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Orbital Stability of Standing Waves for Some Nonlinear Schrödinger Equations

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Abstract. We present a general method which enables us to prove the orbital stability of some standing waves in nonlinear Schrödinger equations. For example, we treat the cases of nonlinear Schrödinger equations arising in laser beams, of time-dependent Hartree equations ....

Introduction

We consider here various nonlinear Schrödinger equations. To explain our results, we will give three examples.

1. Local Nonlinearities

$$i\frac{\partial\Phi}{\partial t}(t,x)+\Delta\Phi(t,x)+|\Phi(t,x)|^{p-1}\Phi(t,x)=0 \quad \text{in} \quad \mathbb{R}_+\times\mathbb{R}^N, \tag{1}$$

where $p>1$.

2. Hartree-Type Time Dependent Equations

$$i\frac{\partial\Phi}{\partial t}(t,x)+\Delta\Phi(t,x)+\sum_{i=1}^{m}\frac{z_i}{|x-x_i|}\Phi(t,x)-\left\{\int_{\mathbb{R}^3}|\Phi(t,y)|^2\frac{1}{|x-y|}dy\right\}\Phi(t,x)$$
$$=0 \quad \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3, \tag{2}$$

where $x_1,\dots,x_m$ are some given points in $\mathbb{R}^3$ and $z_1,\dots,z_m$ are positive constants.

3. Pekar-Choquard Time Dependent Equations

$$i\frac{\partial\Phi}{\partial t}(t,x)+\Delta\Phi(t,x)+\left\{\int_{\mathbb{R}^3}|\Phi(t,y)|^2\frac{1}{|x-y|}dy\right\}\Phi(t,x)$$
$$=0 \quad \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3. \tag{3}$$

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In these three cases we are looking for a complex-valued solution $\Phi(t,x)$ and in order to solve this problem, we will impose an initial condition

$$\Phi(0,x)=\Phi_0(x) \quad \text{in } \mathbb{R}^N; \tag{4}$$

and Φ_0 is a given function in $\mathbb{R}^N$.

In addition, in all three cases, we will require at least $\Phi(t,\cdot)\in L^2(\mathbb{R}^N)$, $\forall t$.

These three types of nonlinear Schrödinger equations arise in various domains of Mathematical Physics ((1) arises for example in the study of propagation of laser beams, see Kelley [16] and Suydam [34]; (2) arises in Quantum Mechanics, see for the original introduction of Hartree equations Hartree [15] and Slater [29]; (3) is a time-dependent version of some equation proposed by Choquard and we refer to Lieb [17] for a brief discussion of the relevance of this problem to Physics).

In these three cases, it is possible to find (under appropriate conditions) standing waves of (1), (2) or (3), i.e. to find solutions $\Phi(t,x)$ of the form: $\Phi(t,x) = e^{i\lambda t}u(x)$ $(\lambda\in\mathbb{R})$. This yields the following equations for $u(x)$:

$$-\Delta u+\lambda u=|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \tag{5}$$

$$-\Delta u-\sum_{i=1}^{m}\frac{z_i}{|x-x_i|}u+\lambda u+\left(|u|^2*\frac{1}{|x|}\right)u=0 \quad \text{in} \quad \mathbb{R}^3, \tag{6}$$

$$-\Delta u+\lambda u-\left(|u|^2*\frac{1}{|x|}\right)u=0 \quad \text{in} \quad \mathbb{R}^3, \tag{7}$$

where u is complex-valued and $u\in L^2(\mathbb{R}^3)$.

Of course 0 is a trivial solution, and we are interested in nontrivial solutions: $u\not\equiv 0$. Equation (5) has been investigated by many authors (Nehari [25]; Ryder [28]; Berger [5]; Strauss [30]; Berestycki and Lions [3, 4]) – in the last reference, the most general results concerning equations of the type (5) are given. The Hartree equation has been studied by many authors, we will mention Reeken [27], Stuart [32], Lieb and Simon [18], and Lions [20, 22]. Finally Eq. (7) has been solved by Lieb [17] and Lions [19].

In these cases, *we will define a ground state solution* $u_0(x)$ *of* (5)–(7) *and we will prove the orbital stability of* u_0, that is we prove:

1. In the case of problem (1): We assume $p<1+\frac{4}{N}$; then for all $\varepsilon>0$, there exists $\delta>0$ such that, if $\inf_{\theta\in\mathbb{R},\, y\in\mathbb{R}^N}\|\Phi_0(\cdot)-e^{i\theta}u_0(\cdot+y)\|_{H^1(\mathbb{R}^N)}<\delta$ then the solution $\Phi(t,x)$ of (1)–(4) satisfies (for all $t\geqq 0$):

$$\inf_{\theta\in\mathbb{R},\, y\in\mathbb{R}^N}\|\Phi(t,\cdot)-e^{i\theta}u_0(\cdot+y)\|_{H^1(\mathbb{R}^N)}<\varepsilon.$$

2. In the case of problem (2): For all $\varepsilon>0$, there exists $\delta>0$ such that if $\inf_{\theta\in\mathbb{R}}\|\Phi_0-e^{i\theta}u_0\|_{H^1(\mathbb{R}^N)}<\delta$, then the solution $\Phi(t,x)$ of (2)–(4) satisfies (for all $t\geqq 0$):

$$\inf_{\theta\in\mathbb{R}}\|\Phi(t,\cdot)-e^{i\theta}u_0(\cdot)\|_{H^1(\mathbb{R}^3)}<\varepsilon.$$

