Commun. Math. Phys. 86, 257–298 (1982)

Communications in
Mathematical
Physics

The Existence of a Non-Minimal Solution to the SU (2) Yang-Mills-Higgs Equations on $\mathbb{R}^3$. Part I

Clifford Henry Taubes ⋆
Harvard University, Cambridge, MA 02138, USA

Abstract. This paper (Part I) and the sequel (Part II) prove the existence of a smooth, non-trivial, finite action solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$ in the Bogomol'nyi-Prasad-Sommerfield limit. The proof uses a simple form of Morse theory known as Ljusternik-Šnirelman theory. Part I establishes that a form of Lusternik-Šnirelman theory is applicable to the SU (2) Yang-Mills-Higgs equations. Here, a sufficient condition for the existence of the aforementioned solution is derived. Part II contains the completed existence proof. There it is demonstrated that the sufficient condition of Part I is satisfied by the SU(2) Yang-Mills-Higgs equations.

I. Introduction

The SU (2) Yang-Mills-Higgs equations on $\mathbb{R}^3$ are the variational equations for a connection (the Yang-Mills potential) and a minimally coupled, associated scalar field which transforms according to the adjoint representation of SU (2) (the Higgs field). These are the variational equations of an action functional [see Eq. (2.1)]. The equations become interesting when one requires the action to be finite, and the boundary condition that the Higgs field have unit norm, asymptotically on $\mathbb{R}^3$, see Eqs. (2.2) and (2.3). This is the Bogomol'nyi-Prasad-Sommerfield limit. In addition there is a first-order system of equations which characterize minima of the functional (2.1); these are called the Bogomol'nyi equations (2.6). As minima, every solution to (2.6) also satisfies the second-order equations (2.2) and (2.3). This general set-up has an analogy with Yang-Mills theory on S^4 [1] and also with Ginzburg-Landau theory [2] ($\lambda = 1$) on $\mathbb{R}^2$. Both these have second-order variational equations and associated first-order equations for minima. The following conjecture has been made for the Yang-Mills-Higgs equations (2.2) and (2.3), the $\lambda = 1$ Ginzburg-Landau equations on $\mathbb{R}^2$ and the Yang-Mills equations on S^4: Every finite action solution to the variational equations is a minima; hence

⋆ Research is supported in part by the Harvard Society of Fellows and the National Science Foundation under Grant PHY 79-16812

0010-3616/82/0086/0257/$08.40

satisfies the associated first-order equations. The conjecture is true for the Ginzburg-Landau theory [3], and unsettled for the Yang-Mills theory on S^4 [1].

The following theorem shows that the conjecture is false for the Yang-Mills-Higgs equations.

Theorem 1.1. *There is a smooth, finite action solution to the* SU(2) *Yang-Mills-Higgs equations in the Bogomol'nyi-Prasad-Sommerfield limit* (2.2), (2.3) *which does not satisfy the first-order Bogomol'nyi equations* (2.6).

As in the case of the Yang-Mills equations on 4-manifolds [4, 5] and 2-manifolds [6], topological aspects play a role in the structure of the solutions. This role is explicit in Theorem 1.1; indeed, the solution is obtained via a mini-max procedure using loops in the function space which are not contractible. These homotopically non-trivial loops arise because the space, $\mathscr{C}$, of finite action pairs of Yang-Mills connections and Higgs fields (Definition 2.1) is homotopically similar to the space Maps $(S^2; S^2)$ of smooth maps from the two sphere to itself. There is a monomorphism of the homotopy groups of Maps $(S^2;S^2)$ into the homotopy groups of $\mathscr{C}$, and the respective path components are in $1-1$ correspondence (Theorem 3.4).

The zero'th homotopy group of $\mathscr{C}$ is the set of path connected components, of which there are a countable number, $\mathscr{C} = \coprod_{k\in\mathbb{Z}} \mathscr{C}_k$. It was proved in [2] that the action functional achieves its infimum on each $\mathscr{C}_k$. These are the solutions to the first-order Bogomol'nyi equations.

The first homotopy group of $\mathscr{C}_0$ contains a subgroup which is isomorphic to $\mathbb{Z}$. The solution of Theorem 1.1 is intimately related to this subgroup, as the proof details.

It should be remarked that spherically symmetric solutions to the monopole equations which are not solutions to the first-order equations are known to exist when the structure group has rank larger than 2 [7]. For the SU (2) case it is known that there are no spherically symmetric solutions other than the solutions to the Bogomol'nyi equations in $\mathscr{C}_{0,\pm 1}$ [8]. The techniques that are developed here should shed light on these other solutions with rank $\geqq 2$ groups.

The proof of Theorem 1.1 uses a mini-max (saddle point) technique known as Ljusternik-Šnirelman theory [9]. For a C^2 function, f, on a compact n-manifold, this technique is easy to describe. Suppose that $p \in M$ is an isolated minima of f. Let $e_0 : (S^k, n) \to (M, p)$ be a generator of the pointed homotopy group, $\Pi_k(M, p)$. Consider the set Λ of maps from (S^k, n) to (M, p) which are homotopic to e_0. For each $e \in \Lambda$, choose $t_e \in S^k$ such that

$$f \cdot e(t_e) = \sup_{t\in S^k} f \cdot e(t). \tag{1.1}$$

Since elements in Λ are not null-homotopic,

$$\operatorname*{Inf}_{\{e\in\Lambda\}} f \cdot e(t_e) = f_\Lambda > f(p). \tag{1.2}$$

As M is compact, a sequence $\{e_i\} \in \Lambda$ with

$$\lim_{i\to\infty} f \cdot e_i(t_{e_i}) \to f_\Lambda \tag{1.3}$$

can be found with the property that

$$\lim_{i\to\infty} e_i(t_{e_i}) \to q(\Lambda) = q \in M;\ q \neq p \tag{1.4}$$

and

$$q \text{ is a critical point of } f,\ df|_q = 0\,. \tag{1.5}$$

This procedure has a generalization to infinite dimensional Banach manifolds (cf. [10]). The gauge invariance of the Yang-Mills-Higgs equations, and the non-compactness of $\mathbb{R}^3$ complicate the application of Ljusternik-Šnirelman theory to the action functional of (2.1). Because of the gauge invariance, the variational problem is not strictly an elliptic one. This is circumvented using *K.* Uhlenbeck's weak compactness theorem [11]. Her theorem states that over a small, bounded domain in $\mathbb{R}^3$, there is a gauge which makes the equations uniformly elliptic.

The non-compactness of $\mathbb{R}^3$ means that Palais-Smale condition C (cf. [10]) does not hold. The papers of Sachs and Uhlenbeck [12] and Schoen and Uhlenbeck [13] on harmonic maps teach that it is often productive to investigate in detail the ways that condition C fails. For the Yang-Mills-Higgs action, a good sequence of configurations (Definition 5.2) fails to converge only by approaching, asymptotically, two exact solutions to the equations which are separated on $\mathbb{R}^3$ by an infinite distance. (This is the analog to the "bubbling off" of harmonic spheres in [12].) This is a manifestation of the physical intuition that the solutions to the equations describe real magnetic monopoles [14], which are localized objects.

The Ljusternik-Šnirelman procedure is applied to non-contractible loops in $\mathscr{C}_0$. In this case, condition C can fail only if the sequence of maxima [corresponding to the sequence in (1.3) and (1.4)] resembles asymptotically a monopole and an anti-monopole [2] which are separated on $\mathbb{R}^3$ by an infinite distance. Such a configuration has action 8π or greater. An explicit, non-contractible loop, in $\mathscr{C}_0$ is exhibited whose maximum action is *less* than 8π. These two facts are used to prove that the mini-max procedure over non-contractible loops in $\mathscr{C}_0$ yields, as in (1.4), a convergent sequence. And, the limit of this sequence is a solution to the second-order Yang-Mills-Higgs equations, but not the first-order Bogomol'nyi equations.

The outline of the proof of Theorem 1.1 is given below. The proof divides into two parts. This paper is Part I. Here the basic technical tools, and *apriori* estimates of Ljusternik-Šnirelman theory on $\mathscr{C}$ are established. Part I consists of Sects. 2–8, where it is proved that the mini-max procedure yields sequences which converge to non-trivial solutions to the second-order Yang-Mills-Higgs equations (2.2) and (2.3).

The second half of the proof is contained in the sequel, Part II [Commun. Math. Phys. **86,** 299–320 (1982)]. In Part II, the mini-max procedure is applied to a specific class of non-contractible loops in $\mathscr{C}_0$. It is shown that the limiting configuration is not a solution to the first-order Bogomol'nyi equations. Below is an outline for both Parts I and II. The symbol $\mathscr{a}(\cdot)$ denotes the Yang-Mills-Higgs action functional (2.1).

I. Ljusternik-Šnirelman Theory on $\mathscr{C}$

(a) Section 2 contains a short review of Yang-Mills-Higgs theory. Here, the space $\mathscr{C}$ of finite action field configurations, and the space of gauge transformations $\mathscr{G}$ are defined.

(b) Section 3 describes the topology of $\mathscr{C}$, $\mathscr{G}$ and $\mathscr{C}/\mathscr{G}$ and the relationship with Maps $(S^2; S^2)$. In particular, it is shown in Sect. 3 that there is a monomorphism of the homotopy groups of Maps $(S^2; S^2)$ into those of $\mathscr{C}$.

(c) The class Λ of non-contractible maps of spheres into $\mathscr{C}$ on which the Ljusternik-Šnirelman procedure is applied is defined in Sect. 4. Here, some useful properties of Λ, in the form of *apriori* estimates, are given.

(d) It is established in Sects. 5 and 6 that for each $k \geqq 0$, there is a sequence of k-spheres $\{c_i(\cdot)\} \in \Lambda$ such that the sequence of configurations, $\{\bar{c}_i\}$ defined for each i to maximize $\mathscr{a}(c_i(\cdot))$ over S^k, converges on $\mathbb{R}^3$. The limit configuration is a smooth, finite action solution to the Yang-Mills-Higgs equations. This is Theorem 5.6.

(e) Theorem 7.1 of Sect. 7 establishes necessary and sufficient conditions on the sequence $\{c_i(\cdot)\}$ for the limit, c to have $\mathscr{a}(c) > 0$.

(f) Theorem 8.1 of Sect. 8 establishes sufficient conditions for the limit c not to satisfy the first order Bogomol'nyi equations.

(g) The first appendix contains the proof that the path components of $\mathscr{C}$ and those of Maps $(S^2; S^2)$ are in $1-1$ correspondence.

(h) The second appendix is a calculation of π_1 (Maps $(S^2;S^2)/\mathrm{SO}(3)$).

II. Minimizing Over Loops

(a) Theorem II.2.1 and Sects. II.2–II.4 contain the proof that there exists a sequence of loops, $\{c_i(\cdot)\} \in \Lambda$ which satisfy the conditions set forth in Theorem 5.6, and Theorems 7.1 and 8.1. Thus, the limit c of the sequence $\{\bar{c}_i\}$ must satisfy the second-order equations but not the first-order equations.

(b) Section II.5 contains the proof, based on ideas of Bourguignon et al. [1], that the configuration c can not be a local minimum of $\mathscr{a}$. Here it is proved that every local minimum of $\mathscr{a}$ on $\mathscr{C}$ satisfies the first-order Bogomol'nyi equations (Theorem II.3.4).

(c) Section II.6 is a summary where the full proof of Theorem 1.1 is exhibited (Theorem II.6.1).

II. Yang-Mills-Higgs Theory

The variables for the static, SU (2) Yang-Mills-Higgs theory are a pair consisting of 1) a connection on the principal bundle $\mathbb{R}^3 \times \mathrm{SU}(2)$ and 2) a section of the vector bundle $\mathscr{g} = \mathbb{R}^3 \times \mathfrak{su}(2)$, called the Higgs field. Let $\Gamma(A)$ denote the space of smooth connections on $\mathbb{R}^3 \times \mathrm{SU}(2)$. The fixed product structure of $\mathbb{R}^3 \times \mathrm{SU}(2)$ identifies $\Gamma(A)$ with $\Gamma(\mathscr{g} \otimes T^*)$ where T^* is the cotangent bundle of $\mathbb{R}^3$. Thus, for $A \in \Gamma(A)$, $A = A_i dx^i$, where $A_i(x)$ is a 2×2, traceless, anti-hermitian matrix. A Higgs field, $\Phi \in \Gamma(\mathscr{g})$, is at each $x \in \mathbb{R}^3$ a 2×2, traceless, anti-hermitian matrix, also.

The Euclidean metric on T^* induces, via the Hodge $*: \underset{p}{\wedge} T^* \to \underset{3-p}{\wedge} T^*$, a positive inner product on $\underset{p}{\wedge} T^*$. The Lie algebra, $\mathfrak{su}(2)$, as the vector space of 2×2 anti-hermitian matrices, has the positive definite inner product $(\sigma^1, \sigma^2) = -2\,\mathrm{trace}(\sigma^1 \sigma^2)$. Together, these metrics induce an inner product on $\mathscr{g} \otimes \underset{p}{\wedge} T^*$.

Thus, for $\omega_1, \omega_2 \in \Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$, the pointwise inner product is $(\omega_1, \omega_2)(x)$ and the pointwise norm is $|\omega_1|(x) = (\omega_1, \omega_1)^{1/2}(x)$. By abusing notation, the bilinear map $\Gamma(\mathscr{g}) \oplus \Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*) \to \Gamma\left(\underset{p}{\wedge} T^*\right)$ is denoted $(\Phi, \omega)(x) = -2\,\mathrm{trace}(\Phi(x)\,\omega(x))$, for $\Phi \in \Gamma(\mathscr{g})$ and $\omega \in \Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$.

In the usual way, the L_2-inner product on $\Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ is defined as $\langle \omega_1, \omega_2 \rangle_2 = \int d^3x\,(\omega_1, \omega_2)(x)$, and $\|\omega_1\|_2 = \langle \omega_1, \omega_1 \rangle_2^{1/2}$.

The Yang-Mills-Higgs action functional is defined on $\Gamma(A) \oplus \Gamma(\mathscr{g})$ to be

$$\mathscr{a}(A, \Phi) = \tfrac{1}{2} \|F_A\|_2^2 + \tfrac{1}{2} \|D_A \Phi\|_2^2 . \tag{2.1}$$

Here $F_A \in \Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ is the curvature of A, so $F_A = dA + A \wedge A$, where d is the usual exterior derivative, and $\wedge$ is the usual exterior product between p-forms (so $A \wedge A = \frac{1}{2}[A_i, A_j]\,dx^i \wedge dx^j$). The 1-form $D_A\Phi \in \Gamma(\mathscr{g} \otimes T^*)$ is the covariant derivative of Φ: $D_A\Phi = d\Phi + [A, \Phi]$.

It should be remarked that the covariant derivative extends to $\Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ in two ways. The first, ∇_A: $\Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*) \to \Gamma((\mathscr{g} \otimes \underset{p}{\wedge} T^*) \otimes T^*)$ is defined by

$$\omega \to \nabla_A \omega = \sum_{i=1}^{3} \left(\frac{\partial \omega}{\partial x^i} + [A_i, \omega] \right) \otimes dx^i .$$

The symbol ∇ will always be used for ∇_0. The second extension, D_A: $\Gamma(\mathscr{g} \otimes \underset{p}{\wedge} T^*) \to \Gamma(\mathscr{g} \otimes \underset{p+1}{\wedge} T^*)$ is the covariant exterior derivative $D_A\omega = d\omega + A \wedge \omega + (-1)^p \omega \wedge A$.

The formal variational equations of $\mathscr{a}(\cdot)$ are the Yang-Mills-Higgs equations:

$$*D_A{*F_A} + [\Phi, D_A\Phi] = 0 , \tag{2.2a}$$

$$*D_A{*D_A\Phi} = 0 , \tag{2.2b}$$

$$D_A F_A = 0 , \tag{2.2c}$$

$$D_A D_A \Phi + [\Phi, F_A] = 0 . \tag{2.2d}$$

Equations (2.2c, d) are the Bianchi identities and they are satisfied by every configuration (A, Φ). These equations are supplemented by the requirement that

$$\lim_{|x| \to \infty} |\Phi|(x) \to 1 . \tag{2.3}$$

The action (2.1) is not finite for every $(A, \Phi) \in \Gamma(A) \oplus \Gamma(\mathscr{g})$. For this reason, restrict attention to the subset

$$\mathscr{C} = \{ c = (A, \Phi) \in \Gamma(A) \oplus \Gamma(\mathscr{g}) \colon \mathscr{a}(c) < \infty, \quad \text{and} \quad \lim_{|x| \to \infty} |\Phi(x)| \to 1 \} . \tag{2.4}$$

This set will now be given, except for two changes, the standard C^∞ topology.

Definition 2.1. The topological space $\mathscr{C}$: Let $\mathscr{C}$ as a set be defined by (2.4). The open neighborhoods of $c=(A,\Phi)\in\mathscr{C}$ are generated by the sets $\mathscr{N}(c;K,\{\varepsilon_j\}_{j=0}^{\infty})$, where $K\subset\mathbb{R}^3$ is a compact set, and $\varepsilon_j>0$, $j=0,1,\ldots$. These sets are defined to be

$$\mathscr{N}(c,K,\{\varepsilon_j\})=\{c'=(A',\Phi')\in\mathscr{C}:$$

$$(1)\qquad |\alpha(c)-\alpha(c')|<\varepsilon_0,$$

$$(2)\qquad \sup_{x\in\mathbb{R}^3}||\Phi|(x)-|\Phi'|(x)|<\varepsilon_0,$$

$$(3)\qquad \text{for each } j=0,1,\ldots,$$

$$\sup_{x\in K}[|\nabla^{(j)}(A(x)-A'(x))|+|\nabla^{(j)}(\Phi(x)-\Phi'(x))|]<\varepsilon_j\}. \tag{2.5}$$

Here $\nabla^{(j)}=\nabla_0\ldots\nabla_0$, j-times.

That the sets $\mathscr{N}(c,K,\{\varepsilon_j\})$ define a topological space is best seen in the following way: The space $\Gamma((\mathfrak{g}\otimes T^*)\oplus\mathfrak{g})$ has the standard C^∞-topology (or C_w^∞-topology in the terminology of [15, Chap. 2]). The set of continuous functions on $\mathbb{R}^3$ is the topological space $C^0(\bar{\mathbb{R}}^3)$ when given the topology that is induced by the supremum-norm [16, Chap. 1]. Now consider $\Gamma((\mathfrak{g}\otimes T^*)\oplus\mathfrak{g})\oplus C^0(\bar{\mathbb{R}}^3)\oplus\mathbb{R}$ with the product topology. The topology on $\mathscr{C}$ of Definition 2.1 is induced by the inclusion $\imath$: $\mathscr{C}\to\Gamma((\mathfrak{g}\otimes T^*)\oplus\mathfrak{g})\oplus C^0(\bar{\mathbb{R}}^3)\oplus\mathbb{R}$ given by $\imath(A,\Phi)=(A,\Phi,|\Phi|,\alpha(A,\Phi))$. The functional $\alpha(\cdot)$, Eqs. (2.2) and (2.3) and the space $\mathscr{C}$ are invariant under the action of the gauge group, $\mathscr{G}$.

Definition 2.2. The gauge group $\mathscr{G}$ is the set $\mathscr{G}=\{g\in C^\infty(\mathbb{R}^3;\mathrm{SU}(2)):g(x=0)=1\}$, with the induced topology.

The topological space $\mathscr{G}$ is a continuous group. The group $\mathscr{G}$ acts continuously on $\mathscr{C}$ with action given by

$$(g,c)=(g,(A,\Phi))\to gc=(gAg^{-1}+gdg^{-1},g\Phi g^{-1}).$$

The group $\mathscr{G}$ acts on $\Gamma(\mathfrak{g}\otimes\underset{p}{\wedge}T^*)$ by pointwise conjugation

$$(g,\psi)\to(g(\psi))(x)=g(x)\,\psi(x)\,g^{-1}(x).$$

The topology of $\mathscr{C}$, $\mathscr{G}$ and $\mathscr{C}/\mathscr{G}$ will be considered in greater detail in Sect. 3. There $\mathscr{C}$ is shown to be the union of path components

$$\mathscr{C}=\coprod_{k\in\mathbb{Z}}\mathscr{C}_k.$$

Formally, $\alpha(\cdot)$ on $\mathscr{C}_k$ is bounded below by $4\pi|k|$ [17]. It is known that every finite action solution to Eqs. (2.2) and (2.3) lies in some $\mathscr{C}_k$ and every such solution does have action greater than or equal to $4\pi|k|$ [2, Chap. IV]. As for the existence of solutions to Eqs. (2.2) and (2.3) on $\mathscr{C}_k$, it was known prior to this date that the functional $\alpha(\cdot)$ attains its infimum on $\mathscr{C}_k$ for all $k\in\mathbb{Z}$ [2]. These solutions to (2.2) and (2.3) on $\mathscr{C}_k$ with $\alpha(\cdot)=4\pi|k|$ necessarily satisfy the Bogomol'nyi equations

$$^*F_A=\pm D_A\Phi\,(+\text{ if }k\geqq 0,\ -\text{ if }k\leqq 0). \tag{2.6}$$

There has been a great deal of literature concerning solutions to (2.6) [2, 18–22].

Concerning non-minimal finite action solutions of (2.2) and (2.3) on $\mathscr{C}$, the only published result up to now is there are no O(3) symmetric solutions which do not satisfy (2.6) for $k=0, \pm 1$ [8]. However, given a solution to (2.2), much about its behavior is known *apriori* [2, Chaps. IV and V].

Formally, the calculus of variations identifies solutions of (2.2) and (2.3) in $\mathscr{C}$ with finite action critical points of $\alpha(\ )$ on $\mathscr{C}$. This is accomplished in practice once a C^1-manifold structure for $\mathscr{C}$ is specified. Here, the non-compactness of $\mathbb{R}^3$ presents a problem. Specifically, a manifold structure which is compatible with the topologies in Definitions 2.2 and 2.3 is not convenient to work with. At the same time, manifold structures based on Sobolev spaces [23] induce topologies which are not compatible with the preceding definitions. [For example $C^\infty(\mathbb{R}^3)$ with the L_2-topology has an uncountable number of path components.] In practice, a Sobolev manifold structure will be employed. Essentially, this works because finite action means that the fields F_A, $D_A\Phi$ are in an L_2 neighborhood of the origin; and using this fact, one can obtain *apriori* estimates which allow one to work with gauge invariant Sobolev norms, as one does over compact manifolds [11, 24].

For the present, in order to be unambiguous, the following definitions are necessary. For $E \to \mathbb{R}^3$ a vector bundle, $\Gamma^c(E)$ denotes the space of smooth, compactly supported sections.

