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Existence and Uniqueness for Random One-Dimensional Lattice Systems

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Abstract. Existence and uniqueness are shown for the fixed point problem pertinent to hopping transport in one-dimension with random transfer rates. Continuity properties of the solution are exhibited. The connection with Dyson's treatment of the linear harmonic chain with random masses is established.

1. Introduction

Diffusion or hopping transport on the one-dimensional lattice $\mathbb{Z}$ is described by the master equation

$$\dot{P}_n = W_{n-1}(P_{n-1} - P_n) + W_n(P_{n+1} - P_n), \tag{1.1}$$

where $P_n(t)$ is the probability of finding a particle at time t on the lattice site n. Randomness is introduced by assuming the transfer rates W_n, $n \in \mathbb{Z}$, to be independent $\mathbb{R}_+$-valued random variables, equally distributed according to a probability measure ν. Thus, one is lead to consider expectations

$$E(f) = \int \prod_{n \in \mathbb{Z}} d\nu(w_n) f(\{w_n\}) \tag{1.2}$$

of measurable functions f on $\mathbb{R}_+^{\mathbb{Z}}$. In [1] it has been shown (by supplementing (1.1) with the initial condition $P_n(0) = \delta_{n0}$), that

$$E(\tilde{P}_0(s)) = \int_0^\infty dt e^{-st} E(P_0(t)) \tag{1.3}$$

is given by

$$E(\tilde{P}_0(s)) = \iint_{\mathbb{R}_+^2} d\mu_s(x) d\mu_s(y) (x + y + s)^{-1} \tag{1.4}$$

for $s \geqq 0$. Here, μ_s, $s \in \mathbb{R}_+$, is a probability measure on $\mathbb{R}_+$ satisfying the integral equation

$$\mu_s([0, x)) = \iint_{A_{s,x}} d\nu(y) d\mu_s(z), \quad x > 0, \tag{1.5}$$

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with $A_{s,x} \subset \mathbb{R}_+^2$ given by

$$A_{s,x} = \{(y,z) \in \mathbb{R}_+^2 \,|\, [y^{-1} + (z+s)^{-1}]^{-1} < x\}. \tag{1.6}$$

In Sect. 2 it is shown that (1.5) has at most one solution. Section 3 is devoted to the existence of a solution; the solution is actually "constructed." In Sect. 4 it is shown that the map $s \to \mu_s$ is vaguely continuous. The connection with the work of Dyson [2] on the linear harmonic chain with random masses is established in Sect. 5. More detailed properties of μ_s and quantities derived thereof have been treated elsewhere [3], [4]; applications are discussed in [5].

2. Uniqueness

Let $\mathscr{P}$ be the set of (regular Borel) probability measures on $\mathbb{R}$, and $\mathscr{D}$ the set of distribution functions, i.e. the set of functions $f: \mathbb{R} \to [0,1]$ which are isotonic, left-continuous and $f(x) - f(-x) \to 1$ as $x \to \infty$. Denote by J the canonical bijection of $\mathscr{P}$ onto $\mathscr{D}$:

$$(J\mu)(x) = \mu((-\infty, x)). \tag{2.1}$$

Let $\mathscr{P}_+ \subset \mathscr{P}$ be the set of probability measures with support in $\mathbb{R}_+$ and $\mathscr{D}_+ \subset \mathscr{D}$ the set of distribution functions f with $f(x) = 0$, $x \leqq 0$. Obviously, J maps $\mathscr{P}_+$ bijectively onto $\mathscr{D}_+$.

Let $\nu \in \mathscr{P}_+$ be fixed and $s \in \mathbb{R}_+ \cup \{\infty\}$. Define the map $T_s: \mathscr{P}_+ \to \mathscr{P}_+$ by

$$(JT_s\mu)(x) = \iint_{A_{s,x}} d\nu(y) d\mu(z), \tag{2.2}$$

with $A_{s,x} \subset \mathbb{R}_+^2$ given by (1.6). By definition, each fixed point of T_s is a solution of the integral equation (1.5) and vice versa.