3. In the case of problem (3): For all $\varepsilon>0$, there exists $\delta>0$ such that if $\inf_{\theta\in\mathbb{R},\, y\in\mathbb{R}^3}\|\Phi_0(\cdot)-e^{i\theta}u_0(\cdot+y)\|_{H^1(\mathbb{R}^3)}<\delta$, then the solution $\Phi(t,x)$ of (3) and (4)

satisfies (for all $t \geqq 0$):

$$\inf_{\theta \in \mathbb{R},\ y \in \mathbb{R}^3} \|\Phi(t,\cdot) - e^{i\theta} u_0(\cdot + y)\|_{H^1(\mathbb{R}^3)} < \varepsilon .$$

[The presence of the infimum over all y of $\mathbb{R}^3$ in 1. and 3. above is due to the invariance of (1) and (3) with respect to translations in space and will be explained with more details below.] We also give various examples and remarks showing that these results are optimal; and we will also indicate various extensions.

In Sect. I below, we give a heuristic presentation of the underlying principle which gives the orbital stability of some specific standing waves in nonlinear Schrödinger equations. Then, in Sect. II, we treat the case of problems (1)–(4); Sect. III is devoted to the study of problems (2)–(4) while we treat the case of (3) and (4) in Sect. IV.

Our results are in some sense an extension of Cazenave [9] and heavily rely on [9] and on the results obtained by the concentration-compactness method introduced in Lions [21, 22].

Finally, for the solutions of the Cauchy problems (1)–(4), (2)–(4), and (3) and (4), we will use the results of Ginibre and Velo [10–13], Cazenave [7, 8], Lin and Strauss [23], and Pecher and von Wahl [26].

I. Orbital Stability of Standing Waves: A Heuristic Presentation

As indicated in the title, we will not give any rigorous argument in this section, but instead we will indicate a general line of argument that will enable us to treat the three examples mentioned in the Introduction.

Let us consider a nonlinear Schrödinger equation of the form:

$$\left\{ \begin{array}{l} i\dfrac{\partial \Phi}{\partial t} + \Delta\Phi + F(\Phi) = 0 \quad \text{in} \quad R_+ \times \mathbb{R}^N \\ \Phi(0,x) = \Phi_0(x), \end{array} \right. \tag{8}$$

where $F(\Phi)$ is some nonlinear map from some Hilbert functional space E into another one H and where Φ, is some given initial condition in E.

1) We will assume that the map F is such that there exists a C^1 real-valued functional $\mathscr{E}$ defined on E such that:

$$\mathscr{E}' = -\Delta - F(\cdot) \quad \text{on } E;$$

that is

$$\mathscr{E} = \frac{1}{2} \int_{\mathbb{R}^N} \nabla\Phi \cdot \nabla\bar{\Phi}\, dx - G(\Phi)$$

and $G' = F$.

2) We will assume that, for some $\mu > 0$, the following minimization problem:

$$I_\mu = \operatorname{Min}\{\mathscr{E}(u),\ u \in E,\ |u|^2_{L^2(\mathbb{R}^N)} = \mu\} \tag{9}$$

can be solved and more precisely that we have:

Case 1

$\mathscr{E}$ is invariant with respect to translation.

Then we assume that, for all minimizing sequences $(u_n)_{n\leqq 1}$ that is:

$$u_n\in E, \quad |u_n|_{L^2(\mathbb{R}^N)}=\mu, \quad \mathscr{E}(u_n)\to I_\mu -, \tag{10}$$

we have: $\exists (y_n)_n\subset \mathbb{R}^N$ such that $u_n(\cdot+y_n)$ is relatively compact in $E\cap L^2(\mathbb{R}^N)$. This implies obviously i) that (9) is solved (I_μ is attained) and ii) that the set S of solutions u of (9) is compact in $E\cap L^2(\mathbb{R}^N)$ (up to translations, obviously).

In Lions [21, 22]; this assumption is shown to be, in very general situations, equivalent to the following *sub-additivity condition*:

$$I_\mu<I_\alpha+I_{\mu-\alpha}, \quad \forall\alpha\in]0,\mu[. \tag{S.2}$$

Case 2

$\mathscr{E}$ is not invariant by translations.

Then we assume that, for all minimizing sequences $(u_n)_{n\leqq 1}$, (u_n) is relatively compact in $E\cap L^2(\mathbb{R}^N)$. Again, this implies that (9) is solved and that the set S of solutions u of (9) is relatively compact in $E\cap L^2(\mathbb{R}^N)$.

Let us mention that, if $\mathscr{E}(u)=\int_{\mathbb{R}^N} e(x,\mathrm{Au}(x))dx$, where $e(x,p)$ is some real function from $\mathbb{R}^N\times\mathbb{R}^p$ into R and where A is some operator invariant by translations from E into M (a functional space of functions defined on $\mathbb{R}^p$) – and if, for example, we have: $e(x,p)\xrightarrow[|x|\to\infty]{} e^\infty(p)$, then the above assumption is equivalent under very general assumptions to the following *generalized subadditivity condition*

$$I_\mu<I_\alpha+I^\infty_{\mu-\alpha}, \quad \forall\alpha\in[0,\mu[, \tag{S.1}$$

where $I^\infty_\lambda=\mathrm{Inf}\left\{\int_{\mathbb{R}^N} e^\infty(\mathrm{Au}(x))dx/u\in E\cap L^2,\ |u|^2_{L^2(\mathbb{R}^N)}=\mu\right\}$ – see Lions [21, 22].