Definition 2.3. The gradient of α: For $c \in \mathscr{C}$, the gradient of α at c is the following linear functional on $\Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$:

$$\nabla \alpha_c(\psi) = \frac{d}{ds}\,\alpha(c+s\psi)\,|_{s=0},$$

where $\psi \in \Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$.

Definition 2.4. A configuration $c \in \mathscr{C}$ will be said to be a critical point of α when $\nabla \alpha_c(\cdot) \equiv 0$ on $\Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$.

For future use, the hessian of α at c needs to be defined too.

Definition 2.5. The hessian of α: This is the bilinear functional on $\Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$, defined for $c \in \mathscr{C}$ by

$$\mathscr{H}_c(\psi) \equiv \frac{d^2}{ds^2}\,\alpha_c(c+s\psi)\,|_{s=0}.$$

Thus, for $c=(A,\Phi)$ and $\psi=(\omega,\eta)$,

$$\nabla \alpha_c(\psi) = \langle D_A\omega, F_A\rangle_2 + \langle [\omega,\Phi], D_A\Phi\rangle_2 + \langle D_A\eta, D_A\Phi\rangle_2, \tag{2.7}$$

while

$$\begin{aligned}\mathscr{H}_c(\psi) = {} & \langle D_A\omega, D_A\omega\rangle_2 + \langle D_A\eta, D_A\eta\rangle_2 + \langle [\omega,\Phi],[\omega,\Phi]\rangle_2 + 2\langle \omega\wedge\omega, F_A\rangle_2 \\ & + 2\langle [\omega,\eta], D_A\Phi\rangle_2 + 2\langle [\omega,\Phi], D_A\eta\rangle_2 .\end{aligned} \tag{2.8}$$

III. The Topology of $\mathscr{C}/\mathscr{G}$

The Yang-Mills-Higgs functional can be considered as a $\mathscr{G}$-invariant functional on $\mathscr{C}$, or as a functional on $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{G}$. Having endowed $\mathscr{C}$ and $\mathscr{G}$ with topologies, the map

$$Q : \mathscr{C} \to \mathscr{C}/\mathscr{G} = \bar{\mathscr{C}} \tag{3.1}$$

gives $\bar{\mathscr{C}}$ the quotient topology. This allows one to consider continuous maps from a topological space X into $\bar{\mathscr{C}}$ or $\mathscr{C}$. Of particular interest are the spaces of continuous maps from k-dimensional spheres. Let $n \in S^k$ denote the north pole, and let $q \in \bar{\mathscr{C}}$ be some fixed orbit under $\mathscr{G}$. Two maps, e_0, $e_1 \in C^0((S^k, n); (\bar{\mathscr{C}}, q))$ are said to be homotopic, written $e_0 \sim e_1$, If there exists a map $F \in C^0(([0,1] \times S^k, [0,1] \times n); (\bar{\mathscr{C}}, q))$ such that 1) $F(0, \cdot) = e_0$, 2) $F(1, \cdot) = e_1$. This notion is standard [25]. For $k \geqq 1$, the homotopy classes of maps in $C^0((S^k, n); (\bar{\mathscr{C}}, q))$ form a group, denoted $\Pi_k(\bar{\mathscr{C}}, q)$. For $k = 0$, $\Pi_0(\bar{\mathscr{C}}, q) = \Pi_0(\bar{\mathscr{C}})$ is the set of path components of $\bar{\mathscr{C}}$.

It is the purpose of this section to provide a description of $\Pi_*(\bar{\mathscr{C}}; q) = \bigoplus_{k=0}^{\infty} \Pi_k(\bar{\mathscr{C}}; q)$. The space $\bar{\mathscr{C}}$ is defined by the map Q of (3.1); and endowing $\bar{\mathscr{C}}$ with the quotient topology insures that Q is continuous. The properties of this quotient are summarized in the following theorems:

Theorem 3.1. *The map $Q : \mathscr{C} \to \bar{\mathscr{C}}$ is a fibration. In fact, there is a continuous map ϱ: $\bar{\mathscr{C}} \to \mathscr{C}$ such that $Q \cdot \varrho = \mathrm{id}_{\bar{\mathscr{C}}}$. (The section ϱ is the polar gauge, Eq.* (3.13).*)*

The map Q induces a homomorphism

$$Q_* : \Pi_*(\mathscr{C}, c) \to \Pi_*(\bar{\mathscr{C}}, Q(c)) \tag{3.2}$$

of the respective k^{th} homotopy groups. In fact, Q_* does more, as the next theorem states.

Theorem 3.2. *The space $\mathscr{G}$ is contractible so $Q_* : \Pi_*(\mathscr{C}, c) \simeq \Pi_*(\bar{\mathscr{C}}, Q(c))$, is an isomorphism and ϱ_* is the inverse.*

As suggested by previous authors, [17, 26, 27], there is a relationship between $\bar{\mathscr{C}}$ and the space Maps $(S^2; S^2)$. The relationship, on the level of homotopy groups is deeper than what is discussed in the literature. There is a map I: Maps $(S^2; S^2) \to \varrho(\bar{\mathscr{C}}) \subset \mathscr{C}$, given by the following definition:

Definition 3.3. The map I: Consider an element $e \in$ Maps $(S^2; S^2)$ as a map from the unit sphere in $\mathbb{R}^3$ to the unit sphere in $\mathfrak{su}(2)$. Then

$$I(e) \equiv (-(1-\beta(x))[e(\hat{x}), de(\hat{x})], (1-\beta(x))\, e(\hat{x})) \in \mathscr{C}, \tag{3.3}$$

where $\hat{x} = x/|x|$ and $0 \leqq \beta(x) \in C_0^\infty(\mathbb{R}^3)$ is a cut-off function such that

$$\begin{aligned} &1) && 1 \geqq \beta(x), \\ &2) && \beta(x) \equiv 1 \quad \text{if} \quad |x| < \tfrac{1}{2}, \\ &3) && \beta(x) \equiv 0 \quad \text{if} \quad |x| > 1. \end{aligned} \tag{3.4}$$

The significance of the map I is summarized in Theorem 3.4.

Theorem 3.4. *Endow* Maps $(S^2; S^2)$ *with the C^∞ topology* [15, Chap. 2]. *The map I of Definition 3.3 is continuous and I induces an exact sequence*

$$0 \to \Pi_*(\text{Maps}(S^2; S^2), e) \xrightarrow{I_*} \Pi_*(\bar{\mathscr{C}}, I(e)). \tag{3.5}$$

In addition, I_* *is a* $1-1$ *correspondence between* $\Pi_0(\mathrm{Maps}(S^2;S^2))$ *and* $\Pi_0(\bar{\mathscr{C}})$.

The conjecture is that I_* is an isomorphism. In fact, define

$$\mathscr{C}_1 = \{(A,\Phi)\in\mathscr{C}: \|\nabla_A\Phi\|_4 < \infty\}, \tag{3.6}$$

with the induced topology. Then I maps into $\mathscr{C}_1$ and I induces an isomorphism

$$I_*^1: \Pi_*(\mathrm{Maps}(S^2;S^2),e) \simeq \Pi_*(\mathscr{C}_1, I(e)). \tag{3.7}$$

For the purposes fo this paper, one could just as well consider the space $\mathscr{C}_1$. For the sake of generality $\mathscr{C}$ will be used, and (3.7) will not be proved here.

The set of groups $\Pi_*(\mathrm{Maps}(S^2;S^2),e)$ is readily described. $\mathrm{Maps}(S^2;S^2)$ has countably many path components; these are labeled by the topological degree. Thus

$$\mathrm{Maps}(S^2;S^2) = \coprod_{k\in\mathbb{Z}} \mathrm{Maps}(S^2;S^2)_k. \tag{3.8}$$

The space $\mathrm{Maps}(S^2;S^2)$ has a distinguished subspace, $\mathrm{Maps}((S^2,n);(S^2,n))$, which is the subspace of maps taking n to n. For each $k\in\mathbb{Z}$, there exists the fibration

$$0\to\mathrm{Maps}((S^2,n);(S^2,n))_k\to\mathrm{Maps}(S^2;S^2)_k\xrightarrow{\pi}S^2\to 0, \tag{3.9}$$

where π is evaluation at the north pole, n. The set of homotopy groups $\Pi_*(\mathrm{Maps}((S^2,n);(S^2,n))_k)$ is independent of k and $\Pi_l(\mathrm{Maps}((S^2,n);(S^2,n))_0)\simeq\Pi_{l+2}(S^3)$ [25, Chap. 1]. Thus, in principle one can compute $\Pi_*(\mathrm{Maps}(S^2;S^2)_k)$ from the fibration (3.9). For $k=0$, this is relatively easy:

Theorem 3.5. *The inclusion of* $\mathrm{Maps}((S^2,n);(S^2,n))_0$ *into* $\mathrm{Maps}(S^2;S^2)_0$ *induces a canonical splitting*

$$\begin{aligned}\Pi_l(\mathrm{Maps}(S^2;S^2)_0) &\simeq \Pi_l(\mathrm{Maps}((S^2,n);(S^2,n))_0)\oplus\Pi_l(S^2),\\ &\simeq \Pi_{l+2}(S^2)\oplus\Pi_l(S^2).\end{aligned} \tag{3.10}$$

Consider $\mathscr{a}(\cdot)$ as a functional on $\bar{\mathscr{C}}$. The group SU(2) acts by conjugation on $\bar{\mathscr{C}}$ by imbedding SU(2) in $C^\infty(\mathbb{R}^3;\mathrm{SU}(2))$ as the subgroup of constant matrices. This action factors through SO(3), and $\mathscr{a}(\cdot)$ is invariant. The SO(3) action is continuous, and for $k\neq 0$, the action is free on $\bar{\mathscr{C}}_k$. So, for $k\neq 0$ $\bar{\mathscr{C}}_k$ is fibred over the quotient, $\tilde{\mathscr{C}}_k\equiv\bar{\mathscr{C}}_k/\mathrm{SO}(3)$. ($\tilde{\mathscr{C}}_k$ is given the quotient topology.) For $k\neq 0$, one may consider $\mathscr{a}(\cdot)$ as a nonlinear functional on $\tilde{\mathscr{C}}_k$, and for this reason the topology of $\tilde{\mathscr{C}}_k$ is interesting.

The group SO(3) acts freely on $\mathrm{Maps}(S^2;S^2)_k$ for $k\neq 0$ by rotations of the image S^2. With this action, the map I: $\mathrm{Maps}(S^2;S^2)_k\to\bar{\mathscr{C}}_k$ is SO(3) equivariant, so I induces the continuous map $\tilde{I}$: $\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3)\to\tilde{\mathscr{C}}_k$.

Theorem 3.6. *For* $k\neq 0$, *the map* $\tilde{I}$ *induces an exact sequence of homotopy groups,*

$$0\to\Pi_*(\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3))\to\Pi_*(\tilde{\mathscr{C}}_k).$$

In addition,

$$\begin{aligned}&\Pi_1(\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3))\simeq\mathbb{Z}_{|k|}\quad \textit{and}\\ &\Pi_l(\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3))\simeq\Pi_{l+2}(S^2)\quad \textit{for}\quad l\geqq 2.\end{aligned}$$

The remainder of this section contains the proofs of the preceding theorems.

Proof of Theorem 3.1. The theorem is proved by exhibiting $\mathscr{C}$ as a product,

$$\mathscr{C} = \bar{\mathscr{C}} \times \mathscr{G}, \tag{3.11}$$

where ϱ becomes the inclusion $\bar{\mathscr{C}} \to \bar{\mathscr{C}} \times 1$.

First, it should be remarked that as in the pure Yang-Mills case, $\mathscr{G}$ acts effectively on $\mathscr{C}$ [4]. Next, the projection $p: \mathscr{C} \to \mathscr{G}$ will be defined. Let $(r = |x|, \theta, \phi)$ be spherical coordinates on $\mathbb{R}^3$. For $c = (A, \Phi) \in \mathscr{C}$, $p(c) \in C^\infty(\mathbb{R}^3; \mathrm{SU}(2))$ is defined to be the unique solution to the following ordinary differential equation:

$$\frac{\partial}{\partial r} p(c)(r, \theta, \phi) - p(c)(r, \theta, \phi)\, A_r(r, \theta, \phi) = 0, \quad \text{and} \quad p(x=0) = 1\,. \tag{3.12}$$

Here $A_r = x^j \frac{\partial}{\partial x^j} \lrcorner A$. This is the polar gauge [28]. The element $p(c)$ is C^∞, and the map $p: \mathscr{C} \to \mathscr{G}$ is readily seen to be continuous with respect to the given topologies. Now define the map $\varrho: \bar{\mathscr{C}} \to \mathscr{C}$ by the following device: Let $Q(c) \in \bar{\mathscr{C}}$ denote the class of $c \in \mathscr{C}$. Then

$$\varrho(c) \equiv p(c)\, c\,. \tag{3.13}$$

It must be established that $\varrho(c)$ depends only on the class of c. By construction, for $\varrho(c) = (A, \Phi)$,

$$A(x=0) = 0, \quad \text{and} \quad A_r \equiv 0\,. \tag{3.14}$$

Let $c, c' \in \mathscr{C}$ with $Q(c) = Q(c')$. Necessarily, one has $\varrho(c) = u\varrho(c')$ for some $u \in \mathscr{G}$. But Eq. (3.14) implies that $\frac{\partial}{\partial r} u = 0$ and so $u = 1$. Therefore, $\varrho(c)$ depends only on $Q(c) \in \bar{\mathscr{C}}$. Because $p(\cdot)$ is continuous, and $\mathscr{G}$ acts continuously on $\mathscr{C}$, the map ϱ: $\bar{\mathscr{C}} \to \mathscr{C}$ is also continuous. The map ϱ is, by inspection, $1-1$ onto its image. Identify $\bar{\mathscr{C}}$ with the image $\varrho(\bar{\mathscr{C}}) \subset \mathscr{C}$. The product structure of $\mathscr{C}$ is exhibited by the homeomorphism $\mathscr{L}$: $\mathscr{C} \to \bar{\mathscr{C}} \times \mathscr{G}$, defined by $\mathscr{L}(c) = (p(c)\,c, p(c))$. This proves Theorem 3.1.

Proof of Theorem 3.2. The contractibility of $\mathscr{G}$ is proved by exhibiting a continuous map, $\mathscr{R}: [0,1] \times \mathscr{G} \to \mathscr{G}$, with

$$\begin{aligned} &1) \qquad \mathscr{R}(0, \cdot) = \mathrm{Id}_{\mathscr{G}}, \\ &2) \qquad \mathscr{R}(1, \cdot) = 1\,. \end{aligned} \tag{3.15}$$

Consider the map $\mathscr{R}$ given for $g \in \mathscr{G}$, $t \in [0,1]$ and $x \in \mathbb{R}^3$ by

$$\mathscr{R}(t, g)(x) = g((1-t)\,x)\,. \tag{3.16}$$

The map $\mathscr{R}$ is continuous with respect to the C^∞ topology on $\mathscr{G}$. It satisfies (3.15) by construction, so $\mathscr{G}$ is contractible, and necessarily, $\Pi_*(\mathscr{G}) = (0)$.

The long exact homotopy sequence of a fibration implies that Q_* is an isomorphism [25, Chap. 7].

Proof of Theorem 3.4. The map I is clearly continuous in the given topologies, hence by functorality, I_* is a homomorphism of the respective homotopy groups. To establish that I_* is a monomorphism, consider, for $l \geqq 0$ and $k \in \mathbb{Z}$, two elements

$$\psi_0, \psi_1 \in \mathrm{Maps}\,((S^l, n); (\mathrm{Maps}\,(S^2; S^2)_k, e_k)), \quad \text{with} \quad I(\psi_0) \sim I(\psi_1)\,. \tag{3.17}$$

Let $F(s,\cdot)\in \mathrm{Maps}(([0,1]\times S^l,[0,1]\times n);(\mathscr{C},I(e_k)))$ be a homotopy between $I(\psi_0)$ and $I(\psi_1)$, and write $F(s,y)=(A(s,y),\Phi(s,y))$ for $(s,y)\in[0,1]\times S^l$. A consequence of the topology given to $\mathscr{C}$ by Definition 2.1 is that the continuous function $|\Phi(s,y)|(x)\in C^0([0,1]\times S^l\times \mathbb{R}^3)$ is continuous in (s,y), uniformly with respect to $x\in\mathbb{R}^3$. In particular, this means that there exists $R<\infty$ such that

$$|\Phi(s,y)|(x)>\tfrac{1}{2}\quad \text{for all}\quad (s,y)\in[0,1]\times S^l\quad \text{if}\quad |x|>R. \tag{3.18}$$

On the two-sphere $\{x\in\mathbb{R}^3: |x|=R+1\}$, the map

$$\hat{\Phi}(s,y)(\cdot)=\Phi(s,y)(\cdot)/|\Phi(s,y)|(\cdot)\in \mathrm{Maps}(([0,1]\times S^l,[0,1]\times n),(\mathrm{Maps}(S^2;S^2),e_k)), \tag{3.19}$$

and is a homotopy between ψ_0 and ψ_1. Hence I_* is a monomorphism. The proof that $I_*\colon \Pi_0(\mathrm{Maps}(S^2;S^2))\twoheadrightarrow \Pi_0(\mathscr{C})$ requires a result from Sect. 4, so this will be proved in Appendix A.

Proof of Theorem 3.5. The projection $\pi\colon \mathrm{Maps}(S^2;S^2)\to S^2$ in (3.9) is given by $\pi(e)=e(n)$. For $k=0$, (3.9) admits a global section,

$$q\colon S^2\to \mathrm{Maps}(S^2;S^2)_0, \tag{3.20}$$

which sends $p\in S^2$ to the constant map $q(p)\colon S^2\to p$. Clearly $\pi\circ q=\mathrm{id}_{S^2}$. Thus the long exact homotopy sequence that is associated to (3.9) for $k=0$ splits and

$$\Pi_l(\mathrm{Maps}(S^2;S^2)_0)\simeq \Pi_l(\mathrm{Maps}((S^2,n);(S^2,n))_0)\oplus \Pi_l(S^2),$$

as claimed.

Proof of Theorem 3.6. The homomorphism $\tilde{I}_*$ is a monomorphism for the same reason that $\tilde{I}_*\colon \Pi_*(\mathrm{Maps}(S^2;S^2))\to \Pi_*(\mathscr{C})$ is a monomorphism.

To calculate $\Pi_*(\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3))$, observe that SO(3) acts naturally on the fibration (3.9). Indeed, the base S^2 is homeomorphic to SO(3)/SO(2) and this implies that $\mathrm{Maps}(S^2;S^2)_k/\mathrm{SO}(3)$ is homeomorphic to $\mathrm{Maps}((S^2,n);(S^2,n))_k/\mathrm{SO}(2)$. Here SO(2) rotates the image sphere around the axis defined by the north and south poles. From the long exact homotopy sequence, one obtains immediately that

$$\Pi_l(\mathrm{Maps}((S^2,n);(S^2,n))_k/\mathrm{SO}(2))\simeq \Pi_l(\mathrm{Maps}((S^2,n);(S^2,n))_k),\quad \text{for}\quad l\geqq 3.$$

For $l\leqq 2$, one has the exact sequence

$$\begin{array}{l} 0\to \Pi_2(\Omega_k)\xrightarrow{\gamma}\Pi_2(\Omega_k/\mathrm{SO}(2))\to \Pi_1(\mathrm{SO}(2)\xrightarrow{\beta}\Pi_1(\Omega_k) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \searrow^{\alpha} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Pi_1(\Omega_k/\mathrm{SO}(2))\to 0. \end{array} \tag{3.21}$$

Here Ω_k is shorthand for $\mathrm{Maps}((S^2,n);(S^2,n))_k$. Recall that $\Pi_2(\Omega_k)\simeq \Pi_4(S^2)\simeq \mathbb{Z}^2$ and $\Pi_1(\Omega_k)\simeq\Pi_3(S^2)\simeq\mathbb{Z}$ [25, Chap. 9], while $\Pi_1(\mathrm{SO}(2))\simeq\mathbb{Z}$. Because γ of (3.21) is an isomorphism if $\beta\neq 0$, Theorem 3.6 is implied by the following lemma:

Lemma 3.7. *The map β in Eq.* (3.21) *is multiplication by k.*

Proof. Fix a configuration $e_k \in \Omega_k$. The orbit of e_k under the SO(2) action is a 1 parameter loop, $e_k(t)$, $t \in S^1$ and hence an element of Maps$((S^1, n); (\Omega_k, e_k))$. The loop $e_k(t)$ is homotopic in Maps$((S^1, n); (\Omega_k, e_k))$ to some multiple, m, of the generator of $\pi_1(\Omega_k, e_k)$. By definition, the map β of (3.21) is multiplication by the integer m. The lemma follows by demonstrating that $m = k$. This task is straightforward, but lengthy and is completed in Appendix B.

IV. Ljusternik-Šnirelman Theory on $\mathscr{C}$

By design, the action functional $\mathscr{a}(\cdot)$ is a continuous map from $\mathscr{C}$ to $\mathbb{R}$, and its derivatives are defined as distributions on $\Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$. The Ljusternik-Šnirelman procedure begins with the definition of the space of maps over which to apply the mini-max procedure on $\mathscr{a}(\cdot)$, as outlined in Sect. 1. The procedure uses non-contractible maps from S^k into $\mathscr{C}$. Here, only homotopy classes induced by the map I: Maps$((S^2, n); (S^2, n)) \to \mathscr{C}$ will be considered, and in particular, only homotopy classes in $\mathscr{C}_0$. The analysis for $\mathscr{a}(\cdot)$ on $\mathscr{C}_k$ is similar. The only difference is that the minima of $\mathscr{a}(\cdot)$ on $\bar{\mathscr{C}}_0$ is a 2-sphere, while the minima of $\mathscr{a}(\cdot)$ on $\bar{\mathscr{C}}_k$, $k \neq 0$ have more complicated topologies [21, 29, 30].

