

Gaussian Ensembles of Random Hermitian Matrices Intermediate Between Orthogonal and Unitary Ones

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Abstract. A Gaussian ensemble of Hermitian matrices depending on a parameter α is considered. When $\alpha = 0$, the ensemble is Gaussian Orthogonal, and when $\alpha = 1$, it is Gaussian Unitary. An analytic expression for the n -level correlation and cluster functions is given for any n and $0 \leq \alpha \leq 1$. This ensemble is of relevance in the study of time reversal symmetry breaking of nuclear interactions.

1. Introduction and Summary

Wigner's Gaussian Orthogonal Ensemble (GOE) of asymptotically large real symmetric matrices has been the model for nuclear energy level fluctuations [1–4]. A large amount of data—mainly on slow-neutron resonances of medium and heavy nuclei, but also on some proton resonances of light nuclei—has been used to test the model. A very recent analysis indicates that the agreement between GOE and the data is remarkably good [5].

It can be shown [6] on general symmetry arguments that if the Hamiltonian governing the system is invariant under time reversal and space rotations, the appropriate ensemble is that of real symmetric matrices. On the other hand, if it contains a time reversal non-invariant part, then the matrices should have an imaginary part as well. One such ensemble, namely Gaussian Unitary Ensemble (GUE) of complex Hermitian matrices in which the real and imaginary parts are sampled independently but with the same weight, has also been studied analytically in great detail [1–4]. It gives very different fluctuation properties than the GOE and, in particular, has stronger level repulsion than observed in nuclear spectra, implying that the nuclear forces are mostly time reversal invariant. Following a suggestion of Wigner [2], the close agreement between GOE and the data may then be used to derive an upper bound on the time reversal non-invariant part of the nuclear interactions by studying ensembles of complex Hermitian matrices with a small imaginary part.

Ensembles of Hermitian matrices in which the dispersions of the real and imaginary parts are arbitrary and unequal have been the subject of several previous studies [7]. It is only recently, however, that approximate forms for the two-level

correlation and cluster functions have been given [8]. It is made plausible that the transition from GOE to GUE in the (asymptotic) eigenvalue fluctuations is governed by a parameter λ which is the dispersion of the imaginary part of the matrix elements expressed in local spacing units; see Ref. [9] for the relevance of this parameter in more general ensembles. In this paper we derive an exact closed expression for the n -level correlation and cluster functions for any n for ensembles intermediate between GOE and GUE, and give, *inter alia*, a rigorous derivation of the transition parameter.

In what follows we consider an ensemble of Hermitian matrices depending on a parameter α . The matrix elements are Gaussian random variables and the relative dispersion of their imaginary to real parts is α . For $\alpha = 0$, the ensemble is GOE, and for $\alpha = 1$, it is GUE. We restrict ourselves to the case $0 \leq \alpha \leq 1$. This restriction enables us to write the ensemble as a sum of GOE and GUE, and consequently the joint probability density of the matrix elements as a convolution (Sect. 2). This plays a key role in the derivation of the joint probability density for the eigenvalues in Sect. 3.

The joint probability density for the eigenvalues follows easily in the two limiting cases from the invariance properties of the ensembles [10]; see Ref. [11] for the third limiting case $\alpha = \infty$. For arbitrary α , however, one needs to integrate over the variables of the unitary transformations which diagonalize the matrices of the ensembles. For the restricted values of α , the formula needed to do this is known [12]. Then, making use of the method of integration over alternate variables [13], we obtain the joint probability density as a pfaffian (Sect. 3).

The same method of integration can be used to derive the one-and-two-level functions [13]. We briefly discuss this line of inquiry in Appendix I. For more general results (Sect. 4), we rely on two theorems on quaternion determinants [14] which have already been used to derive all n -level functions for $\alpha = 0, 1$ [15].

In Sect. 5 we consider the limits when the dimension of the matrices is large. It is shown that the eigenvalue fluctuations (as described by the $n \geq 2$ -level functions) undergo a discontinuous transition at $\alpha = 0$. For all non-zero finite α we obtain results characteristic of GUE. On the other hand, in the limit when $\alpha \rightarrow 0$ but the parameter λ remains finite, the $n \geq 2$ -level functions undergo a smooth transition from GOE ($\lambda = 0$) to GUE ($\lambda = \infty$). Except in the two limiting cases, the functions are non-stationary owing to the variation of λ over the spectrum [8].

In Sect. 6 we study the (asymptotic) two-level functions in detail and derive expressions for the number variance Σ^2 as well as for the ensemble average of the least square statistic Δ_3 which are useful in the analysis of data.

We do not consider here the data analysis for the breaking of time reversal symmetry, as it is already part of a separate study [8]. Some other quantities, such as the spacing distributions which follow from the cluster functions, will be discussed in a later paper.

2. An Ensemble of Random Hermitian Matrices

Consider the ensemble of Hermitian matrices

$$H = A + B, \quad (2.1)$$

where A is an $N \times N$ real symmetric matrix with the probability density

$$p_1(A) = 2^{-N/2} [2\pi(1 - \alpha^2)v^2]^{-N(N+1)/4} \exp\{-\text{Tr } A^2/4(1 - \alpha^2)v^2\}, \quad (2.2)$$

$$dA \equiv \prod_i dA_{ii} \prod_{j>k} dA_{jk}, \quad (2.3)$$

and B an $N \times N$ Hermitian matrix with the probability density

$$p_2(B) = 2^{-N/2} [2\pi\alpha^2v^2]^{-N^2/2} \exp\{-\text{Tr } B^2/4\alpha^2v^2\}, \quad (2.4)$$

$$dB = \prod_i dB_{ii} \prod_{j>k} (\text{Re } dB_{jk})(\text{Im } dB_{jk}). \quad (2.5)$$

The joint probability density for the matrix elements of H is then

$$p(H) = \int p_1(A) p_2(H - A) dA, \quad (2.6)$$

$$dH = \prod_i dH_{ii} \prod_{j>k} (\text{Re } dH_{jk})(\text{Im } dH_{jk}), \quad (2.7)$$

and on the average

$$\frac{\|\text{Im } H\|^2}{\|\text{Re } H\|^2} = \left(\frac{N-1}{N+1} \right) \alpha^2 \xrightarrow{\text{large } N} \alpha^2. \quad (2.8)$$

The ensemble is symmetric in α . We consider therefore $\alpha \geq 0$. Furthermore the definition (2.1) restricts it to $\alpha \leq 1$. When $\alpha = 0$, $B = 0$ with probability one and then $H = A$ forms the GOE. Similarly $\alpha = 1$ gives $H = B$ which is the GUE.

Here v^2 fixes the scale. We shall choose

$$2v^2(1 + \alpha^2) = 1, \quad (2.9)$$

so that the results for GOE and GUE will be identical to those in [3].

We remark that the convolution integral in (2.6) can be evaluated easily and the resultant form, giving explicitly the joint probability density of the real and imaginary parts of H , is then valid for all α . However, it is the integral form in (2.6), which will be used in the next section for the joint probability density of the eigenvalues.

