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Stationary Solutions of the Bogoliubov Hierarchy Equations in Classical Statistical Mechanics, 1

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Abstract. This paper is the first part of the work whose subject is to investigate the set of stationary solutions of *B-B-G-K-Y* hierarchy. We state that under some conditions on the interaction any stationary solution obeying certain restrictions of a general type corresponds to an equilibrium state (in the sense of Dobrushin-Lanford-Ruelle).

0. Introduction

The first mathematically rigorous works related to the theory of non-equilibrium phenomena appeared in Statistical Mechanics at the end of the sixties and the beginning of the seventies. Lanford was the first who has obtained interesting general results in this direction for the case of one-dimensional classical systems [1–2]. The main result of Lanford consists in the construction a natural dynamical system which describes the motion of an infinite number of interacting particles. The next important step was made by Sinai [3–4]. In particular, Sinai has given the rigorous proof of the cluster character of the dynamics for a system of particles in a gas phase. The results of Lanford and Sinai were generalized in the one dimensional case by Zemliakov [5] and Presutti, Pulvirenti and Tirozzi [6]. Combining the methods of Lanford and Sinai, Marchioro, Pellegrinotti and Presutti [7] constructed the dynamics in the multidimensional case for an infinite system in any possible thermodynamic phase.

In connection with the construction of dynamical systems of Statistical Mechanics the problem of studying their ergodic properties arises. For several particular cases this problem was solved in the papers [8–14]. Another problem which is closely connected with the preceding one is that of describing the set of measures, invariant with respect to the constructed dynamics of an infinite system of particles. The last problem was considered in the one-dimensional case in [15] where it was demonstrated that in a natural class of probability measures, defined on the phase space, only the equilibrium states may be invariant with respect to the dynamics constructed in [3].

In this paper we study the stationary solutions of the Bogoliubov hierarchy equations[1] [16]. It is well-known that for systems in a finite volume these equations are equivalent to the Liouville equation and characterize the time evolution of the probability measure on the phase space of a finite number of particles. Performing the thermodynamic limit one obtains the infinite chain of the Bogoliubov hierarchy equations which are related to a system of particles in the whole space. The problem of existence and uniqueness for this chain of equations has not been solved so far (some of the results obtained here are contained in [17–20]).

It is natural to connect the stationary solutions of the Bogoliubov hierarchy equations with the states of an infinite system of particles (i.e., probability measures defined on the phase space) which are invariant with respect to time evolution. In the cases where the dynamics on the phase space has been constructed it is possible to demonstrate that any invariant measure satisfying further conditions of a general type generates a stationary solution of the Bogoliubov hierarchy equations. On the other hand, an immediate analysis of stationary solutions of the Bogoliubov hierarchy equations (unlike the invariant measures) does not require, in general, the use of such delicate dynamical properties as clustering. Apparently, the point is that only functions of a finite (although not bounded) number of variables enter in the Bogoliubov hierarchy equations. One can consider these functions (the correlation functions) as integral characteristics of a measure and their behavior must not necessarily show the influence of singularities arising from the motion of individual configurations of infinitely large number of particles. Thus the approach based on the Bogoliubov hierarchy equations seems not only to be more general but also more natural from the physical point of view.

The main result of the present work consists in the description of all stationary solutions of the Bogoliubov hierarchy equations belonging to a certain class of functions of a finite (but increasing) number of variables[2]. This class corresponds to a set of Gibbs probability measures defined on the phase space. It is shown that any stationary solution from this class corresponds to an equilibrium state associated with the interaction potential appearing in the equations (the interaction between particles is supposed to be described by a finite-range pair potential which depends only on the distance between particles). Thus one can consider this result as a generalization to the multidimensional case of the results of [15].

For defining the equilibrium states we use the approach proposed by Dobrushin [22–24] and Lanford and Ruelle [25]. By definition, any equilibrium state is labelled by three parameters associated with the natural "integrals of motion" of an infinite system: mean energy, density and mean velocity of particles. From this point of view the main result of the present work appears as an assertion of the fact that for certain conditions all integrals of motion are exhausted by those mentioned before.