Let us indicate briefly why (S.1) or (S.2) imply some form of compactness on minimizing sequences. This is proved by the use of the *concentration-compactness method* introduced in [21, 22] that we shall briefly sketch now. It is based upon the following remark: take v_n a sequence in $L^1_+(R^N)$ (or bounded nonnegative measures) such that $\|v_n\|_{L^1(R^N)}=\mu$, then there exists a subsequence (that we still denote by v_n to simplify) such that one of the following possibilities is true for all elements of that subsequence:

i) (vanishing) $\sup_{y\in R^N}\int_{y+B_R} v_n dx\xrightarrow[n]{}0$ for all $R<\infty$,

ii) (compactness) $\exists y_n\in R^N$, $\forall\varepsilon>0$, $\exists R>0$ such that

$$\mu\geqq\int_{y+B_R} v_n dx\geqq\mu-\varepsilon,$$

iii) (dichotomy) $\exists a\in(0,\mu)$, $\forall\varepsilon>0$, $\exists v_n^1, v_n^2\in L^1_+(R^N)$ such that:

$$\overline{\lim_n}\,\|v_n-(v_n^1+v_n^2)\|_{L^1}\leqq\varepsilon, \quad \lim_n\|v_n^1\|_{L^1}=a, \quad \mathrm{dist}(\mathrm{Supp}\,v_n^1,\mathrm{Supp}\,v_n^2)\xrightarrow[n]{}+\infty.$$

Such a lemma is proved in [21, 22] with the use of concentration functions. Now if we take a minimizing sequence u_n in problem (9) and apply the above result to $v_n=(u_n)^2$, one sees easily that cases i) and iii) are ruled out if (S.1) or (S.2) [for $a\in(0,\mu)$] holds.

Finally in the case when $\mathscr{E}$ is not invariant by translations, one proves that y_n in ii) above remains bounded since (S.1) holds (take $a=\mu$). In the two cases we will denote by S the set of solutions to (9).

3) We will assume for any Φ_0 in E, there exists a unique solution $\Phi(t,x)$ of (8) satisfying: for all $t\geqq 0$, we have

$$|\Phi(t,\cdot)|_{L^2(\mathbb{R}^N)}=|\Phi_0|_{L^2(\mathbb{R}^N)}; \tag{11}$$

$$\mathscr{E}(\Phi(t,\cdot))=\mathscr{E}(\Phi_0). \tag{12}$$

We remark that, formally, (11) is obtained by multiplying (8) by $\bar{\Phi}$: indeed we remark that in view of 1. $G(e^{i\theta}\Phi)=G(\Phi)\,(\forall\theta\in\mathbb{R})$ and thus $F(e^{i\theta}\Phi)=e^{i\theta}F(\Phi)$ and $\int_{\mathbb{R}^N} F(\Phi)\bar{\Phi}\in\mathbb{R}$. Thus multiplying (8) by $\bar{\Phi}$, integrating over $\mathbb{R}^N$ and taking the imaginary part, we find: $\frac{d}{dt}(|\Phi(t,\cdot)|_{L^2(\mathbb{R}^N)})=0$. In the same way, multiplying (8) by $\frac{\partial}{\partial t}(\bar{\Phi})$, integrating over $\mathbb{R}^N$ and taking the real part, we obtain

$$\frac{d}{dt}(\mathscr{E}(\Phi(t,\cdot)))=0, \text{ and this yields (12)}.$$

After having described our assumptions 1)–3), we are now able to show that S *is orbitally stable* in the sense that we have

$$\forall\varepsilon>0,\quad \exists\delta>0,\quad \forall\Phi_0\in E,\quad \inf_{u\in S}\|\Phi_0-u\|_{E\cap L^2(\mathbb{R}^N)}<\delta \quad\text{implies:}$$
$$\forall t\geqq 0,\quad \inf_{u\in S}\|\Phi(t,\cdot)-u\|_{E\cap L^2(\mathbb{R}^N)}<\varepsilon.$$

(We remark that if $u\in S$, $e^{i\theta}u\in S$, $\forall\theta\in\mathbb{R}$.) This claim is very easy to prove if $|\Phi_0|^2_{L^2(\mathbb{R}^N)}=\mu$ (we will make this assumption to simplify the presentation): indeed if the above claim were not true there would exist $\varepsilon_0>0$, Φ_0^n, and $t_n\geqq 0$ such that:

$$\begin{cases} \Phi_0^n\in E, \quad |\Phi_0^n|^2_{L^2(\mathbb{R}^N)}=\mu, \quad \mathscr{E}(\Phi_0^n)\xrightarrow[n]{} I_\mu \\ \inf_{u\in S}\|\Phi^n(t_n,\cdot)-u\|_{E\cap L^2(\mathbb{R}^N)}\geqq\varepsilon_0 \end{cases}$$