3. Joint Probability Density of the Eigenvalues

Let x_1, \dots, x_N be the eigenvalues of H :

$$H = U^\dagger x U, x = [x_i \delta_{ij}], U^\dagger U = 1, \quad (3.1)$$

so that [10]

$$dH = \left(\prod_{j=1}^N j! \right)^{-1} \pi^{N(N-1)/2} \Delta^2(x) dx dU, \quad (3.2)$$

where

$$\Delta(x) = \prod_{i>j} (x_i - x_j), dx \equiv dx_1 \dots dx_N. \quad (3.3)$$

Similarly if a_1, \dots, a_N are the eigenvalues of A ,

$$A = \Omega^T a \Omega, a = [a_i \delta_{ij}], \Omega^T \Omega = 1, \quad (3.4)$$

then [10]

$$dA = 2^{-N} \left(\prod_1^N \Gamma \left(1 + \frac{1}{2} j \right) \right)^{-1} \pi^{N(N+1)/4} |\Delta(a)| da d\Omega. \quad (3.5)$$

To get the probability density $p(x)$ of the x_i , one has to further integrate over U in Eq. (2.6). We have

$$\begin{aligned} p(x) &= \int p(H) \frac{dH}{dU} dU \\ &= \text{const } \Delta^2(x) \int \exp \left\{ -\frac{\Sigma a_i^2}{4(1-\alpha^2)v^2} - \frac{\text{Tr}(a - U^\dagger x U)^2}{4\alpha^2 v^2} \right\} |\Delta(a)| da dU, \end{aligned} \quad (3.6)$$

where an intermediate step involves a trivial integration over Ω . The constant in the last step is the product of the constants in (2.2, 2.4, 3.2, 3.5). The integral over U can be performed by using the formula [12]

$$\int dU \exp \left[-\frac{1}{2t} \text{Tr}(R - U^\dagger S U)^2 \right] = t^{N(N-1)/2} \prod_0^{N-1} j! \frac{\det \left[\exp \left\{ -\frac{1}{2t} (r_i - s_j)^2 \right\} \right]}{\Delta(r) \Delta(s)}, \quad (3.7)$$

valid for Hermitian R, S with eigenvalues $\{r_i\}, \{s_i\}$. Note that the determinant has the same sign as $\Delta(r) \Delta(s)$. Then for the final integral over a , we have

$$\begin{aligned} & \int \exp \left(-\frac{\Sigma a_i^2}{4(1-\alpha^2)v^2} \right) \det \left[\exp \left\{ -\frac{(a_i - x_j)^2}{4\alpha^2 v^2} \right\} \right] \text{sign } \Delta(a) da \\ &= N! \exp(-\Sigma x_i^2/4v^2) \int da \det \left[\exp \left\{ -\frac{(a_i - (1-\alpha^2)x_j)^2}{4(1-\alpha^2)\alpha^2 v^2} \right\} \right] \\ & \quad a_1 \leq a_2 \leq \dots \leq a_N \\ &= N! \exp(-\Sigma x_i^2/4v^2) Pf[\varepsilon(x_i, x_j)], \end{aligned} \quad (3.8)$$

where in the last step we have first integrated over alternate variables and then used the theory of Pfaffians [13]. To avoid minor complications, we have taken $N = 2m$, even. The antisymmetric function $\varepsilon(x, y) = -\varepsilon(y, x)$ is a double integral:

$$\begin{aligned} \varepsilon(x, y) &= \int_{-\infty}^{\infty} dz_2 \int_{-x}^{z_2} dz_1 \left\{ \exp \left(-\frac{1}{4(1-\alpha^2)\alpha^2 v^2} [(z_1 - (1-\alpha^2)x)^2 + (z_2 - (1-\alpha^2)y)^2] \right) \right. \\ & \quad \left. - \exp \left(-\frac{1}{4(1-\alpha^2)\alpha^2 v^2} [(z_1 - (1-\alpha^2)y)^2 + (z_2 - (1-\alpha^2)x)^2] \right) \right\} \\ &= 4\pi(1-\alpha^2)\alpha^2 v^2 \text{erf} \left\{ \left(\frac{1-\alpha^2}{8\alpha^2 v^2} \right)^{1/2} (x-y) \right\}, \end{aligned} \quad (3.9)$$

where

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_0^x \exp \left(-t^2\right) d t . \tag{3.10}$$

In the last step of (3.9) the integral over z_1 was performed after introducing the new variable $t=\left(z_2-z_1\right)$.

The Pfaffian in (3.8) is the square root of the determinant of the antisymmetric matrix ε . Collecting all the terms and using the normalization (2.9), we have finally, for even N ,

$$p(x)=C_N \exp \left(-\frac{1+\alpha^2}{2} \Sigma x_i^2\right)|\Delta(x)|\left\{\operatorname{det}\left[f\left(x_i-x_j\right)\right]\right\}^{1 / 2}, \tag{3.11}$$

with

$$f(x)=\operatorname{erf}\left\{\left(\frac{1-\alpha^4}{4 \alpha^2}\right)^{1 / 2} x\right\} \tag{3.12}$$

and

$$C_N^{-1}=2^{3 N / 2}\left(1-\alpha^2\right)^{N(N-1) / 4}\left(1+\alpha^2\right)^{-N(N+1) / 4} \prod_{j=1}^N \Gamma\left(1+\frac{1}{2} j\right) . \tag{3.13}$$

We remark parenthetically that the limit $\alpha=0$ is trivial since, then, $f(x)=\operatorname{sign} x$ and $\operatorname{det} f=1$. On the other hand for $\alpha=1$ one should expand $(\operatorname{det} f)^{1 / 2}$ in powers of $\left(\frac{1-\alpha^4}{4 \alpha^2}\right)^{1 / 2}$ to obtain

$$\lim _{\alpha \rightarrow 1}\left(1-\alpha^2\right)^{-N(N-1) / 4}(\operatorname{det} f)^{1 / 2}=2^{N(N+3) / 4} \pi^{-N / 2} \prod_{j=1}^N\left(\frac{\Gamma\left(1+\frac{1}{2} j\right)}{j !}\right)|\Delta(x)| . \tag{3.14}$$

In both cases the standard results [10] are recovered.

4. Correlation and Cluster Functions

We shall use two theorems from the theory of quaternion matrices to derive the correlation and cluster functions. For a proof of the theorems, and details of the notions involved, one may consult either of the two references in [14]. The same theorems have been used to derive the functions for GOE and GUE [15]. An alternative derivation of the functions is outlined in Appendix I.

Theorem 1. *In an $N \times N$ self-dual quaternion matrix σ , replace its quaternion elements by their 2×2 matrix representatives as follows*

$$1 \rightarrow\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], e_1 \rightarrow\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right], e_2 \rightarrow\left[\begin{array}{cc} 0 & -i \\ -i & 0 \end{array}\right], e_3 \rightarrow\left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right] . \tag{4.1}$$

This gives a matrix $C(\sigma)$ of twice the size with complex elements. Then

$$\operatorname{det} C(\sigma)=\left\{Q \operatorname{det} \sigma\right\}^2, \tag{4.2}$$

where the quaternion-determinant $Q \det \sigma$ is defined as the scalar

$$Q \det \sigma = \sum_P (-1)^{N-l} \prod_1^l (\sigma_{ab} \sigma_{bc} \cdots \sigma_{sa}). \quad (4.3)$$

Here P is any permutation of the integers $(1, 2, \dots, N)$, consisting of l cycles of the form

$$(a \rightarrow b \rightarrow c \rightarrow \cdots \rightarrow s \rightarrow a), \quad (4.4)$$

and $(-1)^{N-l}$ is the parity of P .