The class of Gibbs probability measures under consideration is defined in the same terms as the set of equilibrium states. More precisely, every Gibbs probability measure may be considered as an equilibrium state associated with some interaction of a general (possibly, multiparticle) type. Note that if we confine ourselves

[1] According to another terminology, *B-B-G-K-Y*-hierarchy equations.
[2] This result is announced in [21] under more restrictive conditions.

to the class of Gibbs probability measures we do not lose too much in generality: in fact, one can show (see [26]) that any probability measure satisfying sufficiently general and natural conditions is a Gibbs measure. Thus consideration of the class of Gibbs probability measures only is not the main restriction but it is rather a convenient framework.

The paper consists of four sections. The first one contains the preliminary information on the phase space, states, correlation functions and stationary solutions of the Bogoliubov hierarchy equations. In the second one we give the definition of Gibbs probability measures and formulate the results. The third section contains the proof of the first of two statements in to which the main theorem is divided up. The last, fourth section contains the proofs of a series of auxiliary lemmas.

The statement which makes the second part of the main theorem will be proved in the further papers.

1. Preliminaries

Phase Space. The *phase space* $M=M(R^\nu)$ of a system of particles in R^ν is defined as the set of all finite or countable sets X consisting of pairs (q,v), $q\in R^\nu$, $v\in R^\nu$, and satisfying the following conditions: (a) if $(q,v)\in X$ then $q\neq q'$ for any other $(q',v')\in X$, (b) the set $\{q:(q,v)\in X\}\cap C$ is finite for any compact $C\subset R^\nu$. The vectors q and v are interpreted respectively as the coordinate and velocity of a single particle. For any Borel set $\Omega\subset R^\nu$ define the phase space $M(\Omega)$ of a system of particles in Ω by

$$M(\Omega)=\{X\in M: q\in\Omega \quad \text{for all} \quad (q,v)\in X\}.$$

If Ω is bounded then $M(\Omega)$ consists, of course, only of finite X and may be represented as $M(\Omega)=\bigcup_{n=0}^{\infty} M_n(\Omega)$, where $M_n(\Omega)$ is the phase space of a system of n particles in Ω. The space $M_0(\Omega)$ contains only one element (vacuum) corresponding to the absence of particles in Ω. We denote it by the symbol $\emptyset$. Denote by $(M_1(\Omega)^n)_{\neq}$ the subset of the Cartesian product $M_1(\Omega)^n=(\Omega\times R^\nu)^n$ consisting of all points $((q_1,v_1),(q_2,v_2),\ldots,(q_n,v_n))$, $q_i\in\Omega$, $v_i\in R^\nu$, such that $q_i\neq q_j$ for any $i\neq j$, $i,j=1,\ldots,n$. The space $M_n(\Omega)$ for $n\geqq 1$ is the image of $M_1(\Omega)^n$ under the symmetrization mapping S_n[3]. This mapping realizes an isomorphism of the σ-algebra generated by the symmetric Borel subsets of $(M_1(\Omega)^n)_{\neq}$ and a σ-algebra of subsets of the space $M_n(\Omega)$ which will be denoted by $\mathfrak{C}_n(\Omega)$. For bounded Ω define the σ-algebra

$$\mathfrak{C}(\Omega)=\{A\subseteqq M(\Omega): A\cap M_n(\Omega)\in\mathfrak{C}_n(\Omega)\,,\ n=0,1,\ldots\}.$$

On $\mathfrak{C}(\Omega)$ define the measure λ by

$$\lambda(A)=\sum_{n=1}^{\infty}\frac{1}{n!}m_n(s_n^{-1}[A\cap M_n(\Omega)]),\ \lambda(M_0(\Omega))=1\,, \tag{1.1}$$

where m_n is Lebesgue measure on $M_1(\Omega)^n$ [clearly, $m_n(M_1(\Omega)^n\setminus(M_1(\Omega)^n)_{\neq})=0$].

[3] The mapping S_n is the identification of all $n!$ points of $M_1(\Omega)^n$ belonging to the same image of the permutation group of order n acting on $M_1(\Omega)^n$.

