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Non-Translation Invariant Gibbs States with Coexisting Phases

II. Cluster Properties and Surface Tension*

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Abstract. We prove cluster properties of the spatially inhomogeneous Gibbs states in symmetric two component lattice systems obtained at large (equal) values of the fugacity. We also prove that the surface tension of these systems is given by an integral over the density variation in this state; Gibbs' formula. An alternative formula for the surface tension is also derived.

1. Introduction

In a previous paper [1] we proved for a class of two-component $A-B$ lattice gas systems in three dimensions the existence of Gibbs states in which there is a spatial segregation into an A-rich and a B-rich phase with a "sharp interface". These states are obtained, at high values of the chemical potential μ, $\mu=\mu_A=\mu_B$, by taking the infinite volume limit of a system with boundary conditions favoring $A(B)$ particles in the upper (lower) part of a box. This is entirely analogous to the existence of such nontranslation invariant states for ferromagnetic Ising spin systems at sufficiently low temperatures in three or more dimensions. The latter was first proven by Dobrushin [2] whose methods we used heavily in [1].

The purpose of this paper is to prove further properties of this nontranslation invariant Gibbs state: extremality, exponential clustering of correlation functions (no long range transverse part), and asymptotic behavior far from the interface. We also prove that the surface tension in these states is given by an integral over the correlation functions. This justifies a commonly used expression due originally to Gibbs. The methods used here are, like in [1], based on the work of Dobrushin [3]. For this reason we generally omit details of the proofs. Our results about the surface tension are new and apply also to the Ising model.

We use notation, definitions and results of [1] and we treat only the following model: we have two kinds of particles A and B with chemical potentials $\mu_A=\mu_B=\mu$. There is at most one particle at each point of $\mathbb{Z}^3$ and the presence of a

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particle at x excludes the presence of a particle of the other kind at all y such that $|x-y| = \max_{i=1,2,3} |x^i - y^i| = 1$. The paper is divided into three parts. Section 2 treats the statistics of the interface in terms of groups of walls. Section 3 contains the derivation of the results mentioned above using the results of Sect. 2 and properties of the pure phases which are given in Appendix. Section 4 is devoted to the study of the surface tension.

2. The Ensemble of Groups of Walls

2.1. Definition of the Ensemble. Let $A_{L,M} = \{x \in \mathbb{Z}^3 : |x^1| \leqq M, |x^2| \leqq L, |x^3| \leqq L\}$ be a parallelepiped of base $\sigma_L = \{x : x^1 = 0, |x^2| \leqq L, |x^3| \leqq L\}$ lying in the regular plane $\sigma = \{x : x^1 = 0\}$. We consider $A-B$ boundary condition for $A_{L,M}$: we have an A-particle (respectively a B-particle) at each site outside $A_{L,M}$ with $x^1 \geqq 1$ (respectively $x^1 \leqq -1$). Therefore each configuration in $A_{L,M}$ with $A-B$ boundary condition defines a configuration w in all $\mathbb{Z}^3$. In every configuration w there is exactly one infinite connected component of empty sites, which we call $\Delta(w)$. All adjacent sites to $\Delta(w)$ are occupied by particles. The interface $\bar{\Delta}(w)$ of the configuration w is the couple given by $\Delta(w)$ and all particles of w adjacent to $\Delta(w)$ (I.2) (Paper I, Sect. 2).

The probability of an interface $\bar{\Delta}$, $P_{L,M}(\bar{\Delta})$, is by definition the Gibbs measure of the set of all configurations w in $A_{L,M}$ such that $\bar{\Delta}(w) = \bar{\Delta}$. Taking the limit $M \to \infty$ we obtain a Gibbs state in the set $B_L = \{x \in \mathbb{Z}^3 : |x^2| \leqq L, |x^3| \leqq L\}$. The probability of an interface $\bar{\Delta}$ in B_L computed with this Gibbs state is given by $\lim_{M \to \infty} P_{L,M}(\bar{\Delta}) = P_L(\bar{\Delta})$. Furthermore we know that $P_L(\cdot)$ is concentrated on the set D_L of all interfaces which are contained in bounded sets of B_L (I.3).

To study the probability $P_L(\cdot)$ on D_L we decompose any interface of D_L into pieces of two types called walls and ceilings. There is only one interface which contains no wall. It is the interface of the ground state of the model ($\mu = \infty$) in which the interface is perfectly flat. The next simplest interfaces are those which contain only one wall, e.g., the interface looks like a top-hat with its brim extended to infinity. All walls which come from such an interface are called standard walls. The importance of standard walls comes from the fact that we can associate to every interface a family of disjoint standard walls. We call such a family admissible. Conversely to any admissible family of standard walls corresponds a unique interface $\bar{\Delta}$ (I.4). The probability $P_L(\cdot)$ on D_L then induces a probability on the set of all admissible families of standard walls.

We can now view the standard walls as elementary excitations of the ground state and define many-body long-ranged interactions between these excitations such that the corresponding Gibbs state is the probability $P_L(\cdot)$. These interactions are sufficiently weak so that we can use standard methods of statistical mechanics like equations of the Kirkwood-Salsburg type to study the interface. This is indeed the method of Gallavotti in his paper on the interface of the two dimensional Ising model [4], and we shall use this type of analysis in another paper where we discuss analycity properties of the surface tension and of correlation functions [9].

The standard walls can also be viewed as corresponding to the contours in the description of a pure phase in the Ising model at low temperature. The ground state is the configuration where no contours are present. To obtain a precise analogy we must however modify slightly the description of the interface. For technical reasons it is necessary to group the standard walls so that two standard walls belonging to different groups are not too close to each other. The precise definition of groups of walls is given in (I.5). A family of groups of walls is admissible if two groups of walls F_{t_1}, F_{t_2} of the family are such that $F_{t_1} \cup F_{t_2}$ is not a group of walls. There is a bijection between D_L and the set of all admissible families of groups of walls. The groups of walls are the objects corresponding to the contours. The quantity $\Pi(F_t)$ (I.5) corresponds to the length of a contour. Every group of walls F_t has an origin $t \in \sigma_L$; it is convenient to have a group of walls F_t with origin t for each t of σ_L. We introduce for this purpose the symbol Λ_t, called the *empty group of walls* with origin t, with $\Pi(\Lambda_t)=0$. Let $\mathscr{F}_t^L$ be the set of all groups of walls in B_L with origin t, including Λ_t. Every $\bar{\Delta}$ in D_L is uniquely described by an admissible family $(F_t, t \in \sigma_L)$. We use the notation $\bar{\Delta} = \bar{\Delta}(F_t, t \in \sigma_L)$. When we want to specify that each F_t of $(F_t, t \in \sigma_L)$ belongs to $\mathscr{F}_t^L$ we say that $(F_t, t \in \sigma_L)$ is *admissible in* B_L. We can now give a precise definition of the ensemble of groups of walls. For each $t \in \sigma_L$ we define a random variable $\eta_t^L \in \mathscr{F}_t^L$ with

$$\operatorname{Prob}\{\eta_t^L = F_t, t \in \sigma_L\} = \begin{cases} P_L(\bar{\Delta}) & \text{if} \quad \bar{\Delta} = \bar{\Delta}(F_t, t \in \sigma_L) \quad \text{exists} \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

We can use the same method as used by Dobrushin in [5] for the Ising model to study the random process $(\eta_t^L, t \in \sigma_L)$. We obtain for this process properties similar to those holding for the Gibbs state of a pure phase of the Ising model at low temperature.

