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Global Properties of Radial Wave Functions in Schwarzschild's Space-Time

II. The Irregular Singular Point

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Abstract. Two solutions $\mathscr{R}_5(x, x_s)$ and $\mathscr{R}_6(x, x_s)$ related to the irregular singular point at $x = +\infty$ of the radial wave equation in Schwarzschild's space-time are studied as functions of the independent variable x and the parameter x_s. Analytic continuations of $\mathscr{R}_5$ and $\mathscr{R}_6$ are derived and their relation to the flat-space case solutions is established. Explicit expressions for $\mathscr{R}_3(x, x_s)$ and $\mathscr{R}_4(x, x_s)$ (the solutions about the regular singular point at $x = x_s$) are given. From these expressions and the analytic continuations of $\mathscr{R}_5$ and $\mathscr{R}_6$ the coefficients relating linearly $\mathscr{R}_5$ and $\mathscr{R}_6$ with $\mathscr{R}_i$ $(i = 1, 2, 3, 4)$ are calculated.

1. Introduction

The behavior of weak fields (scalar, electromagnetic or gravitational) around a Schwarzschild black hole is governed by a linear partial differential equation of second order. After separation of the angular variables and the time the resulting linear ordinary second order differential equation has two regular singular points at $x = 0$ and $x = x_s$ and one irregular singular point at $x = +\infty$. The solutions of this differential equation cannot be expressed in terms of any known function of mathematical physics and very few properties of them are known. Thus numerical analysis is introduced sooner or later in the study of wave phenomena around black holes. This situation has been presented in more detail in a previous paper [1], which hereafter will be refered to as paper I. The objectives set in that paper can be described briefly as follows:

(a) Find analytic continuations of the six solutions $\mathscr{R}_i$ $(i = 1, \ldots, 6)$ defined by their expansions at the singular points $x = 0$, $x = x_s$, and $x = +\infty$.

(b) Relate the solutions $\mathscr{R}_i$ of the curved-space case to the solutions of the flat-space case (the spherical Bessel functions).

(c) Determine the analytic expressions of the coefficients $K_{ij}(x_s)$ which relate linearly any three of the solutions $\mathscr{R}_i$.

In paper I we examined four solutions $\mathscr{R}_i$ $(i = 1, 2, 3, 4)$ defined by their converging power series expansions about the regular singular points $x = 0$ and $x = x_s$.

We gave analytic continuations outside the original circles of convergence, we proved that $x_s^l \mathscr{R}_i (i=1,2,3,4)$ becomes proportional to $j_l(x_s)$ when $x_s \to 0$ and we derived explicit expressions for $K_{ij}(i,j=1,2,3,4)$. In this paper we examine two more solutions $\mathscr{R}_5(x, x_s)$ and $\mathscr{R}_6(x, x_s)$ defined by their asymptotic power series expansions as $x \to +\infty$. The objectives are the same as for $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$, and $\mathscr{R}_4$. We are interested in analytic continuations of $\mathscr{R}_5$ and $\mathscr{R}_6$ away from $x=+\infty$, relations with the flat-space solutions and most important for the coefficients $K_{ij}(x_s)$. Since we have only asymptotic expansions for $\mathscr{R}_5$ and $\mathscr{R}_6$, we have to follow a method slightly different from the method followed for $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$, and $\mathscr{R}_4$ in paper I. Some results are easier to get, some more difficult. In Section 2 we find analytic continuations for $\mathscr{R}_5$ and $\mathscr{R}_6$ in the form of power series of x_s and prove that in the limit $x_s \to 0$ $\mathscr{R}_5$ and $\mathscr{R}_6$ reduce to $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ respectively. In Section 3 we derive expressions for the functions $X_n(y)$ defined in paper I. Using these expressions and the expansions of $\mathscr{R}_5$ and $\mathscr{R}_6$ in powers of x_s we calculate in Section 4 the quantities $K_{ij}(x_s)$. In Section 5 we present a few general remarks.

Throughout this paper it is assumed that the reader is familiar with the notation, the methods and the results of paper I. To avoid repetitions of formulas we frequently refer to equations of that paper writing the letter I in front of the number of the equation, e.g. (I.34) means the Equation (34) of paper I. The references given at the end of this paper should be supplemented by those of paper I, if a more complete study is sought.

