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Renormalized G-Convolution of N-Point Functions in Quantum Field Theory: Convergence in the Euclidean Case I

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Abstract. The notion of Feynman amplitude associated with a graph G in perturbative quantum field theory admits a generalized version in which each vertex v of G is associated with a *general* (non-perturbative) n_v-point function H^{n_v}, n_v denoting the number of lines which are incident to v in G. In the case where no ultraviolet divergence occurs, this has been performed directly in complex momentum space through Bros–Lassalle's G-convolution procedure.

In the present work we propose a generalization of G-convolution which includes the case when the functions H^{n_v} are *not* integrable at infinity but belong to a suitable class of slowly increasing functions. A "finite part" of the G-convolution integral is then defined through an algorithm which closely follows Zimmermann's renormalization scheme. In this work, we only treat the case of "Euclidean" r-momentum configurations.

The first part which is presented here contains together with a general introduction, the necessary mathematical material of this work, i.e., Sect. 1 and appendices A and B.

The second part, which will be published in a further issue, will contain the Sects. 2, 3 and 4 which are devoted to the statement and to the proof of the main result, i.e., the convergence of the renormalized G-convolution product.

The table of references will be given in both parts.

Introduction

It has become commonly accepted in Particle Physics that various collision mechanisms may be conveniently described in the language of generalized Feynman amplitudes. These quantities are associated with "fat" Feynman graphs, in which the orthodox point-wise vertices of perturbation theory are replaced by "bubbles", whenever strong interaction processes have to be taken into account (For an up-to-date example of such description, see the well known deep-inelastic scheme [1]).

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According to the situation which is considered, these bubbles are supposed to represent either complete scattering or production amplitudes, or partial hypothetic "resummations" of Feynman amplitudes. Moreover, as far as the momentum configurations are concerned the latter are sometimes taken in Minkowski space and then the Feynman $i\varepsilon$-prescription of integration is used, by reference to the perturbative case; but sometimes for convenience reasons, they are also restricted to the Euclidean space, and the problem of analytic continuation on the mass-shell of the resulting quantities is left unsolved in general.

It was this conceptually confuse situation which led one of us [2] and M. Lassalle [3] to introduce and investigate systematically in complex four-momentum the notion of G-convolution product associated with a graph G.

The main property which was proved by these authors is the following. Let G be a connected graph in which each vertex v is incident to n_v (external or internal) lines of G, and let n be the number of external lines of G. Assume that with each vertex v of G is associated a general n_v-point function $H^{(n_v)}(K^v)$: under this name is meant an analytic n_v-point function whose domain is the primitive[1] axiomatic domain in the complex space $\mathbb{C}^{4(n_v-1)}_{(K^v)}$ of the four-momenta K^v carried by the lines incident to v. Assume that with each internal line i of G is associated a general two-point function $H^{(2)}(\ell_i)$ of the corresponding complex four-momentum ℓ_i.

Then under suitable integrability assumptions on the $H^{(2)}(\ell_i)$ and on the $H^{(n_v)}(K^v)$, it is always possible to define a general n-point function $H^{(n)}_G$ of the set K of (complex) external momenta of G, through the following formula which generalizes the Feynman integral:

$$H^{(n)}_G(K) = \int_{\Gamma(K)} \prod_v H^{(n_v)}(K^v(K,k)) \prod_i H^{(2)}(l_i(K,k)) d_{4m}k \tag{1}$$

In this formula, k denotes a set of $4m$ independent complex integration variables, m being the number of independent loops of G; $K^v(K,k)$ and $\ell_i(K,k)$ are linear functions which are determined by taking into account the four-momentum conservation equations associated with all vertices of G; $\Gamma(K)$ is a $4m$-dimensional contour in the complex space $\mathbb{C}^{4m}_{(k)}$ which lies in the analyticity domain of the integrand and depends continuously on the external configuration.

In Bros–Lassalle's result, the following properties of $H^{(n)}_G$ are specified:

i) Formula (1) is meaningful when K varies in the "Euclidean region" $E^{4(n-1)}$ (i.e., $K = (\vec{P}, iQ^0)$, with $\vec{P} \in \mathbb{R}^{3(n-1)}$, $Q^0 \in \mathbb{R}^{(n-1)}$), provided that $\Gamma(K)$ be also chosen as the Euclidean region $\{k = (\vec{p}, iq^0) \vec{p} \in \mathbb{R}^{3m}, q^0 \in \mathbb{R}^m\}$. This is a trivial consequence of the basic fact [4] that the primitive axiomatic domain of each n_v-point function $H^{n_v}(K^v)$ (resp. $H^{(2)}(\ell_i)$) contains the "Euclidean region" of the corresponding K^v-space (resp. ℓ_i-space). $H^{(n)}_G(K)$ is thus primitively defined and analytic in $E^{4(n-1)}$.

ii) By using appropriate distortions of the contour $\Gamma(K)$ which generalize the Wick rotations, $H^{(n)}_G(K)$ is shown to have an analytic continuation in the whole primitive n-point domain. The various standard boundary values [4] of $H^{(n)}_G$ on the real space (i.e., on "Minkowski-space") are thus defined, and they satisfy the Steinmann relations.

1 Moreover the correct axiomatic analyticity domains for the absorptive parts of $H^{(n_v)}$, or equivalently the Steinmann relations for $H^{(n_v)}$ are also assumed to hold (see ref. 3)

iii) The τ-boundary value[2] $\tau_G^{(n)}$ of $H_G^{(n)}$ is defined by the Ruelle–Araki prescription which generalizes the $i\varepsilon$-prescription of Feynman amplitudes. Moreover formula (1) yields an expression for $\tau_G^{(n)}$ as an integral over Minkowski space in which each factor $H^{(n_v)}$ (resp. $H^{(2)}$) has been replaced by the corresponding τ-boundary value $\tau^{(n_v)}$ (resp. $\tau^{(2)}$) in the integrand.

iv) For i), ii) and the first part of iii) to hold, it is sufficient that the integrability conditions of (1) at infinity be satisfied in every "quasi-Euclidean" region $E(W) = W + E^{4m}$, where W is an arbitrary compact region of $\mathbb{C}^{4m}$. (Of course the second part of iii) requires integrability in Minkowski space).

From the physical point of view, the necessity of imposing integrability conditions as in iv) is not fully satisfactory, since the Green functions $H^{(n)}$ of a polynomially interacting Field Theory in four-dimensional space time may very well have increase properties at infinity which produce ultraviolet divergences in integrals of the type (1), as it is the case in perturbation theory. As a matter of fact, the G-convolution product defined above revealed itself a sufficiently good notion for carrying out the study of the analyticity properties implied by Many Particle Structure Analysis [see refs. 2, 5]: it was because by using analytic cut-offs of the Pauli–Villars type, all structural equations[3] involving p-particle irreducible kernels could be written in a "regularized" form in which all the G-convolution products were well-defined in the above sense. However in this procedure, it was impossible to keep under control the asymptotic properties of the theory and to define the "exact" p-particle irreducible kernels. To do this it would be necessary to write "renormalized" structural equations in which G-products including ultraviolet divergences should be given a precise prescription of integration in order to yield in all cases a finite result.

As another important example for which it could be helpful to have a renormalized version of G-convolution product, we have in mind the study of field equations of a definite four-dimensional Lagrangian theory with polynomial interaction. Indeed in momentum space, these field equations are equivalent to an infinite system of G-convolution equations linking together the various n-point Green functions of the theory[4]; there again, a renormalized form of the equations is required, as it is already indicated by perturbation theory.

Among the known rigorous treatments of renormalization in perturbation theory, Zimmermann's procedure [7] is specially attractive for writing equations in momentum space, since (at least in its Euclidean version) it is exempt from regularization parameter and prescribes to replace the primitive Feynman integrands I_G associated with the graph G by a new rational integrable function R_G, which is called the "renormalized integrand".

It is our purpose in the present work to define a renormalized G-convolution product for an appropriate class of functions $H^{(2)}, H^{(n_v)}$, having slow increase at infinity, by adapting the algebraic procedure of Zimmermann, namely by

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2 For the n-point function associated with a quantum field this is the Fourier transform of the vacuum expectation value of the time-ordered product of n field operators (corrected part)

3 Of the general Bethe–Salpeter type

4 An approach of this type was proposed by J. G. Taylor [6]

replacing the primitive integrand $I_G(K,k)$ of formula (1) by a new renormalized integrand $R_G(K,k)$; the latter should by built up with the same ingredients $H^{(2)}$, $H^{(n_v)}$ as I_G, but its integrability should be assured. We should also like to prove that the resulting integral $H_G^{(n)\,(\mathrm{ren})}(K) = \int\limits_{\Gamma(K)} R_G(K,k)d_{4m}k$ is an analytic function satisfying the above properties i) ii) iii) iv). However, in the present paper, we shall restrict ourselves to the definition of R_G and $H_G^{(n)\,(\mathrm{ren})}$ in the Euclidean region (i.e., to i)). The analyticity properties of $H_G^{(n)\,(\mathrm{ren})}$ and in particular the passage from Euclidean to Minskowski space along the lines of [3] are not investigated here: some results in this connection are given in [12].

As a preliminary step, it will be necessary to define appropriate classes to which the various functions $H^{(2)}$, $H^{(n_v)}$, R_G, will have to belong; such classes will be introduced and studied in Sect. 1. These classes must of course contain the class of rational functions (corresponding to the perturbative case) and be characterized by their precise type of majorization at infinity in terms of powers of the four-momenta in Euclidean space.

A very clever realization of classes of functions of this type had been introduced by Weinberg in 1960 [8]. This author proved a very useful theorem ("power counting" theorem) which provided a set of necessary and sufficient inequalities to be fulfilled by a function of such a class to be an integrable function.

In perturbation theory, it turned out that a simplified and weaker form of the Power Counting Theorem [9] was sufficient for dealing with the case of rational functions. Here however, we will actually *need* to apply Weinberg's theorem in its most general version. As a matter of fact, we shall use classes of functions which are almost as general as those introduced by Weinberg, except for the following important restriction: we shall also impose conditions at infinity to the successive derivatives of our functions. As in other mathematical contexts[5] where one wishes to avoid functions having an oscillatory behaviour at infinity, we shall assume that the powers or "asymptotic coefficients" (in the sense of Weinberg) which govern the behaviour of our functions at infinity decrease by p units when a derivative operator of order p is applied to them. This type of condition will be justified by a technical but crucial result concerning the lowering of certain asymptotic coefficients when an appropriate "Taylor's rest operation" is applied: the proof of this statement which is rather tedious will be given in an appendix. It will be made clear in section IV that it is *this* result which allows Zimmermann's renormalization scheme to be applicable to the present case as well as to perturbation theory.

In Sect. 2 a definition of our generalized renormalized integrand R_G is given: this definition closely follows Zimmermann's algorithm [7] and involves a sum of counterterms which are associated with all the G-forests: a G-forest is a subset of "non-overlapping" subgraphs of G.

In Sect. 3 we introduce the notion of "complete forest with respect to a nested set of subspaces of $E_{(k)}^{rm}$" (this is also an extension of a notion defined in [7]). This notion allows to write new expressions of R_G which are used in the following Sect. 4. The latter contains the proof of our main theorem: R_G satisfies Weinberg's

5 See the definition of symbols in the theory of pseudodifferential operators [10, 11]

convergence criterion, and thus the renormalized integral $H_G^{(n)(\mathrm{ren})}(K)$ is a well-defined function in the euclidean region. In these results the space-time dimension is an arbitrary integer r.

1. Some Classes of Functions with Slow Increase

1.1. The Classes $A_N^{(\alpha)}$

Let E be an N-dimensional vector-space on $\mathbb{R}$. We shall make an extensive use of certain classes $A_N^{(\alpha)}$ of complex valued functions with slow increase on E which have been introduced in 1960 by Weinberg[6] [8]. Each class $A_N^{(\alpha)}$ is characterized by a *bounded* real-valued function α on the set of all the linear subspaces $S \neq \{0\}$ of E. α will be called the "asymptotic indicatrix" of the class $A_N^{(\alpha)}$, by reference to the following fact: each number, or "asymptotic coefficient", $\alpha(S)$ will describe the power of increase (or decrease) which majorizes every function in $A_N^{(\alpha)}$, in "almost all" the directions of the corresponding subspace S; this is made precise in the following definition.

Definition 1a[6]. A complex-valued function f on E belongs to the class $A_N^{(\alpha)}$ if, for every set of $m \leqq N$ independent vectors $L_1 \dots L_m$ and every bounded region W in E there exists a set of numbers $b_1 \dots b_m \geqq 1$ and a constant $M > 0$ (depending on $L_1, \dots, L_m$ and W) such that:

$$\left| f\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C \right) \right| \leqq M \prod_{j=1}^{m} \eta_j^{\alpha(\overline{\{L_1 \dots L_j\}})} \tag{1.1}$$

when the real variables $\eta_j (j = 1, \dots, m)$ belong to the region $\{\eta_j \geqq b_j\}$ and when $C \in W$.

In (1.1), $\overline{\{L_1, \dots, L_j\}}$ denotes the linear closure of the set of vectors $\{L_1, \dots, L_j\}$. From this definition, we easily derive:

Proposition 1.1

a) $A_N^{(\alpha)}$ *is a vector space (on $\mathbb{R}$ or $\mathbb{C}$)*

b) *If $f_1 \in A_N^{(\alpha_1)}, f_2 \in A_N^{(\alpha_2)}$ then $f_1 f_2 \in A_N^{(\alpha_1 + \alpha_2)}$*

c) *If $\alpha \leqq \alpha' : A_N^{(\alpha)} \subset A_N^{(\alpha')}$*

Let us now consider two vector spaces E, E' with respective dimensions N, N', and a linear mapping ρ from E to E'. Then we have the following property.