The homotopy groups of $\mathscr{C}_0$ (and of $\bar{\mathscr{C}}_0$; by identifying $\bar{\mathscr{C}}_0$ with $\varrho(\bar{\mathscr{C}}_0) \subset \mathscr{C}_0$, as in Sect. 3) are defined with respect to the distinguished point

$$c_* = (0, -\tfrac{1}{2}\sigma^3), \quad \text{where} \quad \sigma^3 \in \mathscr{su}(2), \quad \text{and} \quad \sigma^3 \cdot \sigma^3 = -1. \tag{4.1}$$

Note that $\mathscr{a}(c_*) = 0$. The configuration c_* is not in the image of the map I of Definition 3.3. Because change of base point in $\mathscr{C}_0$ induces an isomorphism between $\Pi_*(\mathscr{C}_0, I(e_0))$, and $\Pi_*(\mathscr{C}, c_*)$, this is not a serious problem. An explicit isomorphism is given below. The distinguished point in Maps$(S^2; S^2)_0$ is the constant map: $e_*: S^2 \to -\frac{1}{2}\sigma^3$. Let $y = (t, \hat{y})$ be coordinates on S^k, $k \geqq 1$ where $t \in [0, \pi]$ is the polar angle and $\hat{y}$ are coordinates on the equatorial S^{k-1}. The distinguished point is $n = \{t = 0\}$. Define for $k \geqq 1$, $\mathscr{I}$: $C^0((S^k, n), (\mathscr{C}, I(e_*)) \to C^0((S^k, n), (\mathscr{C}, c_*))$ by

$$\mathscr{I}(c)(t, \hat{y}) = \begin{cases} (0, (1 - 2/\pi\, t\beta(x))(-\tfrac{1}{2}\sigma^3)), & \text{for} \quad t \in [0, \pi/2], \\ c(2t - \pi, \hat{y}), & \text{for} \quad t \in [\pi/2, \pi], \end{cases} \tag{4.2}$$

for $c(\cdot) \in C^0((S^k, n), (\mathscr{C}, I(e_*))$. It is a standard argument that $\mathscr{I}$ induces an isomorphism between $\Pi_*(\mathscr{C}, I(e_*))$ and $\Pi_*(\mathscr{C}, c_*)$ [27, Chap. 7]. Thus, a generator $e \in \Pi_*(\text{Maps}((S^2, n); (S^2, n))_0, e_*)$ induces

$$c(e)(t, \hat{y}) \equiv \begin{cases} (1 - 2/\pi\, t\beta(x))(0, -\tfrac{1}{2}\sigma^3), & \text{for } t \in [0, \pi/2], \\ (1 - \beta(x))(-[e(2t - \pi, \hat{y})(\hat{x}), de(2t - \pi, \hat{y})(\hat{x})], e(2t - \pi), \hat{y})(\hat{x})), & \text{for } t \in [\pi/2, \pi], \end{cases} \tag{4.3}$$

where $\hat{x} = x/|x|$ and $c(e)$ is a generator of $\Pi_k(\mathscr{C}_0, c_*)$ (and $\Pi_k(\bar{\mathscr{C}}_0, c_*)$.)

The sphere $c(e)(\cdot) \in C^0((S^k, n), (\mathscr{C}_0, c_*))$ is the paradigm for the set of functions Λ on which the mini-max procedure takes place.

Definition 4.1. The function set Λ: For $k \geqq 1$, let $e(\cdot) \in C^0((S^k, n)$; (Maps$((S^2, n); (S^2, n)), e_*)$) be a fixed generator of $\Pi_k(\text{Maps}((S^2, n); (S^2, n)), e_*)$.

Define $\Lambda=\Lambda(e)$ to be the set $\Lambda=\{c(\cdot)=c(e)(\cdot)+(\omega(\cdot),\eta(\cdot))\in C^0((S^k,n);(\mathscr{C},c_*))$:

(1) There exists a compact set $K\subset\mathbb{R}^3$ such that $\omega(y)\in\Gamma^c(K;\mathscr{g}\otimes T^*)$ for all $y\in S^k$.

(2) $\lim\limits_{|x|\to\infty}|\eta(y;x)|\to 0$, uniformly with respect to $y\in S^k$.

(3) Let $c(e)(y)=(A_0(y),\Phi_0(y))$. Then $\nabla_{A_0(y)}\eta(y)\in C^0(S^k;L_2(\mathscr{g}\otimes T^*))$.$\}$

It follows from (1) and (3) that $\mathscr{a}(c(\cdot))\in C^0(S^k;\mathbb{R})$ for $c(\cdot)\in\Lambda$.

Definition 4.2. The configuration induced by $c(\cdot)\in\Lambda$: To each $c(\cdot)\in\Lambda$, associate one configuration $\bar{c}=c(y_0)$ where $\mathscr{a}(c(y_0))=\sup\limits_{y\in S^k}\mathscr{a}(c(y))$.

As $\mathscr{a}(c(\cdot))$ is a continuous function on S^k, it achieves its supremum at some $y_0\in S^k$. If there is more than one supremum, the choice is immaterial. One is to think of $c(\cdot)\in\Lambda$ as having associated to it, the induced configuration $\bar{c}$.

Before considering the detailed properties of Λ, some remarks are in order. One may be concerned that for $c(\cdot)=(A(\cdot),\Phi(\cdot))\in\Lambda$, the Higgs field, $\Phi(\cdot)$ is constrained to satisfy

$$\lim_{|x|\to\infty}\Phi(y;|x|\hat{x})=e(y)(\hat{x})\quad\text{for all}\quad y\in S^k. \tag{4.4}$$

This constraint amounts to a choice of gauge, asymptotically in $\mathbb{R}^3$. To put it another way, Eq. (4.4) implies that the gauge group $\mathscr{G}$ has been reduced to $\mathscr{G}^0=\left\{g\in\mathscr{G}:\ \lim\limits_{|x|\to\infty}g(x)\to 1\right\}$. As $\mathscr{G}$ carries no topology, there is no harm in doing this. Indeed, one could reduce $\mathscr{G}$ to 1 and work directly on $\bar{\mathscr{C}}\to\mathscr{C}$, but this complicates the analysis.

The constraint on $\omega(y)=A(y)-A_0(y)$ has to do with the following observations: Consider the subspace $\mathscr{C}'\subset\mathscr{C}$ which is defined for $\delta>0$ by

$$\begin{aligned}\mathscr{C}'=\{c&=(A,\Phi)\in\mathscr{C}: u_c(x)\\ &=\{|x|^2|F_A|(x)+|x|^{3/2+\delta}|\nabla_A\Phi|(x)+|x|^{2+\delta}|F_A,\Phi]|(x))\in L_\infty(\mathbb{R}^3)\}.\end{aligned} \tag{4.5}$$

Give $\mathscr{C}'$ the topology induced by considering it as a subset of $\mathscr{C}\times C^0(\bar{\mathbb{R}}^3)$ as follows: $\mathscr{C}'\in c\to(c,u_c(x))\in\mathscr{C}\times C^0(\bar{\mathbb{R}}^3)$. Every critical point of $\mathscr{a}(\cdot)$ on $\mathscr{C}$ lies in $\mathscr{C}'$[2, Ch. IV]. In addition, I maps Maps$(S^2;S^2)$ into $\bar{\mathscr{C}}'\to\mathscr{C}'$, but here I_* an isomorphism $I_*:\Pi_*(\text{Maps}(S^2;S^2),e_k)\simeq\Pi_*(\mathscr{C};I(e_k))$. Therefore, the Ljusternik-Šnirelman theory applied to homotopy classes induced by I_* takes place in $\mathscr{C}'$, and hence $\bar{\mathscr{C}}'$.

For $(A,\Phi)\in\bar{\mathscr{C}}'$, $A_r=0$, and $A(r\hat{x})=\frac{1}{r}\int\limits_0^r dtt\left(\frac{\partial}{\partial t}\lrcorner F_A\right)(t\hat{x})$, so

$$|A|\leqq\text{const}\cdot\frac{\ln|x|}{|x|+1},\ \text{cf. [28]}. \tag{4.6}$$

In addition, one can show that the

$$\lim_{|x|\to\infty}\Phi(|x|\hat{x})=\hat{\Phi}(\hat{x})\in C^1(S^2;S^2). \tag{4.7}$$

It is a fact, following from (4.5–7), that given $\varepsilon > 0$, and a sphere $c'(\cdot) = (A'(\cdot), \Phi'(\cdot)) \in C^0((S^k, n); (\bar{\mathscr{C}}', c_*))$ which is homotopic to $c_0(e_k)$, there is a sphere $c(\cdot) = (A(\cdot), \Phi(\cdot)) \in \Lambda$ with

$$\sup_{y \in S^k} (\| F_A(y) - F_{A'}(y) \|_2 + \| \nabla_A \Phi(y) - \nabla_{A'} \Phi'(y) \|_2) < \varepsilon . \tag{4.8}$$

As Eqs. (4.6)–(4.8) are explanatory remarks, they will not be proved here.

For the proof of Theorem 1.1, the crucial properties of the space Λ are given by Lemmas 4.3 and 4.6 and Theorems 4.4. and 4.5.

Lemma 4.3. *The set Λ, as a topological subspace of $C^0((S^k; n); (\mathscr{C}, c_*))$ is contractible onto $c(e)(\cdot)$.*

Proof. The space Λ is readily seen to be convex. Thus, every $c(\cdot) \in \Lambda$ is homotopic to $c(e)(\cdot)$.

Theorem 4.4: *Define the number $a_\infty = \inf_{c(\cdot) \in \Lambda} a(\bar{c})$. Then $a_\infty > 0$.*

Theorem 4.4 is crucial in proving that the Ljusternik-Šnirelman procedure doesn't produce the trivial critical point c_*.

The following theorem provides the most useful tool for obtaining *apriori* estimates. It will be invoked again and again in the proof of Theorem 1.1.

Theorem 4.5. *Let $c(\cdot) = (A(\cdot), \Phi(\cdot)) \in \Lambda$. There exists a unique $\hat{c}(\cdot) = (A(\cdot), \hat{\Phi}(\cdot)) \in \Lambda$ with the property that for all $y \in S^k$, (1) $a(\hat{c}(y)) \leqq a(c(y))$, and (2) $\nabla^2_{A(y)} \Phi(y) \equiv {*D_{A(y)}} {*D_{A(y)}} \Phi(y) = 0$.*

The proof of Theorem 4.4 requires the next lemma. This lemma, in some sense, is the heart of the connection between the topology of $\mathscr{C}$ and the critical points of $a(\cdot)$.

Lemma 4.6. *Let e be a generator of $\Pi_k(\mathrm{Maps}((S^2, n); (S^2, n), e_*))$, and let $c(\cdot) = (A(\cdot), \Phi(\cdot)) \in C^0((S^k, n); (\mathscr{C}, c_*))$ be homotopic to $c(e)$. There exists $(y, x) \in S^k \times \mathbb{R}^3$ such that $\Phi(y; x) = 0$.*

Proof of Lemma 4.6. Suppose no such (y, x) existed. The homotopy between $c(\cdot)$ and $c(e)(\cdot)$ must be continuous with respect to the topology of Definition 2.1. Therefore, $R > 0$ exists such that, restircted to the sphere $|x| = R$,

$$e'(y; \hat{x}) \equiv \Phi(y; R\hat{x}) / |\Phi(y; R\hat{x})| \sim e(y; \hat{x}) \tag{4.9}$$

in $C^0((S^k, n);\ (\mathrm{Maps}(S^2; S^2), e_*))$. Let $F(s, y)(\hat{x}) \in C^0(([0,1] \times S^k,\ [0,1] \times n);\ (\mathrm{Maps}(S^2; S^2), e_*))$ be the homotopy of (4.9).

Since Φ never vanishes, $\Phi(y; (1-s) R\hat{x} + sRn) / |\Phi(y; (1-s) R\hat{x} + sRn)|$, $s \in [0, 1]$ defines a homotopy between $e'(y; \hat{x})$ and $q(e'(y; n))(\hat{x})$ in $C^0((S^k, n); (\mathrm{Maps}(S^2; S^2), e_*))$. The map q: $S^2 \to \mathrm{Maps}(S^2; S^2)$ is defined in Eq. (3.20). Meanwhile, $q(F(1-s, y)(n))(\hat{x})$ defines a homotopy between $q(e'(y; n))(\hat{x})$ and $q(e(y; n))(\hat{x}) = e_*$ in $C^0((S^k, n); (\mathrm{Maps}(S^2; S^2), e_*))$. Thus, the chain of homotopies $e(y; \hat{x}) \sim e'(y; \hat{x}) \sim q(e'(y; n))(\hat{x}) \sim q(e(y; n))(\hat{x}) = e_*$ shows that $e(y; \hat{x})$ is homotopic to e_* in $C^0((S^k, n); (\mathrm{Maps}(S^2; S^2), e_*))$. But by Theorem 3.5, this contradicts the assumption that e is a generator of $\Pi_k(\mathrm{Maps}((S^2; n), (S^2; n)), e_*)$. Therefore Φ vanishes on $S^2 \times \mathbb{R}^3$.

The remainder of this section contains the proofs of Theorems 4.4 and 4.5. The essence of the proof of Theorem 4.4 is that due to Lemma 4.6, there exists $y \in S^k$ such that $d|\Phi|(y;x)$ and hence $\| \nabla_A \Phi(y) \|_2$ is not identically zero. Theorem 4.5 is used to obtain a uniform lower bound with the aid of the following *apriori* estimates from [2].

Lemma 4.7. *Let $c = (A, \Phi) \in \mathscr{C}$, and suppose that $\nabla_A^2 \Phi = 0$. There exists a constant $0 < \xi < \infty$, which is independent of (A, Φ), such that*

(1) $\| \nabla |\nabla_A \Phi| \|_2^2 + \| \nabla_A \Phi \|_2^2 \leqq \xi \alpha(c)(1 + \alpha^2(c))$.

(2) *If* $|\Phi|^2(x) = 0$, *then* $|\Phi|^2(y) < \frac{1}{2}$ *whenever* $|x-y| < \xi(\alpha(c)(1+\alpha^2(c)))^{-1}$.

(3) *Let* $V = \{x \in \mathbb{R}^3 \colon |\Phi|^2(x) < \frac{1}{2}\}$. *Then* $v = \int_V d^3x \leqq \xi \alpha^3(c)$.

Proof of Lemma 4.7. Statement (1) is Proposition V.8.1 of [2]. Statement (2) is Lemma IV.16.6 of [2]; while Statement (3) follows from the identity $\Delta |\Phi|^2 = 2 |\nabla_A \Phi|^2$. The argument is proved in Sect. IV.16 of [2], see Eq. IV.16.17.

Proof of Theorem 4.4, assuming Theorem 4.5. Let $c(\cdot) \in \Lambda$ be given, and let $\hat{c}(\cdot) \in \Lambda$ be the k-sphere resulting from Theorem 4.5. By Lemma 4.6, there exists $(y_0, x_0) \in S^k \times \mathbb{R}^3$ such that $\hat{\Phi}(y_0; x_0) = 0$.
Using Statements (2) and (3) of Lemma 4.7 on $\hat{\Phi}(y_0; x)$, one obtains upper and lower bounds for v:

$$\xi \alpha^3(\hat{c}(y_0)) \geqq v \geqq \tfrac{4}{3}\pi \xi^3 [\alpha(\hat{c}(y_0))(1 + \alpha^2(\hat{c}(y_0))]^{-3}. \tag{4.10}$$

These bounds imply, by rearranging terms, the upper bound

$$(1 + \alpha^2(\hat{c}(y_0)))^3 \alpha(\hat{c}(y_0))^6 \geqq \xi' > 0\,. \tag{4.11}$$

Equation (4.11) gives a lower bound for $\alpha(\hat{c}(y_0))$ independent of $\hat{c}(y_0)$, and since $\alpha(\bar{c}) \geqq \alpha(\hat{\bar{c}}) \geqq \alpha(\hat{c}(y_0))$, Theorem 4.4 follows. The crucial fact in the proof was Lemma 4.6.

The proof of Theorem 4.5 is an application of the calculus of variations. One first proves that for each $y \in S^k$, $\hat{\Phi}(y;x)$ exists. Then, with ellipitic regularity theorems, one shows that $\hat{\Phi}(y;x) \in C^0(S^1; C^0(\bar{\mathbb{R}}^3; \mathscr{g}) \cap \Gamma(\mathscr{g}))$. The proof is begun with a proposition that establishes that for each $c = (A, \Phi) \in \mathscr{C}$, there exists $\hat{\Phi} \in \Gamma(\mathscr{g})$ which satisfies $\nabla_A^2 \hat{\Phi} = 0$, and is such that $(A, \hat{\Phi}) \in \mathscr{C}$ also.

Proposition 4.8. *Let $(A, \Phi) \in \mathscr{C}$. There exists a unique, smooth $\eta \in L_6(\mathscr{g})$ such that*: (1) $\nabla_A^2(\Phi + \eta) = 0$. (2) $\| \nabla_A(\Phi + \eta) \|_2^2 = \inf\limits_{\phi \in \Gamma^c(\mathscr{g})} \| \nabla_A(\Phi + \phi) \|_2^2$. (3) $|\Phi + \eta| \leqq 1$ *and* $\lim\limits_{|x| \to \infty} |\Phi + \eta| = 1$.

To facilitate the proof of Proposition 4.8, the following Banach space will be used.

Definition 4.9. Define for $A \in \Gamma(A)$, the Banach space $K_A(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$, $p = 0, 1, 2, 3$, to be the closure of $\Gamma^c(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ in the norm

$$\| \phi \|_K^2 = \| \nabla_A \phi \|_2^2\,. \tag{4.12}$$

This space is modeled after the Banach space $K(\mathbb{R}^3)$ $(K(\underset{p}{\wedge} T^*), p = 1, 2, 3)$ which is the closure of $C_0^\infty(\mathbb{R}^3)$ $(\Gamma^c(\underset{p}{\wedge} T^*))$ in the norm $\| \nabla(\cdot) \|_2$.

Some useful properties of K_A are given in the next lemma:

Lemma 4.10. *There exists a constant $\xi < \infty$ which is independent of A, such that for all $\phi \in K_A(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$,*

$$\|\phi\|_6 \leqq \xi \|\phi\|_{K_A}, \tag{4.13}$$

and

$$\|(1+|x|)^{-1}\phi\|_2 \leqq \zeta \|\phi\|_{K_A}. \tag{4.14}$$

Similar estimates hold on $K(\underset{p}{\wedge} T^*)$.

Proof of Lemma 4.10. Equation (4.13) is Corollary VI.6.2 of [2]. Equation (4.14) follows from Kato's inequality [2, Chap. VI.6] and Lemma 5.4 of [31].

To prove Proposition 4.8, it is useful to be more general, so consider the function $Q(\phi)$ on $K_A(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ or on $K(\underset{p}{\wedge} T^*)$ $(p=0, \ldots 3)$ which is defined as follows. For $G \in L_2((\mathscr{g} \otimes \underset{p}{\wedge} T^*) \otimes T^*)$ and $\phi \in K_A(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$

$$Q(\phi) = \tfrac{1}{2}\|\nabla_A \phi\|_2^2 + \langle \nabla_A \phi, G\rangle_2. \tag{4.15}$$

For $G \in L_2(\underset{p}{\wedge} T^* \otimes T^*)$, and $\phi \in K(\underset{p}{\wedge} T^*)$, $Q(\phi)$ is defined to be

$$Q(\phi) = \tfrac{1}{2}\|\nabla \phi\|_2^2 + \langle \nabla \phi, G\rangle_2. \tag{4.16}$$

In order to simplify notation, K_A will denote $K_A(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$ or $K(\underset{p}{\wedge} T^*)$ and ∇_A will denote ∇ on $\Gamma(\underset{p}{\wedge} T^*)$. The relevant properties of $Q(\cdot)$ are summarized by

Lemma 4.11. *The functional $Q(\cdot)$ on K_A defined by either* (4.15) *or* (4.16) *attains its infimum at a unique $\eta \in K_A$. The section η satisfies*

$$\langle \nabla_A \phi, \nabla_A \eta + G\rangle_2 = 0 \quad \textit{for all} \quad \phi \in K_A. \tag{4.17}$$

If G is a C^∞ section, then η is C^∞ also.

Proof of Lemma 4.11. Since $G \in L_2$, the functional Q is C^∞ on K_A. It is weakly lower semi-continuous, and strictly convex. Further, Q satisfies the coercive estimate, $Q(\phi) \geqq \frac{1}{4}\|\phi\|_{K_A}^2 - \|G\|_2^2$, $\phi \in K_A$. Since K_A is a reflexive Banach space, the calculus of variations [32; 2, Chap. IV.7,8] implies that $Q(\cdot)$ achieves a unique minimum, $\eta \in K_A$ and η satisfies (4.12). The *apriori* estimates in [33, Chap. 5] imply that η is smooth if G is.

Proof of Statements 1) *and* 2) *of Proposition 4.8.* Use Lemma 4.11 with $G = \nabla_A \Phi$.

To prove Statement (3) of Proposition 4.8, it is necessary to establish the following *apriori* estimate:

Lemma 4.12. *Let $w \in L^1_{2,\text{loc}}(\mathbb{R}^3)$ and suppose that $|\nabla w| \in L_2(\mathbb{R}^3)$ and $\lim_{|x|\to\infty} w(x) \to 0$. Then w is in $K(\mathbb{R}^3)$ and hence in $L_6(\mathbb{R}^3)$.*

Proof of Lemma 4.12. Consider the functional $Q(\cdot)$ on $K(\mathbb{R}^3)$ defined by

$$Q(v) = \tfrac{1}{2}\|\nabla v\|_2^2 + \langle \nabla v, \nabla w\rangle_2. \tag{4.18}$$

Applying Lemma 4.11 to Q, above, one concludes that there exists a unique $\bar{v} \in K$ such that for all $v \in K$, $\langle \nabla v, \nabla(\bar{v}+w)\rangle = 0$. By elliptic regularity, $\bar{v}+w \in C^\infty(\mathbb{R}^3)$ and

$$\Delta(\bar{v}+w) = 0. \tag{4.19}$$

The maximum principle implies that $\nabla(\bar{v}+w)=0$, so $\bar{v}+w=0$ and $w=-\bar{v}\in K$ as claimed. Lemma 4.10 states that $w \in L_6(\mathbb{R}^3)$ also.

Lemma 4.12 has the following *apriori* estimate as a corollary:

Corollary 4.13. *There exists $\zeta < \infty$, such that for all $(A, \Phi) \in \mathscr{C}$, $\|(1-|\Phi|)\|_6^2 \leqq \zeta \mathscr{a}(A, \Phi)$.*

Proof of Corollary 4.13. Use Lemma 4.12 and the fact that $|\nabla|\Phi|| \leqq |D_A \Phi|$ [2, Chap. VI.6].

Proof of Statement (3) *of Proposition 4.8.* Let $\hat{\Phi} = \Phi + \eta$. Then $||\hat{\Phi}|-1| \leqq ||\Phi| - 1| + |\eta|$, so $(1-|\hat{\Phi}|) \in L_6(\mathbb{R}^3)$, as $|\eta|$ and $(1-|\Phi|)$ are in L_6. Because $\hat{\Phi}$ satisfies $\nabla_A^2 \hat{\Phi} = 0$, the function $(1-|\hat{\Phi}|) \in K(\mathbb{R}^3)$ satisfies the integral inequality $\langle \nabla v, \nabla(1-|\hat{\Phi}|)\rangle \geqq 0$, for all $0 \leqq v \in C_0^\infty(\mathbb{R}^3)$. By the weak maximum principle, $(1-|\hat{\Phi}|) \geqq 0$ on $\mathbb{R}^3$ (cf. [2, Proposition VI.3.5], or any standard PDE text).