2. The Solutions at $x = +\infty$

To study the solutions of the differential Equation (I.11) near $x=+\infty$ we follow the F-description. If we set

$$\mathscr{R}_5(x, x_s) = e^{i(x + x_s \ln|x - x_s|)} \mathscr{F}_5(x, x_s) \tag{1}$$

and

$$\mathscr{R}_6(x, x_s) = \mathscr{R}_5^*(x, x_s) = e^{-i(x + x_s \ln|x - x_s|)} \mathscr{F}_6(x, x_s) \tag{2}$$

with $\mathscr{F}_6(x, x_s) = \mathscr{F}_5^*(x, x_s)$, the function $\mathscr{F}_5(x, x_s)$ satisfies the differential Equation (I.13). Both (I.11) and (I.13) have an irregular singular point at $x=+\infty$. From (I.13) we can establish for $\mathscr{F}_5$ the asymptotic expansion

$$\mathscr{F}_5(x, x_s) \sim (-i)^{l+1} \sum_n \tau_n x^{-n-1} \tag{3}$$

with $\tau_0 = 1$ and

$$2in\tau_n + (l+n)(l-n+1)\tau_{n-1} + (n-1)^2 x_s \tau_{n-2} = 0\,. \tag{4}$$

From the recurrence relation (4) we observe that the highest power of x_s in the coefficients of x^{-1} and x^{-2} is x_s^0, in the coefficients of x^{-3} and x^{-4} is x_s, and so on. This is an indication that the following is true: If we expand $\mathscr{F}_5(x, x_s)$ in powers of x_s, then the coefficient of x_s^0 starts with a power x^{-1}, the coefficient of x_s starts with a power of x^{-3} and in general the coefficient of x_s^n starts with a power of x^{-2n-1}. To prove this and other properties rigorously we express $\mathscr{F}_5(x, x_s)$ as a contour integral.

Let $G(w, x_s)$ be the solution of the differential Equation [2]

$$w(w+2i)\, d^2G/dw^2 + (x_s w^2 + 2w + 2i)\, dG/dw + [x_s w - l(l+1)]G = 0\,, \tag{5}$$

which for $|w| < 2$ is given by the absolutely convergent power series

$$G(w, x_s) = \sum_n g_n w^n \tag{6}$$

with $g_0 = 1$ and

$$2in^2 g_n - (l+n)(l-n+1) g_{n-1} + (n-1) x_s g_{n-2} = 0\,. \tag{7}$$

If C is a contour in the complex w-plane surrounding clockwise the negative real axis, then

$$\mathscr{F}_5(x, x_s) = ((-i)^l/2\pi) \int_C G(w, x_s) \ln w \cdot e^{xw} dw \tag{8}$$

for every finite x with $\mathrm{Re}\, x > 0$.

The series (6) and its derivative with respect to w satisfy the requirements of Theorem A1 of paper I. Consequently they converge uniformly with respect to x_s in some neighborhood of $w=0$. Since g_n is an entire function of x_s, $G(w, x_s)$ and $dG(w, x_s)/dw$ are entire functions of x_s in some neighborhood of $w=0$. This property combined with the theorem of Section 2.2 of paper I implies that $G(w, x_s)$ *is an entire function of* x_s *for every finite* $w \neq -2i$. Thus we can set

$$G(w, x_s) = \sum_n G_n(w) x_s^n \tag{9}$$

into (8) and obtain

$$\mathscr{F}_5(x, x_s) = \sum_n Z_n(x) x_s^n \tag{10}$$

with

$$Z_n(x) = ((-i)^l/2\pi) \int_C G_n(w) \ln w \cdot e^{xw} dw\,. \tag{11}$$

Expressions (10) and (11) hold for any x_s. Hence $\mathscr{F}_5(x, x_s)$ *is an entire function of* x_s *for every* x *with* $\mathrm{Re}\, x > 0$. Expression (10) is an analytic continuation of $\mathscr{F}_5(x, x_s)$ away from $x = +\infty$. Adding an asterisk to $Z_n(x)$ we obtain an analytic continuation for $\mathscr{F}_6(x, x_s)$. If we multiply by the appropriate factor

$$\exp[\pm i(x + x_s \ln|x - x_s|)]\,,$$

we have analytic continuations of $\mathscr{R}_5$ and $\mathscr{R}_6$ for all $x \neq x_s$ with $\mathrm{Re}\, x > 0$.