As

$$A_{\infty,x} = [0,x) \times \mathbb{R}_+, \tag{2.3}$$

(2.2) yields immediately

$$T_\infty \mu = \nu \tag{2.4}$$

for all $\mu \in \mathscr{P}_+$, i.e. T_∞ has the unique fixed point $\mu_\infty = \nu$. For $s \in \mathbb{R}_+$ the following decomposition of $A_{s,x}$ into disjoint subsets holds

$$A_{s,x} = A_{\infty,x} \cup B_{s,x}. \tag{2.5}$$

The set $B_{s,x} \subset \mathbb{R}_+^2$ is given by

$$B_{s,x} = \{(y,z) \,|\, z < \frac{yx}{y-x} - s, \quad x \leqq y < \phi_s(x)\}, \tag{2.6}$$

where

$$\phi_s(x) = \begin{cases} sx/(s-x), & x < s \\ \infty \quad , & x \geqq s \end{cases}. \tag{2.7}$$

Obviously,

$$B_{t,x} \subsetneqq B_{s,x}, \quad s < t. \tag{2.8}$$

The decomposition (2.5) may easily be read off from the graph of $\psi_s: \mathbb{R}_+^2 \to \mathbb{R}_+$ given by

$$\psi_s(y,z) = [y^{-1} + (z+s)^{-1}]^{-1}. \tag{2.9}$$

From (2.2), (2.5), (2.6) it follows that

$$JT_s\mu = J\nu + K_s J\mu, \tag{2.10}$$

where

$$(K_s J_\mu)(x) = \iint_{B_{s,x}} d\nu(y) d\mu(z), \tag{2.11}$$

or, applying Fubini's theorem,

$$(K_s f)(x) = \int_x^{\phi_s(x)} d\nu(y) f\left(\frac{yx}{y-x} - s\right) \tag{2.12}$$

for $f \in \mathscr{D}_+$. Note that K_s does not depend on $\nu(\{0\})$. In view of (2.8), the inequality

$$K_t f \leqq K_s f, \quad s < t \tag{2.13}$$

holds. The operator K_s has an immediate extension from its "natural" domain $\mathscr{D}_+$ to D, the linear span of $\mathscr{D}_+$. A further extension is obtained by introducing the Banach space

$$\mathscr{B}_\alpha = L^1(\mathbb{R}_+, \rho_\alpha), \quad \frac{d\rho_\alpha}{dx} = \alpha(1+x)^{-1-\alpha}, \quad 0 < \alpha < 1, \tag{2.14}$$

with norm

$$\|f\|_\alpha = \alpha \int_0^\infty dx (1+x)^{-1-\alpha} |f(x)|. \tag{2.15}$$

Using Fubini's theorem and a change of variable

$$x \to z = \frac{yx}{y-x} - s \tag{2.16}$$

yields

$$\|K_s f\|_\alpha \leqq \alpha \int_0^\infty d\nu(y) \int_0^\infty dz\, k_s(y,z) |f(z)|, \tag{2.17}$$

with equality holding for $f \geqq 0$, and

$$k_s(y,z) = y^2(s+y+z)^{-2}\left(1 + \frac{y(s+z)}{s+y+z}\right)^{-1-\alpha} \tag{2.18}$$

The estimates

$$k_s(y,z) < \left(\frac{y}{y+s}\right)^{1-\alpha}(1+z)^{-1-\alpha} \leqq (1+z)^{-1-\alpha}, \tag{2.19}$$

which hold for $y>0, z>0$, lead to

$$\| K_s f \|_\alpha < \| f \|_\alpha \int_0^\infty dv(y) \left(\frac{y}{y+s}\right)^{1-\alpha}, \quad f \neq 0, \tag{2.20}$$

and

$$\| K_s f \|_\alpha < \| f \|_\alpha, \quad f \neq 0. \tag{2.21}$$

By Lebesgue's dominated convergence theorem, (2.20) yields

$$\lim_{s \to \infty} \| K_s f \|_\alpha = 0, \tag{2.22}$$

i.e. K_s is strongly continuous at infinity.

As a consequence of (2.21), the equation

$$f = g + K_s f \tag{2.23}$$

has at most one solution $f \in \mathscr{B}_\alpha$ for any $g \in \mathscr{B}_\alpha$, $g \neq 0$. Thus, in view of (2.10), the following uniqueness theorem holds.