Let us recall now some basic facts we shall need later about the random process $(\eta_t^L, t \in \sigma_L)$. We begin with geometrical properties of the configuration space:

$$\begin{gathered} (F_u, u \in \sigma_L) \text{ admissible} \Rightarrow (F_u, u \in \sigma_L \backslash t, \Lambda_t) \text{ admissible} \\ \Pi(F_t) = 0 \Rightarrow F_t = \Lambda_t . \end{gathered} \tag{2.2}$$

In other words if we remove a group of walls in an admissible family $(F_u, u \in \sigma_L)$ we obtain a new admissible family

The number of distinct groups of walls F_t with origin t and $\Pi(F_t)=k$ does not exceed C^k with C a constant. (2.3)

This is an entropy estimate and it is proved in (I.5).

We define now a function

$$Z_t^L(F_t | F_u, u \in \sigma_L \backslash t) = \begin{cases} \dfrac{P_L(\bar{\Delta})}{P_L(\bar{\Delta}^*)} & \text{if} \quad (F_u, u \in \sigma_L \backslash t, F_t) \quad \text{admissible} \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

where $\bar{\Delta}=\bar{\Delta}(F_u, u\in\sigma_L\backslash t, F_t)$, $\bar{\Delta}^*=\bar{\Delta}^*(F_u, u\in\sigma_L\backslash t, \Lambda_t)$. Using this function we have

$$\begin{aligned}&\mathrm{Prob}(F_t|F_u, u\in\sigma_L\backslash t)\\&=\frac{Z_t^L(F_t|F_u, u\in\sigma_L\backslash t)}{\sum\limits_{F_t^1\in\mathscr{F}_t^L} Z_t^L(F_t^1|F_u, u\in\sigma_L\backslash t)}.\end{aligned}\tag{2.5}$$

$\mathrm{Prob}(F_t|F_u, u\in\sigma_L\backslash t)$ is small for $F_t\neq\Lambda_t$ in the following sense. If $(F_u, u\in\sigma_L\backslash t, \Lambda_t)$ is admissible in B_L and μ large enough

$$\begin{aligned}&Z_t^L(F_t|F_u, u\in\sigma_L\backslash t)\leqq\exp\left(-\frac{\mu}{3}\Pi(F_t)\right),\\&Z_t^L(\Lambda_t|F_u, u\in\sigma_L\backslash t)=1.\end{aligned}\tag{2.6}$$

This is proved in (I.5). Furthermore by an analogous proof we have that Prob $(F_t|F_u, u\in\sigma_L\backslash t)$ depends weakly on F_u, if u and t are far apart in the following sense. If $(F_u, u\in\sigma_L\backslash t, \Lambda_t)$ is admissible in B_L and μ is large enough there exist $\lambda>0$ and $H<\infty$ independent of L and μ such that

$$\left|\log\frac{Z_t^L(F_t|F_u, u\in\sigma_L\backslash t)}{Z_t^L(F_t|F_u, u\in\sigma_L\backslash(t\cup s), \Lambda_s)}\right|\leqq H\exp(-\lambda\mu|t-s|)\tag{2.7}$$

for every $s\in\sigma_L\backslash t$ with $|s-t|\geqq 10(\Pi(F_t)+\Pi(F_s))$.

2.2. Clustering Properties. We define the finite dimensional distributions of the process $(\eta_t^L, t\in\sigma_L)$ by

$$\begin{aligned}\varrho_L(F_{t_1},\ldots,F_{t_n})&\equiv\varrho_L(F_t, t\in C)\\&=\mathrm{Prob}(\eta_t=F_t, t\in C),\end{aligned}\tag{2.8}$$

where $C=(t_1,\ldots,t_n)$ is a finite subset of σ_L. These distributions satisfy the following cluster property.

Proposition 2.1. *For μ large enough and E_1 and E_2 any two disjoint subsets of σ_L, there exist constants $K<\infty$ and $\kappa>0$ independent of E_1, E_2, L and μ, such that*

$$\begin{aligned}&\sum_{F_s, s\in E_1}\sum_{F_p, p\in E_2}|\varrho_L(F_s, s\in E_1, F_p, p\in E_2)\\&\quad-\varrho_L(F_s, s\in E_1)\varrho_L(F_p, p\in E_2)|\\&\leqq K\sum_{s\in E_1}\sum_{p\in E_2}\exp(-\mu\kappa|s-p|).\end{aligned}\tag{2.9}$$

Proof. The proof of (2.9) follows closely Dobrushin's ideas [5]. We therefore present here only an outline of the method of proof.

As L is fixed we will not write it as an index. If $(F_s, s\in E_1, \Lambda_u, u\in\sigma_L\backslash E_1)$ is not admissible, then (2.9) is zero. Therefore we suppose that $(F_s, s\in E_1, \Lambda_u, u\in\sigma_L\backslash E_1)$ is admissible. We can write the left-hand side of (2.9) as follows

$$\begin{aligned}&\sum_{F_s, s\in E_1}\mathrm{Prob}(\eta_s=F_s, s\in E_1)\Bigg(\sum_{F_p, p\in E_2}|\mathrm{Prob}(\eta_p=F_p, p\in E_2)\\&\quad-\mathrm{Prob}(\eta_p=F_p, p\in E_2|\eta_s=F_s, s\in E_1)|\Bigg).\end{aligned}\tag{2.10}$$

Instead of estimating (2.10) directly we estimate

$$\sum_{F_p, p\in E_2} |\mathrm{Prob}(\eta_p = F_p, p\in E_2) - \mathrm{Prob}(\eta_p = F_p, p\in E_2 \,|\, \eta_s = F_s, s\in E_1)|. \tag{2.11}$$