[where $\Phi^n(t)$ is the solution of (8) corresponding to Φ_0^n]. But in view of 3);

$$\mathscr{E}(\Phi^n(t_n,\cdot))\xrightarrow[n]{} I_\mu,\qquad |\Phi^n(t_n,\cdot)|^2_{L^2(\mathbb{R}^N)}=\mu.$$

Therefore we deduce from 2) that either $\Phi^n(t_n,\cdot)$ is relatively compact in $\mathscr{E}\cap L^2(\mathbb{R}^N)$ up to translations, or $\Phi^n(t_n,\cdot)$ is relatively compact in $\mathscr{E}\cap L^2(\mathbb{R}^N)$. In these two cases, this implies:

$$\inf_{u\in S}\|\Phi^n(t_n,\cdot)-u\|_{E\cap L^2(\mathbb{R}^N)}\xrightarrow[n]{}0,$$

and this contradiction proves our claim.

Thus we see that the orbital stability of S is a straightforward consequence of 1) the fact that the minimization problem (9) is well posed and of 2) the conservation laws (11) and (12) (this was first observed in a particular case by Cazenave [9]).

We now explain why we call this phenomena "orbital stability of S". Indeed let us first remark that any u solution of (9) satisfies:

$$-\Delta u - F(u) = \theta u \text{ in } \mathbb{R}^N, \qquad u \in E \cap L^2(\mathbb{R}^N), \qquad |u|_{L^2}^2 = \mu, \tag{13}$$

for some Lagrange multiplier $\theta \in \mathbb{R}$. Therefore $e^{i\theta t}u(x) = \Phi(t,x)$ is the solution of (8) corresponding to $\Phi_0(x) = u(x)$ and thus $e^{i\theta t}u(\cdot)$ is the orbit of u (we remark that $e^{i\theta t}u \in S$, $\forall t \geqq 0$).

We finally explain why this result is in general the best possible: first, very simple examples (linear Schrödinger equations, see also [2]) may be given that show that no stronger forms of stability are in general true. In addition, in general, one cannot replace in the preceding result S by $(e^{i\theta}u(\cdot))_{\theta\in\mathbb{R}}$: indeed suppose F is invariant by translations and following [9] we consider for $u \in S$, $y \in \mathbb{R}^N |y| = 1$, $n \geqq 1$:

$$\Phi_0^n(t,x) = \exp\left\{\frac{i}{n}\left((x,y) - \frac{t}{n}\right)\right\} e^{i\theta t} u\left(x - \frac{2t}{n} y\right).$$

A simple computation shows that $\Phi^n(t,x)$ is the solution of (8) corresponding to the initial condition $\Phi_0^n = \exp\left\{\frac{i}{n}(x,y)\right\} u(x)$. Now, in our examples we will have $E = H^1(\mathbb{R}^N)$, and $\Phi_0^n \xrightarrow[H^1]{} u$ while $\operatorname*{Inf}_{\theta\in\mathbb{R}} \|\Phi^n - e^{i\theta}u\|_{H^1(\mathbb{R}^N)}$ remains strictly positive.

II. Local Nonlinearities

We consider the following nonlinear Schrödinger equations:

$$\begin{cases} i\dfrac{\partial \Phi}{\partial t}(t,x) + \Delta\Phi(t,x) + |\Phi(t,x)|^{p-1}\Phi(t,x) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \quad (1) \\ \Phi(0,x) = \Phi_0(x) & \text{in } \mathbb{R}^N, \quad (4) \end{cases}$$

where $\Phi_0 \in H^1(\mathbb{R}^N)$, $p > 1$.

If $1 < p < 1 + \frac{4}{N}$, it is well-known (see Ginibre and Velo [10, 11]; Cazenave [7, 8]; Lin and Strauss [23]) that there exists a unique solution $\Phi(t,x)$ of (1)–(4) in $C([0,\infty[x; H^1(\mathbb{R}^N))$ and Φ satisfies:

$$|\Phi(t,\cdot)|_{L^2(\mathbb{R}^N)} = |\Phi_0|_{L^2(\mathbb{R}^N)}; \tag{11}$$

$$\mathscr{E}(\Phi(t,\cdot)) = \mathscr{E}(\Phi_0), \tag{12}$$

for all $t \geqq 0$, where $\mathscr{E}$ is defined on $H^1(\mathbb{R}^N)$ by:

$$\mathscr{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}dx.$$

Remark II.1. If $p \geqq 1+\frac{4}{N}$, a solution of (1)–(4) may not exist for all time (see Strauss [31]; Glassey [14]).

We now look for standing waves of (1), that is we consider the equation

$$-\Delta u+\lambda u=|u|^{p-1}u \text{ in } \mathbb{R}^N, \qquad u\in H^1(\mathbb{R}^N) \tag{5}$$

and because of (11), it is natural to prescribe:

$$|u|^2_{L^2(\mathbb{R}^N)}=\mu, \tag{14}$$

where μ is given >0, and λ is a Lagrange multiplier.

A natural way to obtain solutions of (5)–(14) is to look for the following minimization problem:

$$I_\mu=\operatorname{Inf}\{\mathscr{E}(u), u\in H^1(\mathbb{R}^N), |u|^2_{L^2(\mathbb{R}^N)}=\mu\}. \tag{9}$$

We will call any solution of I_μ a *ground state* solution of (5)–(14).