The asymptotic expansion (3) can be obtained also from (6) and (8). To find the asymptotic behavior of $Z_n(x)$ as $x \to +\infty$ we need to know only the asymptotic behavior of $G_n(w)$ as $w \to 0$. From (5) and (9) we conclude that the sequence $G_n(w)$ $(n = 0, 1, \ldots)$ satisfies the differential equations

$$w(w+2i)\, d^2G_n/dw^2 + (2w+2i)\, dG_n/dw - l(l+1)G_n = -w^2\, dG_{n-1}/dw - wG_{n-1}\,. \tag{12}$$

Furthermore, since from (6) and (7)

$$G(0, x_s) = 1\,, \qquad dG(w, x_s)/dw|_{w=0} = -(i/2)l(l+1)\,, \tag{13}$$

we have from (9)

$$\left.\begin{array}{ll} G_0(0)=1\,, & dG_0(w)/dw|_{w=0}=-(i/2)l(l+1)\,, \\ G_n(0)=0\,, & dG_n(w)/dw|_{w=0}=0 \quad \text{for} \quad n>0\,. \end{array}\right\} \tag{14}$$

The differential Equation (12) and the conditions (14) determine uniquely the sequence $G_n(w)$ as given by the formulas

$$G_0(w)=P_l(1-iw)\,, \qquad G_1(w)=i(iw/2+\ln|1-iw/2|)P_l(1-iw) \tag{15}$$

and for $n>0$

$$\begin{aligned} G_n(w)=&\,iP_l(1-iw)\int_0^w wQ_l(1-iw)\,(d/dw)\,(wG_{n-1})\,dw \\ &-iQ_l(1-iw)\int_0^w wP_l(1-iw)\,(d/dw)\,(wG_{n-1})dw\,. \end{aligned} \tag{16}$$

From (15) and (16) we can prove by induction (as in Theorem B3 of paper I) that as $w\to 0$ we have for $n>0$

$$G_n(w)=\zeta'_n w^{2n}+o(w^{2n+1}\ln^n w)\,, \qquad dG_n(w)/dw=2n\zeta'_n w^{2n-1}+o(w^{2n}\ln^n w) \tag{17}$$

with

$$\zeta'_n=i^n(2n)!/2^{4n}(n!)^3\,. \tag{18}$$

Thus the integral (11) can be evaluated [3] to give

$$Z_n(x)=\zeta_n x^{-2n-1}+o(x^{-2n-1}) \tag{19}$$

with

$$\zeta_n=i^{n-l-1}[(2n)!]^2/2^{4n}(n!)^3\,. \tag{20}$$

Hence $Z_n(x)$ starts with a power x^{-2n-1} as $x\to+\infty$.

Substituting (10) into (I.13) we find that the sequence $Z_n(x)$ satisfies the differential equations ($n=0, 1, \ldots$)

$$\begin{aligned} &x^2(d^2Z_n/dx^2)+(2ix^2+2x)\,(dZ_n/dx)+[2ix-l(l+1)]Z_n \\ &=x(d^2Z_{n-1}/dx^2)+(dZ_{n-1}/dx)\,. \end{aligned} \tag{21}$$

For $n=0$ we have a homogeneous differential equation with solutions

$$Z^{(1)}(x)=Z_0(x) \quad \text{and} \quad Z^{(2)}(x)=e^{-2ix}Z_0^*(x)\,, \tag{22}$$

where

$$\begin{aligned} Z_0(x)&=e^{-ix}h_l(x) \\ &=\sum_{m=0}^{l}\left(i^{m-l-1}(l+m)!/2^m m!(l-m)!\right)x^{-m-1}\,. \end{aligned} \tag{23}$$

Combining (1), (2), (10) and (22) we have

$$\lim_{x_s\to 0}\mathscr{R}_5(x, x_s)=h_l^{(1)}(x) \quad \text{and} \quad \lim_{x_s\to 0}\mathscr{R}_6(x, x_s)=h_l^{(2)}(x)\,. \tag{24}$$

These equations relate the curved-space solutions $\mathscr{R}_5$ and $\mathscr{R}_6$ to the flat-space solutions $h_l^{(1)}$ and $h_l^{(2)}$ as given by (I.8) and (I.9).