For the moment F_s, $s\in E_1$ are fixed. We introduce two random processes $(\eta_t^1, t\in\sigma_L)$, $\eta_t^1\in\mathscr{F}_t^L$ with $\mathrm{Prob}(\eta_t^1 = F_t, t\in\sigma_L) = \mathrm{Prob}(\eta_t = F_t, t\in\sigma_L)$ which is our previous process and $(\eta_t^2, t\in\sigma_L\backslash E_1)$, $\eta_t^2\in\mathscr{F}_t^L$ with

$$\mathrm{Prob}(\eta_t^2 = F_t, t\in\sigma_L\backslash E_1) = \mathrm{Prob}(\eta_t = F_t, t\in\sigma_L\backslash E_1 \,|\, \eta_s = F_s, s\in E_1).$$

We can rewrite (2.10) as

$$\sum_{F_p, p\in E_2} |\mathrm{Prob}(\eta_p^1 = F_p, p\in E_2) - \mathrm{Prob}(\eta_p^2 = F_p, p\in E_2)|. \tag{2.12}$$

We put $\sigma_L\backslash E_1 = V_2$ and $\sigma_L = V_1$. We say that $(F_u, u\in V_2)$ is admissible (for the second process) if $(F_u, u\in V_2, F_s, s\in E_1)$ is admissible. It is easy to see that $(\eta_t^2, t\in V_2)$ has the same properties (2.2)–(2.7) as $(\eta_t^1, t\in V_1)$. Let $(\tilde\eta_t, t\in V_i, i=1,2)$ be a joint representation of the two processes. That is $(\tilde\eta_t, t\in V_i, i=1,2)$ is a stochastic process with $\tilde\eta_t^i\in\mathscr{F}_t^L$ and

$$\begin{aligned}&\mathrm{Prob}(\tilde\eta_t^1 = F_t, t\in V_1) = \mathrm{Prob}(\eta_t^1 = F_t, t\in V_1),\\ &\mathrm{Prob}(\tilde\eta_t^2 = F_t, t\in V_2) = \mathrm{Prob}(\eta_t^2 = F_t, t\in V_2).\end{aligned} \tag{2.13}$$

We obtain immediately the inequality

$$\sum_{F_p, p\in E_2} |\mathrm{Prob}(\eta_p^1 = F_p, p\in E_2) - \mathrm{Prob}(\eta_p^2 = F_p, p\in E_2)| \leqq 2\sum_{p\in E_2} \mathrm{Prob}(\tilde\eta_p^1 \neq \tilde\eta_p^2). \tag{2.14}$$

The processes $(\eta_t^i, t\in V_i)$ have properties (2.2)–(2.7) and their conditional probabilities are comparable in the following sense: There exist constants $\lambda>0$, $H<\infty$, and $\hat F_s$, $s\in V_1\backslash V_2 = E_1$, such that

$$\left|\log\frac{Z_t^1(F_t\,|\,F_u, u\in V_1\cap V_2\backslash t, \hat F_u\in V_1\backslash V_2)}{Z_t^2(F_t\,|\,F_u, u\in V_1\cap V_2\backslash t)}\right| \leqq H\exp\left(-\lambda\mu\min_{s\in V_1\backslash V_2}|s-t|\right) \tag{2.15}$$

if $\min_{s\in V_1\backslash V_2}|s-t| \geqq 10\Pi(F_t)$.

We can then find a joint representation with the property: there exist constants $\kappa>0$, $C_1<\infty$ and $C_2<\infty$ such that for μ large enough

$$\mathbb{E}\exp(\kappa\mu\Pi(\eta_t^i)) \leqq C_1, \qquad i=1,2, t\in V_i$$

and (2.16)

$$\mathrm{Prob}(\tilde\eta_p^1 \neq \tilde\eta_p^2) \leqq C_2\sum_{s\in E_1}(1+\exp(\kappa\mu\Pi(\hat F_s)))\exp(-\kappa\mu|s-p|).$$

This follows from Lemmas 1 and 2 of [5] and from the fact that (2.15) is satisfied for any H and λ if $\hat{F}_s = F_s$, $s \in E_1$. Therefore we have proved Proposition 2.1. □

2.3. Dependence on L. Let $(\eta_t^1,\ t\in\sigma_L)=(\eta_t^L,\ t\in\sigma_L)$ and $(\eta_t^2,\ t\in\sigma_{L'})=(\eta_t^{L'},\ t\in\sigma_{L'})$ with $L'>L$. These two processes have the properties (2.2)–(2.7) and if we choose $\hat{F}_s=\Lambda_s$ for $s\in\sigma_{L'}\backslash\sigma_L$ we can prove (2.15) for μ large enough. Therefore we can use the same technique as above to study the dependence on L of the random process $(\eta_t^L,\ t\in\sigma_L)$. We formulate the result using the random variable δ_L with values in D_L defined by $\operatorname{Prob}(\delta_L=\bar{\Delta})=P_L(\bar{\Delta})$. Let $B_M=\{x\in\mathbb{Z}^3 : |x^2|\leqq M,\ |x^3|\leqq M\}$ and $L'>L>M$.

Proposition 2.2. *There exist constants $\kappa>0$, $K<\infty$ such that for μ large enough*

$$\sum_{S\subset B_M} |\operatorname{Prob}(\delta_L\cap B_M=S)-\operatorname{Prob}(\delta_{L'}\cap B_M=S)|$$

$$\leqq G_{M,L}\exp\left(-\frac{\kappa\mu(L-M)}{3}\right) \tag{2.17}$$

with $G_{M,L}\leqq KLM^2$.

The proof is exactly the same as the proof of Lemma 5 in [3]. We mention still another direct consequence of the existence of a joint representation for $(\eta_t^1, t\in\sigma_L)$ and $(\eta_t^2, t\in\sigma_{L'})$ with property (2.16).

Proposition 2.3. *Let D be a finite subset in σ_L and let $F_t\in\mathscr{F}_t^L$ for $t\in D$. If μ is large enough there exist constants $\kappa>0$ and $C<\infty$ such that for all $L'>L$*

$$\sum_{F_t, t\in D} |\varrho_L(F_t, t\in D)-\varrho_{L'}(F_t, t\in D)|$$

$$\leqq C \sum_{\substack{t\in D\\ s\in\sigma_{L'}\backslash\sigma_L}} \exp(-\mu\kappa|s-t|)\,. \tag{2.18}$$

Proof.

$$\sum_{F_t, t\in D} |\varrho_L(F_t, t\in D)-\varrho_{L'}(F_t, t\in D)|$$

$$=\sum_{F_t, t\in D} |\operatorname{Prob}(\tilde{\eta}_t^2=F_t, t\in D)-\operatorname{Prob}(\tilde{\eta}_t^1=F_t, t\in D)|$$

$$=2\sum_{t\in D}\operatorname{Prob}(\tilde{\eta}_t^1 \neq \tilde{\eta}_t^2)$$

$$\leqq \sum_{t\in D}\sum_{s\in\sigma_{L'}\backslash\sigma_L} C\exp(-\kappa\mu|t-s|)\,. \quad \square$$

This proposition shows that the finite dimensional distributions of $(\eta_t^L,\ t\in\sigma_L)$ have properties with respect to L analogous to those of the finite dimensional distributions with respect to Λ of the Ising model with pure boundary condition in a finite volume Λ at low temperature.

