Commun. Math. Phys. 69, 89–98 (1979)

Communications in
Mathematical
Physics

Boundedness of Total Cross-Sections in Potential Scattering

A. Martin
CERN, Geneva, Switzerland

Abstract. If a spherically symmetric potential belongs to the Rollnik class, i.e., if

$$I=\frac{1}{(4\pi)^2}\int\frac{|V(x)|\,|V(x')|}{|x-x'|^2}\,d^3x\,d^3x'$$

is finite, the total cross-section is finite, and an explicit bound on this quantity can be given in terms of I. We also investigate the case of non-spherically symmetric potentials, and show that if I is less than unity, the total cross-section averaged over the directions of the incident beam at a given energy is finite.

1. Introduction

Recently the question of boundedness of total cross-sections in potential scattering has been re-examined by Amrein and Pearson [1]. These authors have shown that if one can be satisfied by statements of the form: "the cross-section is finite for almost all energies" it is no longer necessary to assume local regularity of the potential as was done in some previous work on the subject [2]. Crudely speaking, all the authors [1, 2] find that the cross-sections are finite if the potential decreases sonewhat faster than $|x|^{-2}$ at large distances (in the case of three space dimensions).

Here we want to use a method which, in its essence, is not new, and was applied 15 years ago to get bounds on the scattering amplitude itself [3]. This method is inspired by the work of Froissart to obtain bounds on elementary particle scattering amplitudes [4]. The general idea is that the partial wave amplitudes and cross-sections for small angular momentum can be bounded by unitarity, while for large angular momentum the Born approximation (or nearest singularity contribution in the case of elementary particles!) gives a reasonable estimate.

It is not very difficult to see that the total cross-section, for spherically symmetric potentials, calculated in the Born approximation, will be finite at all energies, except possibly at threshold if the potential belongs to the Rollnik class [5], such that

$$I=\frac{1}{(4\pi)^2}\int\frac{|V(x)|\,|V(x')|}{|x-x'|^2}\,d^3x\,d^3x' \tag{1.1}$$

This is also true for non-spherically symmetric potentials, if one replaces the total cross-section by its *average* over all possible incident directions of the beam.

These facts give a strong presumption that condition (1.1) might be the right one, and this is what we succeed in proving in Sect. 2 for the case of spherically symmetric potentials. Like in [1], (1.1) has the advantage of being an integral condition, not necessitating local regularity. It has also the advantage of indicating rather precisely what kind of decrease of the potential is sufficient. In particular the Sobolev inequality

$$I = \frac{1}{(4\pi)^2} \int \frac{|V(x)|\,|V(x')|}{|x'-x|^2} d^3x d^3x' < C[\int |V(x)|^{3/2} d^3x]^{4/3}, \tag{1.2}$$

where $C < 1.63$ [and, probably, $(2/\pi)^{2/3}$ [6]], indicates that I will be finite if $V(r)$ decreases faster than $r^{-2}(\log r)^{-2/3-\varepsilon}$, $\varepsilon > 0$. In fact, direct examination of I after angular integrations are performed, shows that I will be finite if V decreases like

$$r^{-2}(\log r)^{-1/2-\varepsilon} \qquad \varepsilon > 0. \tag{1.3}$$

In Sect. 3 we have tried to generalize our results to the non-spherically symmetric case, and partially succeeded. In this case the value of the total cross-section depends on the orientation of the incident beam and we shall consider only the total cross-section averaged over incident directions at a *given* energy. We prove that this quantity is finite if I, defined by (1.1) is less than unity. In that case the perturbation expansion of the scattering amplitude is convergent and this is what makes the proof easy. We think that the averaged cross-section remains finite even if I is larger than unity but we have no proof of that statement.

2. The Case of Spherically Symmetric Potentials

For simplicity we set k, centre-of-mass momentum, equal to unity, except in the final inequalities on σ_{total}. The radial Schrödinger equation is obtained by projection over a given angular momentum:

$$u_\ell = u_{\ell 0}(r) + \int_0^\infty K_\ell(r, r') V(r') u_\ell(r') dr', \tag{2.1}$$

where

$$K_\ell(r, r') = -i \frac{rr'}{2} \int_{-1}^{+1} \frac{e^{i|x-x'|}}{|x-x'|} P_\ell(\cos\theta) d\cos\theta, \tag{2.2}$$