For $n>0$ we can express Z_n in terms of the integral recurrence relation

$$Z_n(x) = -(i/2)Z_0 \int_{+\infty}^{x} Z_0^* \frac{d}{dx}\left(x\frac{dZ_{n-1}}{dx}\right)dx$$
$$+(i/2)e^{-2ix}Z_0^* \int_{+\infty}^{x} e^{2ix}Z_0 \frac{d}{dx}\left(x\frac{dZ_{n-1}}{dx}\right)dx\,, \tag{25}$$

where the limits of integration have been fixed by the requirement that $Z_n(x)$ behaves according to (19) as $x\to+\infty$. From (25) we can prove by induction that $Z_n(x)$ can be fully expanded into an asymptotic power series of x^{-1} term by term differentiable. Furthermore expression (25) can be used to evaluate (numerically for $n>0$) the coefficients $Z_n(x)$ in (10) in order to obtain an expansion of $\mathscr{F}_5(x, x_s)$ in powers of x_s at an arbitrary point x. Such a series can be used in the evaluation of $K_{ij}(x_s)$ as in Section 4.

3. Explicit Expressions for $\mathscr{R}_3$ and $\mathscr{R}_4$

To calculate the remaining K_{ij} as functions of x_s we work along the same lines as in Section 6 of paper I. However, a difficulty arises immediately. To calculate, e.g., $K_{35}(x_s)$ we attempt to use the expansions (I.72) and (10), but the coefficients $X_n(y)$ and $Z_n(x)$ depend on different variables and the method does not work. Thus we have to expand both $\mathscr{F}_3(x, x_s)$ and $\mathscr{F}_5(x, x_s)$ in powers of x_s or a more complicated series of functions of x_s with coefficients depending on the *same* variable x or y. In this section we will establish such an expansion for $\mathscr{F}_3(x, x_s)$ with coefficients depending on x only.

To derive such an expansion for $\mathscr{F}_3(x, x_s)$ the asymptotic properties of $X_n(y)$ presented in Theorem B5 of paper I are not sufficient. We need a full and explicit expression for $X_n(y)$. This is given by the following theorem, where p'_r and q'_r are defined by the expansions (I.B7) and (I.B8) written as

$$P_l(1-2y) = \sum_r p'_r y^{l-r}\,, \tag{26}$$

$$Q_l(1-2y) = \sum_r q'_r y^{-l-r-1} \quad \text{for} \quad y>1\,. \tag{27}$$

Theorem. *Under the assumptions of Theorem B5 of paper I, we have for* $n\geqq 0$ *and* $1<y<+\infty$ *the absolutely convergent series*

$$X_n(y) = \sum_r \sum_s A_{nrs} ln^s y \cdot y^{l+n-r} \tag{28}$$

with

$$A_{0r0} = (-1)^l p'_r\,, \qquad A_{0rs} = 0 \quad \textit{for} \quad s>0\,,$$
$$\left.\begin{aligned} A_{1r0} = {} & [1-2\sigma(l)]i(-1)^l p'_{r-1} + 2i(-1)^l q'_{r-2l-2} - i(-1)^l p'_r \\ & + i(-1)^l \sum_m p'_{r-m-2}/(m+1)\,, \end{aligned}\right\} \tag{29}$$
$$A_{1r1} = i(-1)^{l+1} p'_{r-1}\,, \qquad A_{1rs} = 0 \quad \textit{for} \quad s>1\,,$$

and for $n \geqq 2$

$$A_{nrs} = \sum_t 4i(q'_{r-t}B_{nts} - p'_{r-t}C_{nts}), \tag{30}$$

$$B_{nts} = \sum_u \sum_v (l-t+u+1)p'_{t-u}A_{n-1,uv}D_{2l+n-t,vs} - B_n\delta_s^0\delta_t^{2l+n+1}, \tag{31}$$

$$C_{nts} = \sum_u \sum_v (-l-t+u)q'_{t-u}A_{n-1,uv}D_{n-1-t,vs} - C_n\delta_s^0\delta_t^n, \tag{32}$$

$$B_n = \sum_t \sum_u \sum_v (l-t+u+1)p'_{t-u}A_{n-1,uv}D_{2l+n-t,v0}, \tag{33}$$

$$C_n = \sum_t \sum_u \sum_v (-l-t+u)q'_{t-u}A_{n-1,uv}D_{n-1-t,v0}, \tag{34}$$

$$D_{rst} = \begin{cases} (-1)^{s-t}s!/(r+1)^{s-t+1}t! & \textit{for} \quad r \neq -1 \quad \textit{and} \quad 0 \leqq t < s+1 \\ (s+1)^{-1} \quad \textit{for} \quad r=-1 \quad \textit{and} \quad t=s+1 \\ 0 \quad \textit{otherwise}. \end{cases} \tag{35}$$