Lemma 1.1. *Let f' be a function on E' which belongs to a certain class $A_{N'}^{(\alpha')}$. Then its inverse image $f = \rho^* f'$ on E (namely: $\forall K \in E, f(K) = f'(\rho(K))$ belongs to the class $A_N^{(\alpha)}$ which is determined as follows: for every subspace S of E, let us denote by S' its image in $E' : S' = \rho(S)$; then $\alpha(S) = \alpha'(S')$. In the case when S belongs to the Kernel of ρ(i.e., $S' = \{0\}$), $\alpha'(S')$ has to be put equal to zero.*

The proof of this statement makes use of the following

Auxiliary Lemma A. *If g is a Weinberg function in $A_N^{(\alpha)}$, it still satisfies the bound of formula (1.1) for arbitrary sets of vectors $L_1, \dots, L_m$ (i.e. even when the latter are not*

6 In the *original* classes A_N introduced by Weinberg, logarithmic factors $(\ln \eta_j)^{\beta(\overline{\{L_1 \dots L_j\}})}$ were also written at the right-hand side of (1.1); however this refinement is not needed for the present work

linearly independent, or when some of them are equal to $\{0\}$*). The asymptotic coefficients* $\alpha(\overline{\{L_1 L_2 \dots L_j\}})$ *still have the same meaning as before:* $\overline{\{L_1, L_2 \dots L_j\}}$ *denotes the subspace S which is generated by the (possibly redundant) set of vectors* $L_1, L_2, \dots, L_j$. *In the special case* $S = \{0\}$, $\alpha(S)$ *is defined as being equal to zero.*

In view of its technical character, the proof of this auxiliary lemma is given in the appendix A.

Proof of Lemma 1.1. Let $(L_1, L_2 \dots L_m)$ be an arbitrary set of independent vectors in E and let C belong to a compact subset W of E. We put: $L'_j = \rho(L_j)$ $(1 \leqq j \leqq m)$ and $C' = \rho(C)$. Then we have:

$$f\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C\right) = f'\left(\sum_{j=1}^{m} L'_j \eta_j \dots \eta_m + C'\right) \tag{1.2}$$

But in view of Lemma A there exist numbers $b_1, \dots, b_m \geqq 1$ and a constant $M > 0$ such that:

$$\left| f\left(\sum_{j=1}^{m} L'_j \eta_j \dots \eta_m + C'\right)\right| \leqq M \prod_{j=1}^{m} \eta_j^{\alpha'(\overline{L'_1 \dots L'_j})} \tag{1.3}$$

provided that: $\eta_j \geqq b_j \quad 1 \leqq j \leqq m$ and $C' \in W' = \rho(W)$.

By noting that $\overline{\{L'_1 \dots L'_j\}} = \rho(\overline{\{L_1 \dots L_j\}})$, and by taking (1.2) into account we immediately interpret the bound (1.3) as the announced result for $\rho^* f$. In particular, if $L_j \in \operatorname{Ker} \rho$ for $1 \leqq j \leqq m'$, the corresponding asymptotic coefficients $\alpha(\overline{\{L_1 \dots L_j\}})$ are equal to $\alpha'(\{0\}) = 0$. q.e.d.

The most trivial examples of functions in classes $A_N^{(\alpha)}$ are the polynomials on E. In fact, if $E = \mathbb{R}$ it is clear from Definition 1a that the coordinate function K is in the class $A_1^{(\alpha)}$ with (unique) coefficient $\alpha(\mathbb{R}) = 1$. By applying Lemma 1.1 to the case when $E' = \mathbb{R}$, we then obtain that every linear form ℓ on E with null-hyperplane H belongs to the class $A_N^{(\alpha)}$ such that:

$$\alpha(S) = 0 \quad \text{if } S \subset H \quad \text{and } \alpha(S) = 1 \quad \text{if } S \not\subset H.$$

More generally, by applying Proposition 1.1 (b, c and a successively) we obtain:

Proposition 1.2. *a) Let* $\{l_i; i \in I\}$ *be any finite set of linear forms on E with respective null hyperplanes* H_i. *Then each monomial* $\prod_{i \in I} \ell_i^{v_i}$ *belongs to the corresponding class* $A_N^{(\alpha)}$ *which is defined as follows:* $\forall S \subset E$; $\alpha(S) = \sum_{\{i \in I; S \not\subset H_i\}} v_i$

b) Let Q_v *be any polynomial of degree v on E; then* Q_v *belongs to the class* $A_N^{(\alpha)}$ *whose asymptotic indicatrix is the constant* v $(\forall S \subset E,\ v(S) = v)$.

The following non trivial property of the classes $A_N^{(\alpha)}$ has been proved by Weinberg in [8], and will be used as a basic tool in our Sect. 4.

Lemma 1.2. *For a function f to be integrable on E, it is sufficient that it belongs to a class* $A_N^{(\alpha)}$ *whose asymptotic indicatrix* α *satisfies the following inequality:*

$$\sup_{S \subset E} (\alpha(S) + h(S)) < 0 \tag{1.4}$$

In (1.4), the sup. extends over all the subspaces S of E, and $h(S)$ denotes the dimension of S.

1.2. The Classes Σ_n^μ

Let $\mathscr{E}$ be an n-dimensional vector space in which a certain norm (denoted by $\|\cdot\|$) has been chosen. We consider the class $C^\infty(\mathscr{E})$ of all the functions which are infinitely differentiable on $\mathscr{E}$, and we denote by $P_\nu(D)$ any homogeneous polynomial of degree ν in the derivatives with respect to the coordinates of the variable in $\mathscr{E}$. In a more intrinsic way (independent of the choice of a system of coordinates), $P_\nu(D)$ is a linear operator on the space $C^\infty(\mathscr{E})$ which is associated with a given element P_ν of the symmetrized tensor product $\mathscr{E}^{\otimes_s \nu}$ and is defined as follows; if $f \in C^\infty(\mathscr{E})$ and if $f^{(\nu)}(K)$ denotes the derivative application of order ν of f at K, which is a linear form on $\mathscr{E}^{\otimes_s \nu}$, then one has:

$$\forall K \in \mathscr{E}, (P_\nu(D)f)(K) = f^{(\nu)}(K)(P_\nu) \tag{1.5}$$

We then define the following classes Σ_n^μ of functions on $\mathscr{E}$, which are contained in the general class of symbols introduced in ref. [10 and 11].

Definition 1b. Let μ be an arbitrary real number. A function f on $\mathscr{E}$ is said to belong to the class Σ_n^μ if it belongs to $C^\infty(\mathscr{E})$ and if for every integer $\nu \geqq 0$ and every homogeneous polynomial $P_\nu(D)$, there is a constant C_ν such that:

$$|(P_\nu(D)f)(K)| \leqq C_\nu |||P_\nu||| (1 + \|K\|)^{\mu-\nu} \tag{1.6}$$

here $|||P_\nu|||$ denotes a certain norm of P_ν in $\mathscr{E}^{\otimes_s \nu}$, which we do not need to specify.

Remark. Σ_n^μ is independent of the choice of the representative $\|\cdot\|$ in the class of all the (equivalent) norms on $\mathscr{E}$. Indeed in the physical applications of section 2, there will be no canonical choice of the norm on $\mathscr{E}$).

Let now E denote an N-dimensional vector space (as in 1.1) and let λ be a linear mapping from E to $\mathscr{E}$. We shall prove:

Lemma 1.3. *For every function f on $\mathscr{E}$ which belongs to the class Σ_n^μ, the inverse image λ^*f of f belongs to the class $A_N^{(\alpha^{(\mu)})}$ whose asymptotic indicatrix $\alpha^{(\mu)}$ is defined as follows:*

$$\left.\begin{aligned} &\alpha^{(\mu)}(S) = 0 \qquad \text{if } S \subset \operatorname{Ker}\lambda && (1.7.a)\\ &\alpha^{(\mu)}(S) = \mu \qquad \text{if } S \not\subset \operatorname{Ker}\lambda && (1.7.b)\end{aligned}\right\}$$

Moreover, for every integer $\nu > 0$ and every homogeneous polynomial $Q_\nu(D)$ of degree ν on E, the function $Q_\nu(D)(\lambda^*f)$ belongs to the class $A_N^{\alpha(\mu-\nu)}$.

Proof. Let $\{L_1, \ldots, L_m\}$ be an arbitrary set of independent vectors ($m \leqq N$) and W a bounded region in E.

Let $J \leqq m$ be the integer such that:

$$\forall j \leqq J, \quad \lambda(L_j) = \{0\}, \quad \text{and } \lambda(L_{j+1}) \neq \{0\} \tag{1.8}$$

If $J = m$, we have:

$$\left|(\lambda^* f)\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C\right)\right| = |f(\lambda(C))| \leqq M \tag{1.9}$$

where $M = \sup_{C \in W} |f(\lambda(C))|$

If $J < m$, we have:

$$\left|(\lambda^* f)\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C\right)\right| = \left|f\left(\sum_{j=J+1}^{m} \lambda(L_j)\eta_j \dots \eta_m + \lambda(C)\right)\right| \tag{1.10}$$

Then under the conditions:

$$\forall j \leqq m \quad \eta_j \geqq 1, \text{ and } C \in W \tag{1.11}$$

the norm inequality yields:

$$\left\|\sum_{j=J+1}^{m} \lambda(L_j)\eta_j \dots \eta_m + \lambda(C)\right\| \leqq \left(\sum_{j=J+1}^{m} \|\lambda(L_j)\| + \sup_{C \in W} \|\lambda(C)\|\right) \prod_{j=J+1}^{m} \eta_j \tag{1.12}$$

Then by taking into account Definition 1b (namely formula 1.6 in the case $\nu = 0$, $P_0 = 1$), we easily derive from (1.10), (1.12) the following majorization:

$$\left|(\lambda^* f)\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C\right)\right| \leqq M \prod_{j=J+1}^{m} (\eta_j^{\mu}) \tag{1.13}$$

which holds under the conditions (1.11).

In (1.13), the constant M is given by the formula:

$$M = C_0 \left(1 + \sum_{j=J+1}^{m} \|\lambda(L_j)\| + \sup_{C \in W} \|\lambda(C)\|\right)^{\mu}$$

Now if we introduce the asymptotic indicatrix $\alpha^{(\mu)}$ by the formula (1.7), we can easily check that the right-hand sides of both inequalities (1.9) and (1.13) can be rewritten in a unique way as $M \prod_{j=1}^{m} (\eta_j^{\alpha^{(\mu)}(\{\overline{L_1 \dots L_j}\})})$ (since in view of (1.8) the inclusion $\{\overline{L_1 \dots L_j}\} \subset \operatorname{Ker} \lambda$ holds if and only if $j \leqq J$). Thus the first part of Lemma 1.3 is proved.

To prove the second part, we use the fact that for every Q_ν in $E^{\otimes_s \nu}$, there exists an element P_ν in $\mathscr{E}^{\otimes_s \nu}$, such that:

$$Q_\nu(D)(\lambda^* f) = \lambda^*(P_\nu(D) f); \tag{1.14}$$

indeed, if $\lambda_{(\nu)}$ denotes the linear mapping from $E^{\otimes_s \nu}$ to $\mathscr{E}^{\otimes_s \nu}$ which is canonically induced by λ, P_ν is determined by the equation: $P_\nu = \lambda_{(\nu)}(Q_\nu)$, and (1.14) can be derived from (1.5) and from the definition of λ^*.

Now, by applying the same argument as above to the function $P_\nu(D) f$ instead of f (namely, by using formulae (1.9), (1.10), (1.12)) and by taking (1.6) into account, we obtain the second result of Lemma 1.3. q.e.d.

1.3. The Classes $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ of Admissible Weinberg Functions

In this subsection, a vector space $\mathscr{E}_{(K,k)}^{rN} = \mathscr{E}_{(K)}^{r(n-1)} \times E_{(k)}^{rL}$ is considered; the subscripts $(K), (k)$ etc... will always refer to the name of the variable which is used in the corresponding space; the superscripts rL etc... refer to the dimension of the space. The notations are justified by the physical applications of the next section: K and k will correspond respectively to the external and to the integration momentum variables which will be associated with a graph G; these momenta will be supposed to belong to an r-dimensional euclidean vector space (which plays the role of the space of the euclidean four-dimensional energy-momentum vectors). $E_{(k)}^{rL}$ is the space of the sets of L independent euclidean r-vectors, namely: $k = (k_1, \ldots, k_L)$. $\mathscr{E}_{(K)}^{r(n-1)} \approx E^{rn}/E^r$ is the space of the sets of n euclidean r-vectors $K = (K_1 \ldots K_n)$ linked by the relation: $K_1 + K_2 + \ldots + K_n = 0$ ("conservation of total momentum"). In $\underline{\mathscr{E}}_{(K,k)}^{rN}$, $N = L + n - 1$ denotes the total number of independent r-vectors. We shall also make use of the canonical projections π (resp. χ) of $\underline{\mathscr{E}}_{(K,k)}^{rN}$ onto $E_{(k)}^{rL}$[7] (resp. $\mathscr{E}_{(K)}^{r(n-1)}$); namely we have: $\pi(K,k) = (0,k); \chi(K,k) = (K,0)$.