2.4. An Estimation of the Height of the Interface. Let $\gamma_s^L(\bar{\Delta})=\max\{x^1 : x\in\Delta,\ p(x)=s\}$ for each $s=(0,s^2,s^3)\in\sigma_L$. Here $\bar{\Delta}\in D_L$ and $p(x)$ is the orthogonal projection on the regular plane (I.4).

Proposition 2.4. *There exist constants* $\alpha>0$ *and* $K>\infty$ *such that for* μ *large enough and all* $s\in\sigma_L$

$$\mathrm{Prob}(\gamma_s^L \geqq N) \leqq K\exp(-\alpha\mu N). \tag{2.19}$$

Proof. Let $\bar{W}_t$ be a standard wall with origin t and $\bar{\Delta}=\bar{\Delta}(\Lambda_u, u\in\sigma_L\backslash t, F_t=\{\bar{W}_t\})$. Let $h(W_t)$ be the maximum of the modulus of the heights of the ceilings in the interface $\bar{\Delta}$. We have $h(W_t)\leqq \Pi(W_t)$. For each interface $\bar{\Delta}$ in D_L we have that

$$\gamma_s^L(\bar{\Delta}) \leqq \sum_{\substack{\bar{W}_t,\ \text{standard walls}\\ \text{of } \bar{\Delta},\ |t-s|\leqq \Pi(W_t)}} \Pi(W_t). \tag{2.20}$$

Therefore

$$\gamma_s^L(\bar{\Delta}) \leqq \sum_{t\in\sigma_L} \varphi_t(\gamma_t^L)$$

with

$$\varphi_t(F_t)=\begin{cases}\Pi(F_t) & \text{if } |s-t|\leqq \Pi(F_t)\\ 0 & \text{if } |s-t|>\Pi(F_t).\end{cases}$$

From Lemmas of (I.5) we get

$$\begin{aligned}
&\mathbb{E}(\exp(\tfrac{1}{4}\mu\varphi_t(\gamma_t^L))\,|\,\eta_u^L=F_u, u\in\sigma_L\backslash t)\\
&\quad=\sum_{F_t\in\mathscr{F}_t^L} \exp(\tfrac{1}{4}\mu\varphi_t(F_t))\mathrm{Prob}(F_t|F_u, u\in\sigma_L\backslash t)\\
&\quad\leqq \mathrm{Prob}(\Pi(F_t)<|s-t|)\\
&\qquad+\sum_{K\geqq|s-t|} s_3^K \exp(-\mu K(\tfrac{1}{3}-\tfrac{1}{4}))\\
&\quad\leqq 1+C\exp\left(-\frac{\mu|s-t|}{24}\right).
\end{aligned}$$

Applying Lemma 1 of [3] we get

$$\begin{aligned}
&\mathbb{E}\exp\left(\frac{\mu}{4}\gamma_s^L\right)\\
&\quad\leqq \prod_{t\in\sigma_L}\left(1+C\exp\left(-\frac{\mu|s-t|}{24}\right)\right)\leqq K<\infty.
\end{aligned} \tag{2.21}$$

By Chebyshev's inequality

$$\begin{aligned}
\mathrm{Prob}(\gamma_s^L\geqq N)&=\mathrm{Prob}\left(\exp\frac{\mu}{4}\gamma_s^L\geqq \exp\frac{\mu}{4}N\right)\\
&\leqq K\exp\left(-\frac{\mu}{4}N\right). \quad \square
\end{aligned}$$

3. Nontranslation Invariant Gibbs State

3.1. Existence. Let us introduce the random variables S_t with values in $X=\{-1,0,+1\}: S_t=+1(-1)$ if there is an A (B)-particle at t, $S_t=0$ if there is no particle at t. Let P_{AB}^{L} be the Gibbs state in B_L with $A-B$ boundary condition. By a compactness argument there exists a sequence $L_n, L_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} P_{AB}^{L_n} = P_{AB}$ is a Gibbs state for the infinite system on $\mathbb{Z}^3$. This is the type of Gibbs state which we considered in (I.6). Let its finite dimensional distributions be denoted by $r(\hat{S}_t, t \in C) = \text{Prob}(S_t = \hat{S}_t, t \in C)$ where $\hat{S}_t \in X$, and C is a finite subset of $\mathbb{Z}^3$. Before proving new properties of P_{AB} we mention that we can also obtain P_{AB} in a different way. Let $\Lambda_L = \{x \in \mathbb{Z}^3 : |x^i| \leqq L, i = 1, 2, 3\}$. Let $r_{\Lambda_L}(\hat{S}_t, t \in C)$ be the finite dimensional distributions of the Gibbs state in Λ_L with $A-B$ boundary condition.

Proposition 3.1. *For μ large enough and any finite $C \subset \mathbb{Z}^3$ and any $\hat{S}_t \in X$, $t \in C$ we have*

$$r(\hat{S}_t, t \in C) = \lim_{L\to\infty} r_{\Lambda_L}(\hat{S}_t, t \in C) \qquad \textit{exists.} \tag{3.1}$$

The proof is the same as the proof of Theorem 3, Part I in [3].

3.2. Clustering Property and Extremality of P_{AB}. Let D_1 and D_2 be two finite subsets of $\mathbb{Z}^3$ and let $d(D_1, D_2) = \min_{s \in D_1, t \in D_2} |s-t|$.