Proof. For $n=0$ and $n=1$ we derive expression (28) with coefficients given by (29) from the explicit formulas (I.B47) and (I.B48). From (I.B49) we can prove that as $y \to 1$

$$x_n(y) = o[(y-1)\ln|y-1|] \quad \text{for} \quad n>0, \tag{36}$$

which is a stronger version of (I.B57). Using (36) we find from (I.B49) after factorial integration for $n \geqq 2$

$$X_n(y) = 4iQ_l \int_1^y yX_{n-1}(d/dy)(yP_l)\,dy - 4iP_l \int_1^y xX_{n-1}(d/dy)(yQ_l)\,dy. \tag{37}$$

(Note that the second integral in (37) diverges when $n=1$.) Let now (28) be true for $0, 1, \ldots, n-1$. Substituting into (37) the expression for $X_{n-1}(y)$ from (28) and P_l, Q_l, $(d/dy\,(yP_l)$, $(d/dy)(yQ_l)$ from (26) and (27) we arrive after some calculations to expression (28) for $X_n(y)$ with A_{nrs} defined by the relations (30)–(35). The constants D_{rst} have been introduced through the relation

$$\int_1^y y^r \ln^s y\,dy = \sum_t D_{rst}y^{r+1}\ln^t y - D_{rs0} \tag{38}$$

for r and s integers and $s \geqq 0$. The absolute convergence of (28) is a consequence of the fact that the calculations involve only multiplications and integrations of absolutely convergent series. Note that although for convenience the sums in (26) to (35) have been taken with infinite number of terms, many of them contain only a finite number of terms, because the higher order terms are identically zero. Thus, e.g., it can be proved by induction that

$$A_{nrs} = 0 \quad \text{for} \quad s>n \quad \text{or} \quad s>r. \tag{39}$$

We are ready now to expand $\mathscr{F}_3(x, x_s)$ in a series of functions of x_s with coefficients depending on x. Combining (I.72) and (28) we have

$$x_s^l\mathscr{F}_3(x, x_s) = \sum_n (X_n(y)/y^{l+n})\,x^{l+n} = \sum_n \sum_r \sum_s A_{nrs}x^{l+n-r}\ln^s y \cdot x_s^r. \tag{40}$$

But $y = x/x_s$,

$$\ln^s y = (\ln x - \ln x_s)^s = \sum_{t=0}^{s} ((-1)^t s!/t!(s-t)!) \ln^{s-t} x \cdot \ln^t x_s \tag{41}$$

and (40) becomes

$$x_s^l \mathscr{F}_3(x, x_s) = \sum_n \sum_r \sum_s \sum_t A_{nrst} x^{l+n-r} \ln^{s-t} x \cdot x_s^r \ln^t x_s , \tag{42}$$

where

$$A_{nrst} = \begin{cases} A_{nrs}(-1)^t s!/t!(s-t)! & \text{for} \quad 0 \leqq t \leqq s \\ 0 \quad \text{otherwise} . \end{cases} \tag{43}$$

Finally with A_l defined by (I.28) we have

$$(-1)^l A_l x_s^l \mathscr{F}_3(x, x_s) = \sum_r \sum_t Y_{rt}(x) x_s^r \ln^t x_s , \tag{44}$$

where

$$Y_{rt}(x) = (-1)^l A_l \sum_n \sum_s A_{nrst} x^{l+n-r} \ln^{s-t} x . \tag{45}$$

Expression (44) gives the desired expansion for $\mathscr{F}_3(x, x_s)$ and (45) defines the coefficients $Y_{rt}(x)$. Both (44) and (45) will be used in Section 4 to calculate $K_{ij}(x_s)$.