Concerning the solution of (9), the following result, due to Lions [21, 22], is known:

Theorem II.1. *If $p>1+\frac{4}{N}$, $I_\mu=-\infty$. If $p<1+\frac{4}{N}$, we have $I_\mu<0$ for all $\mu>0$ and:*

i) *Let $(u_n)_n$ be a minimizing sequence of* (9): $u_n\in H^1(\mathbb{R}^N)$, $|u_n|^2_{L^2(\mathbb{R}^N)}\xrightarrow[n]{}\mu$ *and* $\mathscr{E}(u_n)\xrightarrow[n]{} I_\mu$; *then there exists $(y_n)_n\subset\mathbb{R}^N$ such that:*

$$u_n(\cdot+y_n) \textit{ is relatively compact in } H^1(\mathbb{R}^N).$$

ii) *Let u be a solution of* (9), *then: $u(\cdot)=e^{i\theta}u_0(\cdot+y)$ for some $\theta\in\mathbb{R}$, $y\in\mathbb{R}^N$ and where u_0 is a solution of* (9) *satisfying:*

$$\begin{cases} u_0(x)=u_0(|x|)\in\mathbb{R},\ u_0(x)>0, \textit{ and } u_0(|x|) \textit{ is decreasing with } |x| \\ -\Delta u_0+\lambda u_0=u_0^p \textit{ in } \mathbb{R}^N, \textit{for some } \lambda>0;\ u_0\in C^2(\mathbb{R}^N). \end{cases}$$

Actually in [21, 22], this result is proved for real valued H^1 functions, but obviously the proof of i) is totally similar, while we show now that if u is a complex-valued solution of (9), then $u(\cdot)=e^{i\theta}v(\cdot)$ where $\theta\in\mathbb{R}$, v is a real-valued solution of (9). Indeed, if $u=u^1+iu^2$, where $u^1,u^2\in H^1(\mathbb{R}^N)$ are real-valued, then $\tilde{u}=|u^1|+i|u^2|$ is still a solution of (9) and this yields:

$$\begin{cases} -\Delta u^i+\lambda u^i=|u|^{p-1}u^i & \text{in } \mathbb{R}^N \\ -\Delta|u^i|+\lambda|u^i|=|u|^{p-1}|u^i| & \text{in } \mathbb{R}^N \end{cases}$$

for some real Lagrange multiplier λ. But this shows that $-\lambda$ is the first eigenvalue of the operator $-\Delta-|u|^{p-1}$ acting over $H^1(\mathbb{R}^N)$ and thus: u^1, u^2, $|u^1|$, $|u^2|$ are all multiples of a positive normalized eigenfunction $\bar{u}$ of $-\Delta-|u|^{p-1}$, i.e.:

$$\begin{cases} -\Delta\bar{u}+\lambda\bar{u}=|u|^{p-1}\bar{u} \text{ in } \mathbb{R}^N,\ \bar{u}\in C^2(\mathbb{R}^N)\cap H^1(\mathbb{R}^N),\ \bar{u}>0 \quad \text{in } \mathbb{R}^N \\ |\bar{u}|^2_{L^2(\mathbb{R}^N)}=\mu. \end{cases}$$

It is now obvious to deduce that: $u=e^{i\theta}\bar{u}$ and that $\bar{u}$ is still a solution of (9). This completes the proof of Theorem II.1.

For all $\mu > 0$, we denote by S_μ the set of all solutions of (9). Then we have the following result of "orbital stability":

Theorem II.2. *Let* $1 < p < 1 + \frac{4}{N}$, *let* $\mu > 0$; *then for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any initial data* Φ_0 *satisfying:*

$$\Phi_0 \in \mathrm{H}^1(\mathbb{R}^N), \qquad \inf_{u \in S_\mu} \|u - \Phi_0\|_{H^1(\mathbb{R}^N)} < \delta$$

then the solution $\Phi(t, x)$ *of* (1)–(4) *satisfies:*

$$\inf_{u \in S_\mu} \|u - \Phi(t, \cdot)\|_{H^1(\mathbb{R}^N)} < \varepsilon, \quad \textit{for all} \quad t \geqq 0.$$

Proof of Theorem II.2. The proof has been already discussed in Sect. I except that we assumed $|\Phi_0|^2_{L^2(\mathbb{R}^N)} = \mu$. But, because of the assumption made, we have:

$$\mu - \delta < |\Phi_0|_{L^2(\mathbb{R}^N)} < \mu + \delta.$$

And using i) of Theorem II.1, we may conclude by a trivial adaptation of the argument given in Sect. I.

Remark II.2. If $p \geqq 1 + \frac{4}{N}$ and if $p < 1 + \frac{4}{N-2}$ ($< \infty$ if $N \leqq 2$) then there exist real, positive solutions of (5)–(14) (for all $\mu > 0$) (see for example Berger [5]; Berestycki and Lions [3, 4]) but these are orbitally instable as it can be seen by a simple argument (see for more general results Berestycki and Cazenave [2]): actually one can exhibit initial data Φ_0 as close to these solutions as one wants such that the solution of (1)–(4) does not exist for all time $t \geqq 0$ (and there is blow-up).