Define the restriction mapping $\pi_\Omega: M \to M(\Omega)$, $\Omega \subseteqq R^\nu$, by

$$\pi_\Omega X = X_\Omega = \{(q, v) \in X : q \in \Omega\}, \quad X \in M. \tag{1.2}$$

For any bounded Borel set Ω denote by $\mathfrak{B}(\Omega)$ the σ-algebra of subsets of the space M which may be represented in the form $\pi_\Omega^{-1} A$, where $A \in \mathfrak{C}(\Omega)$. For any Borel set Ω denote by $\mathfrak{B}(\Omega)$ the smallest σ-algebra containing $\mathfrak{B}(\Omega')$ for any bounded Borel $\Omega' \subseteqq \Omega$ and set for brevity $\mathfrak{B}(R^\nu) = \mathfrak{B}$. It is not hard to verify that the collection of sets $\{\pi_\Omega B : B \in \mathfrak{B}(\Omega)\}$ is a σ-algebra which coincides with $\mathfrak{C}(\Omega)$ for bounded Ω; in the general case we denote it by the same symbol. Clearly, the σ-algebras $\mathfrak{C}(\Omega)$ and $\mathfrak{B}(\Omega)$ are isomorphic (and for $\Omega = R^\nu$ coincide).

The mapping $X \to (X_\Omega, X_{\Omega^c})$[4] [see (1.2)] generates an isomorphism $M \cong M(\Omega) \times M(\Omega^c)$ and a corresponding isomorphism of the σ-algebras:

$$\mathfrak{B} \cong \mathfrak{C}(\Omega) \times \mathfrak{C}(\Omega^c).$$

Now consider the subset $M^0 = M^0(R^\nu) \subset M$ consisting of finite X's. It is not hard to see that $M^0 \in \mathfrak{B}$ and $M(\Omega) \subset M^0$ for any bounded $\Omega \subset R^\nu$. Furthermore, it is obvious that $M^0 = \bigcup_{n=0}^{\infty} M_n$ where $M_n = M_n(R^\nu)$ is the phase space of a system of n particles in R^ν. As before, M_0 contains the unique element $\emptyset$. For $n \geqq 1$ the space M_n is the image of $(M_1^n)_{\neq}$ under the mapping S_n. For points of the space M^0 and particularly for those of $M(\Omega)$ for bounded Ω we shall use the notations $\bar{x}$, $\bar{y}$, $\bar{z}$, etc.; for points belonging to M_1 the bar will be omitted. The same symbols and also X, Y, X_Ω, X_{Ω^c}, etc. will denote, according to the situation, the sets of pairs of vectors (q, v). This convention includes also the meaning of the symbol $\emptyset$. In particular, the notation $\sum_{\bar{x} \in M^0, \bar{x} \subset X, \bar{x} \neq \emptyset}$ indicates the summation over all finite non-empty subsets of the set X; for that the inclusion $\bar{x} \in M^0$ will be omitted.

The formula (1.1) replacing $M_n(\Omega)$ by M_n defines on M^0 a measure which will be denoted by λ as before. For $n \geqq 1$ the space M_n is provided with the natural topology which is induced by the topology of the euclidean space M_1^n. The space M^0 is provided with the topology of the topological union.

We say that a function f defined on a subset $A \subseteqq M^0$ belongs to the class C^k at a point $\bar{x} \in A \cap M_n$, $k = 0, 1, \ldots, n = 1, 2, \ldots$, if there exists a set $W \subseteqq A \cap M_n$ such that $\bar{x} \in W$, $s_n^{-1} W$ is open in M_1^n and $s_n^* f(\cdot) = f(s_n^{-1}(\cdot))$ is a function of the class C^k on $s_n^{-1} W$. Let f be a function of the class C^1 defined on $A \subseteqq M^0$. On the set of pairs $(\bar{x}, x)$ where $\bar{x} \in A$, $x = (q, v) \in \bar{x}$, we shall define two vector-valued functions with values in R^ν which will be denoted respectively by $\nabla_q f(\bar{x})$ and $\nabla_v f(\bar{x})$ and called the gradients of f. For let us fix the set $\bar{x} \setminus x$ and consider the function $f_{\bar{x} \setminus x}(y) = f((\bar{x} \setminus x) \cup y)$ where $y = (q', v')$ is any point of M_1 for which $(\bar{x} \setminus x) \cup y \in A$. It is clear that $f_{\bar{x} \setminus x}(y)$ is a function of the class C^1 at the point $y = x$. This gives us the possibility to set by definition

$$\nabla_q f(\bar{x}) = \nabla_{q'} f_{\bar{x} \setminus x}(y)|_{y=x}, \quad \nabla_v f(\bar{x}) = \nabla_{v'} f_{\bar{x} \setminus x}(y)|_{y=x}.$$

States and Correlation Functions. Now turn to the space M and the σ-algebra $\mathfrak{B}$.