Theorem 2. Let the quaternion elements σ_{jk} of an $N \times N$ self-dual quaternion matrix $\sigma^{(N)}$ satisfy the following conditions:

$$(i) \quad \sigma_{jk} = \Phi(x_j, x_k) = \bar{\sigma}_{kj}, \quad (4.5)$$

(where the bar denotes the operation of quaternion-adjoint), i.e. σ_{jk} depends only on the variables x_j and x_k ,

$$(ii) \quad \int \Phi(x, x) d\mu(x) = c, \quad (4.6)$$

$$(iii) \quad \int \Phi(x, y) \Phi(y, z) d\mu(y) = \Phi(x, z) + \Phi_1(x, z), \quad (4.7)$$

$$\Phi_1(x, z) = \tau \Phi(x, z) - \Phi(x, z) \tau, \quad (4.8)$$

where $d\mu$ is a suitable measure, c a constant scalar and τ a constant quaternion. Then

$$\int Q \det \sigma^{(N)} d\mu(x_N) = (c - N + 1) Q \det \sigma^{(N-1)}, \quad (4.9)$$

where $\sigma^{(N-1)}$ is the $(N-1) \times (N-1)$ matrix obtained from $\sigma^{(N)}$ by removing the row and column containing the variable x_N .

We exhibit now a self-dual quaternion matrix $\sigma^{(N)}(x_1, x_2, \dots, x_N)$ and verify that its elements satisfy Eqs. (4.5–4.8). Moreover, we shall show that

$$\det C(\sigma^{(N)}) = \{p(x)\}^2, \quad (4.10)$$

where $p(x)$ is given by (3.11). We can then immediately write down the n -level correlation function [16]

$$R_n(x_1, x_2, \dots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int p(x_1, \dots, x_N) dx_{n+1} \cdots dx_N \quad (4.11)$$

as a quaternion-determinant.

We write the quaternion elements of σ by their 2×2 matrix representatives

$$\sigma_{jk} = \Phi(x_j, x_k) = \begin{bmatrix} S_N(x_j, x_k) & D_N(x_j, x_k) \\ J_N(x_j, x_k) & S_N^\dagger(x_j, x_k) \end{bmatrix}, \quad (4.12)$$

where

$$\begin{aligned} S_N(x, y) &\equiv S_N^\dagger(y, x) \\ &= \sum_{j=0}^{m-1} \{ \varphi_{2j}(x) \varphi_{2j}(y) - \psi_{2j}(x) A_{2j}(y) \}, \end{aligned} \quad (4.13)$$

$$D_N(x, y) = \sum_{j=0}^{m-1} \{ \varphi_{2j}(x) \psi_{2j}(y) - \psi_{2j}(x) \varphi_{2j}(y) \} = -D_N(y, x), \quad (4.14)$$

$$J_N(x, y) = I_N(x, y) + g(x, y) = -J_N(y, x), \quad (4.15)$$

$$I_N(x, y) = \sum_{j=0}^{m-1} \{ \varphi_{2j}(x) A_{2j}(y) - A_{2j}(x) \varphi_{2j}(y) \} = -I_N(y, x), \quad (4.16)$$

$$g(x, y) = \frac{1}{2} \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{1/2} e^{-(1/2)\alpha^2(x^2 + y^2)} f(x - y) = -g(y, x), \quad (4.17)$$

with

$$\begin{aligned} \varphi_j(x) &= (2^j j! \sqrt{\pi})^{-1/2} e^{-(1/2)x^2} H_j(x) \\ &= (2^j j! \sqrt{\pi})^{-1/2} e^{(1/2)x^2} \left(-\frac{d}{dx} \right)^j e^{-x^2}, \end{aligned} \quad (4.18)$$

$$\psi_j(x) = \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^j e^{(1/2)\alpha^2 x^2} \frac{d}{dx} \{ e^{-(1/2)\alpha^2 x^2} \varphi_j(x) \}, \quad (4.19)$$

and

$$A_j(x) = \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right)^j e^{-(1/2)\alpha^2 x^2} \int_{-\infty}^{\infty} e^{(1/2)\alpha^2 y^2} \epsilon(x - y) \varphi_j(y) dy. \quad (4.20)$$

Here $f(x)$ in (4.17) is as defined in (3.12), the $H_j(x)$ in (4.18) are the Hermite polynomials and $\epsilon(x)$ in (4.20) is given by:

$$\epsilon(x) = \frac{1}{2} \text{sign } x. \quad (4.21)$$

The $\varphi_j(x)$ are orthonormalized harmonic oscillator wave functions:

$$\int_{-\infty}^{\infty} \varphi_i(x) \varphi_j(x) dx = \delta_{ij}. \quad (4.22)$$

By a partial integration we find that

$$\int_{-\infty}^{\infty} A_i(x) \psi_j(x) dx = -\delta_{ij}. \quad (4.23)$$

Moreover, since the parity of $\varphi_j(x)$ is $(-1)^j$, while that of $\psi_j(x)$ or $A_j(x)$ is $(-1)^{j+1}$, we have

$$\int_{-\infty}^{\infty} \varphi_i(x) \psi_j(x) dx = \int_{-\infty}^{\infty} \varphi_i(x) A_j(x) dx = 0, \quad i + j = \text{even}. \quad (4.24)$$

We now define, for any two functions $f_1(x, y)$ and $f_2(x, y)$, the composition

$$f_1 * f_2 \equiv (f_1 * f_2)(x, z) \equiv \int_{-\infty}^{\infty} f_1(x, y) f_2(y, z) dy. \quad (4.25)$$

Then using (4.22–4.24) we find that

$$S_N * S_N = S_N, \quad (4.26)$$

$$S_N * D_N = D_N * S_N^\dagger = D_N, \quad (4.27)$$

$$I_N * S_N = S_N^\dagger * I_N = I_N, \quad (4.28)$$

and

$$I_N * D_N = S_N^\dagger, \quad D_N * I_N = S_N. \quad (4.29)$$

To evaluate $g * S_N$ and $g * D_N$ we need the integrals (see Appendix II):

$$\int_{-\infty}^{\infty} g(x, y) \psi_j(y) dy = \varphi_j(x), \quad (4.30)$$

and

$$\int_{-\infty}^{\infty} g(x, y) \varphi_j(y) dy = A_j(x). \quad (4.31)$$

Using these equations we get:

$$g * S_N = S_N^\dagger * g = -I_N, \quad (4.32)$$

and

$$g * D_N = -S_N^\dagger, \quad D_N * g = -S_N, \quad (4.33)$$

so that

$$J_N * S_N = S_N^\dagger * J_N = 0, \quad (4.34)$$

$$J_N * D_N = D_N * J_N = 0. \quad (4.35)$$

Thus

$$\begin{aligned} \Phi * \Phi &= \begin{bmatrix} S_N * S_N + D_N * J_N & S_N * D_N + D_N * S_N^\dagger \\ J_N * S_N + S_N^\dagger * J_N & J_N * D_N + S_N^\dagger * S_N^\dagger \end{bmatrix} \\ &= \begin{bmatrix} S_N & 2D_N \\ 0 & S_N^\dagger \end{bmatrix} = \Phi + \Phi_1, \end{aligned} \quad (4.36)$$

where

$$\Phi_1 = \begin{bmatrix} 0 & D_N \\ -J_N & 0 \end{bmatrix} = \tau \Phi - \Phi \tau, \quad (4.37)$$

with

$$\tau = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4.38)$$

We also see from (4.22, 4.23) that

$$\int_{-\infty}^{\infty} \Phi(x, x) dx = \int_{-\infty}^{\infty} S_N(x, x) dx = 2m = N. \quad (4.39)$$

Thus Eqs. (4.5–4.8) are satisfied with $c = N$ and τ given by (4.38).