Proposition 3.2. *There exist constants $\kappa > 0$ and $K < \infty$ such that for μ large enough and all D_1, D_2*

$$\sum_{\hat{S}_t, t \in D_1} \sum_{\hat{S}_p, p \in D_2} |r(\hat{S}_t, t \in D_1, \hat{S}_p, p \in D_2) - r(\hat{S}_t, t \in D_1) r(\hat{S}_p, p \in D_2)|$$
$$\leqq |D_1| \cdot |D_2| K \exp(-\mu\kappa d(D_1, D_2)). \tag{3.2}$$

Corollary 3.1. *The Gibbs state P_{AB} is extremal among the set of all Gibbs states.*

Proof of Corollary 3.1. It is sufficient to prove that for any local observable f we have

$$\mathbb{E}(f \langle f \rangle_{\Lambda_n}) \to (\mathbb{E}(f))^2, \qquad n \to \infty, \tag{3.3}$$

where Λ_n is a sequence of finite subsets of $\mathbb{Z}^3$ with $\Lambda_n \to \mathbb{Z}^3$, $n \to \infty$, and where $\langle f \rangle_{\Lambda_n}$ is the conditional expectation value of f given the configuration outside Λ_n. Indeed let us suppose that $P_{AB} = \int P_\alpha d\alpha$ where P_α are extremal Gibbs states α–a.s. Since the P_α are extremal Gibbs states we get by (3.3)

$$\lim_{n\to\infty} \mathbb{E}(f \langle f \rangle_{\Lambda_n}) = \lim_{n\to\infty} \int \mathbb{E}_{P_\alpha}(f \langle f \rangle_{\Lambda_n}) d\alpha$$
$$= \int (\mathbb{E}_{P_\alpha}(f))^2 d\alpha = (\mathbb{E}(f))^2 = \left(\int \mathbb{E}_{P_\alpha}(f) d\alpha\right)^2.$$

Therefore $\mathbb{E}_{P_\alpha}(f)$ is constant α–a.s. Since this is true for all f, $P_{AB} = P_\alpha$, α–a.s. and therefore P_{AB} is extremal. Now in our case we have finite range interaction and therefore $\langle f \rangle_{\Lambda_n}$ depends only on the values of the random variables S_x for x outside Λ_n and close to Λ_n. We obtain (3.3) using (3.2) with $D_1 = \Lambda$, $D_2 = \partial\Lambda_n$ where Λ is such that f depends only on the configuration inside Λ. $\square$

Proof of Proposition 3.1. It is sufficient to prove (3.2) for P_{AB}^{L} with K and κ independent of L. Let $\bar{\Delta} \in D_L$. We introduce the function $\chi(\hat{S}_t | \bar{\Delta})$ for every

$t\in\sigma_L:\chi(\hat{S}_t|\bar{\Delta})=1$ if any of the following conditions are satisfied: $S_t=\hat{S}_t=+1\,(-1)$, the cell t is occupied by an $A(B)$-particle and $t\in\partial\Delta$ or $\hat{S}_t=0$ and $t\in\Delta$ or $t\in S(\bar{\Delta})$. Otherwise $\chi(\hat{S}_t|\bar{\Delta})=0$. We put $\chi(\hat{S}_t, t\in C|\bar{\Delta})=\prod_{t\in C}\chi(\hat{S}_t|\bar{\Delta})$. Using this function we have

$$\begin{aligned} &r_L(\hat{S}_t, t\in C)\\ &=\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(\hat{S}_t, t\in C|\bar{\Delta})r_{\bar{\Delta}}(\hat{S}_t, t\in C\cap S(\bar{\Delta})),\end{aligned}\tag{3.4}$$

where $r_{\bar{\Delta}}(S_t, t\in C\cap S(\bar{\Delta}))$ is a finite dimensional distribution of the Gibbs state of the system enclosed in $S(\bar{\Delta})$ and with pure boundary condition for each connected component $S_i(\bar{\Delta})$ of $S(\bar{\Delta})$. The type, A or B, of boundary condition is given by the kind of particles in $\partial S_i(\bar{\Delta})$ (I.3). To simplify the notation we write $r(C|\bar{\Delta})$ for $r_{\bar{\Delta}}(\hat{S}_t, t\in C\cap S(\bar{\Delta}))$ and $\chi(C|\bar{\Delta})$ for $\chi(\hat{S}_t, t\in C|\bar{\Delta})$. Therefore we have

$$\begin{aligned} &\sum_{\hat{S}_t, t\in D_1\cup D_2} |r(\hat{S}_t, t\in D_1)r(\hat{S}_t, t\in D_2)-r(\hat{S}_t, t\in D_1\cup D_2)|\\ &=\sum_{\hat{S}_t, t\in D_1\cup D_2}\Big|\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(D_1\cup D_2|\bar{\Delta})r(D_1\cup D_2|\bar{\Delta})\\ &\quad -\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(D_1|\bar{\Delta})r(D_1|\bar{\Delta})\sum_{\bar{\Delta}'\in D_L} P_L(\bar{\Delta}')\chi(D_2|\bar{\Delta}')r(D_2|\bar{\Delta}')\Big|\\ &\leqq \sum_{\hat{S}_t, t\in D_1\cup D_2}\Big|\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(D_1\cup D_2|\bar{\Delta})(r(D_1\cup D_2|\bar{\Delta})-r(D_1|\bar{\Delta})r(D_2|\bar{\Delta}))\Big|\\ &\quad +\sum_{\hat{S}_t, t\in D_1\cup D_2}\Big|\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(D_1|\bar{\Delta})r(D_1|\bar{\Delta})\chi(D_2|\bar{\Delta})r(D_2|\bar{\Delta})\\ &\quad -\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(D_1|\bar{\Delta})r(D_1|\bar{\Delta})\sum_{\bar{\Delta}'\in D_L} P_L(\bar{\Delta}')\chi(D_2|\bar{\Delta}')r(D_2|\bar{\Delta}')\Big|.\end{aligned}\tag{3.5}$$

Using (A.2) of Appendix we have

$$\begin{aligned} &\sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\sum_{\hat{S}_t, t\in D_1\cup D_2} |\chi(D_1\cup D_2|\bar{\Delta})(r(D_1\cup D_2|\bar{\Delta})\\ &\quad -r(D_1|\bar{\Delta})r(D_2|\bar{\Delta}))|\leqq |D_1|\cdot K'\exp(-\mu\alpha' d(D_1, D_2)).\end{aligned}\tag{3.6}$$

Let us concentrate on the last term in (3.5). For all x of the projection $P(D_i)$, $i=1,2$, of D_i on the regular plane we choose a square $E_x(\delta)=\{y\in\sigma_L:|x-y|\leqq\delta\}$. Let

$$E_i=\bigcup_{x\in P(D_i)} E_x(\delta).$$

In σ_L we choose the total ordering so that the first points are in E_1 and the next ones in E_2. If $\bar{\Delta}=\bar{\Delta}(F_t, t\in\sigma_L)$ then we put $\bar{\Delta}_i=\bar{\Delta}_i(F_t, t\in E_i, \Lambda_u, u\in\sigma_L\backslash E_i)$, $i=1,2$. Let $\mathscr{E}$ be the set of groups of walls $(F_t, t\in\sigma_L)$ which are admissible and which satisfy the following condition

$$\begin{aligned} &\text{If } \mathrm{Int}(F_t)\cap E_1\neq\emptyset\Rightarrow\begin{cases}\mathrm{Int}\,F_t\cap E_2=\emptyset\\ P(F_t)\cap E_1\neq\emptyset,\end{cases}\\ &\text{If } \mathrm{Int}(F_t)\cap E_2\neq\emptyset\Rightarrow\begin{cases}\mathrm{Int}\,F_t\cap E_1=\emptyset\\ P(F_t)\cap E_2\neq\emptyset.\end{cases}\end{aligned}\tag{3.7}$$