Definition 1.1. A state of a system of particles in R^ν (or, briefly, a state) is any probability measure defined on $\mathfrak{B}$.

[4] $\Omega^c = R^\nu \setminus \Omega$.

Consider the restriction of the state P to the σ-algebra $\mathfrak{B}(\Omega)$. Then by the isomorphism between $\mathfrak{B}(\Omega)$ and $\mathfrak{C}(\Omega)$ we obtain a system of consistent probability measures $\{P_\Omega, \Omega \subseteqq R^\nu\}$ on the σ-algebras $\{\mathfrak{C}(\Omega), \Omega \subseteqq R^\nu\}$. Conversely, if for every bounded Borel set $\Omega \subset R^\nu$ a probability measure P_Ω is defined on the σ-algebra $\mathfrak{C}(\Omega)$ and the condition of consistency holds, then there exists the unique state P on $\mathfrak{B}$ which generates the sytem of measures $\{P_\Omega, \Omega \subset R^\nu\}$ (the Kolmogorov theorem).

Below we shall restrict ourselves to consideration of those states P which are locally absolutely continuous with respect to λ (i.e., $P_\Omega \prec \lambda$ for any bounded Borel set Ω). The Radon-Nikodym derivative $dP_\Omega/d\lambda$ is denoted by P_Ω and called the *local density* of the state P. For any measurable function $f(\bar{x})$, $\bar{x} \in M(\Omega)$, the following equality holds

$$\int\limits_{M(\Omega)} f(\bar{x}) p_\Omega(\bar{x}) d\lambda(\bar{x}) = \int\limits_{M} (\pi_\Omega^* f)(X) dP(X), \tag{1.3}$$

both integrals existing or not existing simultaneously.

Let P be a state of a system of particles in R^ν. For any Borel set $A \subset M^0$ let

$$K_P(A) = \int\limits_M \sum_{\bar{x} \subset X} \chi_A(\bar{x}) dP(X).$$

This formula defines on M^0 a measure K_P, taking its values in $[0, +\infty]$. The measure K_P is called the *correlation measure* of the state P.

Definition 1.2. The Radon-Nikodym derivative (if it exists)

$$\varrho_P(\bar{x}) = \frac{dK_P}{d\lambda}(\bar{x}), \qquad \bar{x} \in M^0, \tag{1.4}$$

is called the *correlation function* of the state P.

A sufficient condition for the existence of correlation function (in the case of a locally absolutely continuous state P) is the σ-finiteness of the measure K_P. The correlation function is expressed in terms of local density by

$$\varrho_P(\bar{x}) = \int\limits_{M(\Omega)} p_\Omega(\bar{x} \cup \bar{y}) d\lambda(\bar{y}), \tag{1.5}$$

where Ω is any bounded Borel set such that $\bar{x} \in M(\Omega)$. The converse formula

$$p_\Omega(\bar{x}) = \int\limits_{M(\Omega)} (-1)^{n(\bar{y})} \varrho_P(\bar{x} \cup \bar{y}) d\lambda(\bar{y}), \qquad \bar{x} \in M(\Omega) \tag{1.6}$$

is true if for any bounded Borel $\Omega \subset R^\nu$

$$\int\limits_{M(\Omega)} 3^{n(\bar{y})} p_\Omega(\bar{y}) d\lambda(\bar{y}) < \infty.$$

[cf. [29]]. Here

$$n(\bar{y}) = n \quad \text{if} \quad \bar{y} \in M_n, \qquad n = 0, 1, \ldots. \tag{1.7}$$

Interaction of Particles. Bogoliubov Hierarchy Equations. Everywhere below we suppose that the interaction of particles is given by a pair potential $U(r)$, $0 \leqq r < \infty$, which satisfies the following conditions:

(I_1) $U(r) \equiv +\infty$ for $0 \leqq r \leqq d_0$ where $d_0 > 0$ (the hard-core condition),

(I_2) $U(r)$ is a function of the class C^3 for $r > d_0$,

(I_3) $\lim_{r \to d_0} U(r) = +\infty$ and $\lim_{r \to d_0^+} d/dr \exp[-cU(r)] = 0$ for any $c > 0$,

(I_4) $U(r) = 0$ for $r \geqq d_1$ where $d_1 > d_0$ (the locality condition).