To find the behavior of $Y_{rt}(x)$ as $x \to +\infty$ we substitute (44) into (I.13) and find for $Y_{rt}(x)$ the differential equation

$$\begin{aligned} &x^2 \, d^2 Y_{rt}/dx^2 + (2ix^2 + 2x) \, dY_{rt}/dx + [2ix - l(l+1)] Y_{rt} \\ &= x \, d^2 Y_{r-1,t}/dx^2 + dY_{r-1,t}/dx . \end{aligned} \tag{46}$$

Because of (39) and (43) we have from (45)

$$Y_{rt}(x) \equiv 0 \quad \text{for} \quad r < t . \tag{47}$$

Thus Y_{tt} satisfies a homogeneous differential equation, which has $Z^{(1)}(x)$ and $Z^{(2)}(x)$ as two linearly independent solutions. Hence for $r > t$ we have

$$\begin{aligned} Y_{rt}(x) = &-\frac{i}{2} Z_0 \int_{x'_{rt}}^{x} Z_0^* \frac{d}{dx}\left(x \frac{dY_{r-1,t}}{dx}\right) dx \\ &+ \frac{i}{2} e^{-2ix} Z_0^* \int_{x''_{rt}}^{x} e^{2ix} Z_0 \frac{d}{dx}\left(x \frac{dY_{r-1,t}}{dx}\right) dx , \end{aligned} \tag{48}$$

where the lower limits of integration x'_{rt} and x''_{rt} must be determined so that the values of Y_{rt} and dY_{rt}/dx at a fixed point are the same if calculated from (45) or (48). The usefulness of (48) lies in the fact that independently of x'_{rt} and x''_{rt} we can show that for $r \geqq t$ we have as $x \to +\infty$

$$\left.\begin{aligned} &Y_{rt}(x) = O(x^{-1} \ln^{r-t} x) , \\ &dY_{rt}(x)/dx = O(x^{-1} \ln^{r-t} x) , \\ &d^2 Y_{rt}(x)/dx^2 = O(x^{-1} \ln^{r-t} x) . \end{aligned}\right\} \tag{49}$$

For $r=t$ the above relations are obvious since $Y_{tt}(x)$ is a linear combination of $Z^{(1)}(x)$ and $Z^{(2)}(x)$. If (49) holds for $Y_{r-1,t}(x)$ then using (23) and (48) we can prove easily that (49) is true for $Y_{rt}(x)$. The order relations (49) will be used in the next section to simplify the expressions for $K_{ij}(x_s)$.

4. The Quantities $K_{ij}(x_s)$

The main objective of this work is the determination of explicit expressions for $K_{ij}(x_s)$. This is so because the quantities $K_{ij}(x_s)$ relate through (I.18) any three solutions of the differential Equation (I.11) and appear in physical problems concerning the behavior of waves near and far from the black hole.

In flat space-time the constants corresponding to K_{ij} are simple and can be easily calculated from relations similar to (I.17). In curved space-time only K_{12}, K_{34} as given by (I.90) and $K_{56}=-2i$ have simple expressions. The remaining K_{ij} do not appear to have simple expressions. K_{13}, K_{14}, K_{23}, and K_{24} have been given in paper I in terms of power series of x_s [see expressions (I.95), (I.104), (I.116), and (I.117)]. The expressions which will be given in this section for the remaining $K_{ij}(x_s)$ are even more complicated, since they contain infinite sums of combinations of powers of x_s and $\ln x_s$.

To evaluate $K_{36}(x_s)$ we use (I.65) and (2). We have

$$K_{36}(x_s)=x(x-x_s)\,(\mathcal{R}_3\, d\mathcal{R}_5^*/dx-\mathcal{R}_5^*\, d\mathcal{R}_3/dx)$$
$$=x(x-x_s)e^{-ix_s}(\mathcal{F}_3\, d\mathcal{F}_5^*/dx-\mathcal{F}_5^*\, d\mathcal{F}_3/dx-2ix(x-x_s)^{-1}\,\mathcal{F}_3\mathcal{F}_5^*\,. \quad (50)$$

Substituting the expansions (10) and (44) into (50) we find after a few calculations that

$$K_{36}(x_s)=(-1)^l A_l^{-1} x_s^{-l} e^{-ix_s} \sum_n \sum_m \xi_{nm} x_s^n \ln^m x_s \quad (51)$$

with

$$\xi_{nm}=\sum_s (x^2 Y_{sm}\, dZ_{n-s}^*/dx-x^2(dY_{sm}/dx)Z_{n-s}^*)$$
$$-xY_{sm}\, dZ_{n-s-1}^*/dx+x(dY_{sm}/dx)Z_{n-s-1}^*-2ix^2 Y_{sm}Z_{n-s}^*)\,. \quad (52)$$