where

$$\frac{x \cdot x'}{rr'} = \cos\theta,$$

and

$$r = |x|, \qquad r' = |x'|,$$

or

$$K_\ell(r,r')=\frac{\pi}{2}\sqrt{rr'}\,J_{\ell+1/2}(r_<)H^{(1)}_{\ell+1/2}(r_>)\;; \tag{2.3}$$

$$u_{\ell 0}(r)=\sqrt{\frac{\pi r}{2}}\,J_{\ell+1/2}(r)\,. \tag{2.4}$$

We shall try to get an inequality on $\int Vu_\ell^2 dr$. To this effect we multiply (2.1) by $u_\ell(r)|V(r)|$ and integrate:

$$\int u_\ell^2|V|dr=\int u_\ell u_{\ell 0}|V|dr+\int drdr' u_\ell(r)|V(r)|K_\ell(r,r')V(r')u_\ell(r')\,. \tag{2.5}$$

We apply the Cauchy-Schwarz inequality to both terms in the right-hand side of (2.5) and get

$$\begin{aligned}&\left[\int u_\ell^2|V|dr\right]\left[1-\sqrt{\int drdr'|V(r')|(K_\ell(r,r'))^2|V(r)|}\right]\\&\quad<\left[\int u_\ell^2|V|dr\int u_{0\ell}^2|V|dr'\right]^{1/2}.\end{aligned} \tag{2.6}$$

We shall now relate the quantity

$$\int |V(r)|(K_\ell(r,r'))^2|V(r')|drdr' \tag{2.7}$$

to I, and show that it decreases with ℓ, so that for ℓ large enough (2.6) allows to control $\int u_\ell^2 Vdr$.

We use definition (2.2) of K_ℓ and apply again Schwarz inequality:

$$\begin{aligned}(K_\ell(r,r'))^2&<r^2r'^2\int_{-1}^{+1}\frac{d\cos\theta}{|x-x'|^2}\int_{-1}^{+1}\frac{d\cos\theta}{2}|P_\ell(\cos\theta)|^2\\&=\frac{r^2r'^2}{2\ell+1}\int_{-1}^{+1}\frac{d\cos\theta}{2|x-x'|^2}\,.\end{aligned} \tag{2.8}$$

Inserting into (2.7) we get

$$\begin{aligned}&\frac{1}{2\ell+1}\int drr^2|V(r)|dr'r'^2|V(r')|\frac{d\Omega_{xx'}}{4\pi}\frac{1}{|x-x'|^2}\\&=\frac{1}{2\ell+1}\int\frac{d\Omega_x}{4\pi}\frac{d\Omega_{x'}}{4\pi}r^2drr'^2dr'\frac{|V(r)|\,|V(r')|}{|x-x'|^2}=\frac{I}{2\ell+1}\,.\end{aligned} \tag{2.9}$$

Therefore, if $\ell>L_0$ such that

$$I=2L_0+1\,, \tag{2.10}$$

we get the inequality

$$\int u_\ell^2|V|dr<\frac{\int u_{\ell 0}^2|V|dr}{\left[1-\sqrt{\dfrac{I}{2\ell+1}}\right]^2}\,. \tag{2.11}$$

is therefore bounded by

$$|e^{i\delta_\ell}\sin\delta_\ell| < [\int u_\ell^2 |V| dr \int u_{\ell 0}^2 |V| dr]^{1/2} < \frac{\int u_{\ell 0}^2 |V| dr}{1 - \sqrt{\frac{I}{2\ell+1}}}. \tag{2.12}$$

To get a bound on the total cross-section

$$\sigma_T = \frac{4\pi}{k^2} \sum (2\ell+1) \sin^2\delta_\ell \tag{2.13}$$

(here k has been inserted!), we can choose arbitrarily L, integer, larger than L_0, and use unitarity for $\ell \leqq L-1$ and (2.12) for $\ell \geqq L$:

$$\sigma_T < \frac{4\pi}{k^2}\left[L^2 + \frac{\sum\limits_{\ell=L}^{\infty} (2\ell+1)[\int u_{\ell 0}^2 |V| dr]^2}{\left[1 - \sqrt{\frac{I}{2L+1}}\right]^2} \cdot \right] \tag{2.14}$$

What is left is to estimate

$$\sum_{\ell=L}^{\infty} (2\ell+1)[\int u_{\ell 0}^2 |V| dr]^2 .$$

We can majorize this sum by extending it from $\ell=0$ to ∞. Then

$$\sum_{\ell=L}^{\infty} (2\ell+1)[\int u_{\ell 0}^2 |V| dr]^2 < \int \frac{d\cos\theta}{2} \left[\sum_{\ell=0}^{\infty} (2\ell+1) \int u_{\ell 0}^2 |V| dr P_\ell(\cos\theta)\right]^2 .$$

