considered systems (4.8), namely for the generalized (finite, nonperiodic) Toda lattice (1.1). The mentioned character of analytic dependence of the solutions [more exactly, the definite dynamical quantities like $\exp(-x_j)$ for (1.1)] admits group interpretation in terms of Inönü-Wigner contraction operation. With the help of the latter one with a contraction parameter λ the internal symmetry algebra of the corresponding dynamical system is transformed into an algebra associated with the non-perturbative part of the Hamiltonian. Then dynamical quantities of the initial system are obtained from the "asymptotical fields"

8 The form of the Hamiltonian (6.2) for some particular cases is encountered in the papers on the inverse scattering method, see e.g. [23]

(solutions when $\lambda=0$) by a tangential Lie-Bäcklund transformation in the classical case and Moeller S-matrix, $S(t,-\infty)$, on the quantum level, which are finite polynomials over λ for exactly integrable systems [21].

7. Solutions for Exactly Integrable Systems

Representation (4.1) means the operators $A_\pm \equiv E_0^{0\pm}+E_0^{f\pm}+E_1^\pm$ are gradients in the sense

$$A_\pm = g^{-1}\partial g/\partial z_\pm\,, \tag{7.1}$$

where g is an element of the complex hull corresponding to the Lie group G with Lie algebra $\mathfrak{g}$ parametrized by the modified Gaussian decomposition

$$g=M_+N_-g_{0+}=M_-N_+g_{0-}\,. \tag{7.2}$$

Here $M_\pm, N_\pm$ belong to the nilpotent subgroups of G related with subspaces $\mathfrak{g}_{\pm a}$, $a\geqq 1$; $g_{0\pm}\in G_0$. In accordance with the general method [1] we shall subject the elements $M_\pm$ to the equations of S-matrix type,

$$\partial M_\pm/\partial z_\pm = M_\pm \sum_{\alpha=1}^{d_{\pm 1}} \varphi_{\pm\alpha}(z_\pm)X_\alpha^\pm \equiv M_\pm L_\pm\,, \tag{7.3}$$

where $\varphi_{+\alpha}(z_+), \varphi_{-\alpha}(z_-)$ are arbitrary functions of its arguments. Introduce

$$\mathfrak{K}\equiv M_+^{-1}M_- = N_-g_{0+}g_{0-}^{-1}N_+^{-1}\,, \tag{7.4}$$

which uniquely defines (for regular g) the elements N_+, N_-, and $g_0\equiv g_{0+}g_{0-}^{-1}$. In this their parameters are the functions of group parameters of the element $\mathfrak{K}$, i.e. $M_+(z_+)$, $M_-(z_-)$.

The elements $M_\pm$, $N_\pm$, and g_0 define general solutions of system (4.2). To prove this statement let us use the relations $\partial\mathfrak{K}/\partial z_- = \mathfrak{K}L_-$, $\partial\mathfrak{K}/\partial z_+ = -L_+\mathfrak{K}$ following from formulae (7.3) and (7.4), or in the equivalent form

$$\begin{aligned} &N_-^{-1}\partial N_-/\partial z_- = g_0L_-g_0^{-1}\,, \quad g_0^{-1}\partial g_0/\partial z_- - N_+^{-1}\partial N_+/\partial z_- = N_+^{-1}L_-N_+ - L_-\,,\\ &N_+^{-1}\partial N_+/\partial z_+ = g_0^{-1}L_+g_0\,, \quad \partial g_0/\partial z_+ g_0^{-1} + N_-^{-1}\partial N_-/\partial z_+ = -N_-^{-1}L_+N_- + L_+ \end{aligned} \tag{7.5}$$

(compare with those of [2]). So taking account of (7.3) the operators $A_\pm$ from (7.1) in decomposition (7.2) are

$$\begin{aligned} A_+ &= g_{0-}^{-1}N_+^{-1}\partial N_+/\partial z_+ g_{0-} + g_{0-}^{-1}\partial g_{0-}/\partial z_+\,,\\ A_- &= g_{0+}^{-1}N_-^{-1}\partial N_-/\partial z_- g_{0+} + g_{0+}^{-1}\partial g_{0+}/\partial z_-\,. \end{aligned} \tag{7.6}$$

The comparison of (7.6) with (7.5) shows that the operators A_+ and A_- take the values in subspaces $\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $\mathfrak{g}_0\oplus\mathfrak{g}_{-1}$, respectively. Therefore formulae (7.3) and (7.4) provide a complete solution of the integration problem for the dynamical systems with the operator form (4.2). Note that it automatically follows from relations (7.5) that the gauge invariant element $\mathfrak{g}_0$ satisfies Eq. (4.7).