As in subsection 1.2, we shall also consider homogeneous polynomials in the derivatives with respect to the variables K, which we shall denote by $P_m(D_k)$; for every $f \in C^\infty(\underline{\mathscr{E}}_{(K,k)}^{rN})$, $P_m(D_k)f$ is a well defined function on $\underline{\mathscr{E}}_{(K,k)}^{rN}$ which does not depend on the choice of the coordinates (see the beginning of 1.2).

Similarly we shall also consider the Taylor expansion of degree d of a function f with respect to variables K at $K = 0$; we denote it by $t_{(K)}^d f$ and the corresponding Taylor rest by $(1 - t_{(K)}^d)f$: these functions are always considered as intrinsically defined on $\underline{\mathscr{E}}_{(K,k)}^{rN}$, even if their explicit expression in coordinates has to be used in some of the arguments given below.

Definition 1c. Let σ be a given set of subspaces of $E_{(k)}^{rL}$ and let ω be a given set of subspaces of $\underline{\mathscr{E}}_{(K,k)}^{rN}$ such that:

a) $\sigma \subset \omega$ (here every subspace $S \in \sigma$ is also considered as a subspace of $\underline{\mathscr{E}}_{(K,k)}^{rN}$; see the previous footnote).
b) $\forall S \in \omega \; \pi(S) \in \sigma$
c) $S \in \omega$ and $S \subset S'$ imply $S' \in \omega$
d) σ and ω do not contain the subspace $\{0\}$.

Remark. From a) and b) one gets that if $S \subset E_{(k)}^{rL}$, the relations $S \in \sigma$ and $S \in \omega$ are equivalent. Then, from c); $S \in \sigma$ and $S \subset S' \subset E_{(k)}^{rL}$ imply $S' \in \sigma$. A couple of sets of subspaces (σ, ω) which satisfy the above conditions a), b), c), d) is said to be an "admissible couple in $\underline{\mathscr{E}}_{(K,k)}^{rN}$".

Definition 1d. Let α be an asymptotic indicatrix on $\underline{\mathscr{E}}_{(K,k)}^{rN}$ such that for every subspace $S \in \omega$ one has:

$$\alpha(S) = \alpha(\pi(S)). \tag{1.15}$$

Then we associate with α, σ, ω a class $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ of "admissible Weinberg functions" $f(k, K)$ by the following conditions:

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7 We shall always identify $E_{(k)}^{rL}$ with the subspace $\{0\}_{(K)} \times E_{(k)}^{rL}$ of $\underline{\mathscr{E}}_{(K,k)}^{rN}$

i) f belongs to the class $A_{rN}^{(\alpha)}$

ii) for every homogeneous derivative polynomial $P_m(D_K)$ of degree m, the function $P_m(D_k)f$ belongs to the class $A_{rN}^{(\alpha_m)}$ which is defined as follows:

$$\forall S \in \omega \qquad \alpha_m(\pi(S)) = \alpha_m(S) = \alpha(S) - m$$
$$\forall S \notin \omega \qquad \alpha_m(S) \leqq \alpha(S)$$

From the above definition and from Proposition 1.1 we easily derive the following.

Proposition 1.3

a) $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ *is a vector space*

b) *if* $f_1 \in \mathscr{A}_{rN}^{(\alpha(1),\sigma,\omega)}$, $f_2 \in \mathscr{A}_{rN}^{(\alpha(2),\sigma,\omega)}$ *then* $f_1 f_2 \in \mathscr{A}_{rN}^{(\alpha(1)+\alpha(2),\sigma,\omega)}$

c) *if* $\sigma' \subset \sigma$ *and* $\omega' \subset \omega$ *then* $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)} \subset \mathscr{A}_{rN}^{(\alpha,\sigma',\omega')}$

d) *if* $\alpha' \geqq \alpha$ *(i.e.* $\forall S : \alpha'(S) \geqq \alpha(S)$*) then:*

$$\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)} \subset \mathscr{A}_{rN}^{(\alpha',\sigma,\omega)}.$$

We now consider two spaces $\underline{\mathscr{E}}_{(K,k)}^{rN} = \underline{\mathscr{E}}_{(K)}^{r(n-1)} \times E_{(k)}^{rL}$, $\underline{\mathscr{E}}_{(K',k')}^{rN'} = \underline{\mathscr{E}}_{(K')}^{r(n-1)} \times E_{(k')}^{rL}$ ($N = L + n - 1$, $N' = L + n' - 1$), and the following linear mapping s from $\underline{\mathscr{E}}_{(K,k)}^{rN}$ into $\underline{\mathscr{E}}_{(K',k')}^{rN'}$:

$$(K', k') = s((K, k)) : \begin{cases} K' = \rho(K, k) \\ k' = k \end{cases} \tag{1.16}$$

where ρ is an arbitrary linear mapping from $\underline{\mathscr{E}}_{(K,k)}^{rN}$ into $\mathscr{E}_{(K')}^{r(n'-1)}$; π and π' denote the respective projections: $(K, k) \to (0, k)$ and $(K', k') \to (0, k')$.

The linear mapping[8] $K' = \rho(K, 0)$ from $\mathscr{E}_{(K)}^{r(n-1)}$ into $\mathscr{E}_{(K')}^{r(n'-1)}$ induces a well-defined linear mapping $\rho_{(m)}$ from $(\mathscr{E}_{(K)}^{r(n-1)})^{\otimes_s m}$ into $(\mathscr{E}_{(K')}^{r(n'-1)})^{\otimes_s m}$. So, to every derivative polynomial $P_m(D_K)$, represented by an element P_m in $(\mathscr{E}_{(K)}^{r(n-1)})^{\otimes_s m}$ there corresponds a derivative polynomial $P'_m(D_{K'})$, (represented by the element $P'_m = \rho_{(m)} P_m$), such that:

$$P_m(D_K)(s^*f) = s^*(P'_m(D_{K'})f) \tag{1.17}$$

Now we can prove:

Lemma 1.4. *Let* (σ, ω) *and* (σ', ω') *be two admissible couples respectively in* $\underline{\mathscr{E}}_{(K,k)}^{rN}$ *and* $\underline{\mathscr{E}}_{(K',k)}^{rN'}$ *such that:*

$$S \in \omega \textit{ implies } s(S) \in \omega' \textit{ and } \sigma' = \{S' \subset E_{(k')}^{rL} : S' = iS ; S \in \sigma\},$$

where i *denotes the isomorphism from* $\{0\} \times E_{(k)}^{(rL)}$ *onto* $\{0\} \times E_{(k')}^{(rL)}$ *defined by* $k' = k$. *Let then* $f(K', k')$ *be an admissible function in a class* $\mathscr{A}_{rN'}^{(\alpha',\sigma',\omega')}$ *then the inverse image* $(s^*f)(K, k) = f(s(K, k))$ *of* f *is an admissible function in the corresponding class* $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ *where* α *is determined by the condition:*

$$\forall S \subset \underline{\mathscr{E}}_{(K,k)}^{rN} \qquad \alpha(S) = \alpha'(s(S)).$$

Remarks.

i) One easily checks the following identity which results from the special form

8 Namely, the tangent mapping to $\rho_{|k=k_0}$ at any point (K_0, k_0)

(1.16) of the mapping s:

$$\pi' \circ s \circ \pi = \pi' \circ s = i\pi \tag{1.18}$$

ii) In the applications (see Sect. 3 and 4) it will be convenient to identify the spaces $E^{rL}_{(k)}$ and $E^{rL}_{(k')}$ (i acting as the identity operator) and to write $\sigma' = \sigma$.
With this convention, Property (1.18) yields:

$$\text{if} \quad S' = s(S) \quad \text{then} \quad \pi(S) = \pi'(S') \tag{1.19}$$

Proof of Lemma 1.4. In view of Lemma 1.1 s^*f belongs to the class $A^{(\alpha)}_{rN}$ such that: $\forall S, \alpha(S) = \alpha'(S')$. Let S be such that $S \in \omega$. Then we have in view of Definition 1*d* applied to the function $f(K', k')$ and formula (1.18)

$$\alpha(S) = \alpha'(S') = \alpha'(\pi'(S')) \text{ and}$$

$$\alpha(\pi(S)) = \alpha'(s \circ \pi(S)) = \alpha'(\pi' \circ s \circ \pi(S)) = \alpha'(\pi'(S')) \text{ i.e. } \alpha(S) = \alpha(\pi(S))$$

so that the indicatrix α satisfies the requirement (1.15) of Definition 1*d*.

Let us now consider an arbitrary derivative polynomial $P_m(D_K)$ and the corresponding polynomial $P'_m(D_{K'})$; then in view of condition ii) of Definition 1d, we can say that $P'_m(D_{K'})f \in A^{(\alpha'_m)}_{rN}$, with:

$$\alpha'_m(S') = \alpha'(S') - m \qquad S' \in \omega'$$

and

$$\alpha'_m(S') = \alpha'(S') \qquad S' \notin \omega'$$

Formula (1.17) and Lemma 1.1 then imply that $P_m(D_K)(s^*f)$ belongs to the class $A^{(\alpha_m)}_{rN}$ where

$$\alpha_m(S) = \alpha'_m(S') = \alpha'(S') - m = \alpha(S) - m, \qquad \text{if} \quad S \in \omega$$

$$\alpha_m(S) = \alpha'_m(S') \leqq \alpha'(S') = \alpha(S) \qquad \text{if} \quad S \notin \omega$$

For proving the following Lemma 5 on Taylor expansions of admissible functions, we need the two following auxiliary propositions.

Proposition 1.4. *Let f belong to a class $A^{(\alpha)}_{rL}$, and let π^*f be the inverse image of f by the projection π (i.e. $f(k)$ considered as a function of (K, k), constant in K). Then for every admissible couple (σ, ω) in $\underline{\mathscr{E}}^{rN}_{(K,k)}$, π^*f belongs to the class $\mathscr{A}^{(\hat{\alpha},\sigma,\omega)}_{rN}$ which is defined as follows:*

$$\textit{for every subspace } S \textit{ of } \underline{\mathscr{E}}^{rN}_{(K,k)}: \hat{\alpha}(S) = \alpha(\pi(S)) \tag{1.20}$$

Proof. In view of Lemma 1.1, the function π^*f belongs to the class $A^{(\hat{\alpha})}_{rN}$, with $\hat{\alpha}$ defined by (1.20). It is clear that for every admissible couple (σ, ω) $\hat{\alpha}$ satisfies the requirement (1.15) of Definition 1d. Moreover, since $P_m(D_K)(\pi^*f) = 0$ for every operator $P_m(D_K)$, the condition ii) of Definition 1d. is trivially satisfied.

Proposition 1.5. *Let $Q_m(K)$ be an arbitrary polynomial of degree m of the coordinates of K and let χ^*Q_m be the inverse image of Q_m by the projection χ (i.e. $Q_m(K)$ is considered as a function on $\underline{\mathscr{E}}^{rN}_{(K,k)}$ constant with respect to k). Then for every admissible couple (σ, ω) in $\underline{\mathscr{E}}^{rN}_{(K,k)}$, χ^*Q_m belongs to the class $\mathscr{A}^{(\alpha^{(m,\sigma)};\sigma,\omega)}_{rN}$ which is* *defined as follows:*

$$\begin{aligned} &\text{if} \quad S \in \omega : \alpha^{(m,\sigma)}(S) = \alpha^{(m,\sigma)}(\pi(S)) = m \\ &\text{if} \quad S \not\subset E_{(k)}^{rL} \text{ and } S \notin \omega : \alpha^{(m,\sigma)}(S) = m \\ &\text{if} \quad S \subset E_{(k)}^{rL} \text{ and } S \notin \sigma : \alpha^{(m,\sigma)}(S) = 0. \end{aligned} \tag{1.21}$$

Proof. Q_m belongs to the class $A_{r(n-1)}^{(m)}$, where the index m denotes the constant function: $m(S) = m$ for every $S \subset \mathscr{E}_{(K)}^{r(n-1)}$ (see Proposition 1.2).

Due to Lemma 1.1., we can say that $\chi^* Q_m$ belongs to a class $\mathscr{A}_{rN}^{(\hat{m},\tau)}$ such that:

$\hat{m}(S) = m \qquad \text{if } S \not\subset E_{(k)}^{rL}$

$\hat{m}(S) = 0 \qquad \text{if } S \subset E_{(k)}^{rL}$

Since $m \geqq 0$, it is clear that:

$\forall \tau, \forall S : \hat{m}(S) \leqq \alpha^{(m,\tau)}(S)$, so that $\chi^* Q_m \in A_{rN}^{(\alpha^{(m,\tau)})}$,

$\forall \tau, \alpha^{(m,\tau)}$ satisfies condition (1.15).

Finally every function $P_{m'}(D_K)(Q_m)$ is a polynomial of degree $m - m'$ and obviously belongs to a class $A^{(\alpha^{(\hat{m},\tau)})}$ which fulfils condition ii) of Definition 1d. We are now in a position to prove:

Lemma 1.5. *Let (σ, ω) be an admissible couple in $\underline{\mathscr{E}}_{(K,k)}^{rN}$, let $f(K,k)$ be an admissible Weinberg function in a class $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ and let $h(K,k) = (t_{(K)}^{d} f)(K,k)$.*

Then for every admissible couple $(\sigma'\omega')$ in $\underline{\mathscr{E}}_{(K,k)}^{rN}$ such that $\sigma' \supset \sigma$, there exists a class $\mathscr{A}_{rN}^{(\alpha',\sigma',\omega')}$ which contains h and which satisfies the following properties:

a) $\forall S$, *with* $\pi(S) \in \sigma$.