In the bracket we recognize the Born approximation for the full amplitude with the potential $|V|$, i.e.,

$$\int \frac{d^3x}{4\pi} |V(x)| e^{i(\mathbf{k}-\mathbf{k}')\cdot x}$$

where $|\mathbf{k}| = |\mathbf{k}'| = 1$ and $\cos\theta = (\mathbf{k}\cdot\mathbf{k}')$. Hence

$$\sum_{\ell=L}^{\infty} (2\ell+1)(\int |u_{\ell 0}|^2 |V| dr)^2$$

$$< \int \frac{d\Omega_k}{4\pi} \frac{d\Omega_{k'}}{4\pi} e^{i(\mathbf{k}-\mathbf{k}')\cdot(x-x')} |V(x)| \, |V(x')| \frac{d^3x}{4\pi} \frac{d^3x'}{4\pi}$$

$$= \int \frac{(\sin|x-x'|)^2}{|x-x'|^2} |V(x)| \, |V(x')| \frac{d^3x}{4\pi} \frac{d^3x'}{4\pi} < I . \tag{2.15}$$

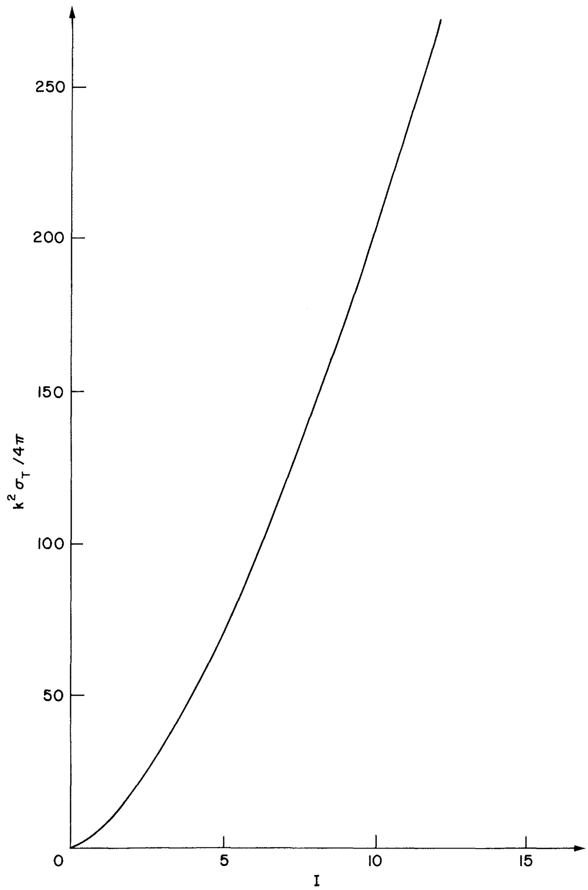

Fig. 1

Hence

$$\sigma_T < \frac{4\pi}{k^2}\left[L^2 + \frac{I}{\left(1-\sqrt{\frac{I}{2L+1}}\right)^2}\right]$$

$$\forall L > L_0 . \tag{2.16}$$

What is left to do is to optimize with respect to L for a given I. This will produce a continuous function of I. For small I the bound behaves like $(4\pi/k^2)I$, for very large I like $\pi I^2/k^2$. The Fig. 1 gives the bound on $k^2\sigma_t/4\pi$ for I between 0 and 12. A uniform bound, valid in $0 < I < \infty$ is

$$\frac{k^2\sigma_T}{4\pi} < 2I + 3.4 I^{4/3} + 2I^{5/3} + \tfrac{1}{4}I^2 . \tag{2.17}$$

To conclude this section let us indicate that it seems to us difficult to get a weaker condition than the existence of I. Indeed if one compares the Born approximation expression of the cross-section and I one sees that the only majorization in (2.15) is the replacement of $(\sin(k|x-x'|))^2$ by 1. For large k $(\sin(k|x-x'|))^2$ averages to $\frac{1}{2}$, so that if I diverges it is difficult to obtain a finite total cross-section, except if V has infinitely many oscillations [7].