For the construction of solutions to the nonlinear system (4.6) with operator form (4.4) containing $d_{\pm 1}$ unknown functions in accordance with the results of

Sect. 2, one can transform to a definite gauge, in which

$$A_+ = E_0^{f+} + J_+, \qquad A_- = E_{-1}. \tag{7.7}$$

Comparing formulae (7.6) and (7.7) we become convinced that $\partial g_{0+}/\partial z_- = 0$, i.e. $g_{0+} = g_{0+}(z_+)$, $N_+^{-1}\partial N_+/\partial z_+ = g_{0-}J_+g_{0-}^{-1}$, and the element $g_{0-}^{-1}\partial g_{0-}/\partial z_+$ takes the values in $\mathfrak{g}_0^f$. So relations (7.5) give the equality

$$L_+ = g_{0+}J_+g_{0+}^{-1}, \tag{7.8}$$

which defines the element g_{0+} up to the right shifts from G_0^0 commuting with $J_\pm$. Thus we assume that the parameters related with G_0^0 are not contained in g_{0+}, i.e. $g_{0+} \equiv g_0^{f+}$ depends only on d_{-1} parameters. The latter ones in their turn define the functional dependence of the operator L_+. The element g_{0-} is equal to $g_0^{-1}g_0^{f+}$.

It follows from equalities (7.5)–(7.7) that

$$A_- = E_{-1} = g_{0+}^{-1}N_-^{-1}\partial N_-/\partial z_- g_{0+} = g_{0-}^{-1}L_-g_{0-}.$$

Using the last relation in (7.5) and formula (7.8) one finds

$$\begin{aligned} -g_{0-}^{-1}\partial g_{0-}/\partial z_+ &+ (N_-g_{0+})^{-1}\partial(N_-g_{0+})/\partial z_+ \\ &= -(g_{0+}^{-1}N_-g_{0+})^{-1}J_+(g_{0+}^{-1}N_-g_{0+}) + J_+, \end{aligned}$$

so $g_{0-}^{-1}\partial g_{0-}/\partial z_+$ takes the values in $\mathfrak{g}_0^f$. Thus the gauge under consideration leads to the correct expressions for the operators $A_\pm$ from (7.7). In this the coefficient functions f_α entering the decomposition

$$A_- = g_{0+}^{-1}g_0L_-g_0^{-1}g_{0+} = \sum_{\alpha=1}^{d_{-1}} f_\alpha X_\alpha^-$$

satisfy the system of equations (4.6) we are interested in.

Therefore the general scheme for the construction of the solutions of the dynamical system described by Eqs. (4.6) is as follows:

i) Introduce two arbitrary elements $g_0^\pm(z_\pm)$ of the complex hull of Lie group G_0 with Lie algebra $\mathfrak{g}_0$.

ii) Construct the operators $L_\pm = g_0^\pm J_\pm(g_0^\pm)^{-1}$ depending functionally on $d_{\pm 1}$ arbitrary functions $\varphi_{\pm\alpha}(z_\pm)$; in terms of these operators find the solutions of two S-matrix-type equations $\partial M_\pm/\partial z_\pm = M_\pm L_\pm$ represented via multiplicative integrals according to the known formulae (see, for instance, [19]).

iii) Define from the equality

$$(M_+g_0^+)^{-1}(M_-g_0^-) = \hat{N}_-\hat{g}_0\hat{N}_+^{-1}, \qquad \hat{N}_\pm \equiv (g_0^\pm)^{-1}N_\pm g_0^\pm, \qquad \hat{g}_0 \equiv (g_0^+)^{-1}g_0g_0^-$$

the element $\hat{g}_0$.

iv) Consider the coefficients of the decomposition of E_{-1}, taking the values in subspace $\mathfrak{g}_{-1}$, $E_{-1} = \hat{g}_0J_-\hat{g}_0^{-1}$, with its generators X_α^-. These coefficients provide the complete solution of the dynamical systems under consideration.