If $P(F_t) \cap E_i \neq \emptyset$, then by our choice of the total ordering in σ_L we have that $t \in E_i$, $i=1,2$. If $(F_t, t \in \sigma_L) \in \mathscr{E}$, then $\bar{\Delta} \cap B_i = \bar{\Delta}_i \cap B_i$ where $B_i = \{x \in \mathbb{Z}^3 : p(x) \in E_i\}$, $i=1,2$. Indeed if $\bar{\Delta}$ and $\bar{\Delta}'$ are two interfaces such that for some $u \in \sigma_L$ the set of standard walls $\bar{W}$ of $\bar{\Delta}$ with $u \in \operatorname{Int} W$ coincide with the set of standard walls $\bar{W}'$ of $\bar{\Delta}'$ with $u \in \operatorname{Int} W'$, then $\{x \in \Delta : p(x) = u\} = \{x \in \Delta' : p(x) = u\}$ (I.4). By (A.2) of Appendix and since $d(D_i; S(\bar{\Delta}), S(\bar{\Delta}_i)) \geqq \delta$ we have

$$\sum_{\hat{S}_t, t \in D_i} |r(D_i|\bar{\Delta}) - r(D_i|\bar{\Delta}_i)| \leqq |D_i| K \exp(-\mu\bar{\alpha}\delta). \tag{3.8}$$

On the other hand it is easy to verify that the conditional finite dimensional distributions $\operatorname{Prob}(\eta_t^L = F_t, t \in C | \mathscr{E})$ satisfy (2.9) of Proposition (2.1). Using this fact and (3.8) we obtain the following upper bound for the last term in (3.5):

$$\tilde{K}|D_1||D_2| \exp(-\mu\tilde{\alpha}\delta) + K \sum_{s \in E_1} \sum_{p \in E_2} \exp(-\mu\kappa|s-p|) + 3P_L(D_L \backslash \mathscr{E}). \tag{3.9}$$

We need an estimate $p_L(D_L \backslash \mathscr{E})$. If $(F_t, t \in \sigma_L)$ does not belong to $\mathscr{E}$ but is admissible in B_L, then either there exists an F_t such that $p(F_t) \cap E_1 \neq \emptyset$ and $p(F_t) \cap E_2 \neq \emptyset$ or there exists and F_t such that $p(F_t) \cap E_i = \emptyset$ but $\operatorname{Int} F_t \cap E_i \neq \emptyset$, $i=1$ or 2. Therefore using Lemmas of (I.5) we get [see (I.6)],

$$p_L(D_L \backslash \mathscr{E}) \leqq |D_1|(2\delta+1)^2 \exp(-\alpha_2 \mu d(D_1, D_2)) + \delta^2 \hat{K}(|D_1| + |D_2|) \exp(-\alpha_1 \mu \delta).$$

Now we choose $\delta = \frac{1}{4} d(D_1, D_2)$ to finish the proof. □

3.3. Asymptotic Property of P_{AB}. Let C be a finite subset of $\mathbb{Z}^3$. Let $d_1(C) = \min_{t \in C} |t^1|$. Let $r(\hat{S}_t, t \in C)$ with $d_1(C)$ large and $t^1 > 0$ for all $t \in C$. We will prove that these distributions are well approximated by the corresponding distributions $r^A(\hat{S}_t, t \in C)$ of the pure phase A. Of course a similar statement holds with B instead of A if $t^1 < 0$ for all $t \in C$.

Proposition 3.3. *There exist two constants $\tilde{\alpha} > 0$ and $\tilde{K} < \infty$ such that for μ large enough, C a finite subset of $\mathbb{Z}^3$ with $t^1 > 0$, $t \in C$,*

$$\sum_{\hat{S}_t, t \in C} |r(\hat{S}_t, t \in C) - r^A(\hat{S}_t, t \in C)| \leqq \tilde{K} \exp(-\tilde{\alpha}\mu d_1(C)). \tag{3.10}$$

Proof. We prove (3.10) for P_{AB}^L with $\tilde{\alpha}$ and $\tilde{K}$ independent of L. Let

$$E = \bigcup_{x \in P(C)} \left\{C\left(x; \frac{d_1(C)}{2}\right) \cap \sigma_L\right\},$$

where $C(x, a) = \{y \in \mathbb{Z}^3 : |x-y| \leqq a\}$. Let $\mathscr{E}$ be the event:

$$\gamma_s^L(\bar{\Delta}) \leqq \frac{d_1(C)}{2}$$

for all $s \in E$.

$$\sum_{\hat{S}_t, t\in C} |r_L(\hat{S}_t, t\in C) - r_L^A(\hat{S}_t, t\in C)|$$

$$= \sum_{\hat{S}_t, t\in C} \left| \sum_{\bar{\Delta}\in D_L} P_L(\bar{\Delta})\chi(\hat{S}_t, t\in C|\bar{\Delta}) r(\hat{S}_t, t\in C|\bar{\Delta}) - r_L^A(\hat{S}_t, t\in C)\right|$$

$$\leqq P(\mathscr{E})|C|\bar{K} \exp\left(-\mu\bar{\alpha}\frac{d_1(C)}{2}\right) + 2P(\notin)$$

since if $\bar{\Delta} \in \mathscr{E}$, then

$$d(C; S(\bar{\Delta}), B_L) \geqq \frac{d_1(C)}{2}$$

[(A.2), Appendix]. Using Proposition 2.4 we get

$$P(\notin) \leqq |C| \left(\frac{d_1(C)}{2}\right)^2 K \exp\left(-\frac{\mu}{4}\frac{d_1(C)}{2}\right). \quad \square$$

4. Surface Tension

4.1. Definition. There are various microscopic definitions of the surface tension τ in the literature and it is not a priori obvious that they are equivalent. A discussion of this question can be found in [6], where it is shown that several different definitions do give the same answer for the two dimensional Ising model, i.e., the one computed by Onsager [7]. Here we use a definition of τ for the Widom Rowlinson model which is similar to the grand canonical surface tension in [6]. This definition appears to us natural and is particularly convenient for cases in which the two coexisting phases whose interfacial tension we desire are related by symmetry.