The *apriori* estimate, Theorem V.8.1 of [2] is now available. The result is this: There exists a constant $\xi < \infty$, independent of A such that

$$\| |\nabla_A \hat{\Phi}| \|_K^2 \leqq \xi(\|F_A\|_2^2 + \|\nabla_A \hat{\Phi}\|_2^2 + \|F_A\|_2^4 \|\nabla_A \hat{\Phi}\|_2^2). \tag{4.20}$$

(Compare with Lemma 4.7.) Using Lemma 4.10 and Eq. (4.20) one obtains that $|\nabla_A \hat{\Phi}| \in L_6(\mathbb{R}^3)$ and hence $\nabla|\hat{\Phi}| \in L_6(\mathbb{R}^3)$. Therefore, $(1-|\hat{\Phi}|) \in L_6^1(\mathbb{R}^3)$ and one can appeal to Proposition III.7.5 of [2] to complete the proof of Proposition 4.8. [Functions in $L_6^1(\mathbb{R}^3)$ decay to zero as $|x| \to \infty$.]

Before turning to the proof of Theorem 4.5, it is necessary to know that the section $\eta \in \Gamma(\mathscr{g})$ of Proposition 4.8 decays to zero as $|x| \to \infty$.

Proposition 4.14. *Let $(A, \Phi) \in \mathscr{C}$ and suppose that $v = \nabla_A^2 \Phi \in L_2(\mathscr{g})$. Let $\eta \in \Gamma(\mathscr{g}) \cap L_6(\mathscr{g})$ be given by Proposition 4.8. Then (1) $\lim_{|x| \to \infty} |\eta|(x) \to 0$, (2) $\nabla_A \eta \in K_A$ and its norm is bounded by a number which depends only on $\mathscr{a}(A, \Phi)$ and $\|v\|_2$.*

Proof of Proposition 4.14. The section $\eta \in \Gamma(\mathscr{g})$ satisfies $\nabla_A^2 \eta = -v$. Since $|\Phi + \eta| < 1$ and $|\Phi|$ is bounded, then $|\eta|$ is bounded. The result now follows from the *apriori* estimate given by Theorem V.8.1 of [2], and Proposition III.7.5 of [2]. The argument is similar to that used to prove Statement (3) of Proposition 4.8.

Proof of Theorem 4.5. The results of Propositions 4.8 and 4.14 will now be applied. Let $c(\cdot) = (A(\cdot), \Phi(\cdot)) \in \Lambda$, and let $c'(\cdot) = (A(\cdot), \Phi_0(\cdot))$. As $A(y) - A_0(y)$ is compactly supported in a fixed, bounded domain in $\mathbb{R}^3$, independent of $y \in S^k$, so is $D_{A(y)}\Phi_0(y)$. Therefore, $c'(\cdot) \in \Lambda$.

Both Propositions 4.8 and 4.14 are applicable to $c'(y)$ for each $y \in S^k$. Define $\hat{\Phi}(y) = \Phi_0(y) + \eta(y)$, with $\eta(y)$ given by these aforementioned propositions. For each $y \in S^k$, $\hat{c}(y) = (A(y), \hat{\Phi}(y)) \in \mathscr{C}$, and by uniquencess, $\hat{c}(n) = c_* = (0, \frac{1}{2}\sigma^3)$. Due to the convexity of the L_2-norm, $\mathscr{a}(\hat{c}(y)) \leqq \mathscr{a}(c(y))$ for all $y \in S^k$.

For the proof of Theorem 4.5, it remains to show that $\hat{c}(y) \in C^0(S^k; \mathscr{C})$ and that Statements (2) and (3) of Definition 4.2 are satisfied by $\hat{\Phi}(y)$. This is done using Statement (1) of Proposition 4.8.

To begin, the equivalence of the norms on $K_{A(y)}$, $y \in S^k$ must be established. This is so that $\eta(y)$ and $\eta(y')$ for $y \neq y'$ can be compared. It is for this purpose that $A(y) - A_0(y)$ is defined to have compact support which is uniform in $y \in S^k$. The topology on $\mathscr{C}$ also plays a crucial role here.

Lemma 4.15. *Let $(A(y), \Phi(y)) \in \Lambda$. The Banach spaces $K_{A(y)}$ and K_0 are equivalent for all $y \in S^k$. Indeed, there exists a constant $0 < z < \infty$, which is independent of $y \in S^k$ such that for all $\phi \in K_0(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$.*

$$z^{-1} \|\phi\|_{K_0} \leqq \|\phi\|_{K_{A(y)}} \leqq z \|\phi\|_{K_0}, \tag{4.21}$$

and for all $y \in S^k$ and $y' \in S^k$,

$$\lim_{y' \to y} \| \nabla_{A(y)}\phi - \nabla_{A(y')}\phi \|_2 \to 0. \tag{4.22}$$

Proof of Lemma 4.15. It is sufficient for the proof of this lemma to establish (4.21) and (4.22) for $\phi \in \Gamma^c(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$. Write $A(y;x) = A_0(y;x) + \omega(y;x)$. By assumption, $R < \infty$ and a ball $B_R \subset \mathbb{R}^3$ exist such that for all $y \in S^k$, $\omega(y;x) = 0$ if $x \notin B_R$. Fix $\phi \in \Gamma^c(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$, and consider the difference in the norms:

$$\begin{aligned} | \|\nabla_{A(y)}\phi\|_2 - \|\nabla\phi\|_2 | &\leqq \|[A(y), \phi]\|_2, \\ &\leqq \|[A_0(y), \phi]\|_2 + \|[\omega(y), \phi]\|_2, \\ &\leqq z_1 (\|(1+|x|)^{-1}\phi\|_2 + \|\phi\|_6 \|\omega(y)\|_3). \end{aligned}$$

The last line follows using Hölder's inequality, and the fact that $|A_0(y)| \leqq z(1+|x|)^{-1}$. By assumption, $\omega(y) \in C^0(S^k; \Gamma^c(B_R; \mathscr{g} \otimes T^*))$, so $\|\omega(y)\|_3$ is uniformly bounded on S^k. Thus

$$\begin{aligned} | \|\nabla_{A(y)}\phi\|_2 - \|\nabla_0\phi\|_2 | &\leqq z_2 (\|(1+|x|)^{-1}\phi\|_2 + \|\phi\|_6), \\ &\leqq 2 z_2 \zeta \|\nabla_0 \phi\|_2, \end{aligned} \tag{4.23a}$$

or

$$\leqq 2 z_2 \zeta \|\nabla_{A(y)}\phi\|_2. \tag{4.23b}$$

Here, the last two steps follow using Lemma 4.10. Equation (4.21) follows directly from (4.23).

The assumptions in Definition 2.1 on $A(y) = A_0(y) + \omega(y)$ imply that a continuous function, $z(y,y')$ on $S^k \times S^k$ exists such that

$$\begin{aligned} &(1)\ \lim_{y \to y'} z(y,y') \to 0. \\ &(2)\ |A_0(y) - A_0(y')| \leqq z(y,y')(1+|x|)^{-1}. \\ &(3)\ |\nabla(A_0(y) - A_0(y'))| \leqq z(y,y')(1+|x|)^{-2}. \end{aligned} \tag{4.24}$$

To prove Eq. (4.22), fix $\phi \in \Gamma^c(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$. Then for all $y, y' \in S^k$,

$$\begin{aligned} \|\nabla_{A(y)}\phi - \nabla_{A(y')}\phi\|_2 &\leqq \|[A_0(y) - A_0(y'), \phi]\|_2 + \|[\omega(y) - \omega(y'), \phi]\|_2 \\ &\leqq (z(y;y') + \|(1+|x|)\,|\omega(y) - \omega(y')|\,\|_\infty) \|(1+|x|)^{-1}\phi\|_2. \end{aligned} \tag{4.25}$$

Equations (4.24) and (4.25), the fact that $\omega(y)\in C^0(S^k;\Gamma^c(B_R;\mathcal{g}\otimes T^*))$ and Lemma 4.10 now establish Eq. (4.22). One may conclude from Proposition 4.14 and Lemma 4.15 that $\eta(y)$ and $\nabla\eta(y)\in K_{A(y')}$ for all $y, y'\in S^k$.

In order to compare $\eta(y)$ with $\eta(y')$ one uses Statement (1) of Proposition 4.8. Thus, for $y, y'\in S^k$, $\eta(y)-\eta(y')$ satisfies

$$\begin{aligned}
&\nabla^2_{A(y)}(\eta(y)-\eta(y'))\\
&= -\nabla^2_{A(y)}\Phi_0(y)+\nabla^2_{A(y')}\Phi_0(y')-2[A_i(y)-A_i(y'),\nabla^i\eta(y')]-[\nabla^i(A_i(y)\\
&\quad -A_i(y')),\eta(y')]\\
&\quad -[A_i(y)-A_i(y'),[A_i(y'),\eta(y')]]+[A_i(y'),[A_i(y')-A_i(y),\eta(y')]], \quad (4.26)
\end{aligned}$$

where $A_i=\dfrac{\partial}{\partial x^i}\,\lrcorner\, A$ and $\nabla^i=\dfrac{\partial}{\partial x^i}\,\lrcorner\, d$. Let $v(y,y')$ denote the right hand side of (4.26). Continuity properties of $v(y,y')$ as $y'\to y$ imply continuity properties of $\eta(y)-\eta(y')$ as $y'\to y$. Elliptic regularity techniques, the "bootstrap" arguments, are used to exhibit this. These begin with

Lemma 4.16. *Let* $v(y,y')$ *denote the right hand side of* (4.26), *then* $\lim_{y'\to y}\|(1+|x|)\,v(y,y')\|_2\to 0$.

Proof of Lemma 4.16. The proof is simplified by splitting $v(y,y')(x)$ into two parts. The part exterior to the ball of radius R [where $\omega(y;x)\equiv 0$] is denoted $v^{\mathrm{ex}}(y,y')$. The part interior to this ball is denoted $v^{\mathrm{in}}(y,y')$.

For $|x|>R$, $A(y)=A_0(y)$ and $\nabla^2_{A(y)}\Phi_0(y)=0$. It follows from (4.24) that

$$|(1+|x|)\,v^{\mathrm{ex}}(y,y')(x)|\leqq z_1\cdot z(y,y')\,\{|\nabla\eta(y')|+(1+|x|)^{-1}\,|\eta(y')|\}. \quad (4.27)$$

Using (4.27), Proposition 4.8, Lemmas 4.15 and 4.10, one obtains the inequality

$$\|(1+|x|)\,v^{\mathrm{ex}}(y,y')(x)\|_2\leqq z_2\cdot z(y,y'). \quad (4.28)$$

Meanwhile, $v^{\mathrm{in}}(y,y')$ satisfies

$$\begin{aligned}
\|(1+|x|)\,v^{\mathrm{in}}(y,y')(x)\|_2 &\leqq z_1(R)\cdot z(y,y')\,\|\eta(y')\|_{K_0} \quad (4.29)\\
&\quad +\left(\frac{4\pi}{3}R^3\right)^{1/2}\sup_{x\in B_R}|\nabla^2_{A(y)}\Phi_0(y)-\nabla^2_{A(y')}\Phi_0(y')|,
\end{aligned}$$

The right hand sides of both (4.28,9) vanish as $y'\to y$, which proves Lemma 4.16.

Completion of the Proof of Theorem 4.5. Take the L_2-inner product of both sides of (4.26) with $(\eta(y)-\eta(y'))$. As both $\eta(y), \eta(y')\in K_{A(y)}(\mathcal{g})$, one can integrate the left hand side of the resulting expresson by parts to obtain

$$\begin{aligned}
\|\nabla_{A(y)}(\eta(y)-\eta(y'))\|_2 &= -\langle\eta(y)-\eta(y'),v(y,y')\rangle_2,\\
&\leqq \|(1+|x|)^{-1}(\eta(y)-\eta(y')\|_2\,\|(1+|x|)\,v(y,y')\|_2,\\
&\leqq \zeta(\|\eta(y)\|_{K_0}+\|\eta(y')\|_{K_0})\,\|(1+|x|)\,v(y,y')\|_2. \quad (4.30)
\end{aligned}$$

Using Eq. (4.30) and Lemmas 4.16 and 4.15, one concludes that

$$\eta(\cdot)\in C^0(S^k;K_0). \quad (4.31)$$

By using (4.26) in conjunction with (4.31), Lemmas 4.15, 4.16, and Theorem V.8.1 of [2], one obtains as well that

$$\nabla\eta(y) \in C^0(S^k; K_0). \tag{4.32}$$

Since $K_0 \to L_6$ continuously, Eqs. (4.32,1) imply that $\eta(y) \in C^0(S^k; L_6^1(\mathscr{g}))$, and since $L_6^1 \to C^0(\bar{\mathbb{R}}^3)$ continuously, $\eta(y) \in C^0(S^k; C^0(\bar{\mathbb{R}}^3; \mathscr{g}))$. Thus $\eta(y)$ is uniformly continuous with respect to the supremum norm on $\mathbb{R}^3$. The continuity of $\nabla_{A_0(y)}\eta(y)$ in $L_2(\mathscr{g} \otimes T^*)$ follows from Eqs. (4.26) and (4.31). The proof that $\eta(y) \in C^0(S^k; \Gamma(\mathscr{g}))$ requires only local estimates. This is standard, and the reader is referred to [33, Chap. 5, 6]. Thus $\eta(y)$ satisfies all the requirements of Definition 4.1 and $\hat{c}(y) = (A(y), \Phi_0(y) + \eta(y)) \in \Lambda$. This completes the proof of Theorem 4.5.

V. The Minimizing Sequence

Presently, a sequence in Λ will be used to construct a solution to Eqs. (2.2) and (2.3). This entails choosing a "good" sequence of spheres $\{c_i(\cdot)\} \in \Lambda$ and proving that the resulting sequence of configurations $\{\bar{c}_i\}$ converges to a solution. This section deals only with the question of convergence over bounded domains, where the convergence follows from *K*. Uhlenbeck's weak compactness theorems. A good sequence has the property that $\nabla\alpha_{\bar{c}_i} \to 0$ as well. As in the finite dimensional case, the existence of "good" sequences is a consequence of the fact the spheres in Λ are not null-homotopic. The following simplified notation will be used: If $\{c_i(y)\} \subset \Lambda$, then $c_i(y) = (A_i(y), \Phi_i(y))$, $\bar{c}_i = (A_i, \Phi_i)$, $\nabla\alpha_{c_i}(y) = \nabla\alpha_{i,y}$, $\nabla\alpha_{\bar{c}_i} = \nabla\alpha_i$ and $\mathscr{H}_{\bar{c}_i} = \mathscr{H}_i$.

Crucial to the proofs in this section is the fact that there exist configurations $c = (A, \Phi) \in \mathscr{C}$ which satisfy

$$D_A * D_A \Phi = 0, \tag{5.1}$$

and spheres $c(y) \in \Lambda$ which satisfy (5.1) for all $y \in S^k$ (Theorem 4.5).

Definition 5.1. Let $c = (A, \Phi) \in \mathscr{C}$. Define the Banach space H_c to be the closure of $\Gamma^c((\mathscr{g} \otimes T^*) \otimes \mathscr{g})$ in the norm $\|\psi\|_c^2 = \|\nabla_A\psi\|_2^2 + \|[\Phi, \psi]\|_2^2$. The symbol $\|\cdot\|_c$ will also denote the above norm on $\Gamma^c(\mathscr{g} \otimes \underset{p}{\wedge} T^*)$, $p = 0, 1, 2, 3$. For $c(y) \in \Lambda$, the following shorthand will often be used: $H_{c(y)} = H_y$, $\|\cdot\|_{c(y)} = \|\cdot\|_y$. For $\{\bar{c}_i\} \in \mathscr{C}$ a sequence: $H_{\bar{c}_i} = H_i$ and $\|\cdot\|_{\bar{c}_i} = \|\cdot\|_i$.

Proposition 5.2. *Let $c \in \mathscr{C}$. Then $\alpha_c(\cdot) \equiv \alpha(c + \cdot)$ extends to a C^∞ functional on H_c. If $c \in \mathscr{C}$ and $\alpha(c) \leqq B$, then the following estimates hold with $\varkappa = \varkappa(B)$: (1) $|\alpha_c(\psi) - \alpha(c) - \nabla\alpha_c(\psi)| \leqq \varkappa \|\psi\|_c^2 (1 + \|\psi\|_c^2)$, (2) $|\alpha_c(\psi) - \alpha(c) - \nabla\alpha_c(\psi) - \frac{1}{2}\mathscr{H}_c(\psi)| \leqq \varkappa \|\psi\|_c^3 (1 + \|\psi_c\|)$. If $c(y) \in \Lambda$, and $\psi \in \Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$, then (3) $\alpha_y(\psi)$, $\nabla\alpha_y(\psi)$, $\mathscr{H}_y(\psi)$ and $\|\psi\|_y$ are all continuous functions of y.*

Next, consider sequences $\{c_i(\cdot)\} \in \Lambda$ which satisfy (5.1) for all $y \in S^k$.

Proposition 5.3. *It is possible to choose a sequence $\{c_i(y)\} \in \Lambda$ which satisfy* (5.1) *for all $y \in S^k$, and satisfy in addition*

$$\begin{aligned} &(1) && \lim_{i\to\infty} \mathscr{a}(\bar{c}_i) \to \mathscr{a}_\infty . \\ &(2) && \mathscr{a}(\bar{c}_i) \geqq \mathscr{a}(\bar{c}_{i+1}) . \\ &(3) && \lim_{i\to\infty} \|\nabla \mathscr{a}_i\|_{H_i^*} \to 0 . \end{aligned} \tag{5.2}$$

The proofs of Propositions 5.2 and 5.3 are deferred to Sect. 6. For the remainder of this section, assume their validity.

Definition 5.4. A good sequence $\{c_i(\cdot)\} = \{(A_i(\cdot), \Phi_i(\cdot)\} \in \Lambda$ is one which satisfies (5.1) and (5.2).

The next theorem states that the sequence $\{\bar{c}_i\}$ that is induced by a good sequence of loops converges. More generally, Theorem 5.6 below states that a form of Condition C of Palais-Smale, (cf. [10, Chap. 6]) is valid locally. The main part of the proof is due to Uhlenbeck [11] (see also [34]). The next definition defines the relevant form of convergence.

Definition 5.5. Let $\{c_i = (A_i, \Phi_i)\}_{i=1}^\infty \in \mathscr{C}$. The sequence $\{c_i\}$ is said to converge strongly in $L^1_{2,\mathrm{loc}}$ to $c = (A, \Phi) \in \mathscr{C}$ if the following is true: (1) There exists a uniform, open cover of $\mathbb{R}^3$ by balls $\{V_\alpha\}$ of radius $r > 0$. (2) There exist, for each i, α, gauge transformations $g_\alpha(i) \in L^2_2(V_\alpha; \mathrm{SU}(2))$. (3) For each α, the sequence $\{g_\alpha(i)\, c_i\}$ converges strongly in $L^1_2(V_\alpha; (\mathscr{g} \otimes T^*) \oplus \mathscr{g})$ to some (A_α, Φ_α). (4) For each α, β, the sequence $\{g_{\alpha\beta}(i) = g_\alpha(i)\, g_\beta^{-1}(i)\}$ converges strongly in $L^2_2(V_\alpha \cap V_\beta; \mathrm{SU}(2))$. (5) In each $V_\alpha \cap V_\beta$; $(A_\alpha, \Phi_\alpha) = g_{\alpha\beta}(A_\beta, \Phi_\beta)$. (6) For each α, there exists $h_\alpha \in L^2_2(V_\alpha; \mathrm{SU}(2))$ such that $h_\alpha c = (A_\alpha, \Phi_\alpha)$ in V_α.

Armed with this definition, the convergence result can be stated.

Theorem 5.6. *Let $\{\bar{c}_i\} = (A_i, \Phi_i) \in \mathscr{C}$ be a sequence that satisfies* (1) $\mathscr{a}(\bar{c}_i) \leqq B$, (2) $\lim_{i\to\infty} \|\nabla \mathscr{a}_i\|_{H_i^*} \to 0$, (3) *Eq.* (5.1). *Then there is a subsequence of $\{\bar{c}_i\}$ which converges strongly in $L^1_{2,\mathrm{loc}}$ to $(A, \Phi) \in \mathscr{C}$, and (A, Φ) is a solution to Eqs.* (2.2) and (2.3).

Proof of Theorem 5.6 assuming Propositions 5.2 and 5.3. The proof is a direct application of K. Uhlenbeck's weak compactness results for gauge fields [11]. For convenience, her relevant results are stated in the following proposition.

Proposition 5.7 (Uhlenbeck [11]). *Let $\{A_i\}_{i=1}^\infty$ be a sequence of C^∞ connections on $\mathbb{R}^3$ with $\|F_{A_i}\|_2^2 \leqq B$. There is an infinite subsequence, also denoted $\{A_i\}$, a constant $\mu > 0$, an $r(B) > 0$, a countable, uniform cover of $\mathbb{R}^3$ by balls $\{U_\alpha\}_{\alpha=1}^\infty$ of radius $r(B)$, and a sequence of gauge transformations $\{g_\alpha(i) \in \Gamma(U_\alpha \cap U_\beta; \mathrm{SU}(2))\}_{i,\alpha=1}^\infty$ such that the following is true*:

(a) *In* U_α, $A_\alpha(i) \equiv g_\alpha(i)\, A_i\, g_\alpha(i)^{-1} + g_\alpha(i)\, dg_\alpha(i)^{-1}$ *satisfies* (1) $d^* A_\alpha(i) = 0$, (2) $i_\alpha^*(*A_\alpha(i)) = 0$, (3) $\|A_\alpha(i)\|_{L^1_2(U_\alpha)} \leqq \mu B$.

(b) *$A_\alpha(i)$ converges weakly in $L^1_2(U_\alpha)$ to A_α.*

(c) *$F_{A_\alpha(i)}$ converges weakly in $L_2(U_\alpha)$ to F_{A_α}.*

(d) $g_\alpha(i)\, g_\beta^{-1}(i)$ *converges weakly in* $L_2^2(U_\alpha \cap U_\beta; \mathrm{SU}(2))$ *to* $g_{\alpha\beta}$ *and* $g_{\alpha\beta} \in C^0(U_\alpha \cap U_\beta; \mathrm{SU}(2))$.

(e) $A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta}\, dg_{\alpha\beta}^{-1}$ *in* $U_\alpha \cap U_\beta$.