We say that the correlation function ϱ_P of a state P is the *stationary solution* of the Bogoliubov hierarchy equations if at each point of the set

$$D^0 = \bar{x} \in M^0 \setminus M_0 : \min_{q, q' \in \bar{x}, q \neq q'} |q - q'| > d_0 \quad [5] \tag{1.8}$$

ϱ_P is a function of the class C^1 [6] and satisfies the equation

$$\{\varrho_P(\bar{x}), H(\bar{x})\} + \int_{M_1 \cap D^0(\bar{x})} \{\varrho_P(\bar{x} \cup y), U(\bar{x}|y)\} dy = 0, \tag{1.9}$$

where $\{\ ,\ \}$ denotes the Poisson brackets, $y = (q, v) \in M_1$, $dy = dm_1(y) = dq dv$, $H(\bar{x})$ is the classical Hamiltonian corresponding to the interaction potential $U(r)$:

$$H(\bar{x}) = \tfrac{1}{2} \sum_{v \in \bar{x}} \langle v, v \rangle + U(\bar{x}) \tag{1.10}$$

(here and below $\langle\ ,\ \rangle$ denotes the inner product in R^ν),

$$U(\bar{x}) = \sum_{q', q'' \in \bar{x}, q' \neq q''} U(|q' - q''|) \tag{1.11}$$

is the potential energy of the system of particles with the coordinates $q' \in \bar{x}$,

$$U(\bar{x}|y) = \sum_{q' \in \bar{x}} U(|q - q'|) \tag{1.12}$$

is the potential energy of the interaction of the particle with the coordinate q and a system of particles with the coordinates $q' \in \bar{x}$ and finally

$$D^0(\bar{x}) = \{\bar{y} \in D^0 : \min_{q' \in \bar{x}, q'' \in \bar{y}} |q' - q''| > d_0\}. \tag{1.13}$$

In conclusion of this section we formulate a simple auxiliary statement which can be easily deduced from the definition of the measure λ [see (1.1).] and will be used repeatedly in future.

Lemma 1.1. *Let $\Phi(\bar{x}, y)$, $\bar{x} \in M^0$, $y \in M_1$, be a measurable function of two variables. Then*

$$\int_{M_1} \int_{M^0} \Phi(\bar{x} \cup y, y) d\lambda(\bar{x}) dy = \int_{M^0 \setminus M_0} \sum_{y \in \bar{x}} \Phi(\bar{x}, y) d\lambda(\bar{x})$$

if the integral in the left hand side converges absolutely.

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[5] The notations $q \in \bar{x}$ and $v \in \bar{x}$ will be used often instead of $(q, v) \in \bar{x}$.

[6] Since the function ϱ_P is defined by the equality (1.4) only almost everywhere (w.r.t. the measure λ), one must find, of course, the smooth version of the correlation function.

2. Gibbs Random Fields. Formulation of Results

Let f be a measurable real-valued function on M^0 which can also take the value $+\infty$ and has the following properties: a) $f(\emptyset)=0$, b) if $f(\bar{x})=+\infty$ and $\bar{x}\subseteqq\bar{y}$, then $f(\bar{y})=+\infty$. For any $\bar{x},\bar{y}\in M^0$, $\bar{x}\cap\bar{y}=\emptyset$ we put

$$h(\bar{x})=\sum_{\bar{z}\subseteq\bar{x}} f(\bar{z}),$$

$$h(\bar{x}|\bar{y})=\begin{cases}\displaystyle\sum_{\bar{z}\subseteq\bar{x}\cup\bar{y},\,\bar{z}\cap\bar{x}\neq\emptyset,\,\bar{z}\cap\bar{y}\neq\emptyset} f(\bar{z}), & \bar{x}\neq\emptyset, \quad \bar{y}\neq\emptyset,\\ 0, \quad \bar{x}=\emptyset \text{ or } \bar{y}=\emptyset. \end{cases}$$