To verify (4.10) we observe that the $2N \times 2N$ matrix

$$\begin{aligned} G &= \begin{bmatrix} S_N(x_j, x_k) & D_N(x_j, x_k) \\ I_N(x_j, x_k) & S_N^\dagger(x_j, x_k) \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{2i}(x_j) & -\psi_{2i}(x_j) & 0 \\ -A_{2i}(x_j) & \varphi_{2i}(x_j) & 0 \end{bmatrix} \begin{bmatrix} \varphi_{2i}(x_k) & \psi_{2i}(x_k) \\ A_{2i}(x_k) & \varphi_{2i}(x_k) \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.40)$$

has rank N . In other words, the last N rows

$$[I_N(x_j, x_k) \ S_N^\dagger(x_j, x_k)] \quad (4.41)$$

are linear combinations of the first N rows

$$[S_N(x_j, x_k) \ D_N(x_j, x_k)]. \quad (4.42)$$

The determinant of $C(\sigma^{(N)})$,

$$C(\sigma^{(N)}) = \begin{bmatrix} S_N(x_j, x_k) & D_N(x_j, x_k) \\ J_N(x_j, x_k) & S_N^\dagger(x_j, x_k) \end{bmatrix}, \quad (4.43)$$

is therefore not changed if we subtract from its last N rows the last N rows of G . Thus

$$\begin{aligned} \det C(\sigma^{(N)}) &= \det \begin{bmatrix} S_N(x_j, x_k) & D_N(x_j, x_k) \\ g(x_j, x_k) & 0 \end{bmatrix} \\ &= (-1)^N \det[g(x_j, x_k)] \det[D_N(x_j, x_k)]. \end{aligned} \quad (4.44)$$

Moreover, from (4.17) we have

$$\det[g(x_j, x_k)] = 2^{-N} \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{N/2} e^{-\alpha^2 \sum x_i^2} \det[f(x_j - x_k)], \quad (4.45)$$

and, since

$$[D_N(x_j, x_k)] = [\varphi_{2i}(x_j) \ \psi_{2i}(x_j)] \begin{bmatrix} \psi_{2i}(x_k) \\ -\varphi_{2i}(x_k) \end{bmatrix}, \quad (4.46)$$

we also have from (4.18, 4.19)

$$\begin{aligned} \det[D_N(x_j, x_k)] &= (-1)^N \{ \det[\varphi_{2i}(x_j) \ \psi_{2i}(x_j)] \}^2 \\ &= (-1)^N \left(\prod_{k=0}^{m-1} \left(\frac{2k+1}{2} \right) \right) \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{N(N-2)/2} (1 + \alpha^2)^N \{ \det[\varphi_i(x_j)] \}^2 \\ &= (-1)^N 2^{-2N} \left(\prod_{j=1}^N \Gamma(1 + \tfrac{1}{2}j) \right)^{-2} \\ &\quad \times (1 - \alpha^2)^{-N(N-2)/2} (1 + \alpha^2)^{N^2/2} e^{-\sum x_i^2} \Delta^2(x), \end{aligned} \quad (4.47)$$

where in the last two steps we have made repeated use of the fact that a determinant is not changed if we add to a column a linear combination of the other columns.

Thus in the second step we have used the relation

$$\sqrt{2}\psi_j(x) = \left(\frac{1+\alpha^2}{1-\alpha^2}\right)^j \left\{ \sqrt{j}(1-\alpha^2)\varphi_{j-1}(x) - \sqrt{j+1}(1+\alpha^2)\varphi_{j+1}(x) \right\} \quad (4.48)$$

to replace the column ψ_{2i} by φ_{2i+1} . (Note that in the first step the index i runs from zero to $m-1$ whereas in the second from zero to $N-1$; in both steps j is from 1 to N .) The last form follows by reducing $\det [H_i(x_j)]$ to $\det [x_j^i]$ which is the Vandermonde determinant $\Delta(x)$. Equations (4.44, 4.45, 4.47) now give (4.10).

All the conditions of the theorem being satisfied, we apply it several times to obtain from (4.9–4.11)

$$R_n(x_1, x_2, \dots, x_n) = Q \det \sigma^{(n)} = \{\det C(\sigma^{(n)})\}^{1/2}, \quad (4.49)$$

where $\sigma^{(n)}$ is the $n \times n$ quaternion matrix

$$\sigma^{(n)} \equiv \sigma^{(n)}(x_1, x_2, \dots, x_n) = [\Phi(x_j, x_k)]_{j,k=1,\dots,n}, \quad (4.50)$$

and $C(\sigma^{(n)})$ is the $2n \times 2n$ matrix

$$C(\sigma^{(n)}) = \begin{bmatrix} S_N(x_j, x_k) & D_N(x_j, x_k) \\ J_N(x_j, x_k) & S_N^\dagger(x_j, x_k) \end{bmatrix}_{j,k=1,2,\dots,n}. \quad (4.51)$$

Thus the level density is

$$\begin{aligned} R_1(x) &= S_N(x, x) \\ &= \sum_{j=0}^{N-1} \varphi_j^2(x) + (1-\alpha^2)\sqrt{N/2}\varphi_{N-1}(x)e^{-(1/2)\alpha^2 x^2} \int_{-\infty}^{\infty} \epsilon(x-z)e^{(1/2)\alpha^2 z^2} \varphi_N(z)dz \end{aligned} \quad (4.52)$$

where the last form follows from the simplified expression for S_N given in (4.63) below. The two-level correlation function is

$$R_2(x, y) = R_1(x)R_1(y) - T_2(x, y), \quad (4.53)$$

where the two-level cluster function T_2 is given by

$$T_2(x, y) = S_N(x, y)S_N(y, x) - D_N(x, y)J_N(x, y). \quad (4.54)$$

As in [14, 15] the n -level cluster function is given by

$$T_n(x_1, x_2, \dots, x_n) = \sum_P \Phi(x_1, x_2)\Phi(x_2, x_3)\dots\Phi(x_n, x_1), \quad (4.55)$$

where the sum is taken over all the $(n-1)!$ distinct cyclic permutations of the indices $(1, 2, \dots, n)$. The functions satisfy the integral relations, namely

$$\int_{-\infty}^{\infty} R_n(x_1, x_2, \dots, x_n)dx_n = (N-n+1)R_{n-1}(x_1, x_2, \dots, x_{n-1}), \quad (4.56)$$

and

$$\int_{-\infty}^{\infty} T_n(x_1, x_2, \dots, x_n)dx_n = (n-1)T_{n-1}(x_1, x_2, \dots, x_{n-1}), \quad (4.57)$$

with

$$\int_{-\infty}^{\infty} R_1(x) dx = \int_{-\infty}^{\infty} T_1(x) dx = N, \quad (4.58)$$

as they should [16].