(f) $d * A_\alpha = 0$ *in* U_α.

(g) $\iota_\alpha^*(*A_\alpha) = 0$.

Here ι_α: $\partial U_\alpha \to U_\alpha$ *is the inclusion.*

The sequence of connections in mind is that defined by the sequence $\{\bar{c}_i = (A_i, \Phi_i)\}$. With no loss of generality, assume that the $\{A_i\}$ satisfy (a)–(g) of Proposition 5.7.

Next consider the Higgs field. It will be shown that in each U_α, the sequence $\{g_\alpha(i)\, \Phi_i\, g_\alpha^{-1}(i)\}$ has a weakly convergent subsequence in $L_2^1(U_\alpha; \mathscr{g})$. This is established by proving that the $L_2^1(U_\alpha; \mathscr{g})$ – norms of the sequence are uniformly bounded in i. Set $\Phi_\alpha(i) = g_\alpha(i)\, \Phi_i\, g_\alpha^{-1}(i)$ in U_α. Notice that because (A_i, Φ_i) satisfies (5.1), $|\Phi_\alpha(i)| \leqq 1$. Now compute the $L_2^1(U_\alpha; \mathscr{g})$ norm of $\Phi_\alpha(i)$:

$$\begin{aligned} \|\Phi_\alpha(i)\|^2_{L_2^1; U_\alpha} &\equiv \int_{U_\alpha} d^3x \{|d\Phi_\alpha(i)|^2 + |\Phi_\alpha(i)|^2\} \\ &\leqq 2 \int_{U_\alpha} d^3x \{|\nabla_{A_\alpha(i)} \Phi_\alpha(i)|^2 + 2(|A_\alpha(i)|^2 + 1)\} \\ &\leqq 4(\mathscr{a}(\bar{c}_i) + \mu B) \leqq B'. \end{aligned} \tag{5.3}$$

The last line follows because by construction, the sequence $A_\alpha(i)$ converges weakly in $L_2^1(U_\alpha)$, and, do to the Rellich Lemma, strongly in $L_2(U_\alpha)$.

Lemma 5.8. *Let* $\{\bar{c}_i = (A_i, \Phi_i)\}$ *as before. There exists an infinite subsequence, also denoted* $\{\bar{c}_i\}$ *such that* $\{A_i\}$ *satisfies* (a)–(g) *of Proposition* 5.7, *and* (a) $\Phi_\alpha(i)$ *converges weakly to* Φ_α *in* $L_2^1(U_\alpha)$, (b) $D_{A_\alpha(i)} \Phi_\alpha(i)$ *converges weakly to* $D_{A_\alpha} \Phi_\alpha$ *in* $L_2(U_\alpha)$, (c) $\Phi_\alpha = g_{\alpha\beta} \Phi_\beta g_{\alpha\beta}^{-1}$ *in* $U_\alpha \cap U_\beta$.

Proof. The proof uses a process of choosing subsequences called diagonalization [34]. Begin in U_1. The unit ball in $L_2^1(U_1)$ is weakly compact; thus (5.3) implies that an infinite subsequence $\{i'\} \in \mathbb{Z}_+$ exists such that $\Phi_1(i')$ converges weakly to Φ_1 in $L_2^1(U_1)$. The map $(A, \Phi) \to D_A \Phi$ from $L_2^1(U_1; (\mathscr{g} \otimes T^*) \oplus \mathscr{g}) \to L_2(U_1; \mathscr{g} \otimes T^*)$ is weakly continuous (use the Rellich lemma [16]), which implies statement (b) above, in U_1. Now relabel $i' \to i = 1, 2 \ldots$ (this is diagonalization) and repeat the procedure in U_2. Since the open cover $\{U_\alpha\}$ is countable, this procedure proceeds by induction, establishing (a), (b) of the lemma. To establish (c) of the Lemma, it is enough to remark that multiplication from $L_2^2(U_\alpha) \times L_2^1(U_\alpha) \to L_2^1(U_\alpha)$ is weakly continuous in three dimensions [16].

It will now be established that the configuration $(A_\alpha, \Phi_\alpha) \in L_2^1(U_\alpha; (\mathscr{g} \otimes T^*) \oplus \mathscr{g})$ is a weak solution to (2.2) in U_α. It is a straightforward exercise to verify that the Bianchi identities (2.2c, d) are automatically satisfied by any $(A, \Phi) \in L_2^1$.

Let $L_{2,c}^1(U_\alpha; \cdot)$ denote the Banach space of L_2^1 sections over U_α which vanish on ∂U_α.

Lemma 5.9. *The imbedding,* $S_{\alpha,i}$: $L_{2,c}^1(U_\alpha; (\mathscr{g} \otimes T^*) \oplus \mathscr{g}) \to H_i$ *given by* $S_{\alpha,i}(\psi) = g_\alpha(i)^{-1} \psi g_\alpha(i)$ *satisfies* $\lim_{i \to \infty} \|S_{\alpha,i}^* \nabla \mathscr{a}_i(\cdot)\|_* \to 0$, *uniformly in* α. *Here* $\|\cdot\|_*$ *is shorthand for the norm on* $L_{2,c}^1(U_\alpha; (\mathscr{g} \otimes T^*) \oplus \mathscr{g})^*$.

Proof of Lemma 5.9. In order to prove the assertion concerning ∇a_i it suffices to show that the norms on $\psi \in L^1_{2,c}$ and $g_\alpha^{-1}(i)\,\psi g_\alpha(i) \in H_i$ are equivalent, uniformly in the index α. Using the inherent gauge invariance,

$$\begin{aligned}\|S_{\alpha,i}(\psi)\|^2_{H_i} &= \|\nabla_{A_\alpha(i)}\psi\|^2_2 + \|[\Phi_\alpha(i),\psi]\|^2_2\\ &\leqq 2\|\nabla\psi\|^2_2 + 4\|\psi\|^2_4\,\|A_\alpha(i)\|^2_4 + 4\|\psi\|^2_2\\ &\leqq 4(1+c\cdot B')(\|\nabla\psi\|^2_2 + \|\psi\|^2_2)\,.\end{aligned}$$

All integrations above are implicitly restricted to U_α. Line 2 follows from line 1 using Hölder's inequality and Lemma 5.9. Line 3 uses the imbedding $L^1_2 \to L_4$ and (a) of Proposition 5.7. On the other hand,

$$\begin{aligned}\|\nabla\psi\|^2_2 + \|\psi\|^2_2 &\leqq 2(\|\nabla_{A_\alpha(i)}\psi\|^2_2 + \|A_{\alpha(i)}\|^2_4\,\|\psi\|^2_4 + \|\psi\|^2_2)\\ &\leqq 2(\|\nabla_{A_\alpha(i)}\psi\|^2_2 + \ell(r)(B'+1)\,\|\psi\|^2_6)\\ &\leqq \ell'(r,B')\,\|S_{\alpha,i}(\psi)\|^2_{H_i}.\end{aligned}$$

Here, line 1 uses Hölder's inequality. Line 2 follows from Proposition 5.7 (a) and the fact that the volume of U_2 is $r^3\,4\pi/3$. Line 3 uses Lemma 4.9.

Notice that $S^*_{\alpha,i}\nabla a_i(\cdot)$ is just $\nabla a_{(A_\alpha(i),\Phi_\alpha(i))}(\cdot)$ restricted to $L^1_{2,c}(U_\alpha;(\mathscr{g}\otimes T^*)\oplus\mathscr{g})$.

If $(A,\Phi)\in L^1_2(U_\alpha;\mathscr{g}\otimes T^*)\oplus\mathscr{g}))$ then it is a straightforward exercise in the Sobolev inequalities to show that $\nabla a_{(A,\Phi)}\in L^{1\,*}_{2,c}(U_\alpha;(\mathscr{g}\otimes T^*)\oplus\mathscr{g})$.

Lemma 5.10. *Let $\{\bar c_i = (A_i,\Phi_i)\}$ satisfy the assumption of Theorem 5.6 and denote by $(A_\alpha,\Phi_\alpha)\in L^1_2(U_\alpha;(\mathscr{g}\otimes T^*)\oplus\mathscr{g})$, the limiting configuration of Proposition 5.7 and Lemma 5.8. Then $\nabla a_{(A_\alpha,\Phi_\alpha)}(\cdot)=0$ on $L^1_{2,c}(U_\alpha;(\mathscr{g}\otimes T^*)\oplus\mathscr{g})$.*

Proof of Lemma 5.10. Using Lemma 5.9, one shows that the map $(A,\Phi)\to\nabla a_{(A,\Phi)}(\cdot)$ from $L^1_2(U_\alpha)$ to $L^{1\,*}_{2,c}(U_\alpha)$ is weakly continuous. For example, for $\eta\in L^1_{2,c}(U_\alpha;\mathscr{g})$,

$$\begin{aligned}&|\nabla a_{(A_\alpha,\Phi_\alpha)}((0,\eta)) - \nabla a_{(A_\alpha(i),\Phi_\alpha(i))}((0,\eta))|\\ &\leqq |\langle[A_\alpha - A_\alpha(i),\eta], D_{A_\alpha(i)}\Phi_\alpha(i)\rangle_2| + |\langle D_{A_\alpha}\eta, D_{A_\alpha}\Phi_\alpha - D_{A_\alpha(i)}\Phi_\alpha(i)\rangle_2|,\\ &\leqq 2a(\bar c_0)\,\|\eta\|_4\,\|A_\alpha - A_\alpha(i)\|_4 + |\langle D_\alpha\eta, D_{A_\alpha}\Phi_\alpha - D_{A_\alpha(i)}\Phi_\alpha(i)\rangle_2|\,.\end{aligned}$$

The first term vanishes as $i\to\infty$ as the imbedding $L^1_2(U_\alpha)\to L_4(U_\alpha)$ is compact. The second term vanishes as $i\to\infty$ because the map $(A_\alpha(i),\Phi_\alpha(i))\to D_{A_\alpha(i)}\Phi_\alpha(i)$ of $L^1_2(U_\alpha)\to L_2(U_\alpha)$ is weakly continuous. The remainder of the proof is similar and it is omitted.

In order to discuss the strong convergence of $\{(A_\alpha(i),\Phi_\alpha(i))\}$ in L^1_2, choose a subcover $\{V_\alpha\subset U_\alpha\}$ of balls of radius $\geqq r/2$.

Lemma 5.11. *For each α, the sequence $\{(A_\alpha(i),\Phi_\alpha(i))\}$ converges strongly in $L^1_2(V_\alpha;(\mathscr{g}\otimes T^*)\oplus\mathscr{g})$ to (A_α,Φ_α). Thus the sequences $\{F_{A_\alpha(i)}, D_{A_\alpha(i)}\Phi_\alpha(i)\}$ converge strongly to $(F_{A_\alpha}, D_{A_\alpha}\Phi_\alpha)$ in $L_2(V_\alpha)$, and the sequence $\{g_{\alpha\beta}(i)\}$ converges strongly to $g_{\alpha\beta}$ in $L^2_2(V_\alpha\cap V_\beta)$.*

Proof. By the Rellich lemma, $\{(A_\alpha(i),\Phi_\alpha(i))\}$ converges strongly to (A_α,Φ_α) in $L_p(U_\alpha)$, $p<6$. Let β be a cut off function which is 1 on V_α and 0 on $\mathbb{R}^3\setminus U_\alpha$. For convenience, the index α will be suppressed. Let $a_i = A - A(i)$. Then $\beta a_i \in L^1_{2,c}(U_\alpha)$. Using Lemma 5.10 and both (a) and (f) of Proposition 5.7,

$$\begin{aligned} 0 &= \langle \nabla^\beta a_i, \nabla A\rangle_2 + \langle \beta(A \wedge a_i + a_i \wedge A) \wedge * F_A + [a_i, \Phi] \wedge {}^*D_A\Phi\rangle_2\,, \\ 0 &= \langle \beta \nabla a_i, \nabla a_i\rangle_2 + \langle \beta a_i \wedge (A \wedge * F_A - * F_A \wedge A - [\Phi, {}^*D_A\Phi] - A_i \wedge {}^*F_{A_i} \\ &\quad + {}^*F_{A_i} \wedge A_i + [\Phi_i, {}^*D_{A_i}\Phi_i])\rangle + \langle a_i \nabla\beta, \nabla a_i\rangle_2 + \nabla \mathscr{a}_{(A_i, \Phi_i)}((\beta a_i, 0))\,. \end{aligned} \tag{5.4}$$

To derive line 2, add and subtract $\nabla \mathscr{a}_{(A_i, \Phi_i)}((\beta a_i, 0))$ from line 1 above. Now, $\nabla\beta \in L_\infty$, A_i converges strongly to A in $L_4(U)$ and ∇A, ∇A_i, F_A and F_{A_i} are uniformly bounded in $L_2(U)$. Hence, from (5.4), one obtains with Hölder's inequality that $\|\nabla(A - A_i)\|^2_{2;V} \leqq \text{constant } (\|A - A_i\|_{4;U} + \|\nabla \mathscr{a}_i\|_*)$. Hence $A_i \to A$ strongly in $L^1_2(V; (\mathscr{g} \otimes T^*))$. The proof that $\Phi_i \to \Phi$ strongly in $L^1_2(V; \mathscr{g})$ is similar. To see that the transition functions converge, note that by the Rellich lemma, $g_{\alpha\beta}(i)$ converges strongly to $g_{\alpha\beta}$ in $L^1_2(V_\alpha \cap V_\beta)$ and $C^0(V_\alpha \cap V_\beta)$. Strong convergence in $L^2_2(V_\alpha \cap V_\beta)$ follows by differentiating

$$A_\alpha(i) = g_{\alpha\beta}(i)\, A_\beta(i)\, g^{-1}_{\alpha\beta}(i) + g_{\alpha\beta}(i)\, dg^{-1}_{\alpha\beta}(i)\,.$$

By construction, $(A_\alpha, \Phi_\alpha) = g_{\alpha\beta}(A_\beta, \Phi_\beta)$ in $V_\alpha \cap V_\beta$. Using Theorem V.2.4 of [2], and Proposition 5.7, one obtains that (A_α, Φ_α) is C^∞ in V_α, and $g_{\alpha\beta}$ is C^∞ in $V_\alpha \cap V_\beta$. Hence, by Theorem V.6.1 of [2], there exist sections $h_\alpha \in L^2_2(V_\alpha; \mathrm{SU}(2))$ such that $h_\alpha(A_\alpha, \Phi_\alpha)$ is C^∞ and $h_\alpha g_{\alpha\beta} h^{-1}_\beta = 1$, for all α, β. In addition, (A, Φ) defined by $(A, \Phi)\big|_{V_\alpha} = h_\alpha(A_\alpha, \Phi_\alpha)$ is a solution to (2.2).

It remains to establish that $(A, \Phi) \in \mathscr{C}$, that is, to establish that $\mathscr{a}(A, \Phi) < \infty$, and that the $\lim\limits_{|x| \to \infty} |\Phi| \to 1$. Let $F_i = F_{A_i}$. The sequence $\{|F_i|\} \in L_2(\mathbb{R}^3)$ converges strongly to $|F|$ in $L_2(U)$ for any open, bounded set $U \subset \mathbb{R}^3$. Since $C^\infty_0(\mathbb{R}^3)$ is dense in $L_2(\mathbb{R}^3)$, the sequence converges weakly to $|F|$ in $L_2(\mathbb{R}^3)$. By the weak-lower semi-dontinuity of the L_2 norm, $\|F\|^2_2 \leqq \lim\limits_{i \to \infty} \|F_i\|^2_2 < \infty$. A similar argument holds for $\|D_A\Phi\|^2_2$. Thus $\mathscr{a}(c) \leqq \lim\limits_{i \to \infty} \mathscr{a}(\bar{c}_i) < \infty$.

To establish Eq. (2.3) for Φ, it is important that Eq. (5.1) hold for each $\bar{c}_i$. Using (4.20), Lemma 4.10 and Kato's inequality one obtains a uniform upper bound for $\|\nabla(1 - |\Phi_i|)\|_6$. Meanwhile, Corollary 4.13 and Lemma 4.10 give a uniform upper bound for $\|(1 - |\Phi_i|)\|_6$. Therefore, $\{(1 - |\Phi_i|)\}$ has weakly convergent subsequences in L^1_6. On any bounded domain $U \subset \mathbb{R}^3$, $\{1 - |\Phi_i|\}$ converges strongly to $(1 - |\Phi|)$ in $L^1_6(U)$. Hence, the weak limits of $\{1 - |\Phi_i|\}$ are the same and they are equal to $(1 - |\Phi|)$ a.e. on $\mathbb{R}^3$. Therefore $(1 - |\Phi|) \in L^1_6$ too. By Proposition III.7.5 of [2],

$$\lim_{|x| \to \infty} (1 - |\Phi|(x)) \to 0\,.$$

VI. The Gradient of $\mathscr{a}$

The existence in Λ of good sequences has the following intuitive basis. Imagine $c(y) \in \Lambda$ with $\mathscr{a}(\bar{c})$ very close to $\mathscr{a}_\infty$. By definition, $c(y)$ is a continuous map from S^k into $\mathscr{C}$. It is here that the topology of $\mathscr{C}$ plays a crucial role. In the C^∞ topology,

$\nabla a_{c(y)}(\psi)$ is continuous in y when ψ is a fixed element of $\Gamma^c((\mathscr{g}\otimes T^*)\oplus\mathscr{g})$ (Proposition 5.2). Because of this continuity, the k-sphere $c(y)$ can be deformed in the direction of the gradient flow to lower the action along the loop. There must exist an $s\in S^k$ with both $a(c(s))>a_\infty$ and $\nabla a_{c(s)}$ small or else this deformation would produce a new $c'(y)\in\Lambda$ with $a(\bar{c}')<a_\infty$, an impossibility. In fact, the gradient flow yields $c'(y)\in\Lambda$ with $\bar{c}'=c(s)$. This new k-sphere is a good one.

The above procedure outlines the proof of Proposition 5.3. There are four steps. The first step is to prove Proposition 5.2, and this step is done last. The remaining steps are done in order. Step 2 is to prove that $\|\nabla a_{c(y)}\|_{H^*_{c(y)}}$ is a continuous function of $y\in S^k$ when $c(y)\in\Lambda$. The third step is the construction of the deformation along the gradient flow. The fourth step is the verification that this deformation has the required effect.

Proposition 5.3 will be shown to be a consequence of

Proposition 6.1. *Let $c(y)\in\Lambda$, with $a(c(y))\leqq B$. Given $\varepsilon>0$, there exists $b(y)\in\Lambda$ which satisfies (1) $a(b(y))\leqq a(c(y))$ and (2) Eq. (5.1) for all $y\in S^k$. (3) $\|\nabla a_{\bar{b}}\|_*^2(1+\|\nabla a_{\bar{b}}\|_*)^{-1}\leqq\max[\varepsilon, v(B)(a(\bar{b})-a_\infty)]$. Here, $\|\cdot\|_*$ denotes the norm on $H_{\bar{b}}^*$, and v is a constant which depends only on B.*

Proof of Proposition 5.3, assuming Proposition 6.1. Choose a sequence $\{c_i(y)\}\in\Lambda$ such that $\lim_{i\to\infty} a(\bar{c}_i)\to a_\infty$. Now apply Proposition 6.1 to obtain the sequence $\{b_i(y)\}\in\Lambda$. A strictly decreasing subsequence will satisfy the requirements of Proposition 5.3.

Proof of Proposition 6.1 assuming Proposition 5.2. The following shorthand will be used throughout: $a_y=a_{c(y)}$, $\nabla a_y=\nabla a_{c(y)}\ldots$ etc.

To begin, it is necessary to consider the continuity of the gradient of a.

Lemma 6.2. *Let $c(y)\in\Lambda$. The function $\|\nabla a_y\|_*$ is a lower semi-continuous function of $y\in S^k$. Here $\|\cdot\|_*$ is the norm on H_y^*.*

Proof. Let $s\in S^k$. As $\Gamma^c=\Gamma^c((\mathscr{g}\otimes T^*)\oplus\mathscr{g})$ is dense in H_s, given $\frac{1}{2}\geqq\varepsilon_1>0$ there exists $\psi\in\Gamma^c$ such that

$$(1)\ \|\psi\|_s=1,\quad\text{and}\quad(2)\ \nabla a_s(\psi)>\|\nabla a_s\|_*-\varepsilon_1. \tag{6.1}$$

The section ψ has compact support. As a consequence of Proposition 5.2, there exists $\delta>0$ such that if $|y-s|<\delta$, then $|\,\|\psi\|_y-1\,|<\varepsilon_1$, and $\nabla a_y(\psi)>\|\nabla a_s\|_*-2\varepsilon_1$. Therefore, when $|y-s|<\delta$,

$$\|\nabla a_y\|_*\geqq\|\psi\|_y^{-1}\nabla a_y(\psi)\geqq(1-\varepsilon_1)\|\nabla a_s\|_*-4\varepsilon_1. \tag{6.2}$$

Let $\varepsilon_1=(4+\|\nabla a_s\|_*)^{-1}\varepsilon_2$. From (6.2), one concludes that given $\varepsilon_2>0$, there exists $\delta(s,\varepsilon_2)>0$ such that $\|\nabla a_s\|_*-\|\nabla a_y\|_*<\varepsilon_2$, whenever $y\in S^k$ satisfies $|y-s|<\delta$. Thus $y\to\|\nabla a_y\|_*$ is lower semi-continuous.

Lemma 6.3. *Let $c(y)\in\Lambda$. Then the function $\|\nabla a_y\|_*$ on S^k is upper semi-continuous.*

Proof. Suppose that the lemma is false. Then there exists $s\in S^k$ and a sequence $\{y_j\}_{j=1}^\infty\in S^k$, converging to s with, $\|\nabla a_j\|_*\geqq\|\nabla a_s\|_*+\delta$, with $\delta>0$. Here,

$\nabla a_j \equiv \nabla a_{y_j}$. Keep in mind that the $\|\cdot\|_*$ norm depends on $y \in S^k$. There is, in this situation, for each y_j, a section $\psi_j \in \Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$,

$$\|\psi_j\|_j = 1, \quad \text{and} \quad \nabla a_j(\psi_j) \geqq \|\nabla a_s\|_* + \delta/2 . \tag{6.3}$$

The Banach space $K_j = K_{A(y_j)}((\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$ is given by Definition 4.9. There is the obvious imbedding $H_j \subset K_j$. By Lemma 4.15, the $\{K_j\}$ are all equivalent to K_s. In fact, using (4.22), there exists $\lambda < \infty$, independent of ψ_j and j such that

$$\|\nabla_{A_s}\psi_j\|_2 \leqq \|\nabla_{A_j}\psi_j\|_2 (1+\lambda) \leqq (1+\lambda) . \tag{6.4}$$

The sequence $\{\psi_j\}$ is, therefore, uniformly bounded in K_s, so it has a weakly convergent subsequence which converges to $\bar{\psi} \in K_s$. Denote this weakly convergent subsequence by $\{\psi_j\}$ also.