Note that the simplicity of the Lie algebra $\mathfrak{g}$ has not been used anywhere in the derivation of the given relations. Thus the formulae obtained in this section give the solutions of dynamical systems associated with a local part of "arbitrary position" Lie algebras containing the $3d$-subalgebra. Under the latter one all the elements of the Lie algebra $\mathfrak{g}$ are organized into the multiplets with the integral values of angular momentum.

Appendix

Here we illustrate by a few examples of the known exactly integrable dynamical systems the general algebraic scheme given above. As a starting point consider the equations in form (4.6). Evidently the problem of its explicit realization is in concretization of a local part $\hat{\mathfrak{g}}$ of the corresponding graded Lie algebra and then in a choice of the suitable parametrization of the functions $\varphi^1_{-\alpha} \equiv f_\alpha(z_+, z_-)$, $1 \leqq \alpha \leqq d_{-1}$, adequate to the systems under consideration. Thus on the first stage one has to define the explicit form of the matrices $R_{\alpha\beta}$ and $A_{\alpha\beta}$.

First let us consider the generalized Toda lattice with fixed endpoints. In the framework of our construction it is connected with the canonical grading (i.e. minimal embedding) of an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$, when

$$\mathfrak{g}_0 \equiv \mathfrak{g}_0^f = \sum_1^{\mathrm{r}} \oplus u(1) = \{h_j, 1 \leqq j \leqq \mathrm{r}\}, \qquad \mathfrak{g}_{\pm 1} = \{X_{\pm j}, 1 \leqq j \leqq \mathrm{r}\}.$$

Then $R_{ij} = -k_{ij} f_i$, $A_{ij} = -\delta_{ij}$ and putting $f_i \equiv \exp(kx)_i$ one comes to system (1.1). Note that infinite-dimensional simple Lie algebra $\tilde{\mathfrak{g}}$ of finite growth (with the Cartan matrix $\tilde{k}$) in the same grading and parametrization leads in accordance with formula (5.3) to the system $\partial^2 x_j/\partial z_+ \partial z_- = \exp(\tilde{k}x)_j$ analogous in a form. This system corresponds to the appropriate periodic problem for the Toda lattice.

The string-type equations constructed in [3] with the Lagrangian

$$\mathfrak{L} = 1/2 \sum_{i,j=1}^{\mathrm{r}} (k_{ij}^{D_\mathrm{r}})^{-1} \partial\varrho_i/\partial z_+ \partial\varrho_j/\partial z_- + \sum_{i=1}^{\mathrm{r}} \exp\varrho_i - \tanh^2(\varrho_{\mathrm{r}-1} - \varrho_\mathrm{r})/2 \cdot \partial\omega/\partial z_+ \partial\omega/\partial z_-$$

,

are connected with such a grading of the algebra B_r when the $3d$-subalgebra has the generators

$$H = \sum_{j=1}^{\mathrm{r}} h_j + h_{\mathrm{r}-1}, \qquad J_\pm = \sum_{j=1}^{\mathrm{r}-2} X_{\pm j} + X_{\pm(\pi_{\mathrm{r}-1} + \pi_\mathrm{r})}.$$

In this $\mathfrak{g}_0 = \left(\sum_1^{\mathrm{r}-1} \oplus u(1)\right) \oplus B_1$, $\mathfrak{g}_0^0 = \{h_\mathrm{r}\}$, $\mathfrak{g}_0^f = \{h_j, 1 \leqq j \leqq \mathrm{r}-2, H, X_{\pm\mathrm{r}}\}$,

$$\mathfrak{g}_{\pm 1} = \{X_{\pm j}, 1 \leqq j \leqq \mathrm{r}-1; X_{\pm(\pi_{\mathrm{r}-1} + \pi_\mathrm{r})}, X_{\pm(\pi_{\mathrm{r}-1} + 2\pi_\mathrm{r})}\},$$