Let $Z_{L,M}^{AB}$ and $Z_{L,M}^{A}$ $(= Z_{L,M}^{B})$ represent the partition functions of our system in $A_{L,M}$ with $A-B$ and pure A (pure B) boundary condition respectively. Define

$$Q_{L,M} = \ln(Z_{L,M}^{AB}) - \ln(Z_{L,M}^{A}).$$

$Q_{L,M}$ is the difference in the Gibbs free energies between the systems with $A-B$ (mixed state) respectively A (pure state) boundary conditions. The surface tension per unit area is then defined as

$$-\lim_{L\to\infty}\lim_{M\to\infty}\frac{1}{(2L+1)^2} Q_{L,M} = \tau \tag{4.1}$$

when the limit exists. Using this definition we show that for μ large enough τ exists, is positive, and that $\frac{d\tau}{d\mu}$ can be expressed as an integral over correlation functions (Gibbs formula) [8]. While these results (even for the Ising model) are not contained in the work of Dobrushin, the elements of the proof are basically similar to what we have already discussed and therefore we shall be very brief. The analogous results for the Ising model at low temperature are stated at the end of this section.

4.2. Integral Representation of τ. Let us consider the system in the finite volume $A_{L,M}$. The hamiltonian is written as

$$H = -\mu \sum_{x \in A_{L,M}} (S_x^2 - 1).$$

Let

$$r_{L,M}^{\alpha}(x) = \langle 1 - S_x^2 \rangle^{\alpha} = \mathrm{Prob}\{S_x = 0\}$$

the probability being computed with $\alpha, \alpha = A$ or AB, boundary conditions. Clearly

$$\frac{d}{d\mu} Q_{L,M}(\mu) = \sum_{x \in A_{L,M}} [r_{L,M}^{A}(x) - r_{L,M}^{AB}(x)].$$

Let $x = (u, v)$ with $u = x^1$ be the vertical height and $v = p(x)$ the projection of x in the regular plane σ_L.

Lemma 4.1. *If μ is large enough*

a) $\lim_{M \to \infty} Q_{L,M}(\mu) = Q_L(\mu)$ *exists;*

b) $\lim_{M \to \infty} \frac{d}{d\mu} Q_{L,M}(\mu) = \sum_{v \in \sigma_L} \sum_{u \in \mathbb{Z}} (r_L^A(u, v) - r_L^{AB}(u, v))$

exists uniformly in μ, where

$$r_L^{\alpha}(x) = \lim_{M \to \infty} r_{L,M}^{\alpha}(x).$$

Proof. We know already that $\lim_{M \to \infty} Q_{L,M}$ exists by results of (I.3). Therefore we have to prove that $\frac{d}{d\mu} Q_{L,M}$ converges uniformly in μ to

$$\sum_{v \in \sigma_L} \sum_{u \in \mathbb{Z}} (r_L^A(u, v) - r_L^{AB}(u, v)).$$

Since the interface is rigid in B_L we expect that Proposition 3.3 is also true for the finite system in $A_{L,M}$ with $M \gg L$. The difficulty of proving this comes from the finiteness of the system: it is not always possible to remove a group of walls from an interface in $A_{L,M}$ and obtain in this way a new interface in $A_{L,M}$. Let $\delta_{L,M}$ be the interface in $A_{L,M}$. We first prove that if $M > L$ then

$$\sum_{\bar{\Delta} \in D_{A_{L,M}}} |\mathrm{Prob}\{\delta_L = \bar{\Delta}\} - \mathrm{Prob}\{\delta_{L,M} = \bar{\Delta}\}| \leq K L^2 \exp(-\bar{a}\mu M/L)$$

with $K < \infty$ and $\bar{a} > 0$ two constants. The proof is similar to the proof of Lemma 3 in [3]. Using this result it is easy to prove Proposition 3.3 for the system in $A_{L,M}$

with a correction term tending to zero exponentially fast when M tends to infinity. The proof of Lemma 4.1 follows then immediately by combining this result with Proposition 3.3 and using properties of the correlation functions in the pure phases. □

Proposition 4.1. *For μ large enough*

$$-\lim_{L\to\infty}\lim_{M\to\infty}\frac{1}{(2L+1)^2}Q_{L,M}(\mu)=\tau(\mu)$$

exists and is a C^1-function whose derivative is given by

$$\frac{d}{d\mu}\tau(\mu)=-\sum_{u\in\mathbb{Z}}(r^A(u,v=0)-r^{AB}(u,v=0)),\tag{4.2}$$

where $r^\alpha(x)=\lim_{L\to\infty}r^\alpha_L(x)$.

Proof. These results follow easily from Lemma 4.1 and the fact that for $L'>L$ and any x in B_L

$$|r^{AB}_L(x)-r^{AB}_{L'}(x)|\leqq K\exp(-\alpha\mu(L-|p(x)|)),$$

where $K<\infty$ and $\alpha>0$ are two constants.

This inequality is a direct consequence of Proposition 2.2 using an idea similar to that of the proof of Proposition 3.2. □

4.3. An Alternative Formula for the Surface Tension. In (I.5) we proved that

$$\exp[Q_L]=\sum_{\bar{\Delta}\in D_L}\exp\left(-\mu\sum_{t\in\sigma_L}\Pi(F_t(\bar{\Delta}))+\sum_{x\in\Delta}f_\mu(x,\Delta,L)\right)\exp(-\mu(2L+1)^2).\tag{4.3}$$

When μ tends to infinity the only term in $\exp[Q_L]\exp[\mu(2L+1)^2]$ which remains nonzero is that with $\bar{\Delta}=\bar{\Delta}_0$ the interface of the ground state. Therefore it is convenient to write $\exp Q_L$ as a product of

$$\exp[-(2L+1)^2b_L(\mu)]=\exp\left(\sum_{x\in\Delta_0}f_\mu(x,\Delta_0,L)\right)\cdot\exp(-\mu(2L+1)^2)\tag{4.4a}$$

and

$$\hat{Z}_L(\mu)=\sum_{\bar{\Delta}\in D_L}\exp\left(-\mu\sum_{t\in\sigma_L}\Pi(F_t(\bar{\Delta}))+\sum_{x\in\Delta}f_\mu(x,\Delta,L)-\sum_{x\in\Delta_0}f_\mu(x,\Delta_0,L)\right).\tag{4.4b}$$

Let us define $r^{AB}_{\bar{\Delta}_0}(u,v)$ as the probability that $S_{(u,v)}=0$ in the state which we obtain as the limit $L\to\infty$ of P^L_{AB} restricted to all configurations which give the interface $\bar{\Delta}_0$ of the ground state. It is then easy to show, using properties of the correlation functions in the pure phases that $\lim_{L\to\infty}b_L(\mu)=b(\mu)$ exists and is a C^1-function whose derivative is

$$\frac{d}{d\mu}b(\mu)=-\sum_{u\in\mathbb{Z}}(r^A(u,v=0)-r^{AB}_{\bar{\Delta}_0}(u,v=0)).\tag{4.5}$$

We now concentrate on (4.4b). Replacing there every $\Pi(F_t(\bar{\Delta}))$ by $\lambda\Pi(F_t(\bar{\Delta}))$ we get a function $\hat{Z}_L(\mu,\lambda)$ such that $\lim_{\lambda\to\infty}\hat{Z}_L(\mu,\lambda)=1$. Let us denote by $\langle\langle G\rangle\rangle_L(\lambda)$ the expectation value of a function G of $\bar{\Delta}$ computed with the probability obtained by replacing $\Pi(F_t)$ by $\lambda\Pi(F_t)$ in $P_L(\bar{\Delta})$ for every $t\in\sigma_L$.