Theorem 2.1. *The map $T_s : \mathscr{P}_+ \to \mathscr{P}_+$, defined by (2.2), has at most one fixed point.*

3. Existence

The functions

$$f_s^{(n)} = \sum_{m=0}^{n} K_s^m J v, \quad n \geqq 0 \tag{3.1}$$

belong to $\mathscr{D}_+$, by induction, as

$$f_s^{(n)} = Jv + K_s f_s^{(n-1)}, \tag{3.2}$$

and, from (2.10), with $f_s^{(n-1)} \in \mathscr{D}_+$ also

$$f_s^{(n)} = J T_s J^{-1} f_s^{(n-1)} \tag{3.3}$$

is in $\mathscr{D}_+$. Furthermore, (3.1) yields

$$f_s^{(n)} = f_s^{(n-1)} + K_s^n J v. \tag{3.4}$$

Hence,

$$0 \leqq f_s^{(0)} \leqq f_s^{(1)} \leqq \ldots \leqq f_s^{(n)} \leqq \ldots \leqq 1, \tag{3.5}$$

as K_s is positivity preserving. Consequently,

$$\lim_{n \to \infty} f_s^{(n)} = f_s \tag{3.6}$$

exists pointwise. As each $f_s^{(n)}$, $n \geqq 0$, is isotonic, also f_s is isotonic. Thus, the limits

$$\lim_{y \uparrow x} f_s(y) = f_s(x_-), \quad \lim_{y \downarrow x} f_s(y) = f_s(x_+) \tag{3.7}$$

exist. Assume f_s not to be left-continuous, i.e.

$$f_s(x) - f_s(x_-) = a > 0. \tag{3.8}$$

For n sufficiently large (say $n > N$)

$$0 \leqq f_s(x) - f_s^{(n)}(x) < a/2, \quad n > N, \tag{3.9}$$

and for $y < x$

$$0 \leqq f_s^{(n)}(y) \leqq f_s(y) \leqq f_s(x_-), \tag{3.10}$$

i.e.

$$f_s^{(n)}(x) - f_s^{(n)}(y) > a/2, \quad n > N. \tag{3.11}$$

Taking the limit $y \uparrow x$ yields

$$f_s^{(n)}(x) - f_s^{(n)}(x_-) \geqq a/2, \quad n > N. \tag{3.12}$$

This contradicts the left-continuity of $f_s^{(n)}$, $n \geqq 0$. Finally (3.5) yields

$$\lim_{x \to \infty} f_s(x) = 1, \tag{3.13}$$

and

$$f_s^{(n)} = 0, \quad x \leqq 0, \quad n \geqq 1 \tag{3.14}$$

yields

$$f_s(x) = 0, \quad x \leqq 0. \tag{3.15}$$

Hence, f is in $\mathscr{D}_+$.

Furthermore, by Lebesgue's dominated convergence theorem

$$\lim_{n \to \infty} \| f_s - f_s^{(n)} \|_\alpha = 0. \tag{3.16}$$

As K_s is bounded, (3.2) combined with (3.16) leads to

$$f_s = Jv + K_s f_s \tag{3.17}$$

or, with (2.10)

$$f_s = J T_s J^{-1} f_s. \tag{3.18}$$

Hence, the following theorem holds.

Theorem 3.1. *The sequence*

$$J T_s^n v = \sum_{m=0}^{n} K_s^m J v \tag{3.19}$$

is in $\mathscr{D}_+$. It converges pointwise and in $\mathscr{B}_\alpha$-norm. Its limit, f_s, defines a probability measure $\mu_s = J^{-1} f_s$ which is a fixed point of T_s.

Remark 1. As $K_\infty = 0$, (3.1) reduces to $f_\infty^{(n)} = Jv$, $n \geqq 1$, i.e. $f_\infty = Jv$, in accordance with Sect. 2.

Remark 2. Let $f_0 \in \mathscr{D}_+$ be given by

$$f_0(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 \end{cases}. \tag{3.20}$$

Applying K_0 to f_0 yields, according to (2.12) and (2.7),

$$K_0 f_0 = f_0 - Jv, \tag{3.21}$$

i.e. $J^{-1} f_0 = \delta_0$ (Dirac measure) is fixed point of T_0.

Remark 3. The ordered case of (1.1) is characterized by $W_n = w$, $w \geqq 0$, $n \in \mathbb{Z}$. This is equivalent to $v = \delta_w$. For $w = 0$, (2.2) yields

$$J T_s \mu = f_0 \tag{3.22}$$

for arbitrary $\mu \in \mathscr{P}_+$, i.e. δ_0 is fixed point of T_s, $s \in \mathbb{R}_+ \cup \{\infty\}$. For $w > 0$, the point $(w, a(s))$ with

$$a(s) = \tfrac{1}{2}[(4ws + s^2)^{1/2} - s], \quad s \in \mathbb{R}_+, \tag{3.23}$$

and

$$a(\infty) = \lim_{s \to \infty} a(s) = w \tag{3.24}$$

is mapped onto itself by ψ_s defined in (2.8). Hence, $\mu_s = \delta_{a(s)}$ is fixed point of T_s, $s \in \mathbb{R}_+ \cup \{\infty\}$.