Remark II.3. We could replace the nonlinearity $|\Phi|^{p-1}$ by a general nonlinearity $f(\Phi)$ where $f \in C(\mathbb{C})$ and $f(re^{i\theta}) = e^{i\theta} f(r)$ for all $r \in \mathbb{R}$, $\theta \in \mathbb{R}$. Then similar results hold provided one assumes:

(i) $\exists u_0 \in H^1(\mathbb{R}^N)$, $|u_0|_{L^2(\mathbb{R}^N)} \leqq \mu$, $\mathscr{E}(u_0) < 0$;

(ii) $\overline{\lim\limits_{t \to +\infty}}\, f(t) t^{-\left(1 + \frac{4}{N}\right)} < \varepsilon_0\ [= \varepsilon_0(\mu)]$;

(iii) $\overline{\lim\limits_{t \to 0_+}}\, F(t) t^{-2} < +\infty \left(F(t) = \int_0^t f(s) ds \right)$.

Under these assumptions, Theorem II.1 is still true (see [21, 22]) and as soon as one can solve (1)–(4), Theorem II.2 holds. For example, if $f(z) = z \log |z|^2$, (i), (ii), and (iii) above are valid and Theorem II.1 still holds. In addition, by a simple application of Gross-Sobolev's inequality – see Adams and Clarke [1], Bialynicki-Birula and Mycielski [6] –, one can prove that there exists a unique positive solution of (9) u_0 and thus:

$$S_\mu = \{e^{i\theta} u_0(\cdot + y), \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.$$

Therefore, the analogue of Theorem II.2 holds in this case (and we answer a conjecture given in Cazenave [9] where a related result was proved under the restrictive assumption that Φ_0 is spherically symmetric).

Remark II.4. If Φ_0 is spherically symmetric, then we may replace in the above result S_μ by the subset of S_μ consisting of spherically symmetric functions [the proof is then identical since $\Phi(t,x)$ remains radial for all $t \geqq 0$ if Φ_0 is radial].

Remark II.5. If $N \leqq 4$ and if $1 < p < 1 + \frac{4}{N}$, then by a uniqueness result due to MacLeod and Serrin [24], we have

$$S_\mu = \{e^{i\theta} u_0(\cdot + y), \theta \in \mathbb{R}, y \in \mathbb{R}^N\},$$

where u_0 is the unique positive radial solution of: $-\Delta u_0 + \lambda u_0 = u_0^p$ in $\mathbb{R}^N$, $u_0 \in H^1(\mathbb{R}^N)$ and $|u_0|_{L^2}^2 = \mu$. Indeed, in view of [24], any positive radial solution of

$$-\Delta u + \lambda u = u^p \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N$$

is given by $u(x) = \lambda^{1/(p-1)} u_1(\sqrt{\lambda} x)$ where u_1 is the unique radial solution of

$$-\Delta u_1 + u_1 = u_1^p \text{ in } \mathbb{R}^N, \quad u_1 \in H^1(\mathbb{R}^N), \quad u_1 > 0 \text{ in } \mathbb{R}^N.$$

Thus

$$|u|_{L^2(\mathbb{R}^N)} = \lambda^{1/(p-1)} \lambda^{-\frac{N}{4}} |u_1|_{L^2(\mathbb{R}^N)} = \lambda^\alpha |u_1|_{L^2(\mathbb{R}^N)},$$

where $\alpha = \frac{1}{p-1} - \frac{N}{4}$. This yields the characterization of S_μ.

III. Hartree-Type Time-Dependent Solutions

We consider the following equation:

$$\begin{cases} i \dfrac{\partial \Phi}{\partial t}(t,x) + \Delta \Phi(t,x) + \sum_{i=1}^m \dfrac{z_i}{|x - x_i|} \Phi(t,x) & \\ \quad - \left\{ \int_{\mathbb{R}^3} |\Phi(t,y)|^2 \dfrac{1}{|x-y|} dy \right\} \Phi(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3; & (2) \\ \Phi(0,x) = \Phi_0(x) \text{ in } \mathbb{R}^3, \qquad \text{in } \mathbb{R}^3, & (4) \end{cases}$$

where $x_1, \ldots, x_m$ are some points in $\mathbb{R}^3$; $z_i > 0$. We denote by $Z = \sum_i z_i$. If $\Phi_0 \in H^1(\mathbb{R}^3)$, there exists a unique solution $\Phi \in C([0, \infty[; H^1(\mathbb{R}^3))$ of (2)–(4) (see Ginibre and Velo [12, 13]); in addition Φ satisfies:

$$|\Phi(t, \cdot)|_{L^2(\mathbb{R}^3)} = |\Phi_0|_{L^2(\mathbb{R}^3)}, \tag{11}$$

$$\mathscr{E}(\Phi(t, \cdot)) = \mathscr{E}(\Phi_0), \tag{12}$$

for all $t \geqq 0$, where $\mathscr{E}$ is defined on $H^1(\mathbb{R}^3)$ by:

$$\mathscr{E}(u) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 - \frac{1}{2} \sum_{i=1}^m \frac{z_i}{|x - x_i|} |u|^2 dx + \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(x)|^2 |u(y)|^2 |x-y|^{-1} dx dy.$$