$k^{\mathfrak{g}}$ is the Cartan matrix of the Lie algebra $\mathfrak{g}$; the matrix function $R_{\alpha\beta}$ and $A_{\alpha\beta}$ have the form

$$R = -\operatorname{diag}(f_1, \ldots, f_{\mathrm{r}+1}) \left\| \begin{array}{ccc|c|c} & & & 0 & 0 \\ & k^{B_{\mathrm{r}-1}} & & 0 & 0 \\ & & & -2f_\mathrm{r}/f_{\mathrm{r}-1} & 0 \\ \hline 0 & -1 & 2 & f_{\mathrm{r}+1}/f_\mathrm{r} & -f_{\mathrm{r}-1}/f_\mathrm{r} \\ \hline 0 & -1 & 2 & 0 & 2f_\mathrm{r}/f_{\mathrm{r}+1} \end{array} \right\|;$$

$$A = -\left\| \begin{array}{cccc} \mathbb{1}_{\mathrm{r}-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\|.$$

Choose a numeration of the indices of f_α, $1 \leqq \alpha \leqq \mathrm{r}+1$, consistent with the arrangement of the root vectors of subspace $\mathfrak{g}_{-1}$ and parametrize them by the relations $f_j \equiv \exp\varrho_j$, $1 \leqq j \leqq \mathrm{r}-2$, $f_{\mathrm{r}-1} \equiv x^{(1)} \exp\sigma$, $f_\mathrm{r} \equiv x_0 \exp\sigma$, $f_{\mathrm{r}+1} \equiv x^{(2)} \exp\sigma$, $x_0^2 + x^{(1)}x^{(2)} = 1$. Then we obtain

$$\partial^2\varrho_i/\partial z_+\partial z_- = \sum_{j=1}^{\mathrm{r}-2} k_{ij}^{A_{\mathrm{r}-2}} \exp\varrho_j, \qquad 1 \leqq i \leqq \mathrm{r}-2;$$

$$\partial^2\sigma/\partial z_+\partial z_- = 2(1-x^{(1)}x^{(2)})^{1/2} \exp\sigma,$$

$$\partial(\partial x^{(\varepsilon)}/\partial z_+/(1-x^{(1)}x^{(2)})^{1/2})/\partial z_- = 2x^{(\varepsilon)} \exp\sigma, \qquad \varepsilon = 1,2.$$

Using the notations $\sigma \equiv (\varrho_{\mathrm{r}-1}+\varrho_\mathrm{r})/2$, $\delta \equiv (\varrho_{\mathrm{r}-1}-\varrho_\mathrm{r})/2$,

$$x^{(\varepsilon)} \equiv (-1)^{\varepsilon-1} \cdot \sinh\delta \cdot \exp[(-1)^{\varepsilon-1}\theta],$$

the substitution $\partial\theta/\partial z_+ = \cosh^{-2}\delta/2 \cdot \cosh\delta \cdot \partial\omega/\partial z_+$, $\partial\theta/\partial z_- = \cosh^{-2}\delta/2 \cdot \partial\omega/\partial z_-$ leads to the desired system of equations for the functions ϱ_j, $1 \leqq j \leqq \mathrm{r}$, and ω. It is obvious that the system identically reduces to the Toda lattice related with the series D_r when ω is a constant.

Consider another class of string-type equations corresponding in the framework of our approach to a different embedding of A_1 in algebra B_r, for which

$$H = 2\sum_{j=1}^{\mathrm{r}-1} h_j + h_\mathrm{r}, \qquad J_\pm = X_{\pm\pi}; \qquad \mathfrak{g}_0 = u(1) \oplus B_{\mathrm{r}-1}; \qquad \pi \equiv \sum_{j=1}^{\mathrm{r}} \pi_j;$$

$$\mathfrak{g}_0^f = \{H, X_{\pm\pi_\mathrm{r}}, X_{\pm(\pi_{\mathrm{r}-1}+\pi_\mathrm{r})}, X_{\pm(\pi_{\mathrm{r}-2}+\pi_{\mathrm{r}-1}+\pi_\mathrm{r})}, \ldots, X_{\pm(\pi_2+\ldots+\pi_\mathrm{r})}\};$$

$$\mathfrak{g}_{\pm1} = \{X_{\pm\pi_1}, X_{\pm(\pi_1+\pi_2)}, \ldots, X_{\pm\pi}; X_{\pm(\pi+\pi_\mathrm{r})}, \ldots, X_{\pm(\pi+\pi_2+\ldots+\pi_\mathrm{r})}\}.$$

