Commun. Math. Phys. 67, 137–146 (1979)

Communications in Mathematical Physics


# Shift Automorphisms in the Hénon Mapping

R. Devaney* and Z. Nitecki

Department of Mathematics, Tufts University, Medford, Massachusetts 02155, USA

**Abstract.** We investigate the global behavior of the quadratic diffeomorphism of the plane given by $H(x,y)=(1+y-Ax^2, Bx)$. Numerical work by Hénon, Curry, and Feit indicate that, for certain values of the parameters, this mapping admits a "strange attractor". Here we show that, for $A$ small enough, all points in the plane eventually move to infinity under iteration of $H$. On the other hand, when $A$ is large enough, the nonwandering set of $H$ is topologically conjugate to the shift automorphism on two symbols.

Several numerical studies have recently appeared [3, 4, 7, 8] on the dynamics of the diffeomorphisms of the plane

$$H(X,Y)=(1+Y-AX^2, BX)\,.$$

Interest in these maps [12, 14, 5] has been prompted by Hénon's numerical evidence [8] for a "strange attractor" when $A=1.4$, $B=0.3$. Feit [4] has shown, for $A>0$ and $0<B<1$, that the non-wandering set $\Omega(H)$ is contained in a compact set, and that all points outside this set escape to infinity. Curry [3] has shown that, for Hénon's values of the parameters, one of the fixed points has a topologically transverse homoclinic orbit, and hence that there is a horseshoe embedded in the dynamics of the map.

The present note is intended to clarify the behavior of the mapping $H$ for parameter values far from those where "strange attractors" have been observed. Hénon and Feit have noted that for $B=0.3$ and $A$ outside a certain interval (roughly $[-0.12, 2.67]$) no attractors are observed; numerically, all points seem to escape to infinity. We exhibit, for any $B\neq 0$, a pair of $A$ values, $A_0<0<A_2$, such that the non-wandering set $\Omega(H)$ is empty for $A<A_0$, but for $A>A_2$, $\Omega(H)$ is the zero-dimensional basic set obtained from Smale's horseshoe construction [9, 11, 13]. We begin by rewriting the map in a more convenient form; then we establish Feit's result (for all $A, B\neq 0$) in a version more suited to our purposes, by

* Partially supported by NSF Grant MCS 77-00430

0010-3616/79/0067/0137/$02.00

constructing a filtration; and finally, for $A > A_2$ we exhibit the elements of the horseshoe construction in our compact set.

Hénon notes that the map $H$ represents one canonical form for quadratic maps with constant Jacobian determinant. We will find it convenient to consider the alternate canonical form for such maps

$$F(x, y) = (A + By - x^2, x) .$$

It is easily verified that for $A$ and $B$ both nonzero, the linear change of coordinates

$$X = x/A \qquad Y = By/A$$

gives a topological conjugacy between $H$ and $F$, with the parameter values $A$ and $B$ unchanged. For Hénon's map $H$, the parameter value $A = 0$ gives a linear map, while in our map $A = 0$ has no special significance. In fact, our results are established without the restrictions $A > 0$, $0 < B < 1$ imposed by earlier papers; we assume only that $B \neq 0$. Thus our analysis includes the orientation-preserving cases $B < 0$, and the area-preserving cases $B = \pm 1$, which were considered by Hénon in an earlier numerical study [6]. Actually, the $B$-values with absolute value greater than one do not exhibit new behaviour, since the inverse map

$$F^{-1}(x, y) = (y, (x - A + y^2)/B)$$

with given parameter values $A = a$, $B = b \neq 0$ is conjugate to the forward map $F$ with $A = a/b^2$, $B = 1/b$ by the linear change of variables

$$x \to -by \qquad y \to -bx .$$

To state our result, we fix $B$ and define three crucial $A$-values

$$A_0 = -(1 + |B|)^2/4$$

$$A_1 = 2(1 + |B|)^2$$

$$A_2 = (5 + 2\sqrt{5})(1 + |B|)^2/4$$

and, for any particular $A$-value, we define $R = R(A)$ by

$$R = (1/2)\{1 + |B| + [(1 + |B|)^2 + 4A]^{1/2}\} .$$

With this notation, our results are summarized in the following theorem.