One can use (4.48) and

$$\sqrt{2} \varphi_j(x) = \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^j \left\{ \sqrt{j}(1 - \alpha^2) A_{j-1}(x) - \sqrt{j+1}(1 + \alpha^2) A_{j+1}(x) \right\} \quad (4.59)$$

to simplify the expressions for S_N and J_N . We have

$$\begin{aligned} \psi_{2j}(x) A_{2j}(y) = & -\varphi_{2j+1}(x) \varphi_{2j+1}(y) + (1 - \alpha^2) \left\{ \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j} \sqrt{j} \varphi_{2j-1}(x) A_{2j}(y) \right. \\ & \left. - \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j+2} \sqrt{j+1} \varphi_{2j+1}(x) A_{2j+2}(y) \right\}, \end{aligned} \quad (4.60)$$

$$\begin{aligned} \varphi_{2j}(x) A_{2j}(y) = & -A_{2j+1}(x) \varphi_{2j+1}(y) + (1 - \alpha^2) \left\{ \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j} \sqrt{j} A_{2j-1}(x) A_{2j}(y) \right. \\ & \left. - \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j+2} \sqrt{j+1} A_{2j+1}(x) A_{2j+2}(y) \right\}, \end{aligned} \quad (4.61)$$

and

$$\begin{aligned} \psi_{2j}(x) \varphi_{2j}(y) = & -\varphi_{2j+1}(x) \psi_{2j+1}(y) + (1 - \alpha^2) \left\{ \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j} \sqrt{j} \varphi_{2j-1}(x) \varphi_{2j}(y) \right. \\ & \left. - \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{2j+2} \sqrt{j+1} \varphi_{2j+1}(x) \varphi_{2j+2}(y) \right\}. \end{aligned} \quad (4.62)$$

Then from (4.13, 4.60)

$$\begin{aligned} S_N(x, y) = & \sum_{j=0}^{2m-1} \varphi_j(x) \varphi_j(y) + (1 - \alpha^2) \sqrt{m} \varphi_{N-1}(x) e^{-(1/2)\alpha^2 x^2} \\ & \times \int_{-\infty}^{\infty} e^{(1/2)\alpha^2 z^2} \varphi_N(z) dz. \end{aligned} \quad (4.63)$$

Also, from (4.31, 4.61) we have

$$\begin{aligned} g(x, y) = & \sum_{j=0}^{\infty} A_j(x) \varphi_j(y) = \frac{1}{2} \sum_{j=0}^{\infty} \{ A_j(x) \varphi_j(y) - \varphi_j(x) A_j(y) \} \\ = & \sum_{j=0}^{\infty} \{ A_{2j}(x) \varphi_{2j}(y) - \varphi_{2j}(x) A_{2j}(y) \} \\ = & \sum_{j=0}^{\infty} \{ A_{2j+1}(x) \varphi_{2j+1}(y) - \varphi_{2j+1}(x) A_{2j+1}(y) \} \end{aligned} \quad (4.64)$$

which along with (4.15, 4.16) gives

$$J_N(x, y) = \sum_{j=m}^{\infty} \{A_{2j}(x)\varphi_{2j}(y) - \varphi_{2j}(y)A_{2j}(x)\}. \quad (4.65)$$

The result (4.55) is already in the form given for $\alpha = 0, 1$ in [15]. For $\alpha = 0$, $g(x, y) = \epsilon(x - y)$, and D_N and I_N can be written, with the help of (4.61, 4.62), as a differential and an integral, respectively, of S_N in (4.63). On the other hand for $\alpha = 1$, while D_N diverges, its product with J_N , (4.65), is zero so that only S_N terms survive in (4.55).

5. Limits when the Matrices are Large

The large- N limits of the n -level functions are known [15, 17] for $\alpha = 0, 1$. We follow here the method of [17] where the limits for $n(> 1)$ -level functions are derived for every region of the spectrum.

We have, from (4.63), for large N

$$\begin{aligned} S_N(x, y) &\approx \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y) \\ &= (\tfrac{1}{2}N)^{1/2} \left\{ \frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_N(y)\varphi_{N-1}(x)}{x - y} \right\} \\ &\approx \frac{(2N)^{1/2} \sin \theta}{\pi} \left\{ \frac{\sin(2N\delta\theta \sin^2 \theta)}{2N\delta\theta \sin^2 \theta} \right\}. \end{aligned} \quad (5.1)$$

Here the second form is obtained by the Christoffel–Darboux formula and the last form, valid for small $\delta\theta$, by using the asymptotic formula [18] for φ_N ,

$$\varphi_N(x) = \left(\frac{2}{N\pi^2 \sin^2 \theta} \right)^{1/4} \left[\sin \left\{ \left(\frac{2N+1}{4} \right) (2\theta - \sin 2\theta) - \left(\frac{4N-1}{4} \right) \pi \right\} + O(N^{-1}) \right], \quad (5.2)$$

where

$$\theta \equiv \theta(x) = \cos^{-1} \left(-\frac{x}{\sqrt{2N+1}} \right), \quad 0 < \theta < \pi, \quad (5.3)$$

and

$$\delta\theta = \theta(x) - \theta(y) \approx (x - y)/(2N)^{1/2} \sin \theta. \quad (5.4)$$

Thus for the asymptotic level density, we obtain from (4.52, 5.1)

$$R_1(x) \approx \frac{(2N)^{1/2} \sin \theta}{\pi} \approx \pi^{-1} (2N - x^2)^{1/2}, \quad (5.5)$$

the famous semicircle of Wigner. Also, as $N \rightarrow \infty$, $(x - y) \rightarrow 0$ while

$$r = (x - y)R_1(x) \approx \frac{2N \delta\theta \sin^2 \theta}{\pi} \quad (5.6)$$

remains finite, we have from (5.1)

$$S_N(x, y) \rightarrow R_1(x) \frac{\sin \pi r}{\pi r}. \quad (5.7)$$

For non-zero finite α , we see from Eqs. (4.14, 4.19, 4.20, 4.65) that

$$D_N(x, y) \rightarrow \infty, \quad J_N(x, y) \rightarrow 0, \quad (5.8)$$

and

$$D_N(x, y) J_N(x, y) \rightarrow 0, \quad (5.9)$$

implying that in the asymptotic- N limit the n -level correlation and cluster functions for $n > 1$ are discontinuous functions of α . For $\alpha = 0$ we have the GOE results and for $\alpha > 0$ the GUE results. However, D_N and J_N have finite limits when $\alpha \rightarrow 0$ and

$$\lambda \equiv \lambda(x) = \frac{(\text{Im } H_{ij})^{\text{RMS}}}{\text{Local Average Spacing at } x} \approx \frac{\alpha}{\sqrt{2}} R_1(x) \quad (5.10)$$

is finite [8]. These limits can be obtained by taking the Fourier transform of Eqs. (4.32, 4.33), using (5.7) for S_N . Using the notation

$$\mathcal{F}(h(x)) \equiv \int_{-\infty}^{\infty} e^{2\pi i k x} h(x) dx \quad (5.11)$$

for the Fourier transform of a function $h(x)$, we have [19] for $b > 0$

$$\mathcal{F}(\text{erf}(bx)) = 2i \int_0^{\infty} \sin(2\pi k x) \text{erf}(bx) dx = \frac{i}{\pi k} \exp(-\pi^2 k^2 / b^2), \quad (5.12)$$

and

$$\mathcal{F}\left(\frac{\sin bx}{x}\right) = 2 \int_0^{\infty} \cos(2\pi k x) \frac{\sin bx}{x} dx = \begin{cases} \pi, & \text{if } 2\pi|k| < b, \\ \pi/2, & \text{if } 2\pi|k| = b, \\ 0, & \text{if } 2\pi|k| > b, \end{cases} \quad (5.13)$$

which, along with (4.32), give for the Fourier transform of I_N .