For the same reasons as in the preceding sections, we consider the following problems:

$$-\Delta u-\sum_{i=1}^{m}\frac{z_i}{|x-x_i|}u+\lambda u+\left(|u|^2*\frac{1}{|x|}\right)u=0 \text{ in } \mathbb{R}^3, \qquad u\in H^1(\mathbb{R}^3), \tag{15}$$

$$|u|^2_{L^2(\mathbb{R}^3)}=\mu. \tag{14}$$

And we introduce again:

$$I_\mu=\operatorname{Inf}\{\mathscr{E}(u), u\in H^1(\mathbb{R}^3), |u|^2_{L^2(\mathbb{R}^3)}=\mu\}. \tag{9}$$

Before stating any result concerning (14)–(15) or (9), we need to introduce λ_1, the first eigenvalue of the operator $-\Delta-\sum_{i=1}^{m}\frac{z_i}{|x-x_i|}$, i.e.

$$\lambda_1=\operatorname{Min}\left\{\int_{\mathbb{R}^3}|\nabla u|^2-\sum_{i=1}^{m}\frac{z_i}{|x-x_i|}|u|^2dx/u\in H^1(\mathbb{R}^3), |u|_{L^2(\mathbb{R}^3)}=1\right\},$$

it is easy to check that $\lambda_1<0$.

We may now state the following result, due to Lieb and Simon [18], and Lions [20, 22]:

Theorem III.1. i) *If $\lambda<0$ or $\lambda\geqq-\lambda_1$, there exists no positive solution of* (15).
ii) *If $\lambda\in[0,-\lambda_1[$, there exists a unique positive solution u_λ of* (15).

In addition $u_\lambda\in C^1(]0,-\lambda_1[;H^1(\mathbb{R}^3))\cap C([0,-\lambda_1];H^1(\mathbb{R}^3))$ and $|u_\lambda|_{L^2(\mathbb{R}^3)}$ is strictly decreasing for $\lambda\in[0,-\lambda_1]$; $u_\lambda \xrightarrow[\lambda\to(-\lambda_1)]{H^1} 0$. We denote by $\mu_0=|u_0|^2_{L^2(\mathbb{R}^3)}$; we have: $\mu_0>Z=\sum_{i=1}^{m}z_i$.

iii) *If $(u_n)_n$ is a minimizing sequence of* (9) *and more precisely if $(u_n)_n$ satisfies:*

$$u_n\in H^1(\mathbb{R}^3), \qquad |u_n|^2_{L^2(\mathbb{R}^3)}\xrightarrow[n]{}\mu\in]0,\mu_0], \qquad \mathscr{E}(u_n)\xrightarrow[n]{}I_\mu, \tag{16}$$

or

$$u_n\in H^1(\mathbb{R}^3), \qquad \varliminf_n |u_n|^2_{L^2(\mathbb{R}^3)}\geqq\mu_0, \qquad \mathscr{E}(u_n)\xrightarrow[n]{}I_{\mu_0}, \tag{17}$$

then $(u_n)_n$ is relatively compact in the Hilbert spaces $X=\{u\in L^6(\mathbb{R}^3), Du\in L^2(\mathbb{R}^3)\}$, and if (16) *holds, (u_n) is relatively compact in $H^1(\mathbb{R}^3)$. In addition if* (17) *holds and $\varliminf_n |u_n|^2_{L^2(\mathbb{R}^3)}>\mu_0$, then all limit points (in the strong topology of X and the weak topology of $L^2(\mathbb{R}^3)$) lie in S_{μ_0}.*

iv) *For any $\mu\in]0,\mu_0]$, the set S_μ of solutions of* (9) *is given by $S_\mu=\{e^{i\theta}u_\lambda, \theta\in\mathbb{R}\}$, where λ is determined by: $|u_\lambda|^2_{L^2(\mathbb{R}^3)}=\mu$.*

Remark III.1. This result is proved in Lions [22] and relies on previous results of Lieb and Simon [18] and Lions [20]. Let us also mention the works of Reeken [27] and Stuart [32, 33].

In fact, in [22], iv) is proved only for real-valued solutions of (9). But by the same argument as in the preceding section we extend iv) to any complex-valued solution of (9).

We may now state our stability result concerning S_μ (or u_λ).

Theorem III.2. *Let* $\mu \in]0, \mu_0]$. *For all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all initial data* Φ_0 *satisfying:*

i) $$\Phi_0 \in H^1(\mathbb{R}^N), \quad \inf_{u \in S_\mu} \|u - \Phi_0\|_{H^1(\mathbb{R}^N)} < \delta,$$

then the solution $\Phi(t,x)$ *of* (2)–(4) *satisfies:*

$$\inf_{u \in S_\mu} \|u - \Phi(t, \cdot)\|_{H^1(\mathbb{R}^3)} < \varepsilon, \quad \textit{for all} \quad t \geqq 0.$$

ii) $$\Phi_0 \in H^1(\mathbb{R}^N), \quad \inf_{u \in S_\mu} \{\|u - \Phi_0\|_X + d_R(u, \Phi_0)\} < \delta, \quad \|\Phi_0\|_{L^2(\mathbb{R}^3)} \leqq R,$$

where d_R *denotes a metric for the weak topology of the ball of radius* R *of* $L^2(\mathbb{R}^3)$ *and* R *is some arbitrary constant larger than* $\sqrt{\mu_0}$, *then the solution* $\Phi(t,x)$ *of* (2)–(4) *satisfies*

$$\inf_{u \in S_\mu} \{\|u - \Phi(t, \cdot)\|_X + d_R(u, \Phi(t, \cdot))\} < \varepsilon \quad \textit{for all} \quad t \geqq 0.$$

We will skip the proof of this result since it is an obvious consequence of Theorem III.1 above and of the argument given in Sect. I.