Remark 4. Replacing $A_{s,x}$ in (2.2) by its closure and taking the limit $x \to 0$ yields $(T_s \mu)(\{0\}) = v(\{0\})$. In particular,

$$\mu_s(\{0\}) = v(\{0\}). \tag{3.25}$$

4. Continuity Properties

In this section continuity properties of K_s and μ_s are discussed. Let $\mathscr{C}(\mathscr{B}_\alpha)$ be the set of bounded operators on $\mathscr{B}_\alpha$.

Theorem 4.1. *The map $s \to K_s$ from $\mathbb{R}_+ \cup \{\infty\}$ to $\mathscr{C}(\mathscr{B}_\alpha)$, defined by (2.10) and (2.7), is strongly continuous.*

Proof. As $K_\infty = 0$, strong continuity at ∞ is equivalent to (2.22). Let $0 \leqq s < t < \infty$ and $f \in \mathscr{B}_\alpha$, $f \geqq 0$. Then

$$\| K_t f - K_s f \|_\alpha = \alpha \int_0^\infty dv(y) \int_0^\infty dz [k_s(y,z) - k_t(y,z)] f(z), \tag{4.1}$$

with k_s given by (2.18). In view of the estimates (2.19) Lebesgue's dominated convergence theorem is applicable yielding

$$\lim_{s \to t} \| K_t f - K_s f \|_\alpha = 0. \tag{4.2}$$

The extension to arbitrary $f \in \mathscr{B}_\alpha$ is trivial as K_s is positivity preserving.

Theorem 4.2. *The map*

$$s \to f_s = \sum_{m=0}^{\infty} K_s^m J v \tag{4.3}$$

from $\mathbb{R}_+ \cup \{\infty\}$ *to* $\mathscr{B}_\alpha$ *is continuous.*

Proof. By Theorem 4.1, $K_s^m Jv$ is continuous. Hence, $f_s^{(n)}, n \geqq 1$, given by (3.1), is continuous. From

$$f_s^{(n)} \leqq f_0^{(n)}, \; n \geqq 1, \tag{4.4}$$

shown below by induction, it follows that $f_s^{(n)}$ converges uniformly to f_s. Hence, f_s is continuous. For $n = 1$, (3.2) and (2.13) yield

$$f_s^{(1)} = Jv + K_s Jv \leqq Jv + K_0 Jv = f_0^{(1)}. \tag{4.5}$$

Assume $f_s^{(n-1)} \leqq f_0^{(n-1)}$. Again using (3.2) and (2.13) leads to

$$f_s^{(n)} = Jv + K_s f_s^{(n-1)} \leqq Jv + K_0 f_s^{(n-1)} \leqq Jv + K_0 f_0^{(n-1)} = f_0^{(n)}. \tag{4.6}$$

This completes the proof.

Let $C_0(\mathbb{R})$ denote the set of $\mathbb{R}$-valued continuous functions on $\mathbb{R}$ with compact support, and $C_0^1(\mathbb{R})$ the subset consisting of the functions in $C_0(\mathbb{R})$ having a continuous derivative.

Theorem 4.3. *The map* $s \to \mu_s = J^{-1} f_s$ *from* $\mathbb{R}_+ \cup \{\infty\}$ *to* $\mathscr{P}_+$ *is vaguely continuous,* i.e.

$$\mu_s(g) = \int g(x) d\mu_s(x), \qquad g \in C_0(\mathbb{R}) \tag{4.7}$$

depends continuously on s.

Proof. It is sufficient to prove the latter statement for $h \in C_0^1(\mathbb{R})$ as $C_0^1(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the sup-norm. Partial integration leads to

$$\mu_s(h) = -\int h'(x) f_s(x) dx, \tag{4.8}$$

where h' is the derivative of h. Hence,

$$|\mu_t(h) - \mu_s(h)| \leqq \int |h'(x)| |f_t(x) - f_s(x)| dx. \tag{4.9}$$

Setting

$$C_\alpha(h) = \sup_{x \in \mathbb{R}_+} |h'(x)| \rho_\alpha(x)^{-1}, \tag{4.10}$$

(4.9) yields the estimate

$$|\mu_t(h) - \mu_s(h)| \leqq C_\alpha(h) \| f_t - f_s \|_\alpha, \tag{4.11}$$

which, together with Theorem 4.2., proves continuity of $\mu_s(h)$ in s.