Definition 2.1. (Dobrushin [21–23], Lanford-Ruelle [24], Kozlov [26]). A state P is called a *Gibbs random field (Gibbs measure)* with the *generating function* f, if for any bounded Borel set $\Omega\subset R^\nu$ the following conditions hold:

1) for almost all (w.r.t. the measure $\lambda\times P$) pairs $(\bar{x},X)\in M(\Omega)\times M$ there exists the limit (being finite or equal to $+\infty$)

$$h(\bar{x}|X_{\Omega^c})=\lim_{\bar{x}'\to X_{\Omega^c}} h(\bar{x}|\bar{x}')$$

where $\bar{x}'\to X_{\Omega^c}$ denotes the net of the finite subsets $\bar{x}'\subset X_{\Omega^c}$ ordered by inclusion,

2) for almost all (w.r.t. the measure P) $X\in M$ there exists the conditional probability measure $[P(\cdot/\mathfrak{B}(\Omega^c))](X)$ defined on the σ-algebra $\mathfrak{B}(\Omega)$ or, equivalently (see Section 1) — on the σ-algebra $\mathfrak{C}(\Omega)$. The last measure is absolutely continuous with respect to the measure λ on $\mathfrak{C}(\Omega)$ and

$$\frac{d[P(\cdot/\mathfrak{B}(\Omega^c))]\,(X)}{d\lambda}(\bar{x})\equiv p_\Omega(\bar{x};X_{\Omega^c})=\Xi^{-1}\exp[-h(\bar{x})-h(\bar{x}|X_{\Omega^c})]^7,\ \bar{x}\in M(\Omega),$$

where $\Xi=\Xi_\Omega(X_{\Omega^c})$ depends on X_{Ω^c} (and on Ω), but does not depend on $\bar{x}\in M(\Omega)$.

From the Definition 2.1 one can derive that for almost all (w.r.t. P) $X\in M$

$$0<\Xi_\Omega(X_{\Omega^c})=\int_{M(\Omega)}\exp[-h(\bar{x})-h(\bar{x}|X_{\Omega^c})]d\lambda(\bar{x})<\infty\,. \tag{2.1}$$

The function $p_\Omega(\bar{x};X_{\Omega^c})$ is called the conditional local density of the Gibbs random field P. The local density p_Ω introduced in Section 1, is connected with the conditional local density by

$$p_\Omega(\bar{x})=\int_M p_\Omega(\bar{x};X_{\Omega^c})dP(X),\qquad \bar{x}\in M(\Omega)\,. \tag{2.2}$$

For any Borel set $\Omega\subseteqq R^\nu$ let us denote by P^Ω the restriction of a Gibbs measure P on the subset $M(\Omega)\subseteqq M$. The measure P^Ω is defined on the σ-algebra $\mathfrak{C}(\Omega)$, but, unlike P_Ω, it is in general not a probability measure. Now let Ω be a bounded set. The double conjugate map of the map $M\to M(\Omega)\times M(\Omega^c)$ transforms the measure P into a measure P' on $M(\Omega)\times M(\Omega^c)$. It is not hard to verify that the measure P' is absolutely continuous with respect to the direct product $\lambda\times P^{\Omega^c}$, and

$$\frac{dP'}{d(\lambda\times P^{\Omega^c})}(\bar{x},Y)=\Xi_\Omega(Y)p_\Omega(\bar{x};Y),\qquad \bar{x}\in M(\Omega),\qquad Y\in M(\Omega^c)\,. \tag{2.3}$$

7 We set $\exp(-\infty)=0$.

From this fact and from (1.4), (2.1) one can easily derive the following formulae for local density and correlation function

$$p_\Omega(\bar{x}) = \int_{M(\Omega^c)} \exp[-h(\bar{x}) - h(\bar{x}|Y)]dP(Y), \qquad \bar{x} \in M(\Omega), \tag{2.4}$$

$$\varrho_P(\bar{x}) = \int_M \exp[-h(\bar{x}) - h(\bar{x}|Y)]dP(Y) \tag{2.5}$$

where

$$h(\bar{x}|Y) = \lim_{\bar{y}\to Y} h(\bar{x}|\bar{y}).$$

There are general sufficient conditions for a state P to be a Gibbs random field (see [26]). In particular, if P is an l-markovian[8] and conditionally locally absolutely continuous (w.r.t. λ)[9] measure on M and if for any bounded Borel set $\Omega \subset R^\nu$

$$P(\pi_\Omega^{-1}\emptyset/\mathfrak{B}(\Omega^c)) > 0$$

almost everywhere on M, then P is a Gibbs random field and its generating function $f(\bar{x})$ vanishes if $\max\limits_{q,q'\in\bar{x}} |q-q'| > l$.