3. The Case of Non-Spherically Symmetric Potentials

Here it is much more difficult to take advantage of unitarity as we did in the case of spherical symmetry. An extra complication is that, a priori, the total cross-section depends on the orientation of the incident beam. Classically we see that a flat reflecting disc will produce a large cross-section when it is perpendicular to the beam, and zero cross-section when it is parallel to the beam. It will appear very natural to consider here the averaged total cross-section

$$\bar{\sigma}_T(k)=\int\frac{d\Omega_k}{4\pi}\,\sigma_T(\mathbf{k})\,. \tag{3.1}$$

The first thing we shall show, which is almost a repetition of what was done in Sect. 2 is that $\bar{\sigma}_T(k)$, evaluated by using the Born approximation for the amplitude, is finite if I is finite. Indeed:

$$\begin{aligned}\bar{\sigma}_{T,\mathrm{Born}}(k)&=4\pi\int\frac{d\Omega_k}{4\pi}\,\frac{d\Omega_{k'}}{4\pi}\,\frac{e^{i(\mathbf{k}-\mathbf{k}')\cdot(x-x')}V(x)V(x')d^3x\,d^3x'}{(4\pi)^2}\\&=\frac{4\pi}{k^2}\int(\sin k|x-x'|)^2\,\frac{V(x)V(x')d^3x\,d^3x'}{(4\pi)^2|x-x'|^2}\\&\leqq\frac{4\pi}{k^2}I\,.\end{aligned} \tag{3.2}$$

Now we shall try to find a situation in which the Born series for the scattering amplitude is convergent, in order to take advantage of (3.2).

For technical reasons, we shall start with an exponentially damped potential $V_\varepsilon(x) = V(x)\exp(-\varepsilon|x|)$. For such a potential $\int|V_\varepsilon(x)|dx$ is convergent if $V(x)$ belongs to the Rollnik class.

We start from the integral equation

$$\psi_{\mathbf{k},\varepsilon}(x)=\psi_{0\mathbf{k}}(x)-\frac{1}{4\pi}\int\frac{e^{ik|x-x'|}}{|x-x'|}\,V_\varepsilon(x')\psi_{\mathbf{k},\varepsilon}(x')d^3x'\,, \tag{3.3}$$

where

$$\psi_{0\mathbf{k}}(x)=e^{i\mathbf{k}\cdot x}\,.$$

Following a procedure similar to that of Sect. 2, we multiply to the left by $\psi^*_{\mathbf{k},\varepsilon}(x)|V_\varepsilon(x)|$, integrate, and use Schwarz inequality. This gives

$$\begin{aligned}&\int|\psi_{\mathbf{k},\varepsilon}(x)|^2|V_\varepsilon(x)|d^3x\left[1-\sqrt{\int\frac{|V_\varepsilon(x)|\,|V_\varepsilon(x')|}{|x-x'|^2}\,\frac{d^3x}{4\pi}\,\frac{d^3x'}{4\pi}}\right]\\&\quad<\left[\int|\psi_{\mathbf{k},\varepsilon}(x)|^2|V_\varepsilon(x)|d^3x\right]^{1/2}\left[\int|\psi_{0,\mathbf{k}}(x)|^2|V_\varepsilon(x)|d^3x\right]^{1/2}.\end{aligned} \tag{3.4}$$

Hence, if $I<1$, $I_\varepsilon<1$, and

$$\int |\psi_{\mathbf{k},\varepsilon}(x)|^2 |V_\varepsilon(x)| d^3x < \frac{\int |V_\varepsilon(x)| d^3x}{(1-\sqrt{I})^2}. \tag{3.5}$$

Here, we see why it was necessary to introduce a cut-off, to ensure the convergence of the integral over $|V_\varepsilon|$. This is because the quantity in the left-hand side of (3.5) controls in fact the full forward scattering amplitude, including the real part, which might become infinite in the limit $\varepsilon \to 0$, but in which we are not really interested. On the other hand, fortunately, the denominator in (3.5) does not depend on ε.