$\alpha'(S) = \alpha(\pi(S))$

b) $\forall S$, *with* $\pi(S) \notin \sigma$, $\pi(S) \in \sigma'$;

$\alpha'(S) = \alpha(\pi(S)) + d.$

c) $\forall S \subset E_{(k)}^{rL}$, *with* $S \notin \sigma'$:

$\alpha'(S) = \alpha(S).$

Proof. By choosing a set of coordinates $(K) = \{K_\lambda ; 1 \leqq \lambda \leqq 4(n-1)\}$ for representing K, we can write:

$$h(K,k) = \sum_{\{\nu; 0 \leqq |\nu| \leqq d\}} \frac{(K)^\nu}{\nu!} D_K^\nu f(0,k) \tag{1.22}$$

where we have used the notations: $\nu = \{\nu_\lambda ; 1 \leqq \lambda \leqq 4(n-1)\}$, $|\nu| = \sum_\lambda \nu_\lambda$, $\nu! = \Pi(\nu_\lambda !)$, $(K)^\nu = \prod_\lambda K_\lambda^{\nu_\lambda}$, and $D_K^\nu f = \left(\prod_\lambda D_{K_\lambda}^{\nu_\lambda}\right) f.$

In view of Proposition (1.3) a), we are led to show that each product $(K)^\nu \cdot D_K^\nu f(0,k)$ occurring at the r.h.s. of (1.22) belongs to the class $\mathscr{A}_{rN}^{(\alpha',\sigma',\omega')}$ for every admissible couple (σ', ω') in $\underline{\mathscr{E}}_{(K,k)}^{rN}$ with $\sigma' \supset \sigma$,

By applying Proposition 1.4. to the function $D_K^\nu f(0\, k)$ which belongs to the class $A_{rL}^{(\alpha_{|\nu|})}$ (see Definition 1d. ii) such that:

$$\alpha_{|\nu|}(S) = \alpha(S) - |\nu| \quad \text{if } S \in \sigma,$$
$$\alpha_{|\nu|}(S) = \alpha(S) \quad \text{if } S \notin \sigma,$$

we conclude that for every admissible couple (σ', ω') in $\mathscr{E}_{(K,k)}^{rN}$, $(\sigma' \supset \sigma)$, the function $\pi^*(D_K^\nu f_{|K=0})(K, k) = D_K^\nu f(0, k)$ belongs to the class $\mathscr{A}_{rN}^{(\hat{\alpha}_\nu, \sigma', \omega')}$, such that:

$$\hat{\alpha}_\nu(S) = \alpha(\pi(S)) - |\nu| \quad \text{if } \pi(S) \in \sigma,$$
$$\hat{\alpha}_\nu(S) = \alpha(\pi(S)) \quad \text{if } \pi(S) \notin \sigma \tag{1.23}$$

By applying Proposition 1.5 to the function $(K)^\nu$, we now see that for every admissible couple (σ', ω'), $(K)^\nu$ belongs to the class $\mathscr{A}_{rN}^{(\alpha(|\nu|,\sigma'), \sigma', \omega')}$, defined by formulae similar to (1.21) (with $m \to |\nu|$).

Then, in view of Proposition 1.3.b), we conclude that the product $(K)^\nu D_K^\nu f(0, k)$ is an admissible Weinberg function in the class $\mathscr{A}_{rN}^{(\hat{\alpha}_\nu + \alpha(|\nu|,\sigma'), \sigma', \omega')}$.

Formulae (1.21) and (1.23) allow us to compute the function $\alpha^\nu = \hat{\alpha}_\nu + \alpha^{(|\nu|, \sigma')}$:

a) if $\pi(S) \in \sigma$:

$$\alpha^\nu(S) = (\alpha(\pi(S)) - |\nu|) + |\nu| = \alpha(\pi(S)) \tag{1.24}$$

b) if $\pi(S) \notin \sigma$, $\pi(S) \in \sigma'$:

$$\alpha^\nu(S) = \alpha(\pi(S)) + |\nu| \leqq \alpha(\pi(S)) + d \tag{1.25}$$

c) if $S \subset E_{(k)}^{rL}$, with $S \notin \sigma'$:

$$\alpha^\nu(S) = \alpha(S). \tag{1.26}$$

Let us call $\alpha' = \sup_{\{\nu; 0 \leqq |\nu| \leqq d\}} \alpha^\nu$; then in view of Proposition 1.3.d), we have:

$\forall \nu$ with $0 \leqq |\nu| \leqq d : (K)^\nu D_K^\nu f(0, k) \in \mathscr{A}_{rN}^{(\alpha', \sigma', \omega')}$.

Moreover, formulae (1.24), (1.25), (1.26) show that α' satisfies the required conditions a), b), c), of the lemma. q.e.d.

Lemma 1.6. *Let $f(K, k)$ be an admissible Weinberg function in a class $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$ and let $g(K, k)$ be the Taylor rest of order d off : $g = (1 - t_{(K)}^d) f$. Then for every admissible couple (σ', ω') in $\mathscr{E}_{(K,k)}^{rN}$ with $\sigma' \subset \sigma$, $\omega' \subset \omega$ there exists a class $\mathscr{A}_{rN}^{(\alpha',\sigma',\omega')}$ which contains g and which satisfies the following properties:*

a) $\forall S \in \omega'$;

$$\alpha'(S) = \alpha'(\pi(S)) = \alpha(S)$$

b) $\forall S \subset E_{(k)}^{rL}$ *with* $S \notin \sigma'$ *and* $S \in \sigma$:

$$\alpha'(S) = \alpha(S) - d - 1$$

c) $\forall S \subset E_{(k)}^{rL}$ *with* $S \notin \sigma$:

$$\alpha'(S) = \alpha(S).$$

In the proof of this statement, it will clearly appear that σ' can be chosen to be the empty set ϕ. With this special choice, Properties b) and c) can be equivalently formulated as follows:

Lemma 1.6.′ *If f belongs to $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$, then $g = (1 - t_{(K)}^{d}) f$ and all the derivatives of g with respect to the variables (K) belong to a certain class $A_{rN}^{(\alpha\phi)}$ such that:*

a) $\forall S \subset E_{(k)}^{rL}$ *with* $S \in \sigma$:

$\alpha_{\phi}(S) = \alpha(S) - d - 1$

b) $\forall S \subset E_{(k)}^{rL}$ *with* $S \notin \sigma$:

$\alpha_{\phi}(S) = \alpha(S)$

Proof of Lemma 1.6. It is a straightforward application of Lemma B.2. Let us consider a fixed admissible couple (σ', ω') with $\sigma' \subset \sigma$, $\omega' \subset \omega$, and an arbitrary derivative $D_K^{\nu} g$ of g, with order $|\nu|$; our argument will include the case of g itself.

$D_K^{\nu} g$ belongs to the Weinberg class $A_{rN}^{(\underline{\alpha}_{|\nu|})}$, with $\underline{\alpha}_{|\nu|}$ defined by formulae (B.44)... (B.50).

We now notice that the following inequalities hold for every value of $|\nu|$:

a) $\forall S \in \omega'$, we have (due to (B.44), (B.45)):

$$\left.\begin{array}{ll} \underline{\alpha}_{|\nu|}(S) = \alpha(S) - |\nu| & \text{if } S \not\subset E_{(k)}^{rL} \\ \underline{\alpha}_{|\nu|}(S) \leqq \alpha(S) - |\nu| & \text{if } S \subset E_{(k)}^{rL} \end{array}\right\} \tag{1.27}$$

b) $\forall S \notin \omega'$ with $S \in \omega$, we have (due to (B.44), (B.45)):

$$\begin{array}{ll} \underline{\alpha}_{|\nu|}(S) = \alpha(S) - |\nu| \leqq \alpha(S), & \text{if } S \not\subset E_{(k)}^{rL} \\ \underline{\alpha}_{|\nu|}(S) \leqq \alpha(S) - d - 1, & \text{if } S \subset E_{(k)}^{rL} \end{array} \tag{1.28}$$

c) $\forall S \notin \omega$, we have (due to (B.46), (B.50)):

$$\begin{array}{ll} \underline{\alpha}_{|\nu|}(S) \leqq \sup[\alpha(S), \alpha(\pi(S)) + d - |\nu|], & \text{if } S \not\subset E_{(k)}^{rL} \\ \underline{\alpha}_{|\nu|}(S) = \alpha(S), & \text{if } S \subset E_{(k)}^{rL} \end{array}$$

Now let us define the index function α' as follows:

$$\alpha'(S) = \alpha'(\pi(S)) = \alpha(S) \text{ for } S \in \omega' \tag{1.29}$$

$$\begin{array}{ll} \alpha'(S) = \alpha(S) & \text{for } S \not\subset E_{(k)}^{rL} \text{ with } S \notin \omega', S \in \omega \\ \alpha'(S) = \alpha(S) - d - 1 & \text{for } S \subset E_{(k)}^{rL} \text{ and } S \in \sigma \backslash \sigma' \end{array} \tag{1.30}$$

$$\begin{array}{ll} \alpha'(S) = \sup[\alpha(S), \alpha(\pi(S)) + d], & \text{for } S \not\subset E_{(k)}^{rL} S \notin \omega \\ \alpha'(S) = \alpha(S) & \text{for } S \subset E_{(k)}^{rL} S \notin \sigma. \end{array} \tag{1.31}$$

The above analysis clearly shows that (in view of Proposition 1.1.c. and Definition 1d.), g belongs to the class $\mathscr{A}_{rN}^{(\alpha',\sigma',\omega')}$. In particular, formulae (1.29), (1.30) and (1.31) exhibit respectively the Properties a), b) and c) of α' which were stated in Lemma 1.6.

q.e.d.

We shall end this section by proving the following lemma which relates the classes $\sum_n^{\mu}$ of subsection 1.2 with the classes of admissible Weinberg functions $\mathscr{A}_{rN}^{(\alpha,\sigma,\omega)}$

Lemma 1.7. *Let λ be a linear mapping from the vector space $\underline{\mathscr{E}}^{rN}_{(K,k)}$ into an n-dimensional euclidean vector space $\mathscr{E}^{\tilde{n}}_{(\tilde{K})}(\tilde{K} = \lambda(K, k))$.*

Then for every function f on $\mathscr{E}^{\tilde{n}}_{(\tilde{K})}$ which belongs to the class $\sum^{\mu}_{\tilde{n}}$, the corresponding inverse image $\lambda^* f$ of f is an admissible Weinberg function on $\underline{\mathscr{E}}^{rN}_{(K,k)}$ which belongs to the following class $\mathscr{A}^{(\alpha,\sigma,\omega)}_{rN}$

a) *σ is the set of all subspaces S of $E^{rL}_{(k)}$ such that $S \not\subset$ Ker λ.*

b) *ω is the set:*

$$\omega = \{S \subset \underline{\mathscr{E}}^{rN}_{(K,k)} : S \not\subset \operatorname{Ker}\lambda\,;\ \pi(S) \in \sigma\} \tag{1.32}$$

c) *the indicatrix α is such that:*

$$-\ \forall S \in \omega : \alpha(S) = \alpha(\pi(S)) = \mu \tag{1.33}$$

$$-\ \forall S \notin \omega : \alpha(S) = 0 \qquad \text{if } S \subset \operatorname{Ker}\lambda \tag{1.34.a}$$

$$\alpha(S) = \mu \qquad \text{if } S \not\subset \operatorname{Ker}\lambda \tag{1.34.b}$$

Proof. We first verify that (ω, σ) is an admissible couple in $\underline{\mathscr{E}}^{rN}_{(K,k)}$ because properties a) b) c) d) of Definition 1c are satisfied.

Let us apply now Lemma 1.3 (with $\underline{\mathscr{E}}^{rN}_{(K,k)}$ playing the role of E). We obtain that $\lambda^* f$ belongs to the class $A^{(\alpha)}_{rN}$ with α defined by (1.34) (1.33). In fact, when $S \in \omega$ i.e. $S \not\subset \operatorname{Ker}\lambda$ and $\pi(S) \not\subset \operatorname{Ker}\lambda$ then 1.7.b implies that $\alpha(S) = \alpha(\pi(S)) = \mu$. Similarly when $S \notin \omega$ then there are two possibilities: either $S \subset \operatorname{Ker}\lambda$ which from 1.7.a implies $\alpha(S) = 0$; or $S \not\subset \operatorname{Ker}\lambda$, $\pi(S) \notin \sigma$ which implies $\alpha(S) = \mu$.

It remains to check that $\lambda^* f$ satisfies condition ii) of Definition 1d: but this is entailed by the second part of Lemma 1.3.

Appendix A

In the Definition 1b of functions belonging to a class $A^{(\alpha)}_N$, it is assumed (as in [8]) that inequalities of the type (1.1) are satisfied for all sets of *linearly independent* vectors $\{L_1, \ldots, L_m\}$. As a matter of fact, we need to use such inequalities even in cases when $L_1, \ldots, L_m$ are *not* independent. We shall then show that this property can be derived from Definition 1b.

Lemma A. *Let f be a function in $\mathbb{R}^N$, which belongs to a definite class $A^{(\alpha)}_N$. Then if $\{L_1, \ldots, L_m\}$ is an arbitrary set of vectors (m being an arbitrary integer), and if W is an arbitrary bounded region in $\mathbb{R}^N$, there exist positive numbers M and $b_j (1 \leqq j \leqq m)$, such that:*

$$\forall C \in W,\ \forall \eta_j \geqq b_j,\ 1 \leqq j \leqq m,$$

one has:

$$\left| f\left(\sum_{j=1}^{m} L_j \eta_j \ldots \eta_m + C \right) \right| \leqq M \prod_{j=1}^{m} \eta_j^{\alpha(\overline{\{L_1 \ldots L_j\}})} \tag{A.1}$$

Remarks

i) As in formula (1.1), $\alpha(\overline{\{L_1 \ldots L_j\}})$ denotes the asymptotic Weinberg coefficient

of the subspace S which is generated by the set $\{L_1, \dots, L_j\}$; but now the dimension of S can be smaller than j.
ii) It is useful to note that formula (A.1) still holds true when some of the vectors $\{L_j\}$ are chosen to be equal to $\{0\}$, provided that one then adds the condition: $\alpha(\{0\}) = 0$.