Because of (47) the summation starts essentially from $s=m$. Hence

$$\xi_{nm}=0 \quad \text{for} \quad n<m\,. \quad (53)$$

Since $K_{36}(x_s)$ is independent of x, each coefficient ξ_{nm} is independent of x, namely a constant number. This can be verified by direct differentiation of (52) with respect to x and use of (21) and (46). One method to find the numerical value of ξ_{nm} is to calculate each term in (52) at an arbitrary point from (25) and (45). Another method is based on a simplified expression for ξ_{nm}. From the asymptotic behavior of $Z_n(x)$ and $Y_{rt}(x)$ as given by (19) and (49) we find easily that

$$\xi_{nm}=\lim_{x\to+\infty}(x^2 Y_{nm}\, dZ^*/dx-x^2\, dY_{nm}/dx-2ix^2 Y_{nm}Z_0^*)\,. \quad (54)$$

Using (23) and (47) we can reduce (54) after some calculations to

$$\xi_{nm}=i^{-l-1}\int_{x'_{nm}}^{\infty} Z_0^* \frac{d}{dx}\left(x\frac{dY_{n-1,m}}{dx}\right)dx\,. \quad (55)$$

This equation suggests another method for evaluating ξ_{nm}.

For $K_{45}(x_s)$ we follow the same procedure. Since $\mathscr{F}_4=\mathscr{F}_3^*$ and $\mathscr{F}_5=\mathscr{F}_6^*$ we have $K_{45}=K_{36}^*$ and from (51)

$$K_{45}(x_s)=(-1)^l A_l^{-1} x_s^{-l} e^{ix_s}\sum_n\sum_m \xi_{nm}^* x_s^n \ln^m x_s\,, \tag{56}$$

where ξ_{nm}^* are the complex conjugates of ξ_{nm}. $K_{45}(x_s)$ is the most important of K_{ij}, since it appears in the definition of the interior solution [4].

To calculate $K_{35}(x_s)$ we use (I.65), (1), (10), and (44). We find

$$\begin{aligned}K_{35}(x_s)&=x(x-x_s)e^{i(2x-x_s+2x_s\ln|x-x_s|)}(\mathscr{F}_3\, d\mathscr{F}_5/dx-\mathscr{F}_5\, d\mathscr{F}_3/dx)\\&=(-1)^l A_l^{-1}x_s^{-l}e^{-ix_s}x(x-x_s)e^{2i(x+x_s\ln|x-x_s|)}\\&\quad\cdot\sum_n\sum_m\sum_s (Y_{sm}\,dZ_{n-s}/dx-Z_{n-s}\,dY_{sm}/dx)x_s^n \ln^m x_s\,.\end{aligned} \tag{57}$$

To expand the factor $\exp[2i(x+x_s\ln|x-x_s|]$ in powers of x_s we use the relation [5]

$$\ln^m(1-x_s/x)=m!\sum_n((-1)^n S_n^{(m)}/n!x^n)\,x_s^n\,, \tag{58}$$

where $S_n^{(m)}$ are the Stirling numbers of the first kind (by definition $S_n^{(m)}=0$ for $n<m$). We have

$$e^{2ix_s\ln|x-x_s|}=e^{2ix_s\ln x}\sum_m (2i)^m(m!)^{-1}x_s^m \ln^m(1-x_s/x) \tag{59}$$

and after a few calculations

$$x(x-x_s)e^{2i(x+x_s\ln|x-x_s|)}=\sum_n V_n(x)x_s^n\,, \tag{60}$$

where

$$V_n(x)=\sum_t\sum_s\frac{(-1)^{n-t-s}(2i)^{t+s}}{t!(n-t-s)!}[S_{n-t-s}^{(s)}+(n-t-s)S_{n-t-s-1}^{(s)}]e^{2ix}x^{t+s-n+2}\ln^t x\,. \tag{61}$$

Substituting into (57) we obtain

$$K_{35}(x_s)=(-1)^l A_l^{-1}x_s^{-l}e^{-ix_s}\sum_n\sum_m \xi'_{nm}x_s^n \ln^m x_s \tag{62}$$

with

$$\xi'_{nm}=\sum_r\sum_s V_{n-r}(Y_{sm}\,dZ_{r-s}/dx-Z_{r-s}\,dY_{sm}/dx)\,. \tag{63}$$

Because of (47) the summation starts with the $s=m$ term. Since $V_{n-r}=0$ for $n<r$ and $Z_{r-s}=0$ for $r<s$, we conclude that

$$\xi'_{nm}=0 \quad \text{for} \quad n<m\,. \tag{64}$$

The coefficients ξ'_{nm} are independent of x and can be obtained from (63) with the functions $Y_{sm}(x)$, $Z_{r-s}(x)$, $V_{n-r}(x)$ and their derivatives evaluated at an arbitrary point x.