Remark III.2. Similar results may be proved for systems of Hartree equations or Hartree-Fock time dependent equations.

IV. Pekar-Choquard Time-Dependent Equations

We consider now the following equation:

$$\begin{cases} i\dfrac{\partial \Phi}{\partial t}(t,x) + \Delta \Phi(t,x) + \left\{ \displaystyle\int_{\mathbb{R}^3} |\Phi(t,y)|^2 \frac{1}{|x-y|} dy \right\} \Phi(t,x) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \quad (3) \\ \Phi(0,x) = \Phi_0(x) & \text{in } \mathbb{R}^3. \quad (4) \end{cases}$$

If $\Phi_0 \in H^1(\mathbb{R}^3)$, there exists a unique solution $\Phi \in C([0,\infty[; H^1(\mathbb{R}^3))$, of (3)–(4) (see Ginibre and Velo [12, 13]); in addition Φ satisfies:

$$|\Phi(t,\cdot)|_{L^2(\mathbb{R}^3)} = |\Phi_0|_{L^2(\mathbb{R}^3)}, \tag{11}$$

$$\mathscr{E}(\Phi(t,\cdot)) = \mathscr{E}(\Phi_0), \tag{12}$$

for all $t \geqq 0$, where $\mathscr{E}$ is defined on $H^1(\mathbb{R}^3)$ by:

$$\mathscr{E}(u) = \int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla u|^2 dx - \tfrac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(x)|^2 |u(y)|^2 |x-y|^{-1} dx dy.$$

Again we introduce the following problems:

$$-\Delta u + \lambda u - \left(|u|^2 * \frac{1}{|x|}\right) u = 0 \text{ in } \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3), \tag{16}$$

$$|u|^2_{L^2(\mathbb{R}^3)} = \mu; \tag{14}$$

and

$$I_\mu = \operatorname{Inf}\{\mathscr{E}(u), u \in H^1(\mathbb{R}^3), |u|^2_{L^2(\mathbb{R}^3)} = \mu\}. \tag{9}$$

We then have:

Theorem IV.1. *Let $\mu>0$.*

i) *Let $(u_n)_n$ be a minimizing sequence of* (9)*: $u_n \in H^1(\mathbb{R}^3)$, $|u_n|^2_{L^2(\mathbb{R}^3)} \underset{n}{\longrightarrow} \mu$ and $\mathscr{E}(u_n) \underset{n}{\longrightarrow} I_\mu$; then there exists $(y_n)_n \subset \mathbb{R}^3$ such that:*

$$u_n(\cdot+y_n) \text{ is relatively compact in } H^1(\mathbb{R}^3).$$

ii) *The set S_μ of solutions of* (9) *is given by:*

$$S_\mu = \{e^{i\theta} u_\lambda(\cdot+y), \theta \in \mathbb{R}, y \in \mathbb{R}^3\},$$

where (λ, u_λ) $(\in]0, +\infty[\times H^1(\mathbb{R}^3))$ is the unique solution of (16)–(14) *such that u_λ is real, positive and radial.*

Remark IV.1. Part ii) of the preceding result is due to Lieb [17] while part i) of the above result is due to Lions [21, 22].

Again, an immediate application of the above result and of Sect. I is the

Theorem IV.2. *Let $\mu>0$; for all $\varepsilon>0$, there exists $\delta>0$ such that for every initial data Φ_0 satisfying:*

$$\Phi_0 \in H^1(\mathbb{R}^3), \qquad \inf_{u \in S_\mu} \|u-\Phi_0\|_{H^1(\mathbb{R}^3)} < \delta,$$

then the solution $\Phi(t,x)$ of (3)–(4) *satisfies for all $t \geqq 0$:*

$$\inf_{u \in S_\mu} \|u-\Phi(t,\cdot)\|_{H^1(\mathbb{R}^3)} < \varepsilon.$$

Remark IV.2. Similar results may be given for equations of the form:

$$i\frac{\partial \Phi}{\partial t}(t,x) + \Delta\Phi(t,x) + V(x)\Phi(t,x)$$
$$+\left\{\int_{\mathbb{R}} |\Phi(t,y)|^2 f(x-y)\,dy\right\}\Phi(t,x) = 0 \quad \text{in } \mathbb{R}_+\times\mathbb{R}^N, \tag{18}$$
$$\Phi(0,x) = \Phi_0(x) \quad \text{in } \mathbb{R}^N, \tag{4}$$

where $V \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $\frac{N}{2} < p,\ q \leqq +\infty$, and $f \in L^\alpha(\mathbb{R}^N) + L^\beta(\mathbb{R}^N)$ with $\frac{N}{2} < \alpha, \beta < +\infty$.

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