It will now be shown that $\bar{\psi} \in H_s$ and $\|\bar{\psi}\|_s \leqq 1$. Indeed, consider the sequence

$$\{\mathscr{G}_j = (\nabla_{A_j}\psi_j, [\Phi_j, \psi_j])\} \in L_2(((\mathscr{g} \otimes T^*) \oplus \mathscr{g}) \otimes T^* \oplus ((\mathscr{g} \otimes T^*) \oplus \mathscr{g})).$$

The sequence $\{\mathscr{G}_j\}$ is uniformly bounded in $L_2(\mathbb{R}^3)$ with norm 1, so it has a weakly convergent subsequence which converges to $\bar{\mathscr{G}} \in L_2(\mathbb{R}^3)$. The norm is weakly-lower semi-continuous, so

$$\|\bar{\mathscr{G}}\|_2 \leqq 1 . \tag{6.5}$$

On the other hand, $(A(y), \Phi(y)) \in C^0(S^k; C^\infty(U; (\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$ for any bounded, open $U \in \mathbb{R}^3$, and as a consequence, $\{(A_j, \Phi_j)\}$ converges strongly to (A_s, Φ_s) in $C^\infty(\bar{U})$. The sequence $\{\psi_j\}$ converges weakly to $\bar{\psi}$ in K_s. Now consider a fixed $E \in L_2(U)$. Then, the inequality

$$\begin{aligned} |\langle \nabla_{A_s}\bar{\psi} - \nabla_{A_j}\psi_j, E\rangle_2; \psi_U| &\leqq |\langle \nabla_{A_s}(\bar{\psi} - \psi_j), E\rangle_2; \psi_U| + |\langle [A_s - A_j, \psi_j], E\rangle_2; \psi_U, \\ &\leqq |\langle \nabla_{A_s}(\bar{\psi} - \psi_j), E\rangle_2; \psi_U| + 2\,\|A_s - A_j\|_3; \psi_U \|\psi_j\|_6 \,\|E\|_2, \end{aligned}$$

allows one to conclude that $\nabla_{A_j}\psi_j$ converges weakly to $\nabla_{A_s}\bar{\psi}$ in $L_2(U)$. Here, one must use Lemmas 4.10 and 4.15 to obtain a uniform bound on $\|\psi_j\|_6$. By a similar argument, $[\Phi_j, \psi_j]$ converges weakly in $L_2(U)$ to $[\Phi_s, \bar{\psi}]$.

Therefore $\bar{\mathscr{G}} = (\nabla_{A_s}\bar{\psi}, [\Phi_s, \bar{\psi}])$ a.e. in $\mathbb{R}^3$, and using (6.5) one concludes that $\bar{\psi} \in H_s$ and

$$\|\bar{\psi}\|_{H_s} \leqq 1 . \tag{6.6}$$

Now consider $\nabla a_s(\bar{\psi})$. Write $\psi_j = (\alpha_j, \eta_j)$ and $\bar{\psi} = (\bar{\alpha}, \bar{\eta})$. Note first that

$$\begin{aligned} \langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j\rangle_2 &= \langle \nabla_{A_j}\eta_j, \nabla_{A_s}\Phi_s\rangle_2 + \langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j - \nabla_{A_s}\Phi_s\rangle_2 \\ &= \langle \nabla_{A_s}\eta_j, \nabla_{A_s}\Phi_s\rangle_2 + \langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j - \nabla_{A_s}\Phi_s\rangle_2 \\ &\quad + \langle [A_j - A_s, \eta_j], \nabla_{A_s}\Phi_s\rangle_2 . \end{aligned} \tag{6.7}$$

Write $A(y;x) = A_0(y;x) + \omega(y;x)$ as in Sect. 4. Then Eq. (6.7) implies that

$$\begin{aligned} |\langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j\rangle_2 - \langle \nabla_{A_s}\eta_j, \nabla_{A_s}\Phi_s\rangle_2| &\leqq z_1(\|\nabla_{A_j}\Phi_j - \nabla_{A_s}\Phi_s\|_2 \\ &+ \|[A_0(y_j) - A_0(s), \eta_j]\|_2 + \|\omega(y_j) - \omega(s)\|_3 \,\|\eta_j\|_6). \end{aligned} \tag{6.8}$$

By assumption, $(\nabla_A\Phi)(y) \in C^0(S^k; L_2(\mathscr{g} \otimes T^*))$, while $\omega(y;x) \in C^0(S^k; L_3(\mathbb{R}^3; \mathscr{g} \otimes T^*))$. (Recall that there exists $R < \infty$ such that $\omega(y;x) \equiv 0$ if

$|x| > R$, for all $y \in S^k$.) Therefore, one can conclude from Eqs. (4.26) and (6.8), and Lemma 4.10 that

$$\lim_{j\to\infty} |\langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j\rangle_2 - \langle \nabla_{A_s}\eta_j, \nabla_{A_s}\Phi_s\rangle_s| \to 0.$$

Because $\{\eta_j\}$ converges weakly to $\bar{\eta}$,

$$\lim_{j\to\infty} |\langle \nabla_{A_s}\eta_j, \nabla_{A_s}\Phi_s\rangle_2 - \langle \nabla_{A_s}\bar{\eta}, \nabla_{A_s}\Phi_s\rangle_2| \to 0,$$

as well. Hence

$$\langle \nabla_{A_s}\bar{\eta}, \nabla_{A_s}\Phi_s\rangle_2 = \lim_{j\to\infty} \langle \nabla_{A_j}\eta_j, \nabla_{A_j}\Phi_j\rangle_2 . \tag{6.9}$$

Similarly, one shows that

$$\langle D_{A_s}\bar{\alpha}, F_{A_s}\rangle = \lim_{j\to\infty} \langle D_{A_j}\alpha_j, F_{A_j}\rangle_2 . \tag{6.10}$$

In addition, one has

$$\lim_{j\to\infty} |\langle [\Phi_j, \alpha_j], D_{A_j}\Phi_j\rangle_2 - \langle [\Phi_j, \alpha_j], D_{A_s}\Phi_s\rangle_2| \leqq \lim_{j\to\infty} \|D_{A_j}\Phi_j - D_{A_s}\Phi_s\|_2 = 0.$$

And by weak continuity,

$$\lim_{j\to\infty} \langle [\Phi_j, \alpha_j], D_{A_s}\Phi_s\rangle_2 = \langle [\Phi_s, \bar{\alpha}], D_{A_s}\Phi_s\rangle_2 .$$

So

$$\langle [\Phi_s, \bar{\alpha}], D_{A_s}\Phi_s\rangle_2 = \lim_{j\to\infty} \langle [\Phi_j, \alpha_j], D_{A_j}\Phi_j\rangle_2 . \tag{6.11}$$

To summarize, Eqs. (6.9)–(6.11), (6.6), and (6.3) imply that

$$\|\nabla \mathscr{a}_s\|_* \geqq \nabla \mathscr{a}_s\left(\frac{\bar{\psi}}{\|\bar{\psi}\|_s}\right) \geqq \lim_{j\to\infty} \nabla \mathscr{a}_j(\psi_j) \geqq \|\nabla \mathscr{a}_s\|_* + \delta/2 .$$

This is a contradiction unless $\delta = 0$. Hence $y \to \|\nabla \mathscr{a}_y\|_*$ is upper semicontinuous, as claimed.

The proof of Proposition 6.1 requires the construction of a deformation along the gradient flow. The deformation will be a y-dependent, compactly supported section of $\Gamma^c((\mathscr{g}\otimes T^*)\oplus \mathscr{g})$.

As a preliminary, define for $\delta \leqq 0$ the sets

$$\Omega(\delta) = \{y \in S^k\colon \|\nabla \mathscr{a}_y\|_*^2 (1 + \|\nabla \mathscr{a}_y\|_*)^{-1} > \delta\},$$

and $\Omega_0 = \Omega(0)$. Since $\|\nabla \mathscr{a}_y\|_*$ is a continuous function of y, the sets $\Omega(\delta)$ are open sets and $\bar{\Omega}(\delta) \subset \Omega_0$ if $\delta > 0$.

Lemma 6.4. *Let $c(y) \in \Lambda$. There exists $\psi_y \in C^0(\Omega_0; \Gamma((\mathscr{g}\otimes T^*)\oplus \mathscr{g}))$ such that:* (1) $\|\psi_y\|_y = 1$. (2) $\nabla \mathscr{a}_y(\psi_y) \leqq -\frac{1}{2}\|\nabla \mathscr{a}_y\|_*$ *for* $y \in \Omega_0$. (3) *If* $\delta > 0$, *there exists* $R(\delta) < \infty$, *such that for* $y \in \bar{\Omega}(\delta)$, $\psi_y \in \Gamma^c(B_{R(\delta)}; (\mathscr{g}\otimes T^*)\oplus \mathscr{g})$.

Proof. For each $y \in \Omega_0$, there exists $\hat{\psi}_y \in \Gamma^c$ which satisfies $\|\hat{\psi}_y\|_y = 1$ and $\nabla \mathscr{a}_y(\hat{\psi}_y) \leqq -3/4\|\nabla \mathscr{a}_y\|_*$. However, $y \to \hat{\psi}_y$ may not be continuous. Since $\hat{\psi}_y$ is compactly supported, Proposition 5.2 implies that there exists $\delta_1 = \delta_1(y)$ such that whenever $s \in S^k$ satisfies $|s-y| < \delta_1$, then

$$\nabla \alpha_s(\hat{\psi}_y) \leqq -11/16\,\|\nabla \alpha_y\|_*, \quad \text{and} \quad |\,\|\hat{\psi}_y\|_s - 1\,| \leqq 1/8. \tag{6.12}$$

As $y \to \|\nabla \alpha_y\|_*$ is continuous, there exists $\delta_2(y) \leqq \delta_1(y)$ such that whenever $s \in \Omega_0$ and $|s-y| < \delta_2$, then

$$\nabla \alpha_s(\hat{\psi}_y) \leqq -5/8\,\|\nabla \alpha_s\|_*. \tag{6.13}$$

There exists a locally finite cover of Ω_0 by open balls $\{D_{r_j}(y_j)\} \subset \Omega_0$ with centers y_j. It follows from Eqs. (6.12,3) that the cover can be chosen so that whenever $y \in D_{r_j}(y_j)$,

$$\nabla \alpha_y(\hat{\psi}_j) \leqq -5/8\,\|\nabla \alpha_y\|_* \quad \text{and} \quad |\,\|\hat{\psi}_j\|_y - 1\,| < 1/8, \tag{6.14}$$

where $\hat{\psi}_j \equiv \hat{\psi}_{y_j}$. Let $\{\beta_j\}$ be a partition of unity subordinate to the open cover $\{B_{r_j}(y_j)\}$. Set

$$\tilde{\psi}_y = \sum_j \beta_j(y)\,\hat{\psi}_j, \quad y \in \Omega_0. \tag{6.15}$$

As $\nabla \alpha(\cdot)$ is a linear functional, one finds using (6.14,5) that for $y \in \Omega_0$,

$$\nabla \alpha_y(\tilde{\psi}_y) = \sum_j \beta_j \nabla \alpha_y(\hat{\psi}_j) \leqq -5/8\,\|\nabla \alpha_y\|_*. \tag{6.16a}$$

The norm of $\tilde{\psi}_y$ is bounded by

$$\|\tilde{\psi}_y\|_y \leqq \sum_j \beta_j \|\hat{\psi}_j\| \leqq 9/8. \tag{6.16b}$$

Now let $\psi_y = \tilde{\psi}_y / \|\tilde{\psi}_y\|_y$. Then $\|\psi_y\|_y = 1$, and from (6.16), $\nabla \alpha_y(\psi_y) \leqq -\frac{1}{2}\|\nabla \alpha_y\|_*$. By construction, $\psi_y \in C^0(\Omega_0; \Gamma((\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$ and if $K \subset \Omega$ is a closed set, there exists $R(K)$ such that $\psi_y \in \Gamma^c(B_{R(K)}; (\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$ for all $y \in K$. Therefore, ψ_y satisfies all the requirements of Lemma 6.4.

The map ψ_y will be used to construct a deformation of $c(y)$ which will satisfy the requirements of Proposition 6.1. It is no loss of generality, however, to assume that the $c(y)$ satisfies (5.1) to begin with (cf. Theorem 4.5). Let $\varepsilon > 0$ be given. The deformation is constructed with the help of a function $0 \leqq f_y \in C^0(S^k)$ which satisfies $f_y \leqq 1$ and $f_y \equiv 0$ on $S^k \setminus \Omega(\varepsilon/4)$. The function f_y will be specified further. Let ψ_y be given by Lemma 6.4 and define the loop

$$b'(y) = c(y) + f_y \psi_y. \tag{6.17}$$

Since $\bar{\Omega}(\varepsilon/4)$ is compact, and contained in Ω_0, Lemma 6.4 insures that $b'(y) \in \Lambda$.

From Proposition 5.2 and Lemma 6.4.,

$$\alpha(b'(y)) \leqq \alpha(c(y)) - \tfrac{1}{2} f_y \|\nabla \alpha_y\|_* + 2\varkappa f_y^2. \tag{6.18}$$

The function f_y will be identically zero outside of the following set:

$$\Omega_1 = \left\{ y \in S^1 \colon \frac{\|\nabla \alpha_y\|_*^2}{(1 + \|\nabla \alpha_y\|_*)} > \max\left[\varepsilon, 4(1+8\varkappa)(\alpha_y - \alpha_\infty)\right] \right\}. \tag{6.19}$$

It is no loss of generality to assume that $\bar{c} \in \Omega_1$, as otherwise $f_y \equiv 0$ and $b'(y) = c(y)$ satisfies the requirements of Proposition 6.1.

The function f_y is defined as follows: Let $\{U_j\}$ be the connected components of Ω_1, and let $\partial U_j = \bar{U}_j \setminus U_j$ denote the boundary of U_j. Denote $\mathscr{a}_j = \sup_{y \in \partial U_j} \mathscr{a}(c(y))$. Let $d(y, \partial U_j)$ denote the geodesic distance between $y \in S^k$ and ∂U_j. Let $\theta(x)$ be the usual step function, so

$$\theta(x) = \begin{cases} 1 & \text{if} \quad |x| \geqq 0, \\ 0 & \text{if} \quad |x| < 0. \end{cases}$$

The function f_y is:

$$\begin{aligned}
&(1) && \text{For } y \in U_j, \\
& && f_y = \|\nabla \mathscr{a}_y\|_* [(1+8\varkappa)(1+\|\nabla \mathscr{a}_y\|_*)]^{-1} \circ r_y, \quad \text{where} \\
& && r_y = \min\{1, [d(y, \partial U_j) + (\mathscr{a}_y - \mathscr{a}_j)\,\theta(\mathscr{a}_y - \mathscr{a}_j)] \\
& && \qquad \cdot 4 \cdot (1+8\varkappa) \cdot \|\nabla \mathscr{a}_y\|_*^{-2} (1+\|\nabla \mathscr{a}_y\|_*)\}. \\
&(2) && \text{For } y \in S^k \setminus \Omega_1, f_y \equiv 0.
\end{aligned} \tag{6.20}$$

The relevant properties off_y are summarized in the following:

Lemma 6.5. *Let f be defined by* (6.20). *Then* (1) $0 \leqq f_y \leqq r_y \leqq 1$. (2) f_y *is continuous.*

Proof. Statement 1 follows by inspection. Consider the continuity statement. The function is clearly continuous in the open sets $\mathrm{Int}(S^k \setminus \Omega_1)$ and Ω_1. It remains to establish that f_y is continuous at points $p \in \partial \bar{\Omega}_1$. It is sufficient to prove that if $\{p_j\} \in \Omega_1$ and $|p_j - p| \to 0$ then $f(p_j) \to 0$. But this follows because $\mathscr{a}(c(y))$ and $\|\nabla \mathscr{a}_{c(y)}\|_*$ are continuous functions on the sphere.

With f_y given by (6.20), let $b'(y)$ be given by (6.17). Using (6.18), one obtains that

$$\mathscr{a}(b'(y) \leqq \mathscr{a}(c(y)) - \tfrac{1}{4}(1+8\varkappa)^{-1}(1+\|\nabla \mathscr{a}_y\|_*)^{-1} \|\nabla \mathscr{a}_y\|_*^2 \, r_y. \tag{6.21}$$

For $y \notin \Omega_1$, $b'(y) = c(y)$. But for $y \in \Omega_1$, Eqs. (6.18, 20, 21) imply that $\mathscr{a}(b'(y)) < \mathscr{a}(c(y))$, and in particular, that there exists $s \in S^k \setminus \Omega_1$ such that

$$\mathscr{a}(b'(s)) > \mathscr{a}(b'(y)) \quad \text{for all} \quad y \in \Omega_1. \tag{6.22}$$

Therefore, the new sphere, $b'(y)$ satisfies Statements (1) and (3) of Proposition 6.1. Now apply Theorem 4.5 to $b'(y)$. Call the result $b(y)$. As $b'(y) = c(y)$ for $y \in S^k \setminus \Omega_1$, and $c(y)$ satisfies (5.1), Theorem 4.5 insures that $b(y) = b'(y) = c(y)$ for $y \in S^k \setminus \Omega_1$. Therefore $\bar{b} = \bar{b}'$, and $b(y)$ satisfies all the requirements of Proposition 6.1.

Proof of Proposition 5.2. Suppose that the space H_c were imbedded in L_2, uniformly with c. Then the proof of Proposition 5.2 would be no different than the proof of the proposition with $\mathbb{R}^3$ replaced by a compact 3-manifold, where the proposition follows using standard Sobolev inequalities, cf. [11, 35]. However H_c does not imbed in L_2. Nonetheless, Proposition 5.2 is true, essentially for the following reason: Let $c = (A, \Phi) \in \mathscr{C}$. The $\|[\Phi, \cdot]\|_2$ component of $\|\cdot\|_c$ provides a bound for the L_2-norm of those $\psi \in \mathrm{H}_c$ which satisfy $(\Phi, \psi)(x) = 0$. This bound depends only on $\mathscr{a}(c)$. Indeed, the volume of the set in $\mathbb{R}^3$ where $|\Phi| < \frac{1}{2}$ is uniformly bounded (cf. Corollary 4.13). Where $\Phi(x) \neq 0$, the stabilizer of $\Phi(x) \in \mathscr{g}$ is 1-dimensional. Since the non-linearities in the action are all commutators this is sufficient to establish Proposition 5.2.

The proof begins with the following observation:

Lemma 6.6. *Let $c = (A, \Phi) \in \mathscr{C}$. Let $V = \{x \in \mathbb{R}^3 : |\Phi|(x) < \frac{1}{2}\}$ and let $v = \int_V d^3x$. There exists a constant $\alpha(v)$ such that for any two $\psi_{1,2} \in H_c$, and $\mu, \delta = 0, 1, 2, 3$*

$$\|[\psi_1^\delta, \psi_2^\mu]\|_2 \leqq \alpha \cdot \|\psi_1\|_c \|\psi_2\|_c. \tag{6.23}$$

Here $\psi_1 \equiv \left(\psi_i^0, \sum_{a=1}^3 \psi_i^a \, dx^a\right)$.

Proof of Lemma 6.6. Let $0 \leqq b \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function which is 1 if $x \in V$ and 0 if distance $(x, V) > \frac{1}{2}$. Given $\eta \in \Gamma^c(\mathscr{g})$, there exists the following linear decomposition: $\eta = \eta^V + \eta^L + \eta^T$, where $\eta^V = b\eta$, $\eta^L = (1-\mathrm{b})|\Phi|^{-2}(\Phi, \eta)\Phi$, $\eta^T = (1-b)(\eta - \eta^L)$.

As $H_c \to L_6$ uniformly, (Lemma 4.10), one has the following inequalities:

$$\begin{aligned} \|\psi^V\|_2 + \|\psi^V\|_6 &\leqq \zeta(1 + v^{2/3}) \|\psi\|_c, \\ \|\psi^L\|_6 &\leqq \zeta \|\psi\|_c, \\ \|\psi^T\|_2 + \|\psi^T\|_6 &\leqq (2 + \zeta) \|\psi\|_c. \end{aligned} \tag{6.24}$$

Using the fact that $[\psi_1^{\delta L}, \psi_2^{\mu L}] = 0$ for all $\mu, \delta = 0, 1, 2, 3$, one obtains

$$\|[\psi_1^\delta, \psi_2^\mu]\|_2 \leqq 2 \| |\psi_1^L| (|\psi_2^T| + |\psi_2^V|) \|_2 + \sum_{S = T, V} \| |\psi_2| |\psi_1^S| \|_2, \tag{6.25}$$

where the subscripts μ, δ have been suppressed. Next, notice that if $v \in L_6$ and $u \in L_2 \cap L_6$, then

$$\|vu\|_2 \leqq \|v\|_6 \|u\|_6^{1/2} \|u\|_2^{1/2}. \tag{6.26}$$

After applying (6.26) to (6.25), and using (6.24) one obtains (6.23).

Armed with Lemma 6.6, the proof that $\mathscr{a}_c(\cdot)$ extends to a C^∞ functional on H_c is an exercise in Hölder's inequality that is left to the reader. The uniform estimates given by Statements (1) and (2) of Proposition 5.2 follow from Lemma 6.6 and Corollary 4.13. Corollary 4.13 gives a uniform estimate for v, depending only on $\mathscr{a}(c)$.

Statement (3) of Proposition 5.2 is a consequence of the choice of topology on $\mathscr{C}$. In particular, $\Lambda \subset C^0(S^1; \Gamma((\mathscr{g} \otimes T^*) \oplus \mathscr{g}))$ and since ψ is compactly supported in some ball $B_R \subset \mathbb{R}^3$, Statement (3) follows readily.

VII. A Nontrivial Limit

Let $\{c_i(y)\} \in \Lambda$ be a good sequence. It follows from Theorem 5.6 that the sequence of configurations, $\{\bar{c}_i = (A_i, \Phi_i)\}$ has a subsequence which converges, modulo gauge transformations. It is possible that the limit configuration has zero action. By

Theorem 5.6, this will occur only if $(F_{A_i}, D_{A_i}\Phi_i)$ converges to zero in every fixed, bounded domain. There are only two ways that this can happen. The first possibility is that there is a fixed $R<\infty$, such that for each i, a ball $B_{R_i}(x_i) \subset \mathbb{R}^3$ exists, with $R_i \leqq R$, and on which $(F_{A_i}, D_{A_i}\Phi_i)$ have uniformly large L_2 norms. But, the sequence of centers, $\{x_i\}$, diverges on $\mathbb{R}^3$. This situation is rectifiable by translating each $\bar{c}_i$ so that x_i becomes the origin. The second possibility is that $\lim\limits_{i\to\infty} R_i \to \infty$. This situation will be shown to be incompatible with the condition $\mathscr{a}_\infty > 0$.