**Theorem.** i) *For* $A < A_0$, $\Omega(F) = \phi$.

ii) *For* $A \geqq A_0$, $\Omega(F)$ *is contained in the square* $S = \{(x, y) \mid |x| \leqq R, |y| \leqq R\}$.

iii) *For* $A \geqq A_1$, $\Lambda = \bigcap_{n \in \mathbb{Z}} F^n(S)$ *is a topological horseshoe; for* $B \neq 0$, *there is a continuous semi-conjugacy of* $\Omega(F) \subset \Lambda$ *onto the 2-shift.*

iv) *For* $A > A_2$, $\Lambda = \Omega(F)$ *has a hyperbolic structure and is conjugate to the 2-shift.*

The value $A_0$ is, as Hénon remarked, precisely the $A$-value at which the first fixed point of $F^2$ appears. Thus, statement i) above follows from the Brouwer translation theorem [1, 2]; however, we shall give a direct proof of this fact. On the other hand, the values we give for $A_1$ and $A_2$ are somewhat larger than the experimental values given by Hénon and Feit (when $B=0.3$, $A_1=3.38$, $A_2=4.00$). In fact, they are clearly not the lowest values that yield the desired conclusions, but they yield these conclusions relatively easily.

The proofs of statements i)–iii) rely on the following technical lemmas:

**Lemma 1.** a) *$R$ is real if and only if $A \geqq A_0$. In this case, $R$ is positive and equals the larger root of*

$$R^2-(|B|+1)R-A=0 .$$

b) $A-|B|R>R$ *if and only if* $A>A_1$ .

*Proof.* a) is trivial; it implies that $A+|B|R=R^2-R$ and the first inequality of b) is equivalent to $R>2(1+|B|)$. Substituting this into the definition of $R$ and solving for $A$ gives the second inequality of b). □

We will find it convenient to denote the image of the point $(x_0, y_0)$ by $(x_1, y_1) = F(x_0, y_0)$, and use negative subscripts for pre-images.

**Lemma 2.** a) *The image under $F$ of the horizontal strip $|y_0| \leqq C$ is the region bounded by the two parabolas*

$$A-|B|C-y_1^2 \leqq x_1 \leqq A+|B|C-y_1^2 .$$

*The image under $F$ of the vertical strip $|x_0| \leqq C$ is the horizontal strip $(y_1| \leqq C$.*

b) *The inverse image of the vertical strip $|x_0| \leqq C$ is the region bounded by the two parabolas*

$$-C-A-x_{-1}^2 \leqq By_{-1} \leqq C-A-x_{-1}^2 .$$

*The inverse image of the horizontal strip $|y_0| \leqq C$ is the vertical strip $|x_{-1}| \leqq C$.*

*Proof.* These are straightforward calculations. □

In the following, we interpret $\min(a, R)$ or $\max(a, R)$ with $R$ complex as equal to $a$.

**Lemma 3.** a) *If $x_0 \leqq \min(-|y_0|, -R)$, then $x_1 \leqq x_0$, with equality only for $x_0=-R$, $y_0=\pm R$.*

b) *If $x_0 \geqq -|y_0|$ and $By_0 \geqq \max(0, |B|R)$, then $By_{-1} \geqq By_0$ and $|y_{-1}| \geqq |y_0|$, with equality only for $x_0=-R$, $y_0=\pm R$.*

*Proof.* For a), by the definition of $x_1$ and, in the last inequality below, our hypothesis on $x_0$,

$$\begin{aligned} x_1-x_0 &= A+By_0-x_0^2-x_0 \\ &\leqq A+|B|\,|y_0|-x_0^2-x_0 \\ &\leqq A-(|B|+1)x_0-x_0^2 . \end{aligned}$$

The last expression is zero for

$$x_0=-(1+|B|)/2 \pm [(1+|B|)^2+4A]^{1/2}/2 .$$

If $R$ is complex, so is $x_0$, and the expression above is negative for all $x_0$, whereas when $R$ is real, the lesser root is $x_0=-R$, so that the expression remains negative