5. The Disordered Harmonic Chain

A fixed point problem similar to the one of Sect. 1 was posed by Dyson [2] in the context of the mass-disordered infinite linear harmonic chain. Results analogous to those of Sects. 2–4 are obtained, and the connection between the two fixed point problems is exhibited.

An infinite linear harmonic chain is described by the equations of motion

$$M_n\ddot{Q}_n = W_{n-1}(Q_{n-1} - Q_n) + W_n(Q_{n+1} - Q_n), \quad n\in\mathbb{Z}. \tag{5.1}$$

Here M_n is the mass of the n^{th} particle, Q_n its displacement from its equilibrium position and W_n the spring constant of the spring between particle n and $n+1$.

Several variants of disorder may be envisaged, involving randomness of masses and spring constants. The case considered here is case II of Dyson, where M_n, $n\in\mathbb{Z}$, are independent equally distributed $\mathbb{R}_+$-valued random variables, whereas the spring constants W_n have a common fixed value. Let $\tau\in\mathscr{P}_+$ denote the probability measure describing the distribution of the masses. It is assumed that

$$\tau(\{0\}) = 0, \tag{5.2}$$

i.e. there are no zero-mass particles, or more stringent,

$$\tau([0,m)) = 0, \qquad m > 0, \tag{5.3}$$

i.e. a mass gap.

Dyson's fixed point problem consists in finding a probability measure $\rho_s\in\mathscr{P}_+$ satisfying

$$\rho_s = R_s\rho_s, \qquad s\in\mathbb{R}_+. \tag{5.4}$$

The map $R_s:\mathscr{P}_+ \to \mathscr{P}_+$ is given by

$$(JR_s\rho)(x) = \iint_{C_{s,x}} d\tau(y)d\rho(z), \tag{5.5}$$

with

$$C_{s,x} = \{(y,z)\in\mathbb{R}_+^2 \mid sy + z/(1+z) < x\}. \tag{5.6}$$

From (5.6) it follows that (5.5) may be rewritten as

$$JR_s\rho = J\tau_s + H_sJ\rho. \tag{5.7}$$

For $s > 0$ the two parts of (5.7) are given by

$$(J\tau_s)(x) = (J\tau)\left(\frac{x-1}{s}\right), \tag{5.8}$$

and, with $f\in\mathscr{D}_+$,

$$(H_sf)(x) = \int_{\beta_s(x)}^{x/s} d\tau(y) f\left(\frac{x-sy}{1-x+sy}\right), \tag{5.9}$$

where

$$\beta_s(x) = \max\left\{0, \frac{x-1}{s}\right\}. \tag{5.10}$$

The case $s=0$ is obtained either directly from (5.5), (5.6), i.e. from

$$(JR_0\rho)(x) = \begin{cases} J\rho\left(\dfrac{x}{1-x}\right), & x<1, \\ 1 \quad , & x \geqq 1, \end{cases} \tag{5.11}$$

or as limits from (5.8) and (5.9), yielding

$$(J\tau_0)(x) = \begin{cases} 0, & x \leqq 1, \\ 1, & x > 1, \end{cases} \tag{5.12}$$

and

$$(H_0 f)(x) = \begin{cases} f\left(\dfrac{x}{1-x}\right), & x<1, \\ 0 \quad , & x>1, \end{cases} \tag{5.13}$$

supplemented by

$$(H_0 f)(1) = \lim_{x \uparrow 1} f\left(\frac{x}{1-x}\right) \qquad (=1 \text{ for } f \in \mathscr{D}_+). \tag{5.14}$$

Extension of (5.9), (5.13) and (5.14) to $f \in D$, the linear span of $\mathscr{D}_+$, is immediate. A further extension of (5.9) and (5.13) to the Banach space $\mathscr{B}_\alpha$, defined in (2.14), leads to the estimate (equality holding for $f \geqq 0$)

$$\|H_s f\|_\alpha \leqq \alpha \int_0^\infty d\tau(y) \int_0^\infty dz\, h_s(y,z)|f(z)|, \qquad s \in \mathbb{R}_+, \tag{5.15}$$

with

$$h_s(y,z) = (1+z)^{-2}\left(1+\frac{z}{1+z}+sy\right)^{-1-\alpha}. \tag{5.16}$$

The inequalities

$$h_s(y,z) < (1+z)^{-1-\alpha}(1+sy)^{-1-\alpha} < (1+z)^{-1-\alpha}, \tag{5.17}$$

holding for $y>0$, $z>0$, imply for $f \neq 0$

$$\|H_s f\|_\alpha < \|f\|_\alpha \int_0^\infty d\tau(y)(1+sy)^{-1-\alpha} < \|f\|_\alpha. \tag{5.18}$$

This yields uniqueness for $f \in \mathscr{B}_\alpha$, satisfying $f = g + H_s f$, $g \in \mathscr{B}_\alpha$, $g \neq 0$. In particular, there is at most one solution of (5.4). For $s=0$, there is a solution, namely

$$\rho_0 = \delta_0, \tag{5.19}$$

as may be verified with (5.11).