Condition (3.5) allows us to show that the Born series expansion for ψ converges:

$$\psi = \psi_0 + KV\psi_0 + \ldots + \underbrace{KVK \ldots KV\psi}_{n \text{ terms}} \tag{3.6}$$

where K in x space is

$$-\frac{e^{ik|x-x'|}}{|x-x'|}.$$

The last term can be bounded by

$$I^{\frac{n-2}{2}} [\int |V_\varepsilon| \, |\psi|^2 d^3x]^{1/2} [\int |K(x,x')|^2 |V(x')| d^3x']^{1/2},$$

i.e.,

$$[\int |K(x,x')|^2 |V(x')| d^3x']^{1/2} \frac{I^{\frac{n-2}{2}} [\int |V_\varepsilon(x)| d^3x]^{1/2}}{(1-\sqrt{I})^2}. \tag{3.7}$$

Hence the scattering amplitude

$$T_\varepsilon = \int \psi^*_{0\mathbf{k}'}(x) V_\varepsilon(x) \psi_{\mathbf{k}}(x) d^3x \tag{3.8}$$

can be written as

$$T_\varepsilon = \sum_1^{n-1} T_{n,\varepsilon} + R_{n,\varepsilon}, \tag{3.9}$$

with

$$|R_n| \leqq \frac{I^{\frac{n-1}{2}} \int |V_\varepsilon(x)| d^3x}{(1-\sqrt{I})^2}. \tag{3.10}$$

Hence if $I \leqq \alpha < 1$, R_n goes to zero for $n \to \infty$, for ε fixed, arbitrarily small. The series expansion of $T(\mathbf{k}', \mathbf{k})$ is therefore convergent. The total cross-section, averaged over angles, is therefore given by

$$\overline{\sigma_{T,\varepsilon}} = 4\pi \int \frac{d\Omega_k}{4\pi} \frac{d\Omega_{k'}}{4\pi} \sum_{m,n} T^*_{m,\varepsilon}(\mathbf{k}', \mathbf{k}) T_{n,\varepsilon}(\mathbf{k}', \mathbf{k}). \tag{3.11}$$

The general term of this double series is of the form

$$\frac{4\pi}{k^2}\int \frac{d^3x_1}{4\pi}\cdots\frac{d^3x_m}{4\pi}\frac{d^3y_1}{4\pi}\cdots\frac{d^3y_n}{4\pi}\frac{\sin k|x_1-y_1|}{|x_1-y_1|}$$
$$\cdot\frac{\sin k|x_m-y_n|}{|x_m-y_n|}V_\varepsilon(x_1)\frac{e^{ik|x_1-x_2|}}{|x_1-x_2|}\cdots V_\varepsilon(x_n)$$
$$\cdot V_\varepsilon(y_1)\frac{e^{ik|y_1-y_2|}}{|y_1-y_2|}\cdots V_\varepsilon(y_m)\,. \tag{3.12}$$

It is easy to see that it is bounded by

$$\frac{4\pi}{k^2}\operatorname{Tr}(H_\varepsilon^{m+n}), \tag{3.13}$$

where

$$H_\varepsilon(x,x')=\frac{1}{4\pi}|V_\varepsilon(x)|^{1/2}\frac{1}{|x-x'|}|V_\varepsilon(x')|^{1/2}\,.$$

However $|V_\varepsilon(x)|\leqq|V(x)|$, and

$$H_\varepsilon<H_0=H\,.$$

So if

$$\lim_{N\to\infty}(\operatorname{Tr}H^N)^{\frac{1}{N}}=\alpha<1 \tag{3.14}$$

the double series defining $\bar\sigma_T$ converges uniformly with respect to ε as ε approaches unity. A sufficient condition for (3.14) is

$$I\leqq\alpha^2<1\,. \tag{3.15}$$

Indeed

$$\operatorname{Tr}(H^N)\leqq I^{N/2} \tag{3.16}$$

Furthermore, we get a bound on $\bar\sigma_T$ which is *independent* of ε:

$$\bar\sigma_T<\frac{4\pi}{k^2}\frac{I}{(1-\sqrt{I})^2}\,. \tag{3.17}$$

Rigorous minded people might worry that even if $\lim_{\varepsilon\to 0}\bar\sigma_T$ is finite this is not a proof that $\bar\sigma_T$ is finite. One can in fact use another trick which is to average the Schrödinger equation, with $\varepsilon=0$, over some small angular interval in $\mathbf{k}/k$. One can define $\psi_{\mathbf{k},\eta}$ and $\psi_{0,\mathbf{k}\eta}$ as

$$\psi_{\mathbf{k},\eta}=\frac{\int\frac{d\Omega_{k'}}{2\pi}\psi_{\mathbf{k}'}(x)\Theta\left[\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2}-\cos\eta\right]}{1-\cos\eta}, \tag{3.18}$$

$$\psi_{0\mathbf{k},\eta}=\frac{\int\frac{d\Omega_{k'}}{2\pi}\psi_{0\mathbf{k}'}(x)\Theta\left[\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2}-\cos\eta\right]}{1-\cos\eta}\,. \tag{3.19}$$