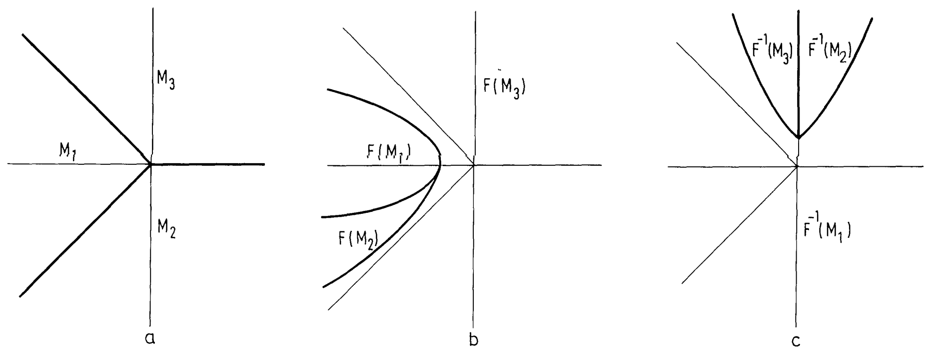


**Fig. 1a–c.** Filtration for $A<A_0$. Situation shown is for $B>0$; $B<0$ is obtained by reflection about the $x$-axis. **a** Filtration. **b** Image under $F$. **c** Image under $F^{-1}$

for $x_0<-R$. On the other hand, when $x_0=-R$ but $|y_0|<R$, the last inequality above is strict, so that equality only holds for $x_0=-|y_0|=-R$.

The proof of b) is similar. Consider

$$B(y_{-1}-y_0)=y_0^2+x_0-A-By_0$$
$$\geqq y_0^2-(1+|B|)y_0-A\ .$$

The last expression is positive provided $|y_0|>\max(0,R)$; if $B$ is positive, this says $y_{-1}>y_0\geqq 0$, whereas for $B$ negative, it says $y_{-1}<y_0\leqq 0$. The equality statements follow as before. □

Now, to prove statement i) of the theorem, we define a partition of the plane by

$$M_1=\{(x,y)\,|\,x\leqq -|y|\}$$
$$M_2=\{(x,y)\,|\,x\geqq -|y| \text{ and } By\leqq 0\}$$
$$M_3=\{(x,y)\,|\,x\geqq -|y| \text{ and } By\geqq 0\}$$

(see Fig. 1a).

**Proposition 1.** *For* $A<A_0$,

a) $F(M_1\cup M_2)\subset$ *interior* $M_1$.
b) $x$ *is strictly decreasing along $F$-orbits in* $M_1$.
c) $F^{-1}(M_2\cup M_3)\subset$ *interior* $M_3$.
d) $|y|$ *is strictly increasing along $F^{-1}$-orbits in* $M_3$.

*Proof.* By Lemma 3a, if $(x_0, y_0)\in M_1$, then $y_1=x_0>x_1$. The inequality is strict because $R$ is complex. Also, since $x_0\leqq 0$, $y_1=-|y_1|$. This shows $F(M_1)\subset$ interior $M_1$ and statement b). Moreover, by Lemma 2a, the $x$-axis maps to a parabola opening left with vertex at $(A,0)$; this lies to the left of the boundary of $M_1$. Furthermore, the image of the line $By=-\varepsilon$ is a parabola to the left of the previous one, so that $F(M_2)$ lies to the left of the boundary of $M_1$ (see Fig. 1b).

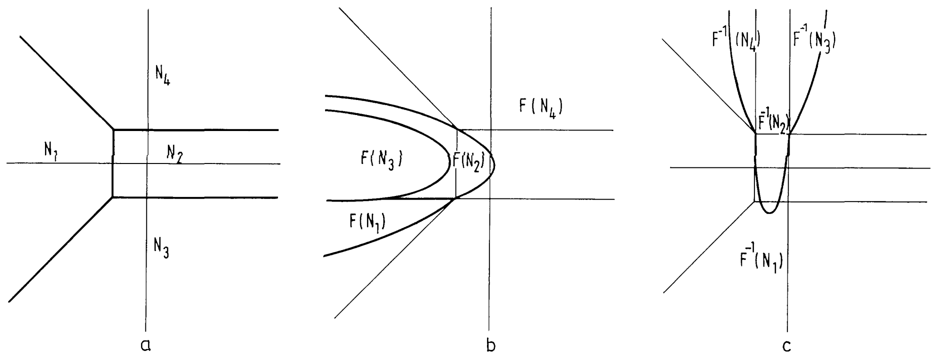


**Fig. 2a–c.** Partition for $A \geqq A_0$. Situation shown is for $B>0$. **a** Partition. **b** Image under $F$. **c** Image under $F^{-1}$