Existence of a solution for $s>0$ is obtained by introducing the sequence

$$g_s^{(n)} = \sum_{m=0}^{n} H_s^m J\tau_s = JR_s^n \tau_s, \tag{5.20}$$

$n=0,1,2,\ldots$. As in Sect. 3 one shows that $g_s^{(n)} \to g_s \in \mathscr{D}_+$ pointwise, and in $\mathscr{B}_\alpha$, as $n \to \infty$, with g_s satisfying $g_s = JR_s J^{-1} g_s$, i.e. $\rho_s = J^{-1} g_s$ is fixed point of R_s.

For the ordered case $\tau = \delta_m$, the solution is given by

$$\rho_s = \delta_{b(s)}, \quad b(s) = \tfrac{1}{2}\{ms + (4ms + m^2 s^2)^{1/2}\}. \tag{5.21}$$

As in Sect. 4 one show that $s \to g_s$ is $\mathscr{B}_\alpha$-continuous and $s \to \rho_s$ is vaguely continuous. There is, however, a difference in behaviour of f_s and g_s with respect to the limit $s \to \infty$. The former satisfies

$$\lim_{s \to \infty} f_s = f_\infty = Jv \ \text{ in } \mathscr{B}_\alpha, \tag{5.22}$$

the latter

$$\lim_{s \to \infty} g_s = 0 \ \text{ in } \mathscr{B}_\alpha. \tag{5.23}$$

At a first glance, the two fixed point problems $\mu_s = T_s \mu_s$ and $\rho_s = R_s \rho_s$ of Sects. 1 and 5, respectively, seem to be similar only with respect to their general structure, but there is a deeper relationship. Actually, ρ_s may be obtained from μ_s by choosing v appropriately.

Set $f(x)=0$ for $x \leqq 0$ and

$$f(x) = \tau((x^{-1}, \infty)), \qquad x>0. \tag{5.24}$$

One verifies easily $f \in \mathscr{D}_+$, taking (5.2) into account. Hence,

$$J^{-1} f = v \in \mathscr{P}_+, \tag{5.25}$$

and

$$v(\{0\}) = \lim_{x \downarrow 0} f(x) = \lim_{x \downarrow 0} \{1 - \tau([0, x^{-1}]\} = 0. \tag{5.26}$$

Solving $\mu_s = T_s \mu_s$, with v given by (5.24), (5.25) yields $\mu_s \in \mathscr{P}_+$, which satisfies $\mu_s(\{0\}) = 0$ in view of (5.26) and (3.25). This implies that g_s with $g_s(x)=0$ for $x \leqq 0$ and

$$g_s(x) = \mu_s((sx^{-1}, \infty)) \tag{5.27}$$

is in $\mathscr{D}_+$. It satisfies $JR_s J^{-1} g_s = g_s$, as shown below, i.e. $\rho_s = J^{-1} g_s \in \mathscr{P}_+$ is fixed point of R_s. Now,

$$\begin{aligned} g_s(x) &= \mu_s((sx^{-1}, \infty)) = (T_s \mu_s)((sx^{-1}, \infty)) \\ &= \iint_{\mathbb{R}_+^2 \setminus \bar{A}_{s,sx^{-1}}} dv(u) d\mu_s(v), \end{aligned} \tag{5.28}$$

with $\bar{A}$ being the closure of A.

Now, for $u \neq 0$ and $v \neq 0$,

$$(u,v) \in \mathbb{R}_+^2 \setminus \bar{A}_{s,sx^{-1}} \Leftrightarrow (u^{-1}, sv^{-1}) \in C_{s,x}. \tag{5.29}$$

Hence, in view of (5.24) and (5.27),

$$g_s(x) = \iint\limits_{C_{s,x}} d\tau(y)\, d\rho_s(z), \tag{5.30}$$

which proves the invariance of $\rho_s = J^{-1} g_s$.

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