$$\mathcal{F}(I_N) \approx \frac{i}{2\pi k} \exp(-8\pi^2 \lambda^2 k^2) \begin{cases} 1, & \text{if } |k| < 1/2, \\ 1/2, & \text{if } |k| = 1/2, \\ 0, & \text{if } |k| > 1/2. \end{cases} \quad (5.14)$$

An inverse Fourier transform then gives

$$I_N(x, y) \approx -\frac{1}{\pi} \int_0^r dt \int_0^{\pi} dk \cos kt e^{-2\lambda^2 k^2} = -\frac{1}{\pi} \int_0^{\pi} dk \frac{\sin kr}{k} e^{-2\lambda^2 k^2}, \quad (5.15)$$

so that

$$J_N(x, y) \approx J(r; \lambda) = -\frac{1}{\pi} \int_0^r dt \int_{\pi}^{\infty} dk \cos kt e^{-2\lambda^2 k^2} = -\frac{1}{\pi} \int_{\pi}^{\infty} dk \frac{\sin kr}{k} e^{-2\lambda^2 k^2}. \quad (5.16)$$

Similarly from (4.33) we have

$$D_N(x, y) \approx (R_1(x))^2 D(r; \lambda), \quad (5.17)$$

$$D(r; \lambda) = \frac{d}{dr} \left\{ \frac{1}{\pi} \int_0^\pi dk \cos kr e^{2\lambda^2 k^2} \right\} = -\frac{1}{\pi} \int_0^\pi dk k \sin kr e^{2\lambda^2 k^2}. \quad (5.18)$$

Using the new variables

$$r_i = \int_{-\infty}^{x_i} R_1(x) dx, \quad (5.19)$$

we get, from (4.49–4.51, 4.55, 5.7, 5.16–5.18), for the n -level correlation function

$$\begin{aligned} \mathcal{R}_n(r_1, r_2, \dots, r_n; \lambda) &\equiv \lim_{N \rightarrow \infty} \{R_1(x_1)R_1(x_2) \cdots R_1(x_n)\}^{-1} R_n(x_1, x_2, \dots, x_n) \\ &= Q \det [\Phi(r_i - r_j; \lambda)]_{i,j=1,2,\dots,n}, \end{aligned} \quad (5.20)$$

and for the n -level cluster function

$$\begin{aligned} Y_n(r_1, r_2, \dots, r_n; \lambda) &\equiv \lim_{N \rightarrow \infty} \{R_1(x_1)R_1(x_2) \cdots R_1(x_n)\}^{-1} T_n(x_1, x_2, \dots, x_n) \\ &= \sum_P \Phi(r_1 - r_2; \lambda) \Phi(r_2 - r_3; \lambda) \cdots \Phi(r_n - r_1; \lambda), \end{aligned} \quad (5.21)$$

where

$$\Phi(r; \lambda) = \begin{bmatrix} \frac{\sin \pi r}{\pi r} & D(r; \lambda) \\ J(r; \lambda) & \frac{\sin \pi r}{\pi r} \end{bmatrix}, \quad (5.22)$$

with $J(r)$ and $D(r)$ given above, Eqs. (5.16, 5.18).

The transition parameter λ , Eq. (5.1), varies over the spectrum because of its dependence on x . As a consequence, the fluctuation properties, described by the set of \mathcal{R}_n or Y_n functions for $n > 1$, are non-stationary, the exceptions [17] being the two limiting cases. In particular, the $\text{GOE}(\lambda = 0) \rightarrow \text{GUE}(\lambda = \infty)$ transition then is seen to be faster in the middle of the semicircle than at the ends [8].

6. Two-Point Fluctuation Measures

The two-level cluster function, namely

$$Y_2(r; \lambda) = \left(\frac{\sin \pi r}{\pi r} \right)^2 - J(r; \lambda) D(r; \lambda) = Y_2(r; \infty) - J(r; \lambda) D(r; \lambda), \quad (6.1)$$

is of particular interest in the analysis of data. Its asymptotic expansion can be derived by partial integration in (5.16, 5.18). We have for $|r| \gtrsim 1$

$$Y_2(r; \lambda) - Y_2(r; \infty) = \frac{r^2 \cos^2 \pi r - 16\pi^2 \lambda^4 \sin^2 \pi r}{\pi^2 (r^2 + 16\pi^2 \lambda^4)^2} + \dots \quad (6.2)$$

valid for any value of λ whether large or small compared to r (see appendix III). For the Fourier transform,

$$b(k; \lambda) = \int_{-\infty}^{\infty} e^{2\pi i k r} Y_2(r; \lambda) dr, \quad (6.3)$$

we have [20]

$$b(k; \infty) = \int_{-\infty}^{\infty} \left(\frac{\sin \pi r}{\pi r} \right)^2 e^{2\pi i k r} dr = \begin{cases} 1 - |k|, & |k| \leq 1, \\ 0, & |k| \geq 1, \end{cases} \quad (6.4)$$

and

$$\begin{aligned} b(k; \lambda) - b(k; \infty) &= - \int_{-\infty}^{\infty} J(r; \lambda) D(r; \lambda) \exp(2\pi i k r) dr \\ &= - \int_{-\infty}^{\infty} dr \int_1^{\infty} dk_1 \int_0^1 dk_2 \frac{k_2}{k_1} \sin(\pi k_1 r) \sin(\pi k_2 r) \exp(2\pi i k r) \\ &\quad \times \exp[-2\pi^2 \lambda^2 (k_1^2 - k_2^2)] \\ &= \frac{1}{4} \int_0^{\infty} dk_1 \int_{-1}^1 dk_2 \frac{k_2}{k_1} \delta\left(|k| - \frac{k_1 + k_2}{2}\right) \exp[-2\pi^2 \lambda^2 (k_1^2 - k_2^2)]. \end{aligned} \quad (6.5)$$

Integrating over k_1 we get

$$\begin{aligned} b(k; \lambda) - b(k; \infty) &= \begin{cases} -\frac{1}{2}|k| \exp(-8\pi^2 \lambda^2 |k|) \int_{-1}^1 \frac{t - 1 + 1/|k|}{t + 1 + 1/k} \exp(-8\pi^2 \lambda^2 k^2 t) dt, & |k| \leq 1, \\ -\frac{1}{2} \exp(-8\pi^2 \lambda^2 k^2) \int_{-1}^1 dt \frac{t}{t + 2|k|} \exp(-8\pi^2 \lambda^2 |k| t), & |k| \geq 1, \end{cases} \end{aligned} \quad (6.6)$$

while integrating over k_2 we get a different form,

$$b(k; \lambda) - b(k; \infty) = -k e^{-8\pi^2 \lambda^2 |k|} \{1 + h(k; \lambda)\}, \quad (6.7)$$

where

$$1 + h(k; \lambda) = \begin{cases} \frac{sh(8\pi^2 \lambda^2 k^2)}{8\pi^2 \lambda^2 k^2} + E(1, 2|k| + 1), & |k| \leq 1, \\ \frac{sh(8\pi^2 \lambda^2 |k|)}{8\pi^2 \lambda^2 k^2} e^{8\pi^2 \lambda^2 (|k| - k^2)} + E(2|k| - 1, 2|k| + 1), & |k| \geq 1, \end{cases} \quad (6.8)$$

and

$$E(x, y) = \exp(8\pi^2 \lambda^2 (k^2 + |k|)) \int_y^x \exp(-8\pi^2 \lambda^2 |k| t) \frac{dt}{t}. \quad (6.9)$$

which can of course be expressed in terms of the standard E_1 functions.