The proof of c) and d) is similar. If $(x_0, y_0) \in M_3$, then by Lemma 3b,

$$|y_{-1}| > |y_0| = |x_{-1}|$$

and

$$By_{-1} > By_0 \geqq 0 \, .$$

An analysis similar to the above, using Lemma 2b, completes the proof of c) (see Fig. 1c). □

Proposition 1 gives a filtration for $F$ with Liapunov functions for the extreme sets $M_1$, $M_3$, and $M_2 \cap F(M_2) = \phi$, so that $\Omega(F) = \phi$ for $A < A_0$. Statement ii) of the theorem is proven by an analogous construction, with the positive $x$-axis above expanded into another element of the filtration. Specifically, when $R$ is real (i.e., $A \geqq A_0$), define four sets by Fig. 2a

$$N_1 = \{(x, y) | x \leqq \min(-|y|, R)\}$$
$$N_2 = \{(x, y) | x \geqq -R, |y| \leqq R\}$$
$$N_3 = \{(x, y) | x \geqq -|y|, By \leqq |B|R\}$$
$$N_4 = \{(x, y) | x \geqq -|y|, By \geqq |B|R\} \, .$$

**Proposition 2.** *For* $A \geqq A_0$,

a) $F(N_1) \subset N_1$.

b) $F(N_2 \cup N_3) \subset N_1 \cup N_2$.

c) $x$ *is decreasing along* $F$*-orbits in* $N_1$ *(strictly decreasing except at the two points* $x = -|y| = -R$*).*

d) $F^{-1}(N_3 \cup N_4) \subset N_4$.

e) $F^{-1}(N_2) \subset N_2 \cup N_3 \cup N_4$.

f) $|y|$ *is increasing along* $F^{-1}$*-orbits in* $N_4$ *(strictly increasing except at the point* $(-R, R\,\mathrm{sign}(B))$*).*

*Proof.* The only statements whose proof differs from Proposition 1 are b) and e). b) is proven by invoking Lemma 2a with $C = R$, and by noting that the right boundary of $F(N_2)$ intersects $x = -R$ at $|y| = R$ by Lemma 1a (see Fig. 2b).

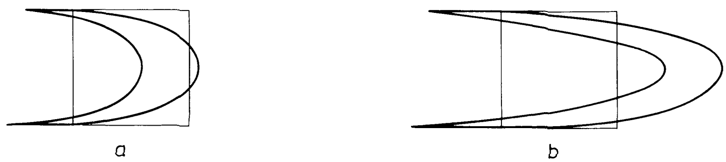


**Fig. 3a and b.** Image of $S$. **a** $A_0 \leqq A < A_1$. **b** $A_1 \leqq A$

Similarly, we note that the boundary curves $-x_0 = |y_0| > R$ of $N_1$ map under $F^{-1}$ to curves in the interior of $N_4$. This follows from Lemma 3b and proves d). Also, the line segment $x_0 = -R$, $|y_0| \leqq R$ maps to the parabolic segment

$$By_{-1} = x_{-1}^2 - R - A, \quad |x_{-1}| \leqq R$$

which is disjoint from $N_1$. This proves e) (see Fig. 2c).

Proposition 1 does not, strictly speaking, give a filtration for $F$, since one of the two left corners of the square $S$ maps to the boundary of $N_1$ under $F$, and to the boundary of $N_4$ under $F^{-1}$. This could be taken care of by a slight modification of $N_1$ and $N_2$, but for our purposes, it suffices to note that, by c) and f), all points in the interior of $N_1 \cup N_4$ are wandering. In fact, they escape to infinity in at least one time direction. By b) and e), this implies that $\Omega(F) \subset N_2 \cap F^{-1}(N_2)$. But from Lemma 2, we see immediately that

$$N_2 \cap F^{-1}(N_2) \subset S = \{(x, y) \mid |x| \leqq R, |y| \leqq R\}$$

thus proving statement ii) of the Theorem.