Proof. From the set $\{L_1, L_2, \dots, L_m\}$, we extract a well-defined subset of n independent vectors $\{L_{i_1}, L_{i_2}, \dots, L_{i_n}\}$ with $i_1 < i_2 < \dots i_n \leqq m$, through the following recursive construction:

a) i_1 is the smallest integer such that $L_{i_1} \neq \{0\}$.

b) Suppose that $i_1, \dots, i_{k-1}$ have been determined. Then i_k is the smallest of the set of integers $j > i_{k-1}$ such that $\{L_{i_1}, L_{i_2}, \dots, L_{i_{k-1}}, L_j\}$ be a set of independent vectors.

c) n is the largest value of the index k such that b) be applicable, i.e. n is the dimension of the subspace generated by $\{L_1, \dots, L_m\}$.

For any integer $j \leqq m$, there exists a unique set of real numbers a_{jk} such that:

$$L_j = \sum_k a_{jk} L_{i_k}$$

From the above construction, we even deduce that:

$$a_{jk} = 1 \quad \text{if } j = i_k \quad (1 \leqq k \leqq n)$$

and $\quad a_{jk} = 0 \quad$ if $j < i_k$,

so that we can write:

$$\sum_{j=1}^{m} L_j \eta_j \dots \eta_m = \sum_{k=1}^{n} \left(1 + \sum_{j > i_k} \frac{a_{jk}}{\eta_{i_k} \dots \eta_{j-1}} \right) \eta_{i_k} \dots \eta_m L_{i_k}. \tag{A.2}$$

We now wish to determine numbers $\eta'_1 \dots \eta'_n$ as functions of $\eta \equiv (\eta_1, \dots, \eta_m)$, in such a way that:

$$\sum_{j=1}^{m} L_j \eta_j \dots \eta_m + C = \sum_{k=1}^{n} L_{i_k} \eta'_k \dots \eta'_n + C.$$

By taking (A.2) into account, we are led to a unique solution which is defined as follows:

$$\eta'_k(\eta) = \frac{\left(1 + \sum\limits_{j > i_k} \dfrac{a_{jk}}{\eta_{i_k} \dots \eta_{j-1}} \right)}{\left(1 + \sum\limits_{j > i_{k+1}} \dfrac{a_{jk}}{\eta_{i_{k+1}} \dots \eta_{j-1}} \right)} \times \prod_{j = i_k}^{i_{k+1} - 1} \eta_j \tag{A.3}$$

Let us now apply definition 1b to the set of independent vectors $\{L_{i_k};$ $1 \leqq k \leqq n\}$. There exist constants M' and b'_k, for $1 \leqq k \leqq n$, such that:

$\forall C \in W$, $\forall \eta = (\eta_1, \dots, \eta_m)$ such that

$\eta'_k(\eta) \geqq b'_k$ for $1 \leqq k \leqq n$, one has:

$$\left| f\left(\sum_{j=1}^{m} L_j\eta_j \ldots \eta_m + C \right) \right| \leqq M' \prod_{k=1}^{n} \eta_k'^{\,\alpha(\overline{\{L_{i_1} \ldots L_{i_k}\}})} \tag{A.4}$$

Now, if we put $a = \sum_{(j,k)} |a_{jk}|$ (note that $a > 1$) and if we impose the conditions:

$$\left.\begin{array}{ll} \eta_{i_k} \geqq 2a & \text{for } 1 \leqq k \leqq n \\ \eta_j \geqq 1 & \text{for every } j (1 \leqq j \leqq m) \end{array}\right\} \tag{A.5}$$

we easily deduce from (A.3) the following inequalities:

$$\frac{1}{3}\eta_{i_k} \leqq \frac{1}{3} \prod_{j=i_k}^{i_{k+1}-1} \eta_j \leqq \eta_k'(\eta) \leqq 3 \prod_{j=i_k}^{i_{k+1}-1} \eta_j. \tag{A.6}$$

From the left inequalities (A.6), we infer that the conditions $\eta_k'(\eta) \geqq b_k'$ are fulfilled as long as (A.5) holds together with the additional conditions:

$$\eta_{i_k} \geqq 3b_k' \quad \text{for } 1 \leqq k \leqq n.$$

From the right inequality (A.6), we conclude that the right-hand side of (A.4) is majorized by:

$$3^n M' \prod_{j=1}^{m} \eta_j^{\alpha(\overline{\{L_1 \ldots L_j\}})}.$$

To obtain this, we have also noticed that for $i_k \leqq j < i_{k+1}$ one has: $\alpha(\overline{\{L_1 \ldots L_j\}}) = \alpha(\overline{\{L_{i_1} \ldots L_{i_k}\}})$, and that for $j < i_1$, one has: $\alpha(\overline{\{L_1, \ldots, L_j\}}) = \alpha(\{0\}) = 0$ (see remark ii) above).

To summarize, we have proved the following result:

$$\text{if we put: } \left| \begin{array}{ll} b_j = 1 & \text{for } j \neq i_k \\ b_{i_k} = \sup(2a, 3b_k') & \text{for } 1 \leqq k \leqq n \end{array} \right.$$

and $\quad M = 3^n M'$,

then the conditions: $C \in W$, $\eta_j \geqq b_j$, $1 \leqq j \leqq m$ imply the inequality (A.1.) q.e.d.

Appendix B

Taylor Rests of Graded Weinberg Functions

In this appendix, we consider Weinberg functions in a definite class $A_N^{(\alpha_0)}$ on a space $\mathbb{R}^N_{(x,y)} = \mathbb{R}^p_{(x)} \times \mathbb{R}^q_{(y)} (N = p + q)$, which are infinitely differentiable and satisfy the following additional conditions.

Let f be such a function; we assume that each partial derivative $D_x^\nu f$ of f with respect to the variables x, with total order $n = |\nu|$ belongs to a definite class $A_N^{(\alpha_n)}$. So, each asymptotic indicatrix $\alpha_n (n = 0, 1, 2 \ldots)$ governs, in the sense of formula (1.1), the behaviour at infinity on every subspace S of $\mathbb{R}^N_{(x,y)}$ of *all* the derivatives $D_x^\nu f$ of f with total order n. We shall also say that f is *graded with respect to* x and that it is *asymptotically governed by the sequence* $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots,\}$.

Of course this sequence is by no means unique, since in particular f is governed by every sequence $\{\alpha'_n\}$ such that: $\forall n \geqq 0, \alpha'_n > \alpha_n$ (see Proposition 1.1.c).

We shall prove a general property of the Taylor rests with respect to the variables x of these graded functions (Lemma B.1). This result will then be applied to the case of *admissible* Weinberg functions (see Definition 1d) which are graded functions of a special type (Lemma B.2).

Lemma B.1. *Let f be a Weinberg function in $\mathbb{R}^N_{(x,y)}$ which is graded with respect to x, and asymptotically governed by the sequence $\{\alpha_n\}$.*

Then the Taylor rest $g = (1 - t^d_{(x)})f$ of order d of f is also a Weinberg function which is graded with respect to x; it is asymptotically governed by a sequence $\{\underline{\alpha}_n\}$ which is determined as follows:

a) *For $n > d$, one has:*

$$\forall S \subset \mathbb{R}^N_{(x,y)}, \ \underline{\alpha}_n(S) = \alpha_n(S) \tag{B.1}$$

b) *For $n \leqq d$, two cases occur:*

$-$ *if* $S \subset \mathbb{R}^q_{(y)} : \underline{\alpha}_n(S) = \alpha_{d+1}(S)$

$$- \textit{if } S \not\subset \mathbb{R}^q_{(y)}: \tag{B.2}$$

$$\underline{\alpha}_n(S) = \sup_{\{S'; S' \succ_y S\}} [\alpha_{d+1}(S')] + d - n + 1 \tag{B.3}$$

Here we have used the notation $S' \succ_y S$ which is defined as follows: we say that two subspaces S, S' of $\mathbb{R}^N_{(x,y)}$ satisfy the relation $S' \succ_y S$ if the following conditions hold:
i) $S \cap \mathbb{R}^q_{(y)} \subset S' \cap \mathbb{R}^q_{(y)}$ ($\mathbb{R}^q_{(y)}$ being always identified with the subspace $\{0\} \times \mathbb{R}^q_{(y)}$ of $\mathbb{R}^N_{(x,y)}$).
ii) $\pi(S) = \pi(S')$, π denoting the projection $(x, y) \to (0, y)$ of $\mathbb{R}^N_{(x,y)}$ onto $\mathbb{R}^q_{(y)}$. In (B.3), the "sup" runs over all subspaces S' of $\mathbb{R}^N_{(x,y)}$ which are such that $S' \succ_y S$.

Remark. Since each indicatrix α_n is a bounded function (see Sect. 1.1) taking this "sup" always yields a finite number.

Proof. There will be two parts in this proof. The longer part (A) will be concerned with proving that for every function $f(x, y)$ which is graded with respect to x and governed by a sequence $\{\alpha_n\}$, the corresponding function $g = (1 - t^d_{(x)})f$ belongs to a class $A^{\underline{\alpha}}_N$, with the indicatrix $\underline{\alpha} = \underline{\alpha}_0$, given by formulae (B.2) and (B.3), namely:

$$\underline{\alpha}(S) = \alpha_{d+1}(S) \qquad \text{if } S \subset \mathbb{R}^q_{(y)} \tag{B.2'}$$

$$\alpha(S) = \sup_{\{S'; S' \succ_y S\}} [\alpha_{d+1}(S')] + d + 1, \ \text{if } S \not\subset R^q_{(y)} \tag{B.3'}$$

In the shorter part (B), this first result will be used to show that each derivative $D^\nu_x g$ of g, with order $|\nu| = n$ belongs to the corresponding class $A^{(\underline{\alpha}_n)}$, with $\underline{\alpha}_n$ given by formulae (B.1), (if $n > d$) or (B.2), (B.3) (if $n \leqq d$).

A) To prove that g belongs to $A^{(\underline{\alpha})}_N$, we shall first treat the case when x is a *single* variable, namely: $p = 1$, q arbitrary. Then we shall treat the general case through a recursive argument over p.

1. *The Case* $p=1$

To prove that $g(x,y)$ is a Weinberg function, we have to consider an arbitrary set of m independent vectors $\{L_1,\dots,L_m\}$ in $\mathbb{R}_{(x)}\times\mathbb{R}^q_{(y)}$ and an arbitrary bounded region W in this space. Let $\mathscr{R}(L_1,L_2,\dots,L_m;W)$ be the region of $\mathbb{R}_{(x)}\times\mathbb{R}^q_{(y)}$ which is parametrized as follows:

$$(x,y)=L_1\eta_1\dots\eta_m+\dots+L_j\eta_j\dots\eta_m+\dots+L_m\eta_m+C$$
$$\text{with } C\in W \qquad \text{and } \forall j\leqq m:\eta_j\geqq 0 \tag{B.4}$$

We have to prove that in $\mathscr{R}(L_1,\dots,L_m;W)$ the function g satisfies a majorization of the following form:

$$\left|g\left(\sum_{j=1}^{m}L_j\eta_j\dots\eta_m+C\right)\right|\leqq M\eta_1^{\alpha(\{\overline{L_1}\})}\dots\eta_j^{\alpha(\{\overline{L_1\dots L_j}\})}\dots\eta_m^{\alpha(\{\overline{L_1\dots L_m}\})} \tag{B.5}$$

provided that $\eta_1>b_1\dots\eta_m>b_m$; where $b_1,\dots,b_m$ are suitably chosen positive numbers.

We shall use the following basic expression of the Taylor rest of f:

$$g(x,y)=\frac{x^{d+1}}{d!}\int_0^1 D_x^{d+1}f((1-t)x,y)t^d\,dt \tag{B.6}$$

We thus need to extend the parametrization (B.4) to the region

$$\{((1-t)x,y),\quad \text{with } 0\leqq t\leqq 1,\quad (x,y)\in\mathscr{R}(L_1,\dots,L_m;W)\}$$

Let L be the unit vector of $\mathbb{R}_{(x)}$, identified with $(x=1,y=0)$; for every $j\leqq m$, we can write in a unique way:

$$L_j=\mu_jL+L'_j \text{ with } L'_j\in\mathbb{R}^q_{(y)}$$

and $C=\mu L+C'$ with $C'\in\mathbb{R}^q_{(y)}$.

For (x,y) in $\mathscr{R}(L_1,\dots,L_m;W)$, we then have:

$$((1-t)x,y)=\sum_{j=1}^{m}L_j\eta_j\dots\eta_m-Ltx+C \tag{B.7}$$

with

$$x=\sum_{j=1}^{m}\mu_j\eta_j\dots\eta_m+\mu \tag{B.8}$$

and if we choose $v=tx$ as a new integration variable in (B.6), the latter can be rewritten:

$$g\left(\sum_{j=1}^{m}L_j\eta_j\dots\eta_m+C\right)=\frac{1}{d!}\int_0^x v^d\,dv\,D_x^{d+1}f\left(\sum_{j=1}^{m}L_j\eta_j\dots\eta_m-Lv+C\right) \tag{B.9}$$

Following closely the method used by Weinberg in the proof of his general "power counting" theorem (see [8]), we shall cover the integration interval $[0,x]$ of (B.9) by a finite number of sets, in each of which a particular majorization of the Weinberg function $|D_x^{d+1}f|$ holds. These sets can be clustered into families according to the following recursive scheme.