A quantity often used to measure the rigidity of the spectrum is the number variance $\Sigma^2(\bar{n})$, namely the variance of the number of levels in a given interval containing on the average \bar{n} levels [21, 22]. We have in the present case

$$\Sigma^2(\bar{n}; \lambda) = 2 \int_0^{\bar{n}} (\bar{n} - r) (\delta(r) - Y_2(r; \lambda)) dr$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\sin^2 \pi \bar{n} t}{\pi^2 t^2} (1 - b(t; \lambda)) dt \\
&= \frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1 - \cos \bar{n} t}{t} \right) \left(\frac{1 - b(t/2\pi; \lambda)}{(t/2\pi)} \right) dt \\
&= \Sigma^2(\bar{n}; \infty) + [\Sigma^2(\bar{n}; \lambda) - \Sigma^2(\bar{n}; \infty)]. \tag{6.10}
\end{aligned}$$

Here in the formal decomposition in the last form, the λ -independent term, namely the GUE result [22], is

$$\begin{aligned}
\Sigma^2(\bar{n}; \infty) &= \frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1 - \cos \bar{n} t}{t} \right) \left(\frac{1 - b(t/2\pi; \infty)}{(t/2\pi)} \right) dt \\
&= \frac{1}{\pi^2} \int_0^{2\pi} \left(\frac{1 - \cos \bar{n} t}{t} \right) dt + \frac{1}{\pi^2} \int_{2\pi}^{\infty} \left(\frac{2\pi}{t} \right) \left(\frac{1 - \cos \bar{n} t}{t} \right) dt \\
&= \frac{1}{\pi^2} \left[\log(2\pi \bar{n}) + \gamma + 1 - \text{Ci}(2\pi \bar{n}) + 2\pi \bar{n} \left(\frac{\pi}{2} - \text{Si}(2\pi \bar{n}) \right) - \cos 2\pi \bar{n} \right] \\
&\rightarrow \frac{1}{\pi^2} [\log(2\pi \bar{n}) + \gamma + 1], \tag{6.11}
\end{aligned}$$

whereas the λ -dependent term is

$$\begin{aligned}
\Sigma^2(\bar{n}; \lambda) - \Sigma^2(\bar{n}; \infty) &= \frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1 - \cos \bar{n} t}{t} \right) \left(\frac{b(t/2\pi; \infty) - b(t/2\pi; \lambda)}{t/2\pi} \right) dt \\
&= \frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1 - \cos \bar{n} t}{t} \right) e^{-4\pi \lambda^2 t} (1 + h(t/2\pi; \lambda)) dt \\
&= \frac{1}{2\pi^2} \log \left(\frac{\bar{n}^2 + 16\pi^2 \lambda^4}{16\pi^2 \lambda^4} \right) + \xi(\lambda) + \rho(\bar{n}; \lambda) \\
&\rightarrow \frac{1}{2\pi^2} \log \left(\frac{\bar{n}^2 + 16\pi^2 \lambda^4}{16\pi^2 \lambda^4} \right) + \xi(\lambda). \tag{6.12}
\end{aligned}$$

The final asymptotic forms in (6.11, 6.12) are valid, with good approximations as in (6.2), for $\bar{n} \gtrsim 1$ and λ of any size compared to \bar{n} . The functions $\xi(\lambda)$ and $\rho(\bar{n}; \lambda)$ are defined by

$$\xi(\lambda) = \frac{1}{\pi^2} \int_0^{\infty} e^{-4\pi \lambda^2 t} h(t/2\pi; \lambda) \frac{dt}{t} \tag{6.13}$$

and

$$\rho(\bar{n}; \lambda) = -\frac{1}{\pi^2} \int_0^{\infty} e^{-4\pi \lambda^2 t} \cos(\bar{n} t) h(t/2\pi; \lambda) \frac{dt}{t} \tag{6.14}$$

$$= -\frac{4\lambda^2}{\bar{n}^2 + 16\pi^2 \lambda^4} + \dots \tag{6.15}$$

For small λ we can use an expansion [23] of the E -function of (6.9). After some tedious calculations we get

$$\xi(\lambda) = \frac{1}{\pi^2} \log 4\pi\lambda^2 + \frac{1}{\pi^2} \left(\log 2\pi + \gamma + 1 - \frac{\pi^2}{4} \right) - 4\lambda^2 + \dots \quad (6.16)$$

which in (6.10), along with (6.11, 6.12), reproduces for $\lambda = 0$ the well-known GOE result [21]. We mention moreover that the final result in (6.9) has also been derived in [8] but without an exact form for $\xi(\lambda)$.

Another commonly-used fluctuation measure is the Δ_3 -statistic which for a given \bar{n} measures the least square deviation of the staircase energy function from the best straight line fitting it [21, 22]. Its ensemble average, related to Σ^2 [24], is given by

$$\begin{aligned} \bar{\Delta}_3(\bar{n}; \lambda) &= \frac{2}{\bar{n}^4} \int_0^{\bar{n}} (\bar{n}^3 - 2\bar{n}^2 r + r^3) \Sigma^2(r; \lambda) dr \\ &\rightarrow \frac{1}{2} \Sigma^2(\bar{n}; \lambda) - \frac{9}{4\pi^2} + \frac{4\lambda^4}{\bar{n}^2} \left[1 + \frac{2\bar{n}}{\pi\lambda^2} \tan^{-1} \left(\frac{\bar{n}}{4\pi\lambda^2} \right) \right. \\ &\quad \left. - 4 \left(1 + \frac{4\pi^2\lambda^2}{\bar{n}^2} \right) \log \left(\frac{\bar{n}^2 + 16\pi^2\lambda^4}{16\pi^2\lambda^4} \right) \right], \end{aligned} \quad (6.17)$$

where the last term in the last form goes from zero to $9/8\pi^2$ as λ increases from zero to infinity, agreeing with the known limiting results [21, 22]. As indicated by the numerical integrations [5] for GOE and GUE, the term ignored, namely the integral involving $\rho(\bar{n}; \lambda)$, may not however be very small for $\bar{n} \lesssim 10$.