To prove statement iii) of the Theorem, we note that, by Lemma 2a, $F(S)$ is the region bounded by the parabolas

$$x_1 = A \pm |B|R - y_1^2, \quad |y_1| \leqq R$$

and the two horizontal line segments

$$y_1 = \pm R, \ -(1+2|B|)R \leqq x_1 \leqq -R.$$

The latter follows from the identities

$$A + |B|R - R^2 = -R$$

$$A - |B|R - R^2 = -(1+2|B|)R$$

which follow immediately from Lemma 1. The vertex of the left boundary of $F(S)$ is at $(A+|B|R, 0)$; as $A$ increases, this moves to the right. When $A$ passes $A_1$, it crosses the right edge of $S$, by Lemma 1b (see Fig. 3). Thus, for $A > A_1$, we have the topological part of the horseshoe: the image of any horizontal line segment in $S$ is a parabola which cuts across $S$ in two segments. By a standard analysis [9, 11, 13] points in the invariant set $\Lambda = \bigcap_{-\infty \leqq n \leqq \infty} F^n(S)$ can be coded by a bisequence of 0's and 1's according to which of the two components of $S_{-1} = S \cap F(S)$ contain successive backward iterates and which of the two components of $S_1 = S \cap F^{-1}(S)$ contain successive forward iterates. The coding gives a continuous orbit-preserving map of $\Lambda$ onto the Cantor set underlying the shift automorphism on two symbols. To see that $\Omega(F)\subset\Lambda$ maps onto the shift space, we note that every periodic sequence corresponds to a nested intersection of closed disks, and hence to at least one periodic orbit of $F$; but the closure of the set of these periodic orbits is a set of non-wandering points in $\Lambda$ mapping onto the shift space. This establishes statement iii) of the Theorem.

To prove statement iv), we first note that the matrix of partial derivatives of $F$

$$JF=\begin{pmatrix}-2x & B\\ 1 & 0\end{pmatrix}$$

and of $F^{-1}$

$$JF^{-1}=\begin{pmatrix}1/B & 2x/B\\ 0 & 1\end{pmatrix}$$

are independent of $A$ and $y$. Thus, for $B\neq 0$ fixed, the hyperbolicity of an invariant set depends only on the projection of this set onto the $x$-axis. We will show that orbits which stay away from a band about the $x$-axis (whose width depends on $B$) have two constant bundles of sectors

$$S_\lambda^+=\{(\xi,\eta)\,|\,|\xi|\geqq\lambda|\eta|\}$$

$$S_\lambda^-=\{(\xi,\eta)\,|\,\lambda|\xi|\leqq|\eta|\}$$

with $\lambda>1$ which are invariant, respectively, under the Jacobians $JF$ and $JF^{-1}$. We will then show that the invariant set $\Lambda=\cap F^n(S)$ is disjoint from this band when $A>A_2$.

**Lemma 4.** *Suppose that, for some* $\lambda>1$, $x$ *satisfies*

$$|x|\geqq\lambda(1+|B|)/2\,.$$

*Then:* a) *For any vector* $(\xi_0,\eta_0)\in S_\lambda^+$, *the vector* $(\xi_1,\eta_1)=JF_x(\xi_0,\eta_0)$ *satisfies* $|\xi_1|>\lambda|\xi_0|$.

b) *For any vector* $(\xi_0,\eta_0)\in S_\lambda^-$, $(\xi_{-1},\eta_{-1})=JF_x^{-1}(\xi_0,\eta_0)$ *satisfies* $|\eta_{-1}|\geqq\lambda|\eta_0|$.

*Proof.* By hypothesis,

$$2|x|\geqq\lambda+\lambda|B|$$

so that

1) $2|x|-|B|/\lambda>2|x|-\lambda|B|\geqq\lambda$,

2) $2|x|-\lambda\geqq\lambda|B|$.