One first covers $[0,x]$ by an open set $\mathcal{T}_h = \bigcup_{i \in \sigma_h} I_i : h$ is a fixed integer $\leqq m$ which will be determined below; each I_i is an open interval whose label varies in a finite family σ_h. One then successively defines sets $\mathcal{T}_r, \mathcal{J}_r, \mathcal{H}_r$, of $[0,x]$ for $h \leqq r \leqq m$, in such a way that:

$$\mathcal{T}_r = \mathcal{J}_r \cup \mathcal{H}_r, \tag{B.10}$$

$$\mathcal{H}_r \subset \mathcal{T}_{r+1} \qquad (\text{for } r \leqq m-1) \tag{B.11}$$

with

$$\left.\begin{aligned} \mathcal{T}_r &= \bigcup_{(i_h \ldots i_r) \in \sigma_r} I_{i_h \ldots i_r} \\ \mathcal{J}_r &= \bigcup_{(i_h \ldots i_r) \in \sigma_r} J_{i_h \ldots i_r} \\ \mathcal{H}_r &= \bigcup_{(i_h \ldots i_r) \in \sigma_r} H_{i_h \ldots i_r} \end{aligned}\right\} \tag{B.12}$$

Each set $I_{i_h \ldots i_r}$ is an open interval, from which a closed interval or "hole" $H_{i_h \ldots i_r}$ is taken away; the corresponding hollowed interval is:

$$J_{i_h \ldots i_r} = I_{i_h \ldots i_r} \backslash H_{i_h \ldots i_r}.$$

Each finite family σ_r of labels $(i_h \ldots i_r)$ is determined recursively by conditions of the following form:

$$\forall i_h \ldots i_{r-1} \quad H_{i_h \ldots i_{r-1}} \subset \bigcup_{i_r \in \sigma(i_h \ldots i_{r-1})} I_{i_h \ldots i_{r-1} i_r}.$$

(the latter obviously imply (B.11), if (B.12) is taken into account).

Let $F(v)$ be the integrand in the right-hand side of (B.9). From (B.10), (B.11), it follows that:

$$[0,x] \subset \mathcal{T}_h \subset \left(\bigcup_{h \leqq r \leqq m} \mathcal{J}_r \right) \cup \mathcal{H}_m;$$

and correspondingly:

$$\left| \int_{[0,x]} F(v) dv \right| \leqq \sum_{h \leqq r \leqq m} \int_{\mathcal{J}_r} |F(v)| dv + \int_{\mathcal{H}_m} |F(v)| dv. \tag{B.13}$$

In each set $J_{i_h \ldots i_r}$, (for $r \leqq m$), and $H_{i_h \ldots i_m}$, a definite Weinberg majorization of $|F(v)|$ will be produced; then by taking (B.12) into account, one will obtain bounds on all the terms of the r.h.s. of (B.13): all these bounds will turn out to have the required form (i.e. that of the r.h.s. of (B.5)).

It is only for $r = h$, that $\mathcal{J}_r$ needs a definition which is slightly different from that of [8]. In a) we shall define $\mathcal{J}_h$ and derive the corresponding majorization of $\int_{\mathcal{J}_h} |F(v)| dv$. In b), we shall rest upon [8] for the detailed definition of all the other sets $\mathcal{J}_r, \mathcal{H}_r, \mathcal{T}_r$ and indicate briefly the majorizations of the corresponding terms in (B.13).

a) Contribution of the set $\mathscr{I}_h$

Let h be the integer ($h \leqq m$) for which:

$$\mu_h \neq 0, \quad \text{and} \quad \mu_j = 0, \quad \text{for every } j \leqq h-1;$$

h is also the integer for which each subspace $S_j = \overline{\{L_1, \dots, L_j\}}$, with $j \leqq h-1$, is contained in $\mathbb{R}^q_{(y)}$, while each subspace S_j, with $j \geqq h$, is *not* contained in $\mathbb{R}^q_{(y)}$.

In the argument of $D_x^{d+1} f$ in formula (A.9), we replace v by a new variable z which is defined by:

$$v = \eta_h \dots \eta_m u + \eta_{h+1} \dots \eta_m z, \tag{B.14}$$

u being a fixed number.

This allows us to write:

$$\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C = \sum_{j=1}^{h-1} L_j \eta_j \dots \eta_{h-1} \left(\frac{\eta_h}{|z|} \right) |z| \eta_{h+1} \dots \eta_m$$
$$+ (L_h - Lu) \left(\frac{\eta_h}{|z|} \right) |z| \eta_{h+1} \dots \eta_m - \varepsilon(z) L |z| \eta_{h+1} \dots \eta_m + \sum_{j=h+1}^{m} L_j \eta_j \dots \eta_m + C$$

where $\varepsilon(z)$ denotes the sign of z.

This expression is convenient for taking into account the assumption that $D_x^{d+1} f$ belongs to the Weinberg class $A_N^{\alpha_{d+1}}$ and therefore satisfies the following majorization (note that here it may be necessary to use the result of the auxiliary lemma proved in Appendix A, if sets of non independent vectors $\{L_1, \dots, L_j, L\}$ are produced):

$$\left| D_x^{d+1} f \left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C \right) \right| \leqq M_h(u) \prod_{j=1}^{h-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1, \dots, L_j\}})}$$
$$\left(\frac{\eta_h}{|z|} \right)^{\alpha_{d+1}(\overline{\{L_1 \dots L_{h-1}, L_h - Lu\}})} |z|^{\alpha_{d+1}(\overline{\{L_1 \dots L_h, L\}})} \prod_{j=h+1}^{m} \eta_j^{\alpha_{d+1}(\overline{\{L_1, \dots, L_j, L\}})} \tag{B.15}$$

provided that:

$$\eta_j \geqq b_j(u) \quad \text{for } j = 1 \dots m, \ j \neq h \tag{B.16}$$

$$\frac{\eta_h}{|z|} \geqq b_h(u) \tag{B.17}$$

$$\text{and } |z| \geqq b_0(u); \tag{B.18}$$

here $M_h(u)$, and the $b_j(u)$ ($0 \leqq j \leqq m$) denote appropriate positive numbers which—a priori—depend on u; the $b_j(u)$ are always supposed to be larger than 1.

The set of points v for which the majorization (B.15) is satisfied is given by plugging conditions (B.17), (B.18) into (B.14): condition (B.17) yields the interval:

$$I(u) = \{v; \eta_h \dots \eta_m (u - b_h^{-1}(u)) < v < \eta_h \dots \eta_m (u + b_h^{-1}(u)\} \tag{B.19}$$

adding condition (B.18) yields the "hollowed interval"

$$J(u) = I(u) \cap \{v; |v - \eta_h \dots \eta_m u| > \eta_{h+1} \dots \eta_m b_0(u) \tag{B.20}$$

Let us determine a positive number u_0 such that the integration interval $[0, x]$ in (B.9) be covered by $\bigcup_{0 \leqq u \leqq u_0} I(u)$. This is readily obtained by choosing $u_0 = \sum_{j=h}^{m} \mu_j + \mu$; in fact, in view of (B.8) we have:

$$x = \eta_h \dots \eta_m \left(\mu_h + \sum_{j=h+1}^{m} \frac{\mu_j}{\eta_h \dots \eta_{j-1}} + \frac{\mu}{\eta_h \dots \eta_m} \right),$$

and if we impose the conditions:

$$\eta_h \geqq 1, \dots \eta_m \geqq 1 \tag{B.21}$$

we obtain the inequality:

$$x \leqq \eta_h \dots \eta_m u_0$$

which yields (in view of (B.15)):

$$[0, x] \subset [0, \eta_h \dots \eta_m u_0] \subset \bigcup_{0 \leqq u \leqq u_0} I(u)$$

By applying the Heine–Borel lemma, we can extract from $\{I(u); 0 \leqq u \leqq u_0\}$ a finite covering by intervals $I_i = I(u_i)$ (i belonging to a finite set σ_h); we also introduce the corresponding family of "hollowed" intervals $J_i = J(u_i)$ through formula (B.20).

We then conclude that when v is restricted to vary in the set $\mathscr{J}_h = \bigcup_{i \in \sigma_h} J_i$, the following majorization can be deduced from (B.15) (the latter being applied to each case: $u = u_i$):

$$\int_{\mathscr{J}_h} |F(v)| dv \leqq \prod_{j=1}^{h-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j\}})} \times \prod_{j=h+1}^{m} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j, L\}}) + d + 1} \dots\dots$$

$$\dots \times \sum_{i \in \sigma_h} \frac{M_h(u_i)}{d!} \eta_h^{\alpha_{d+1}(\overline{\{L_1 \dots L_{h-1}, L_h - Lu_i\}})} \times \dots$$

$$\dots \int_{b_0(u_i) \leqq |z| \leqq \eta_h b_h^{-1}(u_i)} |z|^{\alpha_{d+1}(\overline{\{L_1 \dots L_h, L\}}) - \alpha_{d+1}(\overline{\{L_1, \dots L_{h-1}, L_h - Lu_i\}})} (\eta_h u_i + z)^d dz \tag{B.22}$$

In view of (B.16) and (B.21), this majorization holds provided that one has:

$$\eta_h \geqq 1 \quad \text{and} \quad \forall j \neq h, 1 \leqq j \leqq m : \eta_j \geqq \sup_{i \in \sigma_h} (b_j(u_i)) \tag{B.23}$$

It remains to estimate the dependence of the right-hand side of (B.22) with respect to η_h, by performing the integration over z. The dominant contribution of the term labelled by "i" in the summation $\sum_{i \in \sigma_h}$ can have the alternative forms: $\eta_h^{\hat{\alpha}_i}$ or $\eta_h^{\hat{\alpha}_i - 1} \ln \eta_h$; we take this as a definition of $\hat{\alpha}_i$, the occurrence of either form being linked with the sign of the difference of the exponents:

$$\left.\begin{aligned} \alpha' &= \alpha_{d+1}(\overline{\{L_1 \dots, L_h, L\}}) \\ \alpha'_i &= \alpha_{d+1}(\overline{\{L_1 \dots, L_{h-1}, L_h - Lu_i\}}) \end{aligned}\right\} . \tag{B.24}$$

It is easy to check that:

i) if $\alpha' \geqq \alpha'_i : \hat{\alpha}_i = \alpha' + d + 1$

ii) if $\alpha' < \alpha'_i : \hat{\alpha}_i = \sup_{0 \leqq r \leqq d} [(\alpha'_i + r) + \sup(1, \alpha' - \alpha'_i + d - r + 1)]$

which yields $\hat{\alpha}_i \leqq \sup(\alpha', \alpha'_i) + d + 1$

From this it follows (in view of (B.24) that in all cases:

$$\sup_{i \in \sigma_h} \hat{\alpha}_i \leqq \sup[\alpha_{d+1}(\overline{\{L_1, \dots L_h, L\}}), \sup_{i \in \sigma_h} \alpha_{d+1}(\overline{\{L_1 \dots L_{h-1}, L_h - Lu_i\}})] + d + 1$$
$$\leqq \alpha(\overline{\{L_1, \dots L_h\}}) \tag{B.25}$$

For deriving the latter inequality, we have made use of (B.3) and of the fact that for every number u, one has:

$$\overline{\{L_1, \dots L_h, L\}} \succ_y \overline{\{L_1 \dots L_h\}} \quad \text{and} \quad \overline{\{L_1, \dots L_{h-1}, L_h - Lu\}} \succ_y \overline{\{L_1 \dots L_h\}}$$

Similarly, we notice (by referring to (B.2′), (B.3′)) that:

$$\text{—for } j \leqq h-1, \alpha_{d+1}(\overline{\{L_1, \dots, L_j\}}) = \underline{\alpha}(\overline{\{L_1, \dots, L_j\}}), \tag{B.26}$$

since then $\overline{\{L_1, \dots, L_j\}} \subset \mathbb{R}^q_{(y)}$;

$$\text{—for } j \geqq h+1, \alpha_{d+1}(\overline{\{L_1, \dots, L_j, L\}}) + d + 1 \leqq \underline{\alpha}(\overline{\{L_1, \dots, L_j\}}), \tag{B.27}$$

since then $\overline{\{L_1, \dots, L_j, L\}} \succ_y \overline{\{L_1, \dots, L_j\}}$.

From (B.25, B.26, B.27) we conclude that the majorization (B.22) entails the following one:

$$\int_{\mathscr{I}_h} |F(v)| \, dv \leqq M_h \prod_{j=1}^{m} \eta_j^{\underline{\alpha}(\overline{\{L_1, \dots, L_j\}})} \tag{B.28}$$

for a certain constant M_h, and provided that the parameters η_j satisfy the conditions (B.23).

b) Contributions of the sets $\mathscr{I}_r$, for $h < r \leqq m$, and of the set $\mathscr{H}_m$.