We consider the Gibbs random fields whose generating functions satisfy the following conditions:

(G_1) $\{\bar{x} \in M^0 : f(\bar{x}) < +\infty\} = D^0$ [see (1.8)];

(G_2) a) the function f belongs to the class C^2 at each point $\bar{x} \in D^0$,

b) for each $n = 1, 2, \ldots$ the functions $\exp[-f(\bar{x})]$ and $\max\limits_{(q,v)\in\bar{x}} [|\nabla_q \exp[-f(\bar{x})]| + |\nabla_v \exp[-f(\bar{x})]|]$ are bounded on $M_n \cap D^0$;

c) let $f(q, v)$ be the restriction of f to M_1, then $\lim\limits_{|v|\to\infty} \exp[-f(q, v)] = 0$ for any $q \in R^\nu$, and the integrals

$$\int_{R^\nu} |v| \exp[-f(q,v)]dv, \quad \int_{R^\nu} |v||\nabla_q \exp[-f(q,v)]|dv, \quad \int_{R^\nu} |\nabla_v \exp[-f(q,v)]|dv$$

converge and are bounded uniformly over $q \in R^\nu$;

(G_3) there exists $n_0 \geqq 2$ such that $f(\bar{x}) \equiv 0$ for $\bar{x} \in D^0$ and $n(\bar{x}) > n_0$(i.e., the number of "interacting" particles is finite);

(G_4) for any (q, v), $(q', v') \in M_1$ such that $|q - q'| = d_0$

$$\begin{aligned}\lim_{q''\to q'} & \exp[-f((q,v)\cup(q'',v'))] \\ &= \lim_{q''\to q'} |\nabla_q \exp[-f((q,v)\cup(q'',v'))]| \\ &= \lim_{q''\to q'} |\nabla_{q''} \exp[-f((q,v)\cup(q'',v'))]| = 0\,;\end{aligned}$$

(G_5) there exists a constant $d_2 \geqq d_0$ such that if $\bar{x} \in D^0$, $n(\bar{x}) > 1$ and diam $\bar{x} > d_2$ then

$$|f(\bar{x})| \leqq \Psi(\operatorname{diam}\bar{x}), \quad \max_{(q,v)\in\bar{x}} [|\nabla_q f(\bar{x})| + |\nabla_v f(\bar{x})|] \leqq \Psi(\operatorname{diam}\bar{x}),$$

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8 A measure P is called l-markovian ($l > 0$) if for any $\Omega \subset R^\nu$ and $B \in \mathfrak{B}(\Omega)$ with the probability 1(w.r.t. P) $P(B/\mathfrak{B}(\Omega^c)) = P(B/\mathfrak{B}(W_l(\Omega)))$ where $W_l(\Omega) = \left\{q \in \Omega^c : \inf\limits_{q'\in\Omega} |q - q'| \leq l\right\}$.

9 The conditional local absolute continuity means that $P(B/\mathfrak{B}(\Omega^c)) = 0$ with the probability 1(w.r.t. P) if $\lambda(\pi_\Omega B) = 0$, $B \in \mathfrak{B}(\Omega)$.

where

$$\operatorname{diam}\bar{x}=\max_{q,q'\in\bar{x}}|q-q'| \tag{2.6}$$

and $\Psi(r)$, $r\geqq 0$, is a non-increasing function such that

$$\sum_{k\in\mathbb{Z}^+} k^{n_0\nu-2}\Psi(k)<\infty(\mathbb{Z}^+=\{1,2,\ldots\})\,;$$

(G_6) for any $\bar{x}\in D^0$ and $(q,v)\in\bar{x}$

$$\int_{M_1\cap D^0(\bar{x})}[|\nabla_{v'}\exp[-f(q',v')-h((q',v')|\bar{x})]|+\exp[-f(q',v')]$$

$$\cdot(|\nabla_v\exp[-h((q',v')|\bar{x})]|+\exp[-h((q',v')|\bar{x})])]\,|U'(|q-q'|)|dq'dv'\leqq c_1$$

where c_1 is an absolute constant.