(This case coincides naturally with the previous one when $r=2$.) Denoting $x_i^{(1)} \equiv f_i \exp(-\sigma)$, $x_i^{(2)} \equiv \bar{f}_i \exp(-\sigma)$, $x_0 \equiv f_\mathrm{r} \exp(-\sigma)$, $\bar{f}_i \equiv f_{2\mathrm{r}-i}$, $1 \leqq j \leqq \mathrm{r}-1$, $x_0^2 + (\mathbf{x}^{(1)}\mathbf{x}^{(2)}) = 1$, one comes from Eqs. (4.6) to the system

$$\partial^2\sigma/\partial z_+\partial z_- = 2[1-(\mathbf{x}^{(1)}\mathbf{x}^{(2)})]^{1/2} \exp\sigma,$$

$$\partial(\partial\mathbf{x}^{(\varepsilon)}/\partial z_+/[1-(\mathbf{x}^{(1)}\mathbf{x}^{(2)})]^{1/2})/\partial z_- = 2\mathbf{x}^{(\varepsilon)} \exp\sigma, \qquad \varepsilon = 1,2.$$

In parametrization $x_i^{(\varepsilon)} = (-1)^{\varepsilon-1} \sinh\delta \cdot \exp[(-1)^{\varepsilon-1}\theta_i] \cdot n_i$, $\mathbf{n}^2 = 1$, these equations take the form

$$\partial^2\sigma/\partial z_+\partial z_- = 2\cosh\delta \cdot \exp\sigma, \qquad \partial^2\delta/\partial z_+\partial z_- = 2\sinh\delta \cdot \exp\sigma - \tanh\delta \cdot \Delta,$$

$$n_i\hat{D}\theta_i + (\partial\theta_i/\partial z_+ \partial n_i/\partial z_- + \partial\theta_i/\partial z_- \partial n_i/\partial z_+) = 0,$$

$$\hat{D}n_i + n_i\partial\theta_i/\partial z_+ \partial\theta_i/\partial z_- - n_i\Delta = 0,$$

where

$$\hat{D} \equiv \partial^2/\partial z_+\partial z_- + 2\sinh^{-1} 2\delta\partial\delta/\partial z_- \partial/\partial z_+ + \coth\delta\partial\delta/\partial z_+ \partial/\partial z_-,$$

$$\Delta \equiv \sum_j [\partial\theta_j/\partial z_+ \partial\theta_j/\partial z_- n_j^2 - \partial n_j/\partial z_+ \partial n_j/\partial z_-].$$

Using the representation of the $(\mathrm{r}-1)$-dimensional unit vector $\mathbf{n}$ in generalized Euler angles it is easy to carry out further reduction over the dimensionality and to receive a symmetrical form (with respect to z_+ and z_-). The system of equations for the series D_r supplied by the grading with $\mathfrak{g}_0 = u(1) \oplus D_{r-1}$ can be constructed absolutely analogously.

It is important to note that in the framework of the geometrical construction this system represents the parametric formulation of the Gauss, Peterson-Codazzi, and Ricci equations for $2d$-minimal surfaces in N-dimensional Euclidean or pseudo-Euclidean space ($N = 2r + 1$ for B_r and $N = 2r$ for D_r)[9]. The components of the fundamental tensor, torsion vector and second fundamental forms are expressed just via solutions of the system constructed in Sect. 7. On the whole this interpretation of the corresponding subclass of systems (4.6) is connected with the intrinsic geometry of surfaces in Euclidean, pseudo-Euclidean or affine space (minimal and constant curvature two-dimensional surfaces).

In the same way starting from Eqs. (5.3) one can obtain the corresponding equations of completely integrable systems associated with infinite-dimensional Lie algebras of finite growth, in particular, the Lund-Regge model.

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9 In the simplest particular case ($r = 1, 2$) this relationship has been established in an explicit form in [10]

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