To define an interval $I_{i_h \dots i_r}$, one considers a certain sequence U_r of $r - h + 1$ positive numbers: $\{u_{i_h}, \dots, u_{i_h \dots i_j}, \dots, u_{i_h \dots i_r}\}$ and the associated change of variables:

$$v = v_{U_r}(z) \equiv \sum_{j=h}^{r} \eta_j \dots \eta_r \dots \eta_m u_{i_h \dots i_j} + \eta_{r+1} \dots \eta_m z \tag{B.29}$$

This definition extends to the case $r = m$, provided that one adds the convention that "$\eta_{m+1} \dots \eta_m$" is then equal to 1. From (B.29), one obtains:

$$\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C = \sum_{j=1}^{h-1} L_j \eta_j \dots \eta_{r-1} \left(\frac{\eta_r}{|z|} \right) |z| \eta_{r+1} \dots \eta_m$$
$$+ \sum_{j=h}^{r} (L_j - Lu_{i_h \dots i_j}) \eta_j \dots \eta_{r-1} \left(\frac{\eta_r}{|z|} \right) |z| \eta_{r+1} \dots \eta_m - \varepsilon(z) L |z| \eta_{r+1} \dots \eta_m$$
$$+ \sum_{j=r+1}^{m} L_j \eta_j \dots \eta_m + C$$

The assumption $D_x^{d+1} f \in A_N^{\alpha_{d+1}}$ then implies the following majorization:

$$\text{For } v = v_{U_r}(z),\quad \frac{\eta_r}{|z|} \geqq b_r(U_r),\quad |z| \geqq b_0(U_r),$$
$$\eta_j \geqq b_j(U_r),\quad j = 1 \dots m, j \neq r, \tag{B.30}$$

$$\left| D_x^{d+1} f\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C \right) \right| \leqq M_r(U_r) \prod_{j=1}^{h-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j\}})} \times \dots$$
$$\cdot \prod_{j=h}^{r-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_{h-1}, L_h - Lu_{i_h}, \dots, L_j - Lu_{i_h \dots i_j}\}})}$$
$$\cdot \left(\frac{\eta_r}{|z|} \right)^{\alpha_{d+1}(\overline{\{L_1 \dots L_{h-1}, L_h - Lu_{i_h}, \dots, L_r - Lu_{i_h \dots i_r}\}})} \tag{B.31}$$
$$\cdot |z|^{\alpha_{d+1}(\overline{\{L_1, \dots L_r, L\}})} \prod_{j=r+1}^{m} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j, L\}})}$$

The numbers $M_r(U_r)$ which occur in (B.30) are appropriate constants such that: $b_j(U_r) \geqq 1$. One then defines:

$$I_{i_h \dots i_r} = \{ v = v_{U_r}(z) ; |z| < \eta_r b_r^{-1}(U_r) \}$$
$$J_{i_h \dots i_r} = I_{i_h \dots i_r} \cap \{ v = v_{U_r}(z) ; |z| > b_0(U_r) \}$$

so that (B.31) holds in $J_{i_h \dots i_r}$.

Assuming that a sequence $U_{r-1} = \{ u_{i_h} \dots u_{i_h \dots i_{r-1}} \}$ has been defined recursively, a set of numbers $\{ u_{i_h \dots i_r} ; i_r \in \sigma(i_1 \dots i_{r-1}) \}$ is determined through the Heine–Borel lemma in such a way that:

$$H_{i_h \dots i_{r-1}} = I_{i_h \dots i_{r-1}} \backslash J_{i_h \dots i_{r-1}} \subset \bigcup_{i_r \in \sigma(i_1 \dots i_{r-1})} I_{i_1 \dots i_{r-1} i_r};$$

Taking as its starting point the set $\{u_i, i \in \sigma_h\}$ of a), this procedure achieves the definition of all the sequences $\{u_{i_h \dots i_r}\}$ and sets $I_{i_h \dots i_r}, J_{i_h \dots i_r}, H_{i_h \dots i_r}$, for every $r \leqq m$.

From (B.30), (B.31) and from the definition (B.12) of $\mathscr{I}_r$, we then conclude that:

$$\forall r \leqq m : \int_{\mathscr{I}_r} |F(v)| dv \leqq \prod_{j=1}^{h-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j\}})} \times \prod_{j=r+1}^{m} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j, L\}}) + d + 1} \times \dots$$
$$\dots \sum_{(i_h \dots i_r) \in \sigma_r} M_r(U_r) \left(\prod_{j=h}^{r} \eta_j^{\alpha'_{i_h \dots i_j}} \right) \times \dots \tag{B.32}$$
$$\dots \int_{b_0(U_r) \leqq |z| \leqq \eta_r b_r^{-1}(U_r)} |z|^{\alpha' - \alpha'_{i_h \dots i_r}} \left(\sum_{j=h}^{r} \eta_j \dots \eta_r u_{i_h \dots i_j} + z \right)^d dz$$

where we have put:

$$\alpha' = \alpha_{d+1}(\overline{\{L_1, \dots, L_r, L\}})$$
$$\alpha'_{i_h \dots i_j} = \alpha_{d+1}(\overline{\{L_1, \dots, L_{h-1}, L_h - Lu_{i_h}, \dots, L_j - Lu_{i_h \dots i_j}\}}) \tag{B.33}$$

(B.32) holds provided that all $\eta_j \geqq b_j^{(r)} \geqq 1$, the $b_j^{(r)}$ being appropriate numbers.

The estimate of the powers of η_j for $h \leqq j \leqq r$ in the r.h.s. of (B.32) are given by an analysis which is similar to the one described in a). It yields the desired result

$$\int_{\mathcal{J}_r} |F(v)| dv \leqq M_r \prod_{j=1}^{m} \eta_j^{\alpha(\overline{\{L_1 \dots L_j\}})} \tag{B.34}$$

To obtain this, we had to take into account (as in a)) the fact that:

$$\forall j; h \leqq j \leqq r: \overline{\{L_1 \dots L_{h-1}, L_h - Lu_{i_h}, \dots, L_j - Lu_{i_h \dots i_j}\}} \succ_{y} \overline{\{L_1 \dots L_j\}}$$

so that (in view of (B.3′)):

$$\left[\sup_{(i_h \dots i_j) \in \sigma_j} (\alpha'_{i_h \dots i_j}) \right] + d + 1 \leqq \underline{\alpha}(\overline{\{L_1 \dots L_j\}}) \tag{B.35}$$

and similarly: $\alpha_{d+1}(\overline{\{L_1, \dots, L_r, L\}}) + d + 1 \leqq \underline{\alpha}(\overline{\{L_1 \dots L_r\}})$

Finally, it remains to majorize $\int_{\mathcal{H}_m} |F(v)| dv$. On each interval $H_{i_h \dots i_m}$, we have:

$$v = v_{U_m}(z); \text{with } |z| \leqq b_0(U_m) \tag{B.36}$$

which implies:

$$\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C = \sum_{j=1}^{h-1} L_j \eta_j \dots \eta_m + \sum_{j=h}^{m} (L_j - Lu_{i_h \dots i_j}) \eta_j \dots \eta_m + C'$$

where $C' = C - Lz$ varies (in view of (B.36)) in a compact set W'.

The majorization (B.31) is thus replaced by:

$$\left| D_x^{d+1} f\left(\sum_{j=1}^{m} L_j \eta_j \dots \eta_m - Lv + C \right) \right| \leqq M_m(U_m) \prod_{j=1}^{h-1} \eta_j^{\alpha_{d+1}(\overline{\{L_1 \dots L_j\}})} \times \dots$$

$$\cdot \prod_{j=h}^{m} \eta_j^{\alpha_{d+1}(\overline{\{L_1, \dots, L_{h-1}, L_h - Lu_{i_h}, \dots, L_j - Lu_{i_h \dots i_j}\}})}$$

Since no z-integration remains in this bound(valid on $H_{i_1 \dots i_m}$,) the end of the above argument (see (B.35)) applies directly and allows us to conclude that:

$$\int_{\mathcal{H}_m} |F(v)| dv \leqq M_m \prod_{j=1}^{m} \eta_j^{\underline{\alpha}(\overline{\{L_1 \dots L_j\}})} \tag{B.37}$$

We can now conclude that in the case $p = 1$, the announced result (B.5) is a trivial consequence of (B.13), (B.28), (B.34), (B.37).

2. *The General Case*

We split arbitrarily the space $\mathbb{R}^p_{(x)}$ into two factors:

$$R^p_{(x)} = R^r_{(u)} \times R^s_{(v)} \text{ (i.e. } x = (u, v) \text{ with } r \leqq p - 1, s \leqq p - 1).$$

We then use the following identity which links together the Taylor rests of f at $x = 0$ and at $u = 0$, and the Taylor rests of the partial derivatives $D_u^\nu f_{|u=0}$

at $v=0$:

$$[(1-t_x^d)f](x,y)=[(1-t_u^d)f](u,v,y)+\dots$$
$$\cdot\sum_{|\nu|=0}^{d}\frac{u^\nu}{\nu!}[(1-t_v^{d-|\nu|})D_u^\nu f_{|u=0}](v,y) \tag{B.38}$$

Since all the functions $f(u,v,y), D_u^\nu f_{|u=0}(v,y)$ are graded Weinberg functions with respect to variables u and v and since $r,s\leqq p-1$, it is possible to apply—as a recursive assumption—the statement of the present lemma to each Taylor rest which occurs at the right-hand side of formula (B.38). We will show that each term of this formula belongs to the class $A^{\underline{\alpha}}_{(p+q)}$, with:

$$\underline{\alpha}(S)=\alpha_{d+1}(S) \qquad \text{if } S\subset R^q_{(y)}$$
$$\underline{\alpha}(S)=\sup_{\{S';S'\succ_y S\}}[\alpha_{d+1}(S')]+(d+1) \qquad \text{if } S\not\subset R^q_{(y)}$$

i) $(1-t_u^d)f$ is a Weinberg function in the class $A^{\underline{\alpha}'}_{(p+q)}$ such that:

$$\underline{\alpha}'(S)=\alpha_{d+1}(S), \qquad \text{if } S\subset R^{s+q}_{(v,y)}$$
$$\underline{\alpha}'(S)=\sup_{\{S';S'\succ_{(v,y)} S\}}[\alpha_{d+1}(S')]+d+1, \qquad \text{if } S\not\subset R^{s+q}_{(v,y)}$$

where $S'\succ_{(v,y)}S$ if: i) $S\cap\mathbb{R}^{s+q}_{(v,y)}\subset S'\cap\mathbb{R}^{s+q}_{(v,y)}$, and ii) $\pi_{(u)}(S)=\pi_{(u)}(S')$, $\pi_{(u)}$ denoting the projection of $\mathbb{R}^{p+q}_{(x,y)}$ onto $\mathbb{R}^{s+q}_{(v,y)}$ parallel to $\mathbb{R}^r_{(u)}$. If we also introduce the projection $\pi_{(v)}$ of $\mathbb{R}^{s+q}_{(v,y)}$ onto $\mathbb{R}^q_{(y)}$ parallel to $\mathbb{R}^s_{(v)}$, we have: $\pi=\pi_{(v)}\circ\pi_{(u)}$; from this we deduce that the property:

$$S'\succ_{(v,y)}S \text{ implies } S'\succ_y S$$

Then we can easily check that for every S, one has:

$$\underline{\alpha}'(S)\leqq\underline{\alpha}(S)$$

ii) For each multiple index ν with $|\nu|\leqq d$, $D_u^\nu f_{|u=0}$ is a Weinberg function on $\mathbb{R}^{s+q}_{(v,y)}$ which is graded with respect to v and asymptotically governed by the sequence $\{\alpha_{|\nu|},\alpha_{|\nu|+1},\dots,\alpha_{|\nu|+n}\dots\}$ (the functions $\alpha_{|\nu|+n}(S)$ being here restricted to the set of subspaces S of $\mathbb{R}^{s+q}_{(v,y)}$). Then from our recursive assumption, we can say that $(1-t_v^{d-|\nu|})D_u^\nu f_{|u=0}$ belongs to the class $A^{(\alpha'|\nu|)}_{(s+q)}$ which is defined as follows:

$$\alpha'_{|\nu|}(S)=\alpha_{d+1}(S) \qquad \text{if } S\subset R^q_{(y)} \tag{B.39}$$
$$\alpha'_{|\nu|}(S)=\sup_{S';S'\succ_y S}[\alpha_{d+1}(S')]+d-|\nu|+1$$
$$\text{if } S\subset\mathbb{R}^{s+q}_{(v,y)} \text{ and } S\not\subset R^q_{(y)}$$

Now, we have to consider the function $(1-t_v^{d-|\nu|})D_u^\nu f_{|u=0}=h_\nu$ as a function on $\mathbb{R}^{p+q}_{(x,y)}$ (constant in v), namely $\pi^*_{(u)}h_\nu$: Lemma 1.1 implies that this function belongs to the class $A^{(\alpha|'\nu|)}_{(p+q)}$ such that:

$$\forall S\subset\mathbb{R}^{p+q}_{(x,y)},\quad \alpha''_{|\nu|}(S)=\alpha'_{|\nu|}(\pi_{(u)}(S)). \tag{B.40}$$

Then from (B.39) and (B.40) we can conclude that $\forall S \subset \mathbb{R}^N_{(x,y)}$ such that $S \not\subset \mathbb{R}^q_{(y)}$

$$\alpha''_{|\nu|}(S) \leqq \sup_{\{S';S' \underset{y}{\succ} S\}} [\alpha_{d+1}(S')] + d - |\nu| + 1 \tag{B.41}$$

Indeed this results from the following (easy to check) geometrical property:

Lemma. *If $\underline{S}, \underline{S}' \subset \mathbb{R}^{s+q}_{(v,y)}$ with $\underline{S}' \underset{y}{\succ} \underline{S}$ and if $\underline{S} = \pi_u(S)(S \subset \mathbb{R}^N_{(x,y)})$, then $\underline{S}' \underset{y}{\succ} S$.*

Finally, since u^ν is a Weinberg function[9] with asymptotic coefficients $\alpha(S) = |\nu|$ for every $S \not\subset \mathbb{R}^{s+q}_{(v,y)}$ and $\alpha(S) = 0$ for $S \subset \mathbb{R}^{s+q}_{(v,y)}$, we can conclude from (B.39), (B.41), and from Proposition 1.1.b, that each term $\frac{u^\nu}{\nu!}[(1 - t_v^{d-|\nu|}) D_u^\nu f_{|u=0}](v, y)$ occurring in (B.38) is a Weinberg function which belongs to the class $A^{\underline{\alpha}}_{(p+q)}$. Therefore, in view of Proposition 1.1.a, we conclude that $(1 - t_x^d) f(x, y)$ belongs to $A^{\underline{\alpha}}_{(p+q)}$.