Let $a \in \mathbb{R}^3$ and $c \in \mathscr{C}$. Denote the translated configuration

$$c(x+a) \equiv (T_a c)(x). \tag{7.1}$$

Theorem 7.1. *Let $\{\bar{c}_i\} \in \mathscr{C}$ be a sequence which satisfies* (5.2) *with $\mathscr{a}_\infty > 0$. Then there exists a sequence of points $\{x_i\} \in \mathbb{R}^3$ with the following properties*: (1) *The sequence $\{T_{x_i} c_i\}$ has a subsequence which converges strongly in $L^1_{2,\mathrm{loc}}$ to $(A, \Phi) \in \mathscr{C}$.* (2) *$(A, \Phi)$ satisfies Eqs.* (2.2,3). (3) *$\mathscr{a}(A, \Phi) > 0$.*

The proof of Theorem 7.1 is based on the physical intuition that monopoles are localized objects. This intuition is affirmed by the next proposition:

Proposition 7.2. *Let $\{\bar{c}_i = (A_i, \Phi_i)\} \in \mathscr{C}$ satisfy* (5.2) *with $\mathscr{a}_\infty > 0$. Define for each i,*

$$\gamma_i = \sup_{x\in\mathbb{R}^3} \left[\int_{|x-x'|<1} d^3x' (|F_{A_i}|^2 + |\nabla_{A_i}\Phi_i|^2) \right]. \tag{7.2}$$

Then

$$\lim_{i\to\infty} \gamma_i > \gamma > 0. \tag{7.3}$$

The remainder of this section contains the proof of Theorem 7.1 and Proposition 7.2.

Proof of Theorem 7.1 assuming Proposition 7.2. Let $a \in \mathbb{R}^3$ and $c \in \mathscr{C}$. Then $\mathscr{a}(T_a c) = \mathscr{a}(c)$, $\|\nabla \mathscr{a}_{T_a c}\|_* = \|\nabla \mathscr{a}_c\|_* \dots$ etc. If c satisfies (5.1), then $T_a c$ does also. For any choice of $\{x_i\}$, the sequence $(T_{x_i} c_i\}$ satisfies (5.2), and appealing to Theorem 5.6 establishes Statements (1) and (2) of Theorem 7.1.

By Proposition 7.2, there exists i_0, and for all $i > i_0$, there exist $x_i \in \mathbb{R}^3$ such that

$$\int_{|x-x_i|<1} d^3x (|F_{A_i}|^2 + |\nabla_{A_i}\Phi_i|^2) > \gamma > 0. \tag{7.4}$$

For $i \leqq i_0$, set $x_i = 0$ and for $i > i_0$ choose x_i so that (7.4) holds. Denote the new sequence $\{T_{x_i} c_i\}$ by $\{(A_i, \Phi_i)\}$ as well. Then for all $i > i_0$,

$$\int_{|x|<1} d^3y (|F_{A_i}|^2 + |\nabla_{A_i}\Phi_i|^2) > \gamma > 0. \tag{7.5}$$

By Theorem 5.6, $(F_{A_i}, D_{A_i}\Phi_i)$ converges strongly in L_2 of the unit ball. Hence if (A, Φ) denotes the limit in Theorem 5.6, $\mathscr{a}(A, \Phi) > \gamma > 0$.

Proof of Proposition 7.2. The proof is by contradiction. Suppose there exists a subsequence, $\{c_i = (A_i, \Phi_i)\}$ which satisfies the assumptions of Theorem 7.1 and such that $\lim_{i\to\infty}\gamma_i\to 0$. The strategy is to show that the three conditions, $\mathscr{a}_\infty>0$, $\|\nabla\mathscr{a}_i\|_*\to 0$ and $\gamma_i\to 0$ are incompatible. The last two conditions will imply that (1) $\|[\Phi_i,\nabla_{A_i}\Phi_i]\|_2\to 0$, (2) $\|1-|\Phi_i|\|_\infty\to 0$, (3) $\|(\Phi_i,\nabla_{A_i}\Phi_i)\|_2\to 0$, (4) $\|[\Phi_i,F_{A_i}]\|_2\to 0$, and finally that (5) $\|(\Phi_i,F_{A_i})\|_2\to 0$. As the stabilizer of a nonzero $\sigma\in\mathfrak{su}(2)$ is 1-dimensional, properties (1)–(5) above of the sequence $\{(A_i,\Phi_i)\}$ contradict the assumption that $\mathscr{a}_\infty>0$.

To begin, let $g_i=\nabla_{A_i}\Phi_i$. As $c_i=(A_i,\Phi_i)$ satisfies (5.1), one can infer from Lemma 4.7 that $\|g_i\|_{c_i}$ is finite. An estimate of $\|g_i\|_{c_i}$ that is more useful than that in Lemma 4.7 is provided by

Lemma 7.3. *Let $c=(A,\Phi)\in\mathscr{C}$ satisfy* (5.1). *Then*

$$\|g\|_c^2\leqq\varkappa(\bar\gamma(\|g\|_c^2+\mathscr{a}_c)+\|\nabla\mathscr{a}_c\|_*(1+\|g\|_c)\|g\|_c),\tag{7.6}$$

where $\varkappa$ is a constant that is independent of (A,Φ) and $\bar\gamma$ is given by the right hand side of (7.2) *with $(A_i,\Phi_i)=(A,\Phi)$.*

Proof. Since $\nabla_A^2\Phi=0$, one obtains by commuting covariant derivatives:

$$\nabla_A^2 g-2*(*F\wedge g+g\wedge *F)+[*D*F,\Phi]=0.$$

Next, take the L_2 inner product with g. As $g\in L_2$, one can integrate by parts to obtain

$$-\|\nabla_A g\|_2^2-4\langle F,g\wedge g\rangle_2+\langle D_A*F,*[\Phi,g]\rangle_2=0.$$

A second integration by parts yields

$$-\|\nabla_A g\|_2^2-4\langle F,g\wedge g\rangle_2+\langle F,D_A[\Phi,g]\rangle_2=0.$$

Together with the definition of $\nabla\mathscr{a}_c$, this last equation implies that

$$-\|g\|_c^2-4\langle F,g\wedge g\rangle_2+\nabla\mathscr{a}_c(([\Phi,g],0))=0.\tag{7.7}$$

The fact that $[\Phi,\mathrm{g}]\in H_c$ follows from the estimates of Lemma 4.7, and Proposition 4.8.

In fact, since $|\Phi|<1$,

$$\begin{aligned}\|[\Phi,g]\|_c&\leqq 2\|g\|_c+2\|g\wedge g\|_{L_2},\\&\leqq 2(1+\varkappa_1\|g\|_c)\|g\|_c;\end{aligned}$$

and the last line follows from Lemma 6.6. Thus

$$\nabla\mathscr{a}_c(([\Phi,g],0))\leqq 2\|\nabla\mathscr{a}_c\|_*(1+\varkappa_1\|g\|_c)\|g\|_c.\tag{7.8}$$

Let $\{V_\nu\}$ be a uniform, countable open cover of $\mathbb{R}^3$ by balls of radius 1. Let $\{\beta_\nu^2\}$ be a subordinate partition of unity. Using a trick due to Morrey [33, Lemma 5.2.1], one obtains:

$$\begin{aligned}\langle F,g\wedge g\rangle_2=\sum_\nu\langle F,\beta_\nu^2 g\wedge g\rangle_2&\leqq\bar\gamma\cdot\sum_\nu\|\beta_\nu g\|_4^2,\\&\leqq\bar\gamma\cdot\varkappa_2\sum_\nu\|\nabla_A\beta_\nu g\|_2^2,\\&\leqq\bar\gamma\cdot\varkappa_3\sum_\nu(\|\beta_\nu\nabla_A g\|_2^2+\|\beta_\nu g\|_2^2).\end{aligned}$$

Here, line 1 follows by Hölder's inequality, line 2 by Lemma 4.10 and line 3 because the cover is locally uniform. From line 3, one concludes that

$$\langle F, g \wedge g\rangle_2 \leqq \bar{\gamma} \cdot \varkappa_4 \cdot (\|g\|_c^2 + \alpha_c). \tag{7.9}$$

Equations (7.7)–(7.9) establish the lemma.

Continuing the proof of Proposition 7.2, one concludes using Lemma 7.3 that

$$\lim_{i\to\infty} \|\nabla_{A_i}\Phi_i\|_{c_i} \to 0, \tag{7.10}$$

and by Lemma 4.10, that

$$\lim_{i\to\infty} \|\nabla_{A_i}\Phi_i\|_6 \to 0. \tag{7.11}$$

In particular, (7.10) implies that

$$\lim_{i\to\infty} \|[\Phi_i, \nabla_{A_i}\Phi_i]\|_2 \to 0. \tag{7.12}$$

The next step is to prove that $\|1-|\Phi_i|\|_\infty \to 0$. This result is obtained with the aid of the next Lemma.

Lemma 7.4. *Let* $(A,\Phi)\in\mathscr{C}$, *and satisfy* (5.1). *Let* $w(x)=\frac{1}{2}(1-|\Phi|^2(x))$. *There exists a constant* $\varkappa<\infty$ *which does not depend on* (A,Φ) *such that* $\|w\|_\infty \leqq \varkappa(\|g\|_2^{1/2}\|g\|_6^{3/4}+\|g\|_2^{5/4})\|g\|_6^{3/4}$.

Proof of Lemma 4.7. Since (A,Φ) satisfies (5.1), $w(x)$ satisfies

$$-\Delta w(x) = |g|^2(x). \tag{7.13}$$

The function $w(x)$ is the unique solution to (7.13) which vanishes at infinity. Using the Green's function for $(-\Delta)$, one concludes from Corollary VI.4.2 of [2] that

$$\begin{aligned} w(x) &= \frac{1}{4\pi}\int d^3y\,\frac{|g|^2}{|x-y|}(y) \\ &\leqq \frac{1}{4\pi}\left(\int_{|y|\leqq 1}\frac{d^3y}{|y|^2}\right)^{1/2}\|g\|_4^2 + \frac{1}{4\pi}\left(\int_{|y|>1}\frac{d^3y}{|y|^4}\right)^{1/4}\|g\|_{8/3}^2. \end{aligned} \tag{7.14}$$

Meanwhile Hölder's inequality yields

$$\|g\|_4^2 \leqq \|g\|_2^{1/2}\|g\|_6^{3/2}, \quad \text{while} \quad \|g\|_{8/3}^2 \leqq \|g\|_4\|g\|_2 \leqq \|g\|_6^{3/4}\|g\|_2^{5/4}. \tag{7.15}$$

Equations (7.14,5) establish the lemma.

Continuing with the analysis of $(1-|\Phi_i|)$, Lemma 7.4 and (7.11) imply that $\|w_i\|_\infty \leqq \varkappa\cdot(\|g_i\|_2^{1/2}\|g_i\|_6^{3/4}+\|g_i\|_2^{5/4})\|g_i\|_6^{3/4}$; and so the $\lim_{i\to\infty} w_i(x)\to 0$ uniformly with x. Therefore it has now been established that

$$\lim_{i\to\infty} \|(1-|\Phi_i|)\|_\infty \to 0. \tag{7.16}$$

As $w_i\in K$ (Lemma 4.12), one can multiply both sides of (7.13) by w_i, integrate over $\mathbb{R}^3$, and integrate by parts on the left hand side to obtain

$$\|\nabla w_i\|_2^2 = \langle w_i, |g_i|^2\rangle_2 \leqq \|w_i\|_\infty 2\cdot\alpha(c_i). \tag{7.17}$$

Since $\|\nabla w_i\|_2 = \|(\Phi_i, \nabla_{A_i}\Phi_i)\|_2$, Eqs. (7.16,7) imply that

$$\lim_{i\to\infty} \|(\Phi_i, \nabla_{A_i}\Phi_i)\|_2 \to 0\,. \tag{7.18}$$

To summarize, Eqs. (7.12), (7.16), and (7.18) state that

$$\lim_{i\to\infty} \|\nabla_{A_i}\Phi_i\|_2 \to 0\,. \tag{7.19}$$

As a bonus from (7.10), one has

$$\lim_{i\to\infty} \|[\Phi_i, F_{A_i}]\|_2 \to 0\,, \tag{7.20}$$

as $\|[F,\Phi]\|_2 = \|D_A D_A \Phi\|_2 \leqq \|\nabla_A \Phi\|_c$. Now turn attention to $f_i = (\Phi_i, F_{A_i})$. By assumption, $f_i \in L_2(\wedge_2 T^*) \cap \Gamma(\wedge_2 \Gamma^*)$. The fact that $\|\nabla a_i\|_* \to 0$ has the following consequence:

Lemma 7.5. *Given* $\varepsilon > 0$, *there exists* $i(\varepsilon) < \infty$ *such that for all* $i > i(\varepsilon)$, *and for all* $\omega \in K(T^*)$ *and* $u \in K(\mathbb{R}^3)$ *(cf. Eq.* (4.8)*),*

$$|\langle d\omega, f_i\rangle_2| < \varepsilon \|\nabla\omega\|_2\,, \quad |\langle * du, f_i\rangle_2| < \varepsilon \|\nabla u\|_2\,. \tag{7.21}$$

Proof of Lemma 7.5. It is sufficient to establish (7.21) for compactly supported ω and u. Let $\omega \in \Gamma^c(\wedge_p T^*)$, $p = 0,1$. Let $(A, \Phi) \in \mathscr{C}$, and suppose that $\|\nabla_A \Phi\|_3 < \infty$. Let $\hat\omega = \omega\Phi \in \Gamma^c(\mathscr{g} \otimes \wedge_p T^*)$, $p = 0,1$. Then

$$\begin{aligned} \|\nabla_A \hat\omega\|_2 &\leqq \|\nabla_A \omega\|_2 + \|\omega|\nabla_A\Phi|\,\|_2 \leqq \|\nabla\omega\|_2 + \|\omega\|_6 \|\nabla_A \Phi\|_3 \\ &\leqq \|\nabla\omega\|_2 (1 + \zeta \|\nabla_A\Phi\|_3)\,. \end{aligned} \tag{7.22}$$

Here, the last line uses Lemma 4.10. Using (7.22), (7,11), and (7.19) one establishes that there exists $i_0 < \infty$ such that for all $i > i_0$, and all $\omega \in \Gamma^c(\wedge_p T^*)$ $(p = 0,1)$,

$$\|\nabla_{A_i}\hat\omega\|_2 \leqq 2\|\nabla\omega\|_2\,. \tag{7.23}$$

Let $\omega \in \Gamma^c(T^*)$. By integrating by parts, one has $\langle d\omega, f_i\rangle_2 = \nabla a_i((\hat\omega, 0)) - \langle \omega \wedge \mathrm{D}_{A_i}\Phi_i, F_{A_i}\rangle_2$. Therefore, using (7.23) one obtains

$$\begin{aligned} |\langle d\omega, f_i\rangle_2| &\leqq 2\|\nabla a_i\|_* \|\nabla\omega\|_2 + \|\omega\|_6 \|F_i\|_2 \|\nabla_{A_i}\Phi_i\|_3 \\ &\leqq \|\nabla\omega\|_2 (2\|\nabla a_i\|_* + \zeta a_i^{1/2} \|\nabla_{A_i}\Phi_i\|_3)\,. \end{aligned} \tag{7.24}$$

Here, line 2 is obtained with Hölder's inequality, and Lemma 4.10.

If $u \in C_0^\infty(\mathbb{R}^3)$, then integration by parts establishes, via the Bianchi identities, that $\langle * du, f_i\rangle_2 = \langle * u, D_{A_i}\Phi_i \wedge F_i\rangle_2$. Therefore, using (7.23) one obtains that

$$|\langle * du, f_i\rangle_2| \leqq \|u\|_6 \|F_i\|_2 \|\nabla_{A_i}\Phi_i\|_3 \leqq \|\nabla u\|_2 \cdot \zeta a_i^{1/2} \|\nabla_{A_i}\Phi_i\|_3\,. \tag{7.25}$$

Because of (7.11) and (7.20), $\lim_{i\to\infty} \|\nabla_{A_i}\Phi_i\|_3 \to 0$. Hence, the lemma is an immediate consequence of (7.24,5).

The Hodge theorem for $\mathbb{R}^3$, with Lemma 7.5 imply that $\|f_i\|_2 \to 0$. Indeed, let

$$K^T(T^*) = \{\omega \in K(T^*)\colon d^*\omega = 0\}\,.$$

Proposition 7.6. (Hodge Theorem.) *There exists a unique, orthogonal decomposition of* $L_2(\underset{2}{\wedge} T^*)$ *as follows*: $L_2(\underset{2}{\wedge} T^*) = dK^T(T^*) \oplus *dK(\mathbb{R}^3)$, *and both* d *and* $*d$ *are isometric.*

Proof. The proof is essentially Lemma 4.11, cf. [36].

The significance of Proposition 7.6 is that it implies that one can write $f_i = d\omega_i + *du_i$, with $(\omega_i, u_i) \in K^T(T^*) + K(\mathbb{R}^3)$. As the decomposition is orthogonal and isometric

$$\|f_i\|_2^2 = \|\nabla\omega_i\|_2^2 + \|\nabla u_i\|_2^2 . \tag{7.26}$$

On the other hand, Lemma 7.5 implies that given $\varepsilon > 0$, for all $i > i(\varepsilon)$,

$$\|f_i\|_2^2 \leqq \varepsilon(\|\nabla\omega_i\|_2^2 + \|\nabla u_i\|_2^2)^{1/2} \leqq \varepsilon\|f_i\|_2 ,$$

and so

$$\|f_i\|_2 \leqq \varepsilon . \tag{7.27}$$

Therefore

$$\lim_{i\to\infty} \|(\Phi_i, F_i)\|_2 \to 0 . \tag{7.28}$$

Equations (7.16), (7.19), (7.20), and 7.27) contradict the fact that $a_\infty > 0$; thus Proposition 7.2 is true.

VIII. The Monopole Number

Consider the convergence of a sequence $\{\bar{c}_i\} \in \mathscr{C}_0$ induced by good sequence $\{c_i(y)\} \in A$. One must demonstrate that the limiting configuration, as given by Theorems 5.6 and 7.1, is an element of $\mathscr{C}_0$. Indeed, as the convergence is in $L^1_{2,\mathrm{loc}}$, an independent argument is necessary to prove this. A sufficient condition for the limit to lie in $\mathscr{C}_0$ is given below. This condition uses the fact that for sufficiently well-behaved $(A, \Phi) \in \mathscr{C}_k$, the index k is given by [2, Chap. IV]

$$k = \frac{1}{4\pi} \langle D_A\Phi, *F_A\rangle_2 . \tag{8.1}$$

Theorem 8.1. *Let* $\{c_i\} \in \mathscr{C}_0$ *be a sequence which satisfies* (5.2), *and converges strongly in* $L^1_{2,\mathrm{loc}}$ *to a solution,* $c \in \mathscr{C}$, *of Eqs.* (2.2,3). *If* $a_\infty < 8\pi$, *then* $c \in \mathscr{C}_0$.

The strategy for the proof is to demonstrate that the statement $c \notin \mathscr{C}_0$ leads to a contradiction. The physical intuition is that for $c \notin \mathscr{C}_0$, the sequence $\{\bar{c}_i\}$ must correspond to monopole, anti-monopole pairs, which have an infinite separation in the limit. This requires energy $(a) \geqq 8\pi$.

To begin, suppose that $(A, \Phi) \in \mathscr{C}_k$ is a solution to Eqs. (2.2,3). Then by Theorem IV.1.5 of [2], Eq. (8.1) holds. On the other hand, for $c \in \mathscr{C}_0$, one has

Lemma 8.2. *Let* $(A, \Phi) \in \mathscr{C}_0$. *Then*

$$\langle D_A\Phi, *F_A\rangle_2 = 0 . \tag{8.2}$$

Proof of Theorem 8.1 assuming Lemma 8.2. As $D_A\Phi$, $F_A \in L_2$, the integral in (8.1) is absolutely convergent. A consequence of this fact is that given $\varepsilon > 0$, there exists $R < \infty$ such that

$$\left| \int_{|x|<R} (D_A\Phi, {}_*F_A) - 4\pi k \right| < \varepsilon/2 . \tag{8.3}$$

The integrand in (8.3) is gauge invariant, so by the strong convergence assertion of Theorem 5.6, there exists $i(\varepsilon) < \infty$ such that if $i > i(\varepsilon)$, $\bar{c}_i = (A_i, \Phi_i)$ satisfies

$$\left| \int_{|x|<R} (D_{A_i}\Phi_i, *F_{A_i}) - 4\pi k \right| < \varepsilon/2 , \tag{8.4}$$

as well. By Lemma 8.2, if $i > i(\varepsilon)$,

$$\left| \int_{|x|>R} (D_{A_i}\Phi_i, *F_{A_i}) + 4\pi k \right| < \varepsilon/2 , \tag{8.5}$$

also.

If $U \subset \mathbb{R}^3$ is any open set, the triangle inequality gives

$$\tfrac{1}{2} \int_U (|F_{A_i}|^2 + |D_{A_i}\Phi_i|^2) \geqq \left| \int_U (D_{A_i}\Phi_i, *F_{A_i}) \right| .$$

Therefore (8.4,5) imply that given $\varepsilon > 0$, there exists $i(\varepsilon) < \infty$ such that for all $i > i(\varepsilon)$, $\mathscr{a}(c_i) > 8\pi|k| - \varepsilon \geqq 8\pi - \varepsilon$. Since $\lim_{i\to\infty} \mathscr{a}(c_i) \to \mathscr{a}_\infty$ by hypothesis, $\mathscr{a}_\infty \geqq 8\pi$, which is a contradiction.

Proof of Lemma 8.2. By assumption, there exists $R_0 < \infty$ such that $||\Phi|(x) - 1| < \frac{1}{2}$ if $|x| > R_0$. In addition, as a map from $S_R^2 = \{x \in \mathbb{R}^3 \colon |x| = R\}$ to $S^2 = \{\sigma \in \mathfrak{su}(2) \colon |\sigma| = \mathbf{1}\}$, $\Phi/|\Phi|$ is null homotopic for all $R > R_0$. The group SU(2) acts on $\mathfrak{su}(2)$ by conjugation which, when restricted to $S^2 \subset \mathfrak{su}(2)$ is the Hopf fibration $0 \to S^1 \to \mathrm{SU}(2) \to S^2 \to 0$. The Hopf fibration is a Serre fibration: A consequence of this fact is that if $\Phi/|\Phi| \colon S_R^2 \to S^2$ is null homotopic, there exists a smooth gauge transformation $g \in C^\infty(\mathbb{R}^3; \mathrm{SU}(2))$ such that $g\Phi/|\Phi| g^{-1} = \frac{1}{2}\sigma^3$ if $|x| > R_0$ (cf. [25, Chap. 2].)