It is possible to show that (3.19) behaves like $|x|^{-1}$ at large distances. Therefore, we can repeat the previous argument to show that if $I \leqq \alpha < 1$, the Born series for $\psi_{\mathbf{k},\eta}$ converges, because $\int |\psi_{0\mathbf{k},\eta}|^2 (V) d^3x$ contains an extra convergence factor $1/(|x|^2)$, and is guaranteed to converge if V belongs to the Rollnik class. One can define a pseudo-averaged cross-section

$$\bar{\sigma}_{T,\eta} = 4\pi \int \frac{d\Omega_k}{4\pi} \frac{d\Omega_{k'}}{4\pi} \left| \int d^3x \psi^*_{0\mathbf{k}',\eta}(x) V(x) \psi_{\mathbf{k},\eta}(x) \right|^2 .$$

As $\eta \to 0$ the integrand tends to the integrand appearing in the definition of $\bar{\sigma}_T$ pointwise. However, the integral is dominated by an η independent integral and tends to $\bar{\sigma}_T$, which is indeed finite.

One can slightly improve condition (3.15). In fact, what is essential for the finiteness of the cross-section is condition (3.14). It is easy to recognize, following the Birman-Schwinger approach to the bound state problem [8], that (3.14) is nothing but the condition for the absence of bound states with negative or zero energy of the potential $-|V(x)|$. Indeed, (3.14) means that the largest eigenvalue of the positive operator H is less than unity. So we get the following result:

if (i) $V(x)$ belong to the Rollnik class,

(ii) $-|V(x)|$ admits no negative or zero energy bound state, $\bar{\sigma}_T(k)$ is finite.

We believe that the present results could be generalized and that the only really crucial condition is the existence of I. Possible ways are:

1. Replacing the iterative solution of the Schrödinger equation by the Fredholm method;
2. Subtracting explicitly the bound states from the symmetrized kernel of the Schrödinger equation (in the case of a purely attractive potential).

References

1. Amrein, W.O., Pearson, D.C.: Preprint, University of Geneva. UGVA-DPT 1978/09-179 (1978)
2. Amrein, W.O., Jauch, J.M., Sinha, K.B.: Scattering theory in quantum mechanics. Reading: Benjamin (1977)
   Green, T.A., Lanford, O.E.: J. Math. Phys. **1**, 139 (1960)
   Misra, B., Speiser, D., Targonski, G.: Helv. Phys. Acta **16**, 963 (1963)
   Villarroel, D.: Nuovo Cimento **70** A, 175 (1970)
   Jauch, J.M., Sinha, K.B.: Helv. Phys. Acta **45**, 580 (1972)
   Martin, Ph., Misra, B.: In: Scattering theory in mathematical physics. J. A. LaVita and J. P. Marchand (eds.), Reidel: Dordrecht (1974)
   Hunziker, W.: Helv. Phys. Acta **36**, 838 (1963)
   Kupsch, W., Sandhas, W.: Commun. Math. Phys. **2**, 147 (1966)
3. Martin, A.: Lectures at the University of Washington. Preprint University of Washington (1964), unpublished
   See also: Martin, A.: Nuovo Cimento **23**, 641 (1962); and **31**, 1229 (1964)
4. Froissart, M.: Phys. Rev. **123**, 1053 (1961)
5. Rollnik, H.: Z. Physik **145**, 639 (1956)
6. See, for instance: Martin, A.: CERN Preprint TH. 2085 (1975); and Comptes Rendus de la RCP No. 25, Vol. 23. University of Strasbourg, Mathematics Department (1976)

7. In the case of oscillating potentials one can expect many surprises as was indicated in the pioneer work of Baeteman, M., Chadan, K.: Annales de l'Institut Henri Poincaré A **24**, 1 (1976). We do not want to give a complete list of references. Let us just indicate that the latest developments, concerning potentials oscillating at large distances are:
Reignier, M.: Preprint, Vrije Univ. Brussels (1979)
Chadan, K., Martin, A.: CERN Preprint TH. 2655 (1979) to be published in Commun. Math. Phys.
8. Birman, M.: Mat. Sb. **55**, 125 (1961)
Schwinger, J.: Proc. Nat. Acad. Sci. (US) **47**, 122 (1961)

Communicated by J. Ginibre

Received May 10, 1979