To determine the behavior of K_{35}, K_{36}, K_{45}, and K_{46} near $x_s=0$ we observe that, since $\xi_{nm}=\xi'_{nm}=0$ for $n<m$ we have from (51) and (62) that as $x_s\to 0$

$$K_{35}(x_s)=(-1)^l A_l^{-1}x_s^{-l}e^{-ix_s}[\xi'_{00}+o(1)] \tag{65}$$

and

$$K_{36}(x_s)=(-1)^l A_l^{-1}x_s^{-l}e^{-ix_s}[\xi_{00}+o(1)]\,. \tag{66}$$

From (43) and (45) we find after some calculations that

$$Y_{00}(x) = e^{-ix} j_l(x), \tag{67}$$

which could also be obtained by taking the limit of (44) as $x_s \to 0$. From (61) we find

$$V_0(x) = x^2 e^{2ix} \tag{68}$$

The sums (52) and (63) have only the $r = s = 0$ terms which give

$$\xi_{00} = -i \quad \text{and} \quad \xi'_{00} = i. \tag{69}$$

Thus we obtain from (65) and (66) as $x_s \to 0$

$$K_{35}(x_s) = K^*_{46}(x_s) = ((-1)^l i / A_l x_s^l)\, [1 + o(1)] \tag{70}$$

and

$$K_{36}(x_s) = K^*_{45}(x_s) = ((-1)^{l+1} i / A_l x_s^l)\, [1 + o(1)]. \tag{71}$$

Following the same procedure we can determine analytically ξ_{11} and ξ'_{11} namely the coefficients of $x_s \ln x_s$ in the expansions (51) and (62). For higher order terms, however, the coefficients ξ_{nm} and ξ'_{nm} can be calculated only numerically through a computer.

The remaining K_{ij}, namely K_{15}, K_{16}, K_{25}, K_{26} can be calculated using the methods followed up to now for K_{35}, K_{36}, K_{45}, and K_{46}. This procedure requires the expansion of $\mathscr{F}_1(x, x_s)$ and $\mathscr{F}_2(x, x_s)$ in forms similar to (44). To avoid the complicated calculations and results associated with this method and since K_{15}, K_{16}, K_{25}, and K_{26} are not of much interest in physical problems, we will content ourselves with indirect expressions. Since the quantities K_{ij} satisfy the relations [2]

$$K_{ij} K_{kl} + K_{ik} K_{lj} + K_{il} K_{jk} = 0 \tag{72}$$

we have using $K_{34}(x_s)$ from (I.90)

$$K_{ij} = (i/2x_s^2)(K_{3i} K_{4j} - K_{4i} K_{3j}). \tag{73}$$

This relation for $i = 1, 2$ and $j = 5, 6$ expresses K_{15}, K_{16}, K_{25}, and K_{26} in terms of K_{ij} already calculated. Obviously K_{15}, K_{16}, K_{25}, and K_{26} as functions of x_s are of the same form as K_{35}, K_{36}, K_{45}, and K_{46}, namely infinite sums of terms of the form $x_s^n \ln^m x_s$.

5. General Remarks

With the results presented in this paper we complete a step towards a better understanding of the solutions of the radial wave equation in Schwarzschild's space-time. Since the questions set in the introduction arise in the studies of weak fields around black holes, it is expected that the answers given will be useful in establishing rigorously certain properties in black hole physics which up to now have been supported by numerical calculations only. In a future paper we will examine the effect of our results on the physically interesting solutions of the radial wave equation, namely the interior and exterior solutions. Applications on the "baldness" property and the stability problem of a Kerr black hole can also be considered.

It seems that the analytic continuations given for $\mathscr{R}_i$, the relations of $\mathscr{R}_i$ to the spherical Bessel or Hankel functions and the expressions derived for K_{ij} are all the informations we can get by analytical methods. Further progress is quite probably possible only through numerical analysis.

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