To show a), we use the formula for $JF_x$, the inequality $|a+b|\geqq|a|-|b|$, the hypothesis $|\eta_0|\leqq|\xi_0|/\lambda$, and 1) in succession to conclude

$$\begin{aligned}|\xi_1|&=|-2x\xi_0+B\eta_0|\geqq 2|x|\,|\xi_0|-|B|\,|\eta_0|\\ &\geqq(2|x|-|B|/\lambda)|\xi_0|>\lambda|\xi_0|\,.\end{aligned}$$

Similarly, to show b), we use the formula for $JF_x^{-1}$ and 2) to obtain

$$\begin{aligned}|\eta_{-1}| &= |\xi_0 + 2x\eta_0|/|B| \\ &\geqq (2|x| - \lambda)|\eta_0|/|B| \\ &\geqq \lambda|\eta_0| . \quad \square\end{aligned}$$

As a consequence, we can prove the following.

**Proposition 3.** *If* $(x_0, y_0)$ *and* $(x_1, y_1) = F(x_0, y_0)$ *both satisfy*

$$|x| > \lambda(1 + |B|)/2$$

*for some* $\lambda > 1$, *then*

a) *for any vector* $(\xi_0, \eta_0) \in S_\lambda^+$, $(\xi_1, \eta_1) = JF_{x_0}(\xi_0, \eta_0)$ *belongs to* $S_\lambda^+$, *and*

$$|(\xi_1, \eta_1)| \geqq \lambda |(\xi_0, \eta_0)| .$$

b) *For any vector* $(\xi_1, \eta_1) \in S_\lambda^-$, $(\xi_0, \eta_0) = JF_x^{-1}(\xi_1, \eta_1)$ *belongs to* $S_\lambda^-$ *and*

$$\lambda |(\xi_1, \eta_1)| \leqq |(\xi_0, \eta_0)| .$$

*Proof.* Our notation above is such that in both a) and b),

$$(\xi_1, \eta_1) = JF_{x_0}(\xi_0, \eta_0) .$$

In particular, $\eta_1 = \xi_0$. To show a), we invoke Lemma 4a to get

$$\lambda|\eta_1| = \lambda|\xi_0| \leqq |\xi_1|$$

so that $(\xi_1, \eta_1) \in S_\lambda^+$, while by the definition of $S_\lambda^+$

$$\lambda|\eta_0| \leqq |\xi_0| = |\eta_1| .$$

This proves a). Similarly, to establish b), we use Lemma 4b to conclude

$$|\eta_0| \geqq \lambda|\eta_1| = \lambda|\xi_0|$$

and the definition of $S_\lambda^-$ to conclude

$$|\xi_0| = |\eta_1| \geqq \lambda|\xi_1| . \quad \square$$

Our last step in establishing statement iv) is to verify the hypotheses of Proposition 3 for all points of $\Lambda = \cap F^n(S)$ when $A > A_2$.

**Proposition 4.** *If* $A > A_2$, *there exists* $\lambda > 1$ *such that*

$$|x| \geqq \lambda(1 + |B|)/2$$

*for all* $(x, y) \in S \cap F^{-1}(S)$.

*Proof.* From Lemma 1b, we see that the pre-image $F^{-1}(S)$ lies outside the parabola

$$By_1 = R - A + x_1^2$$

which intersects the edge of $S$ at a pair of points with $x$-coordinates $\pm x_*$ where

$$x_*^2 = By - R + A = A - (1 + |B|)R .$$

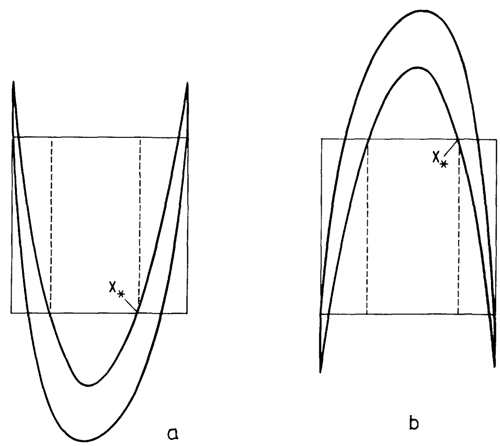


**Fig. 4a and b.** Preimage of $S$; $A > A_1$. **a** $B > 0$. **b** $B < 0$

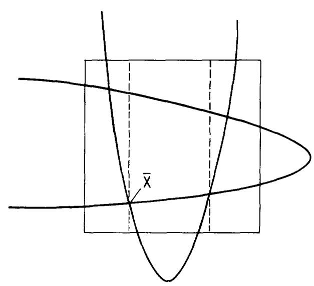


**Fig. 5**

It is clear that $F^{-1}(S) \cap S$ lies in the region $|x| \geqq |x_*|$ (see Fig. 4).