B) For every derivative operator D_x^ν with total order $|\nu| = n$, the function $D_x^\nu g$ is related with the corresponding derivative $D_x^\nu f$ of f through the following formula:

$$D_x^\nu g = (1 - t_{(x)}^{d-n}) D_x^\nu f \tag{B.42}$$

Notice that for $n > d$, (B.42) reduces to $D_x^\nu g = D_x^\nu f$. Now $D_x^\nu f$ is, as f, a graded function; it is asymptotically governed by the sequence $\{\forall m \geqq 0 : \alpha'_m = \alpha_{m+n}\}$. So when $n > d$, we readily obtain that $D_x^\nu g$ belongs to the class $A_N^{(\alpha'_0)} = A_N^{(\alpha_n)}$; namely, formula (B.1) holds true.

When $n \leqq d$, we can apply the result of part A to the function $D_x^\nu f$ and to its Taylor rest of order $d - n$. In view of (B.42), we conclude that $D_x^\nu g$ belongs to a class $A_N^{(\underline{\alpha}_n)}$ with $\underline{\alpha}_n$ expressed as follows:

—if $S \subset \mathbb{R}^q_{(y)}$:

$$\underline{\alpha}_n(S) = \alpha'_{d-n+1}(S) = \alpha_{d+1}(S) \tag{B.2}$$

—if $S \not\subset R^q_{(y)}$:

$$\underline{\alpha}_n(S) = \sup_{\{S';S' \underset{y}{\succ} S\}} [\alpha'_{d-n+1}(S')] + d - n + 1 = \sup_{\{S';S' \underset{y}{\succ} S\}} [\alpha_{d+1}(S')] + d - n + 1 \tag{B.3}$$

This achieves the proof of Lemma B.1.

Application to Admissible Weinberg Functions (see Definition 1d)

Let ω (resp. σ) be a set of subspaces of $\mathbb{R}^N_{(x,y)}$ (resp. $\mathbb{R}^q_{(y)}$) enjoying the properties of Definition 1c and let α be an index function on $\mathbb{R}^N_{(x,y)}$, such that:

$$\forall S \in \omega, \quad \alpha(S) = \alpha(\pi(S)).$$

The class $\mathscr{A}_N^{(\alpha,\sigma,\omega)}$ can be alternatively defined as follows: it is the class of all functions $f(x, y)$ which are graded with respect to x, and asymptotically governed

9 See Proposition 1.2

by the following sequence $\{\alpha_n\}$:

$$\left.\begin{aligned}\forall n \geqq 0: \quad & \alpha_n(S) = \alpha(S) - n \qquad && \text{if } S \in \omega \\ & \alpha_n(S) = \alpha(S) && \text{if } S \notin \omega\end{aligned}\right\} \tag{B.43}$$

We shall prove:

Lemma B.2. *If f belongs to a class $\mathscr{A}_N^{(\alpha,\sigma,\omega)}$, then its Taylor rest $g = (1 - t_{(x)}^d)f$ is a graded function with respect to x; it is asymptotically governed by a sequence $\{\underline{\alpha}_n\}$ which is defined (for every integer $n \geqq 0$) by the following formulae:*

If $S \not\subset \mathbb{R}_{(y)}^q$ and $S \in \omega$:

$$\underline{\alpha}_n(S) = \alpha(S) - n \tag{B.44}$$

If $S \subset \mathbb{R}_{(y)}^q$ and $S \in \sigma$:

$$\underline{\alpha}_n(S) = \alpha(S) - \sup(n, d+1) \tag{B.45}$$

If $S \not\subset \mathbb{R}_{(y)}^q$, $S \notin \omega$ and $\pi(S) \in \sigma$

$$\underline{\alpha}_n(S) = \sup[\alpha(S), \alpha(\pi(S)) - n] \text{ for } n \leqq d \tag{B.46}$$

$$\underline{\alpha}_n(S) = \alpha(S) \text{ for } n > d \tag{B.47}$$

If $S \not\subset \mathbb{R}_{(y)}^q$, $S \notin \omega$ and $\pi(S) \notin \sigma$

$$\underline{\alpha}_n(S) = \sup[\alpha(S), \alpha(\pi(S)) + d - n] \text{ for } n \leqq d \tag{B.48}$$

$$\underline{\alpha}_n(S) = \alpha(S) \text{ for } n > d \tag{B.49}$$

If $S \subset \mathbb{R}_{(y)}^q$ and $S \notin \omega$

$$\underline{\alpha}_n(S) = \alpha(S) \tag{B.50}$$

Proof. We shall first prove that $f \in A_N^{(\underline{\alpha}_0)}$.

Let us consider an arbitrary sequence of vectors $L_1, \ldots, L_m (m \leqq N)$ in $\mathbb{R}_{(x,y)}^N$ and the associated nested subspaces $S_j = \overline{\{L_1, \ldots, L_j\}}$, with $1 \leqq j \leqq m$. We look for a Weinberg type majorization (1.1) for $g\left(\sum_{j=1}^{m} L_j \eta_j \ldots \eta_m + C\right)$ (when $\eta_1 > 0, \ldots, \eta_m > 0$, and C varies in a compact set W).

With the sequence $L_1, \ldots, L_m$ are associated the two following integers h and $l (1 \leqq h, l \leqq m)$:

h is such that : $\forall j < h$, $S_j \subset \mathbb{R}_{(y)}^q$

and $\forall j \geqq h$, $S_j \not\subset \mathbb{R}_{(y)}^q$

l is such that[10] : $\forall j < l$, $S_j \notin \omega$,

and $\forall j \geqq l$, $S_j \in \omega$.

Then two cases occur and have to be studied separately:

a) $l < h$. Then we have (property b) of the set ω)

10 The existence of such an l is assured by the properties of the set ω (see Definition 1c)

for $j < l$, $S_j \subset \mathbb{R}^q_{(y)}$ and $S_j \notin \sigma$

for $l \leqq j < h$, $S_j \subset \mathbb{R}^q_{(y)}$ and $S_j \in \sigma$

for $j \geqq h$, $S_j \not\subset \mathbb{R}^q_{(y)}$ and $S_j \in \omega$

Let us apply Lemma B.1 to the graded function f; it tells us in particular that $g \in A_N^{(\underline{\alpha}^{(a)})}$, where the index $\underline{\alpha}^{(a)}$ satisfies (in view of (B.2)′–(B.3)′):

$$\text{for } S_j \subset \mathbb{R}^q_{(y)}, \quad S_j \notin \sigma: \quad \underline{\alpha}^{(a)}(S_j) = \alpha(S_j) \tag{B.51}$$

$$\text{for } S_j \subset \mathbb{R}^q_{(y)}, \quad S_j \in \sigma: \quad \underline{\alpha}^{(a)}(S_j) = \alpha(S_j) - d - 1 \tag{B.52}$$

$$\text{for } S_j \not\subset \mathbb{R}^q_{(y)}, \quad S_j \in \omega: \quad \underline{\alpha}^{(a)}(S_j) = \sup_{\{S'; S' \succ S_j\}} [\alpha_{d+1}(S')] + d + 1 \tag{B.53}$$

But in the latter case the relation $S' \succ S_j$ implies $S' \in \omega$ (indeed $S' \supset S' \cap \mathbb{R}^q_{(y)} \supset S_j \cap \mathbb{R}^q_{(y)} = S_{h-1}$ and since $S_{h-1} \in \omega, S' \in \omega$). So we always have: $\alpha_{d+1}(S') = \alpha(S') - d - 1$ and $\alpha(S') = \alpha(\pi(S')) = \alpha(\pi(S_j)) = \alpha(S_j)$. It follows that (B.53) can be replaced by

$$\underline{\alpha}^{(a)}(S_j) = \alpha(S_j). \tag{B.53$'$}$$

b) $l \geqq h$. Then we have:

for $j < h$: $S_j \subset \mathbb{R}^q_{(y)}$, $S_j \notin \sigma$

for $h \leqq j < l$: $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \notin \omega$

for $j \geqq l$: $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \in \omega$

Let us then use directly the formula which defines g:

$$g(x, y) = f(x, y) - \sum_{|\nu|=0}^{d} f_x^{(\nu)}(0, y) x^\nu \tag{B.54}$$

and let us apply Lemma 1.1 and Propositions 1.1b) and 1.2b) to each term in the summation $\sum_{|\nu|=0}^{d}$ of (B.54).

We then obtain that each function $f_x^{(\nu)}(0, y) x^\nu$ belongs to a class $A_N^{(\underline{\alpha}^{(\nu)})}$ such that:

If $S_j \subset \mathbb{R}^q_{(y)}$, $S_j \notin \sigma$; $\underline{\alpha}^{(\nu)}(S_j) = \alpha(S_j)$

If $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \notin \omega$; $\underline{\alpha}^{(\nu)}(S_j) = \alpha_{|\nu|}(\pi(S_j)) + |\nu|$

If $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \in \omega$; $\underline{\alpha}^{(\nu)}(S_j) = \alpha_{|\nu|}(\pi(S_j)) + |\nu| = \alpha(S_j)$

Then we apply Propositions 1.1 (a) and c)) to the sum of all the terms at the right-hand side of (B.54). As a result, we obtain that $g \in A_N^{(\underline{\alpha}^{(b)})}$ where the index $\underline{\alpha}^{(b)}$ satisfies the following conditions:

$$\text{If } S_j \subset \mathbb{R}^q_{(y)}, \quad S_j \notin \sigma; \quad \underline{\alpha}^{(b)}(S_j) = \alpha(S_j) \tag{B.55}$$

If $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \notin \omega$, $\pi(S_j) \notin \sigma$

$$\underline{\alpha}^{(b)}(S_j) = \sup[\alpha(S_j), \alpha(\pi(S_j)) + d] \tag{B.56}$$

If $S_j \not\subset \mathbb{R}^q_{(y)}$, $S_j \notin \omega$, $\pi(S_j) \in \sigma$;

$$\underline{\alpha}^{(b)}(S_j) = \sup[\alpha(S_j), \alpha(\pi(S_j))] \tag{B.57}$$

$$\text{If } S_j \not\subset \mathbb{R}^q_{(y)}, \quad S_j \in \omega; \quad \underline{\alpha}^{(b)}(S_j) = \alpha(S_j) \tag{B.58}$$

By comparing formulae (B.51) and (B.55) (resp. (B.53)′ and (B.58)), we see that when a subspace S can belong to both types a) and b) of sequences $L_1 \dots L_m$, one always has $\alpha^{(a)}(S) = \alpha^{(b)}(S)$.

So we have proved that g satisfies all the Weinberg type majorizations (1.1) of a class $A_N^{(\underline{\alpha}_0)}$, if one defines $\underline{\alpha}_0$ through the following conditions:

If $S \subset \mathbb{R}^q_{(y)}$, $S \notin \sigma$: $\underline{\alpha}_0(S) = \underline{\alpha}^{(a)}(S) = \underline{\alpha}^{(b)}(S) = \alpha(S)$

If $S \subset \mathbb{R}^q_{(y)}$, $S \in \sigma$: $\underline{\alpha}_0(S) = \underline{\alpha}^{(a)}(S) = \alpha(S) - d - 1$

If $S \not\subset \mathbb{R}^q_{(y)}$, $S \notin \omega$, $\pi(S) \in \sigma$:

$\underline{\alpha}_0(S) = \underline{\alpha}^{(b)}(S) = \sup[\alpha(S), \alpha(\pi(S))]$

If $S \not\subset \mathbb{R}^q_{(y)}$, $S \notin \omega$, $\pi(S) \notin \sigma$:

$\underline{\alpha}_0(S) = \underline{\alpha}^{(b)}(S) = \sup[(\alpha(S), \alpha(\pi(S)) + d]$

If $S \not\subset \mathbb{R}^q_{(y)}$, $S \in \omega$

$\underline{\alpha}_0(S) = \underline{\alpha}^{(a)}(S) = \underline{\alpha}^{(b)}(S) = \alpha(S)$

These conditions on $\underline{\alpha}_0$ coincide with those of formulae (B.44) to (B.50) in the case $n = 0$.

It remains to be checked that for every derivative operator D_x^ν with $|\nu| = n > 0$, $D_x^\nu g$ belongs to $A_N^{\underline{\alpha}_n}$, with $\underline{\alpha}_n$ defined by formulae (B.44)…(B.50).

The case $n > d$ is trivial, since then: $D_x^\nu g = D_x^\nu f$. The case $1 \leqq n \leqq d$ is treated by using formula (B.42), which allows to repeat the argument given above (case $n = 0$), f being simply replaced by $D_x^\nu f$ and d by $d - n$. The detailed estimates of the $\underline{\alpha}_n(S)$ which yield formulae (B.44)…(B.50) are then straight-forward and left to the reader.

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