We introduce now for every v in the regular plane the random variable n_v defined by

$$n_v+1 \quad \text{is the number of empty sites } y \text{ in the interface with } p(y)=v\,, \tag{4.6}$$

i.e., n_v+1 is the "thickness" of the interface above the horizontal position v.

Proposition 4.2. *If μ is large enough then*

i) $\tau(\mu)=b(\mu)-\mu a(\mu)$ *where $b(\mu)$ is differentiable and its derivative is given by* (4.5) *and*

$$a(\mu)=\lim_{L\to\infty}\int_1^\infty d\lambda \langle\!\langle n_0\rangle\!\rangle_L(\lambda)$$
$$=\int_1^\infty d\lambda \lim_{L\to\infty}\langle\!\langle n_0\rangle\!\rangle_L(\lambda)\,.$$

ii) *Both $\mu a(\mu)$ and $b(\mu)-\mu$ tend to zero exponentially fast when μ tends to infinity. In particular there exists a constant $K<\infty$ such that for all L*

$$\langle\!\langle n_0\rangle\!\rangle_L(\lambda)\leqq K\exp\left(-\frac{\mu\lambda}{3}\right).$$

The proof of Proposition 4.2 uses the same arguments as before and therefore we omit it.

4.4. Final Comments. i) Let $\langle\varrho(u)\rangle^\alpha=1-r^\alpha(u,v=0)$, be the density of particles at $(u,v=0)\in\mathbb{Z}^3$ in the A or $A-B$ phase. Formula (4.2) of Proposition 4.1 becomes

$$\frac{d\tau}{d\mu}=\sum_{u\in\mathbb{Z}}(\langle\varrho(u)\rangle^A-\langle\varrho(u)\rangle^{AB})\,. \tag{4.7}$$

This formula is essentially due to Gibbs [8].

ii) Let us now discuss briefly the analogous result for the Ising model. In this case the $+$ boundary condition corresponds to the A boundary condition of the Widom Rowlinson model and the $\pm$ boundary condition to the AB boundary condition. We define the microscopic energy density at x by

$$e_x=-\tfrac{1}{2}\sigma_x J\sum_{y:\|x-y\|=1}\sigma_y\,,$$

where $J>0$ is the ferromagnetic coupling constant and $\|\cdot\|$ the Euclidean distance. Let $\langle e_u\rangle^+$ and $\langle e_u\rangle^\pm$ be the expectation values of e_x at $x=(u,0,0)$. Then we can do exactly the same analysis as before and we obtain instead of (4.7) the formula

$$\frac{d\tau}{d\beta}=\sum_{u\in\mathbb{Z}}(\langle e_u\rangle^+-\langle e_u\rangle^\pm)$$

valid for large values of β, the inverse temperature.

iii) Concerning Proposition 4.2, we see that $\tau(\mu)$ is expressed as a sum of two terms. The first one $b(\mu)$ is the surface tension that we would obtain if we had a completely flat interface. This is the limit of $b_L(\mu)$; $(2L+1)^2 b_L(\mu)$ being the difference, when $M \to \infty$, between the free energy of a system with pure A b.c. in $\Lambda_{L,M}$ and the "same" system when it is divided in two by a "rigid horizontal wall" at $x^1=0$. The term $a(\mu)$ then takes account of the surface tension due to the deformations of the interface itself. Notice, however, that in the integral over λ we do not take the usual expectation value, since λ multiplies μ in front of $\Pi(F_t)$ and not in $f_\mu(x, \Delta, L)$.

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Appendix. The Pure Phases at High Activity

Let V_i be any finite subset of $\mathbb{Z}^3$. We consider only the Gibbs state of the system in V_i with pure boundary condition, say of kind A. The corresponding Gibbs random process is $(\eta_t^i, t \in V_i)$, $\eta_t^i \in X = \{-1, 0, +1\}$. The finite dimensional distributions of the process are $r_{V_i}(\hat{S}_t, t \in C) = \mathrm{Prob}(\eta_t^i = \hat{S}_t, t \in C)$ where $\hat{S}_t$ are fixed values in X.

Lemma A.1. *There exist a constant $K < \infty$ such that for μ large enough, $C \subset V_1 \cap V_2$*

$$\sum_{\hat{S}_t, t \in C} |r_{V_1}(\hat{S}_t, t \in C) - r_{V_2}(\hat{S}_t, t \in C)|$$

$$\leqq K \sum_{t \in C} \sum_{s \in V_1 \Delta V_2} \exp\left(-\frac{\mu}{2}|t-s|\right). \tag{A.1}$$

Remark. Let $d(t_1, V_1; t_2, V_2) = \sup\{d : V_1 \cap C(t_1, d)$ is congruent to $V_2 \cap C(t_2; d)\}$ (I.3). Let $d(C; V_1, V_2) = \min_{t \in C} d(t, V_1; t, V_2)$. Then there exist two constants $\alpha > 0$ and $\bar{K} < \infty$ such that for μ large enough

$$\sum_{\hat{S}_t, t \in C} |r_{V_1}(\hat{S}_t, t \in C) - r_{V_2}(\hat{S}_t, t \in C)|$$

$$\leqq |C| \bar{K} \exp(-\mu \bar{\alpha} d(C; V_1, V_2)).$$

Lemma A.2. *There exist two constants $\alpha' > 0$ and $K' < \infty$ such that for μ large enough, B and C finite subsets of V_1*

$$\sum_{\hat{S}_t, t \in B \cup C} |r_{V_1}(\hat{S}_t, t \in B \cup C) - r_{V_1}(\hat{S}_t, t \in B) r_{V_1}(\hat{S}_t, t \in C)|$$

$$\leqq |B| K' \exp(-\mu \alpha' d(B, C)). \tag{A.2}$$

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