Appendix I. An Alternative Derivation of the Correlation and Cluster Functions

An alternative way to derive the n -level functions is to consider the integral

$$L_N(u) = \int_{-\infty}^{\infty} \cdots \int p(x_1, x_2, \dots, x_N) \prod_{i=1}^N u(x_i) dx_i \quad (I.1)$$

for an arbitrary function $u(x)$. This integral can be evaluated by expanding the pfaffian in (3.11) in terms of its elements and then integrating term by term by the method of integration over alternate variables as in [13]. We omit the details. The result for $N = 2m$ is

$$L_N(u) = [\det \eta]^{1/2}, \quad (I.2)$$

where η is an $N \times N$ antisymmetric matrix with elements ($i, j = 0, 1, \dots, m-1$)

$$\eta_{2i, 2j} = 2 \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{i+j} \int_{-\infty}^{\infty} \int dx dy \varphi_{2i}(x) \varphi_{2j}(y) g(x-y) u(x) u(y), \quad (I.3)$$

$$\eta_{2i, 2j+1} = -\eta_{2j+1, 2i} = \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{i-j} \int_{-\infty}^{\infty} \int dx dy \varphi_{2i}(x) \psi_{2j}(y) g(x-y) u(x) u(y), \quad (I.4)$$

$$\eta_{2i+1, 2j+1} = \frac{1}{2} \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{-i-j} \int_{-\infty}^{\infty} \int dx dy \psi_{2i}(x) \psi_{2j}(y) g(x-y) u(x) u(y). \quad (I.5)$$

The correlation functions now follow as functional derivatives of $L_N(u)$ with respect to u at $u = 1$ [13]. In particular the one and two level functions, Eqs. (4.52, 4.54), were derived in this way which facilitated the task of guessing the $N \times N$ quaternion matrix $\sigma^{(N)}$ in (4.12).

Appendix II. Proof of Equations (4.30, 4.31)

Consider the integral

$$\begin{aligned} & \int_{-\infty}^{\infty} \exp \left\{ (2yz - z^2) - \frac{1}{2}(1 + \alpha^2)y^2 - \left(\frac{1 - \alpha^4}{4\alpha^2} \right) (x - y)^2 \right\} dy \\ &= \frac{2\alpha\sqrt{\pi}}{1 + \alpha^2} \exp \left\{ - \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right)^2 z^2 + 2 \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right) xz - \frac{1}{2}(1 - \alpha^2)x^2 \right\}. \end{aligned} \quad (\text{II.1})$$

Expanding both sides of this equation in powers of z , for which we use the generating function for the Hermite polynomials [25]

$$\sum_{j=0}^{\infty} \frac{H_j(x)}{j!} z^j = e^{2xz - z^2}, \quad (\text{II.2})$$

and using the definition (4.18), we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \varphi_j(y) \exp \left\{ -\frac{1}{2}\alpha^2 y^2 - \left(\frac{1 - \alpha^4}{4\alpha^2} \right) (x - y)^2 \right\} dy \\ &= \left(\frac{2\alpha\sqrt{\pi}}{1 + \alpha^2} \right) \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right)^j e^{(1/2)\alpha^2 x^2} \varphi_j(x). \end{aligned} \quad (\text{II.3})$$

Then a partial integration along with the definitions (4.17, 4.19) yields (4.30).

For (4.31), it is convenient to write it as

$$\frac{1}{2} \left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{j + (1/2)} \int_{-\infty}^{\infty} f(x - y) e^{-(1/2)\alpha^2 y^2} \varphi_j(y) dy = \int_{-\infty}^{\infty} \epsilon(x - y) e^{(1/2)\alpha^2 y^2} \varphi_j(y) dy, \quad (\text{II.4})$$

where we have used the definition (4.20). Differentiating both sides of this relation with respect to x , we obtain (II.3). Therefore the relation is true if it is valid for one value of x . At $x = \infty$, $f(x - y) = 2\epsilon(x - y) = 1$, so that we have to prove that

$$\left(\frac{1 + \alpha^2}{1 - \alpha^2} \right)^{j + (1/2)} \int_{-\infty}^{\infty} e^{-(1/2)\alpha^2 y^2} \varphi_j(y) dy = \int_{-\infty}^{\infty} e^{(1/2)\alpha^2 y^2} \varphi_j(y) dy. \quad (\text{II.5})$$

When j is odd, the truth of this equation is evident. For even j , we consider the integral

$$\int_{-\infty}^{\infty} \exp \left\{ (2yz - z^2) - \frac{1}{2}(1 \pm \alpha^2)y^2 \right\} dy = \left(\frac{2\pi}{1 \pm \alpha^2} \right)^{1/2} \exp \left\{ \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right)^{\pm 1} z^2 \right\}, \quad (\text{II.6})$$

which along with (II.2) yields

$$\int_{-\infty}^{\infty} e^{\pm(1/2)\alpha^2 y^2} \varphi_{2j}(y) dy = \left(\frac{2\pi}{1 \pm \alpha^2} \right)^{1/2} (2^{2j}(2j)! \sqrt{\pi})^{-(1/2)} \frac{(2j)!}{j!} \left(\frac{1 - \alpha^2}{1 + \alpha^2} \right)^{\pm j}, \quad (\text{II.7})$$

verifying (II.5). This completes the proof of (4.31).

Appendix III

Asymptotic forms of $J(r; \lambda)$ and $D(r; \lambda)$.

By partial integration

$$\begin{aligned} \int_0^{\pi} \frac{\sin kr}{k} \exp(-2\lambda^2 k^2) dk &= \exp(-r^2/8\lambda^2) \operatorname{Im} \int_{\pi}^{\infty} \exp \left\{ -2\lambda^2 \left(k - \frac{ir}{4\lambda^2} \right)^2 \right\} \frac{dk}{k} \\ &= \exp(-r^2/8\lambda^2) \operatorname{Im} \left[\exp \left\{ -2\lambda^2 \left(k - \frac{ir}{4\lambda^2} \right)^2 \right\} \right. \\ &\quad \times \left. \left\{ -\frac{1}{4\lambda^2 k \left(k - \frac{ir}{4\lambda^2} \right)} + \dots \right\} \right]_{\pi}^{\infty} \\ &= \exp(-2\lambda^2 \pi^2) \operatorname{Im} e^{inr} \left\{ \frac{-1}{\pi(4\pi\lambda^2 - ir)} + \dots \right\} \\ &= \frac{1}{\pi} \exp(-2\lambda^2 \pi^2) \left\{ \frac{r \cos \pi r + 4\pi\lambda^2 \sin \pi r}{16\pi^2 \lambda^4 + r^2} + \dots \right\}. \quad (\text{III.1}) \end{aligned}$$

Similarly

$$\begin{aligned} &\int_0^{\infty} k \sin kr \exp(2\lambda^2 k^2) dk \\ &= \pi \exp(2\lambda^2 \pi^2) \left\{ \frac{r \cos \pi r - 4\pi\lambda^2 \sin \pi r}{16\pi^2 \lambda^4 + r^2} + \dots \right\}. \quad (\text{III.2}) \end{aligned}$$

Eqs. (5.16, 5.18, 6.1) then give (6.2).

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Note added in proof. The ensemble of Hermitian matrices H with $\| \text{Im} H \|^2 > \| \text{Re} H \|^2$, corresponding to $\alpha^2 > 1$, has now been studied. We find the same joint probability density for the eigenvalues as in (3.11), and hence all the results which follow from it.