Let us write $A$ in the form

$$A = k(1 + |B|)^2/2 \ .$$

A quick calculation from the formula for $R$ then gives

$$R = (1 + \sqrt{1+2k})(1 + |B|)/2$$

so that

$$x_*^2 = (k - 1 - \sqrt{1+2k})(1 + |B|)^2/2 \ .$$

For $k > 0$, this quantity increases with $k$ and the left hand factor equals 1 when $2k = 5 + 2\sqrt{5}$ or $A = A_2$. Thus when $A > A_2$, we can take

$$\lambda^2 = (k - 1 - \sqrt{1+2k}) > 1$$

to get the conclusion of the proposition, and hence hyperbolicity of $\Lambda$. $\square$

We close with the observation that the conclusion of Proposition 4 for points of $\Lambda$ could be obtained with lower values of $A$ if, instead of estimating the intersection of the edge of $F^{-1}(S)$ with the edge of $S$, we considered the smallest $x$-value, $|\bar{x}|$, among the four intersections of the edge of $F^{-1}(S)$ with the edge of $F(S)$ (see Fig. 5). This point, which is a solution of the two quadratic equations

$$By = R - A + x^2$$
$$x = A + |B|R - y^2$$

is a lower bound for $|x|$ on $\Lambda$ which satisfies the hypotheses of Proposition 4 for values of $A$ somewhat lower than $A_2$.

We finally remark that our theorem shows that the phenomena of the "Hénon attractor" are part of a bifurcation occurring in the creation of a horseshoe from nothing. This gives another perspective on the significance of these mappings for dynamical systems theory.

*Note.* After this paper was written, it came to our attention that S. Newhouse has outlined a proof of a similar result [10].

## References


1. Andrea, S.A.: On homeomorphisms of the plane, and their embeddings in flows. Bull. AMS **71**, 381–383 (1965)
2. Brouwer, L.: Beweis des ebenen Translationssatzes. Math. Ann. **72**, 37–54 (1912)
3. Curry, J.: On the structure of the Hénon attractor. Preprint, National Center for Atmospheric Research (1977)
4. Feit, S.: Characteristic exponents and strange attractors. Commun. Math. Phys. **61**, 249–260 (1978)
5. Guckenheimer, J.: Bifurcations of dynamical systems. Preprint, University of California at Santa Cruz (1978)
6. Hénon, M.: Numerical study of quadratic area preserving mappings. Q. Appl. Math. **27**, 291–312 (1969)
7. Hénon, M.: A two-dimensional mapping with a strange attractor. Commun. Math. Phys. **50**, 69–77 (1976)
8. Hénon, M., Pomeau, Y.: Two strange attractors with a simple structure. In: Turbulence and the Navier-Stokes equation. Lecture notes in mathematics, Vol. 565, pp. 29–68. Berlin, Heidelberg, New York: Springer 1975
9. Moser, J.: Stable and random motions in dynamical systems. In: Annals of mathematical studies 77. Princeton: Princeton University Press 1973
10. Newhouse, S.: Lectures on dynamical systems. C.I.M.E. Summer in dynamical systems, Bressanone, Italy, 1978
11. Nitecki, Z.: Differentiable dynamics. Cambridge, MA, London: MIT Press 1971
12. Ruelle, D.: Dynamical systems with turbulent behavior. In: mathematical problems in theoretical physics. Lecture notes in physics. Vol. 80, pp. 341–360. Berlin, Heidelberg, New York: Springer 1978
13. Smale, S.: Diffeomorphisms with many periodic points. In: Differential and combinatorial topology. Princeton: Princeton University Press 1965
14. Ford, J.: Computer experiments and dynamical systems. Asterisque **49**, 75–92 (1977)


Communicated by J. L. Lebowitz

Received December 12, 1978