Without loss of generality, one may now assume that if $|x| > R_0$, then

$$\Phi = \tfrac{1}{2} |\Phi| \sigma^3 . \tag{8.6}$$

The assumption of finite action implies that the integral in (8.2) is absolutely convergent. Let $\beta(x)$ be the cut-off function introduced in (3.4) and $\beta_R(x) = \beta(x/R)$. Given $\varepsilon > 0$, there exists $R_0 < \infty$, such that if $R > R_0$,

$$\left| \int \beta_R (D_A\Phi, *F_A) - \int (D_A\Phi, *F_A) \right| < \varepsilon/2 . \tag{8.7}$$

Integrating by parts and using the Bianchi identities (2.2c), one obtains that $\int \beta_R (D_A\Phi, *F_A) = -\int d\beta_R \wedge (\Phi, F_A)$; and using (8.6) and the definition of F_A:

$$\int \beta_R (D_A\Phi, *F_A) = -\int \tfrac{1}{2} d\beta_R |\Phi| \wedge (\sigma^3, dA + A \wedge A) . \tag{8.8}$$

Define $A^T = A - \frac{1}{4}\sigma^3(\sigma^3, A)$; $A^L = \frac{1}{2}(\sigma^3, A)$. By assumption, A^T is C^∞. In addition,

$$\int_{|x|>R_0} |\nabla_A \Phi|^2 = \int_{|x|>R_0} \{|d|\Phi||^2 + 2|\Phi|^2 |A^T|^3\},$$

so $A^T \in L_2(\mathbb{R}^3; \mathcal{g} \otimes T^*)$ and $d|\Phi| \in L_2(\mathbb{R}^3; T^*)$. Using these facts, one obtains from (8.8) that

$$|\int \beta_R(D_A\Phi, *F_A)| \leqq -\int (d\beta_R |\Phi| \wedge dA^L) + \frac{2}{R} \|d\beta\|_\infty \|A^T\|_2^2, \tag{8.9}$$

where it is assumed that R is sufficiently large so that $|\Phi| < 2$. Note that $\|d\beta_R\|_\infty = R^{-1} \|d\beta\|_\infty$.

Since $\int d\beta_R \wedge dA^L = 0$, the right side of (8.9) is

$$= \int d\beta_R (1 - |\Phi|) \wedge dA^L + \frac{\varkappa}{R},$$

$$\leqq \frac{1}{2} \int d\beta_R (1 - |\Phi|) \wedge (\sigma^3, F_A) + \frac{2\varkappa}{R}, \tag{8.10}$$

where $\varkappa = 2\|d\beta\|_\infty \|A^T\|_2^2 < \infty$. Now use the fact that $d\beta_R = d\beta_R(1 - \beta_{R/2})$ to obtain from (8.10) that

$$|\int \beta_R(D_A\Phi, *F_A)| \leqq \frac{1}{2} \|d\beta_R\|_3 \|(1 - |\Phi|)\|_6 \|(1 - \beta_{R/2}) F_A\|_2 + \frac{2\varkappa}{R}. \tag{8.11}$$

By Corollary 4.13, $\|1 - |\Phi|\|_6 \leqq \alpha a(c)$, and by rescaling, $\|d\beta_R\|_3 = \|d\beta\|_3$. One concludes from (8.11) that given $\varepsilon > 0$ there exists $R_1 < \infty$ such that for all $R > R_1$,

$$|\int \beta_R(D_A\Phi, *F_A)| < \varepsilon/2. \tag{8.12}$$

Together with (8.7), Eq. (8.12) implies Lemma 8.2.

Appendix A

The purpose of this appendix is to complete the proof of Theorem 3.4 by proving

Proposition A.1. *The map I:* Maps $(S^2; S^2) \to \mathscr{C}$ *of Definition 3.3 is a* $1-1$ *map of the set* Π_0 (Maps $(S^2; S^2)$) *onto* $\Pi_0(\mathscr{C})$.

To prove the proposition, one must exhibit, given $c \in \mathscr{C}$, a path $c(t) \in C^0([0,1]; \mathscr{C})$, such that $c(0) = c$ and $c(1) = I(e)$, for $e \in \text{Maps}(S^2; S^2)_k$.

By Proposition 4.8, it is no loss of generality to assume that $c = (A, \Phi)$ satisfies

$$\nabla_A^2 \Phi = 0. \tag{A.1}$$

As a consequence of (A.1) and Lemma 4.7, $\nabla_A \Phi \in L_4(\mathcal{g} \otimes T^*)$. Choose $R < \infty$ so that $|\Phi|(x) > 1/2$ for $|x| > R$. Then

$$e(\hat{x}) = \Phi(R\hat{x})/|\Phi|(R\hat{x}) \in \text{Maps}(S^2; S^2), \tag{A.2}$$

and for $|x| > R$, the map $\Phi(|x|\hat{x})/|\Phi|(|x|\hat{x}) \in \text{Maps}(S^2; S^2)$ is homotopic to $e(\hat{x})$.

The homotopy lifting property of fibrations implies the existence of $g(x) \in \mathscr{G}$ satisfying

$$\begin{array}{llll} (1) & g(x) = 1, & \text{for} \quad |x| \leqq R, & \\ (2) & g(x)\,\Phi(x)\,g^{-1}(x) = |\Phi|(x)\,e(x/|x|), & \text{for} \quad |x| \geqq R. & \text{(A.3)} \end{array}$$

As $\mathscr{G}$ is contractible, it is path connected. Then a consequence of (A.3) is that it is no loss of generality to assume that

$$\Phi(x) = |\Phi|(x)\,e(x/|x|) \quad \text{for} \quad |x| \geqq R. \tag{A.4}$$

Note that

$$|\nabla_A \Phi|^2 = |\nabla|\Phi||^2 + |\Phi|^2\,|\nabla_A e|^2 \quad \text{for} \quad |x| > R, \tag{A.5}$$

and as a consequence,

$$\nabla_A((1-\beta(x))e) \in L_2(\mathfrak{g} \otimes T^*) \cap L_4(\mathfrak{g} \otimes T^*). \tag{A.6}$$

Now consider the path

$$c^1(t) = (A, (1-t)\,\Phi + t(1-\beta)\,e). \tag{A.7}$$

A consequence of (A.6) is that $c^1(t) \in C^0([0,1], \mathscr{C})$ and

$$c^1(0) = (A, \Phi), \quad \text{while} \quad c^1(1) = (A, (1-\beta)\,e). \tag{A.8}$$

Next consider the curve

$$c^2(t) = ((1-t\beta)\,A, (1-\beta)\,e). \tag{A.9}$$

The t-dependence of $c^2(t)$ is compactly supported in the unit ball in $\mathbb{R}^3$, so $c^2(t) \in C^0([0,1]; \mathscr{C})$. Meanwhile,

$$c^2(0) = (A, (1-\beta)\,e), \quad \text{and} \quad c^2(1) = ((1-\beta)\,A, (1-\beta)\,e). \tag{A.10}$$

For notational convenience denote $(1-\beta)A$ by A again, keeping in mind that A now vanishes in the unit ball.

Examining $\nabla_A e$, one observes that $A^T = [e, [A, e]]$ satisfies

$$(1-\beta)\,(A^T + [e, de]) \in L_4(\mathfrak{g} \otimes T^*) \cap L_2(\mathfrak{g} \otimes T^*). \tag{A.11}$$

As $|[e, de]| \leqq \text{constant} \cdot |x|^{-1}$, one concludes that $A^T \in L_4(\mathfrak{g} \otimes T^*)$ also.

For notational convenience, let

$$\begin{aligned} a &= A^T + (1-\beta)\,[e, de], \\ a^L &= (e, A). \end{aligned} \tag{A.12}$$

A short calculation reveals that for $|x| > 1$,

$$[e, F_A] = [e, da] - a^L \wedge a, \tag{A.13}$$

$$\nabla_A e = -[e, a], \tag{A.14}$$

$$(e, F_A) = da^L + \tfrac{1}{2}\operatorname{tr}(de \wedge A^T) + (e, A^T \wedge A^T). \tag{A.15}$$

It follows from (A.11) and (A.15) that $da^L \in L_2(\underset{2}{\wedge} T^*)$. Now let

$$c^3(t) = (a^L e + (1-\beta)[e, de] + (1-t)a, (1-\beta)e). \tag{A.16}$$

Due to (A.12),

$$\begin{aligned} c^3(0) &= (A, (1-\beta)e), \\ c^3(1) &= (a^L e + (1-\beta)[e, de], (1-\beta)e). \end{aligned} \tag{A.17}$$

It is a consequence of (A.13)–(A.15) and (A.11) that $\mathscr{a}(c^3(t)) < \infty$. Thus $c^3(t) \in C^0([0,1]; \mathscr{C})$ also. Finally, let

$$c^4(t) = ((1-t)a^L e + (1-\beta)[e, de], (1-\beta)e). \tag{A.16}$$

Then $c^4(0) = c^3(1)$, and $c^4(1) = I(e)$. Meanwhile, $\mathscr{a}(c^4(t))$ is finite as $da^L \in L_2(\underset{2}{\wedge} T^*)$. Therefore, the path $c(t) = (c^4 \circ c^3 \circ c^2 \circ c^1)(t)$ connects (A, Φ) with $I(e)$ and proves Proposition A.1.

Appendix B

The purpose of this second appendix is to complete the proof of Theorem 3.6 by establishing that $\Pi_1(\mathrm{Maps}((S^2, n); (S^2, n))_k/\mathrm{SO}(2)) \simeq \mathbb{Z}_{|k|}$. This follows from Lemma 3.7 which will now be proved.

To begin, recall that $\Omega = \mathrm{Maps}((S^2, n); (S^2, n))$ has a natural operation, $\#$ which is defined as follows: Represent S^2 as the unit square $I^2 = [0,1] \times [0,1]$ with the boundary, I^2, identified as the distinguished point n. For $e_1, e_2 \in \Omega$, define

$$(e_1 \# e_2)(t_1, t_2) = \begin{cases} e_1(2t_1, t_2), & 0 \leqq t \leqq \frac{1}{2}, \\ e_2(2t_1 - 1, t_2), & \frac{1}{2} \leqq t_1 \leqq 1. \end{cases} \tag{B.1}$$

By inspection, $e_1 \# e_2 \in \Omega$, so $\#$ is well defined. The operation $\#$ endows the point set $\Pi_0(\Omega)$ with the structure of an abelian group. In fact, degree $[e_1 \# e_2]$ = degree $[e_1]$ + degree $[e_2]$, so that the degree is a group isomorphism between $(\Pi_0(\Omega), \#)$ and $(\mathbb{Z}, +)$ [25, Chap. 1,7]. In addition, for $e_k \in \Omega_k$, it is relatively easy to check that $e_k \# (\cdot): \Omega_l \to \Omega_{l+k}$ is a homotopy equivalence.

Choose e_0 to be the constant map $S^2 \to n$. Then $h \in \mathrm{Maps}((S^1, n); (\Omega_0, e_0))$ is a map from $(S^1 \times S^2)/((S^1 \times n) \cup (n \times S^2))$ to S^2, which is to say, h defines a map from S^3 to S^2. [$(S^1 \times S^2)/((S^1 \times n) \cup (n \times S^2))$ is homeomorphic to S^3.] The converse of the last statement is also true. One concludes that $\Pi_1(\Omega_0, e_0) \simeq \Pi_3(S^2) \simeq \mathbb{Z}$.

Let $[h] \in \Pi_3(S^2)$. The class $[h]$ is a multiple, $\alpha[h]$, of the generator of $\Pi_3(S^2)$. The integer $\alpha[h]$ is the Hopf invariant of h and it can be calculated in the following way [37, Chap. 4]: Let $\omega \in \Gamma(\underset{2}{\wedge} T^*_{S^2})$ satisfy

$$\int_{S^2} \omega = 1. \tag{B.2}$$

Let $h \in [h]$ be a C^1 representative. Since $H^2(S^3) = (0)$,

$$h^* \omega = dv_h \tag{B.3}$$

for some $v_h \in \Gamma(T^*_{S^3})$. Then

$$\alpha[h] = \int_{S^3} (v_h \wedge h^* \omega). \tag{B.4}$$

Since $H^1(S^3) = (0)$, v_h is unique up to the image of d, and this ambiguity does not affect the integral (B.4).

The action of SO(2) on Ω_k generates a class $[\hat{e}_k] \in \Pi_1(\Omega_k, e_k)$. Using the operation, $\#$, this gives a class $[e_{-k} \# \hat{e}_k] \in \Pi_1(\Omega_0, e_{-k} \# e_k)$. As Ω_0 is connected, there exists a path $b: \left[0, \frac{\pi}{2}\right] \to \Omega_0$ with $b(0) = e_0$ and $b\left(\frac{\pi}{2}\right) = e_{-k} \# e_k$. By conjugating the loop $e_{-k} \# \hat{e}_k$ by $b(t)$, one obtains a class $[b \circ (e_{-k} \# \hat{e}_k) \circ b^{-1}] \in \Pi_1(\Omega_0, e_0) \simeq \Pi_3(S^2)$. The map β of the fibration (3.21) is multiplication by the Hopf invariant, $\alpha[b \circ (e_{-k} \# \hat{e}_k) \circ b^{-1}]$. Therefore, Lemma 3.7 follows upon establishing that $\alpha[b \circ (e_{-k} \# e_k) \circ b^{-1}) = k$.

The calculation of $\alpha[\cdot]$ is facilitated by fixing a basis $(\sigma^1, \sigma^2, \sigma^3) \in \mathfrak{su}(2)$ with the properties: (1) $\sigma^i \sigma^j = -\delta^{ij} - \varepsilon^{ijk} \sigma^h$, and (2) $\{\frac{1}{2}\sigma^i\}$ is an orthonormal frame. The distinguished point of the image S^2 is the point $-\frac{1}{2}\sigma^3$. The distinguished point of the domain S^2 is the point $\theta = 0$ in polar coordinates. Let

$$e_k = \tfrac{1}{2}[-\cos\theta\sigma^3 + \sin\theta(\cos k\phi\sigma^1 - \sin k\phi\sigma^2)]. \tag{B.5}$$

The map $e_k \in \Omega_k$ and the loop that is generated by the action of SO(2) on Ω_k is

$$\hat{e}_k = \tfrac{1}{2}[-\cos\theta\sigma^3 + \sin\theta(\cos(k\phi - t)\sigma^1 - \sin(k\phi - t)\sigma^2)]. \tag{B.6}$$

Up to homotopy, the loop $(e_{-k} \# \hat{e}_k)$ is given by

$$\hat{\mu}_k = \begin{cases} \tfrac{1}{2}[-\cos 2\bar{\theta}\sigma^3 + \sin 2\bar{\theta}(\cos(\mathrm{k}\phi - \tau)\sigma^1 - \sin(k\phi - \tau)\sigma^2)] & \text{for} \quad 0 \leqq \theta \leqq \pi/2; \\ \tfrac{1}{2}[-\cos 2\bar{\theta}\sigma^3 + \sin 2\bar{\theta}(\cos k\phi\sigma^1 - \sin k\phi\sigma^2)], & \text{for} \quad \pi/2 \leqq \theta \leqq \pi. \end{cases} \tag{B.7}$$

Here, $\bar{\theta}(\theta)$ is a smooth function of θ which satisfies (1) $\frac{d\bar{\theta}}{d\theta} \geqq 0$, (2) $\bar{\theta} = \theta$ for $\theta \notin \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right]$, (3) $\bar{\theta} = \frac{\pi}{2}$ for $\theta \in \left[\frac{7\pi}{16}, \frac{9\pi}{16}\right]$. Meanwhile, $\tau(t)$ is a smooth function of t which satisfies (1) $\frac{d\tau}{dt} \geqq 0$, (2) $\tau = 0$ for $t \in \left[0, \frac{\pi}{8}\right]$, (3) $\tau = t$ for $t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$ and (4) $\tau = 2\pi$ for $t \in \left[\frac{15\pi}{8}, 2\pi\right]$. For the curve $b(t)$, $t \in \left[0, \frac{\pi}{2}\right]$, take

$$\begin{aligned} b_k(t) = \tfrac{1}{2}[&-(\cos 2\bar{\theta}\sin^2\tau + \cos^2\tau)\sigma^3 + \sin\tau\sin 2\bar{\theta}(\cos k\phi\sigma^1 - \sin k\phi\sigma^2) \\ &- \cos\tau\sin\tau(1 - \cos 2\bar{\theta})(\cos k\phi\sigma^2 + \sin k\phi\sigma^1)]. \end{aligned} \tag{B.8}$$

It is now a straightforward calculation to obtain that the integer k is the Hopf invariant for the following loop in $\mathrm{Maps}((S^1, n); (\Omega_0, e_*))$:

$$(b_k^{-1} \circ \hat{\mu}_k \circ b_k)(t) = \begin{cases} b_k(t), & t \in \left[0, \frac{\pi}{2}\right], \\ \hat{\mu}_k(2t - \pi), & t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right), \\ b_k(2\pi - t), & t \in \left[\frac{3\pi}{2}, 2\pi\right). \end{cases} \tag{B.9}$$

Except for remarking that the contributions from the intervals $\left[0, \frac{\pi}{2}\right)$ and $\left[\frac{3\pi}{2}, 2\pi\right)$ cancel, this calculation is left to the reader.

Acknowledgements. The author wishes to acknowledge the many valuable conversations over the past months with Professors R. Bott, A. Jaffe, T. Parter and K. Uhlenbeck.

References

1. Bourguignon, J.P., Lawson, B., Simons, J.: Stability and gap phenomena for Yang-Mills fields. Proc. Natl. Acad. Sci. USA **76,** 1550 (1979)
2. Jaffe, A., Taubes, C.H.: Vortices and Monopoles. Boston: Birkhäuser 1980
3. Taubes, C.H.: On the equivalence of first and second order equations for gauge theories. Commun. Math. Phys. **75,** 207 (1980)
4. Atiyah, M.F., Jones, J.D.S.: Topological aspects of Yang-Mills theory. Commun. Math. Phys. **61,** 97 (1978)
5. Singer, I.M.: Some remarks on the Gribov ambiguity. Commun. Math. Phys. **60,** 7 (1978)
6. Atiyah, M.F., Bott, R.: On the Yang-Mills equations over Riemann surfaces (to appear)
7. Burzlaff, J.: A finite energy SU(3) solution which does not satisfy the Bogomol'nyi equations. Acta. Phys. Austr. Suppl. (1981)
8. Maison, D.: Uniqueness of the Prasad-Sommerfield monopole solution. Nucl. Phys. **B182,** 114 (1981)
9. Ljusternik, L.: Topology of the calculus of Variations in the large. (Amer. Math. Soc. Transl. **16**) Providence, R.I.: Amer. Math. Soc. (1966)
10. Berger, M.: Nonlinearity and functional analysis. New York: Academic Press 1977. See also Palais, R.S.: Ljusternik-Šnirelman theory on Banach manifolds. Topology **5,** 115 (1966)
11. Uhlenbeck, K.: Connections with L^p bounds on curvature. Commun. Math. Phys. **83,** 31 (1981)
12. Sachs, J., Uhlenbeck, K.: The Existence of minimal 2-spheres. Ann. Math. **113,** 1 (1981)
13. Schoen, R., Uhlenbeck, K.: Regularity of minimizing harmonic maps (to appear)
14. Carrigan, R.A., Jr., Trower, W.P.: Superheavy magnetic monopoles. Sci. Am. **246,** 106 (1982)
15. Hirsh, M.W.: Differential topology. Berlin, Heidelberg, New York: Springer 1976
16. Adams, R.A.: Sobolev spaces. New York: Academic Press 1975
17. Bogomol'nyi, E.B.: The stability of classical solutions. Sov. J. Nucl. Phys. **24,** 449 (1976)
18. Prasad, M., Sommerfield, C.: Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon. Phys. Rev. Lett. **35,** 760 (1975)
19. Ward, R.: A Yang-Mills Higgs monopole of charge 2. Commun. Math. Phys. **79,** 317 (1981)
20. Prasad, M.: Yang-Mills Higgs monopole solutions of arbitrary topological charge. Commun. Math. Phys. **80,** 137 (1981)
21. Hitchin, N.: Monopoles and geodesics. Commun. Math. Phys. **83,** 579 (1982)
22. Corrigan, E., Goddard, P.: An n-monopole solution with $4n-1$ degrees of freedom. Commun. Math. Phys. **80,** 575 (1981)

23. Palais, R.: Foundations of global analysis, New York: W.A. Benjamin Co. 1968
24. Taubes, C.H.: Self-dual Yang-Mills fields on non-self-dual 4-manifolds. J. Diff. Geom. **17,** 139 (1982)
25. Spanier, E.H.: Algebraic topology. New York: McGraw-Hill, 1966. See also Steenrod, N.: Topology of fibre bundles. Princeton, NJ: Princeton University Press 1951
26. Goddard, P., Nuyts, J., Olive, D.: Gauge theories and magnetic charge. Nucl. Phys. **B125,** 1 (1977)
27. Taubes, C.H.: Surface integrals and monopole charges in non-abelian gauge theories. Commun. Math. Phys. **81,** 299 (1981)
28. Uhlenbeck, K.: Removable singularities in Yang-Mills fields. Commun. Math. Phys. **83,** 11 (1982)
29. Weinberg, E.: Parameter counting for multimonopole solutions. Phys. Rev. **D20,** 936 (1979)
30. Taubes, C.H.: In preparation
31. Parker, T., Taubes, C.H.: On Witten's proof of the positive energy theorem. Commun. Math. Phys. **84,** 224 (1982)
32. Vainberg, M.M.: Variational methods and method of monotone operators in the theory of nonlinear equations. New York: Wiley 1973
33. Morrey, C.B.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
34. Sedlacek, S.: A direct method for minimizing the Yang-Mills functional over 4-manifolds. Commun. Math. Phys. (to appear)
35. Parker, T.: Gauge theories on 4-dimensional, Riemannian manifolds. Commun. Math. Phys. **85,** 563–602 (1982)
36. Ladyzhenskaya, O.A.: The mathematical theory of viscous, incompressible flow. London: Gordon and Breach 1963
37. Vick, J.W.: Homology theory. New York: Academic Press 1973

Communicated by A. Jaffe

Received May 5, 1982