The condition (G_6), unlike the precedents, is formulated in terms of the function $h(\cdot|\cdot)$. However, it is not hard to see that on account of (G_2), (G_3), and (G_5), (G_6) follows from the condition (G_6'), imposed directly on the function f:

(G_6') for any $\bar{x}=(q,v)\cup(q',v')\in M_2\cap D^0$

$$(\exp[-f(\bar{x})]+|\nabla_v\exp[-f(\bar{x})]|+|\nabla_q\exp[-f(\bar{x})]|)\,|U'(|q-q'|)|\leqq c_2$$

where c_2 is an absolute constant.

Remarks. 1. The condition (G_1) means that the generating function f has a "hard core" of length equal to that of the potential U. The conditions (G_3) and (G_5) are, from our point of view, the most essential restrictions imposed on f. All other conditions are needed only for the possibility to interpret in the usual way each term on the left hand side of (1.8) and to apply known theorems about changing the order of the integration, the derivation of an integral depending on a parameter etc. (see Sect. 3).

2. Using the method, developed in [21–23, 24], one can prove that for any function f obeying (G_1–G_6), there exists at least one Gibbs random field whose generating function is f (for this one needs only conditions (G_1–G_3) and (G_5)). There is a widespread hypothesis that, generally speaking, a Gibbs random field is not uniquely defined by its generating function (for a discussion of this problem see [21–24]), although, as far as we know, there is no concrete example of the non-uniqueness so far[10].

Fix the numbers $\beta>0$ and $\mu\in R^1$ and the vector $v_0\in R^\nu$ and let

$$f_0(\bar{x})=\begin{cases}(\beta/2)\langle v-v_0,v-v_0\rangle+\beta\mu\,, & \bar{x}=x\in M_1\,,\quad x=(q,v)\,,\\ \beta U(|q-q'|)\,, & \bar{x}\in M_2\cap D^0\,,\quad \bar{x}=(q,v)\cup(q',v')\,,\\ 0\,, & \bar{x}\in M_n\cap D^0\,,\quad n>2\,,\end{cases} \tag{2.7}$$

and $f_0(\bar{x})=+\infty$ for $\bar{x}\notin D^0$. Using (I_1-I_4), it is easy to verify that the function f_0 satisfies the conditions (G_1–G_5) and (G_6').

Definition 2.2. A Gibbs random field with generating function f_0 given by (2.7) is called an *equilibrium measure (state)* corresponding to the Hamiltonian $H(\bar{x})$ [see (1.10)].

[10] We emphasize that only systems with one type of particles are considered. Outside of this class such examples are known (see [27–28]).

The parameters β, μ and v_0 in (2.7) have a physical meaning: β and μ correspond to the mean energy and the particle density in an equilibrium state, and v_0 is just the mean velocity of the particles.

The non-uniqueness of the equilibrium state corresponding to a Hamiltonian H and given values of the parameters β, μ and v_0 (if it holds) is usually interpreted as the presence of a phase transition in the system of classical particles with interaction described by the potential.

Main Theorem. *Let P be a Gibbs random field with generating function f obeying* (G_1–G_6). *Suppose that the correlation function ϱ_P is a stationary solution of the Bogoliubov hierarchy equations*[11]. *Then f is given by* (2.7), *i.e., P is an equilibrium state, corresponding to the Hamiltonian H.*

This theorem seems to be an essential extension of the recent result of Gallavotti and Verboven [30].

The proof of the main theorem follows from two theorems below.

Theorem 1. *If the conditions of the main theorem hold then the generating function f of the Gibbs random field P satisfies the equation*

$$\{f(\bar{x}), H(\bar{x})\} + \sum_{y \in \bar{x}} \{f(\bar{x} \setminus y), U(\bar{x} \setminus y | y)\} = 0, \qquad \bar{x} \in D^0 \tag{2.8}$$

(if $\bar{x} = x \in M_1$ then the second term on the left-hand side of (2.8) *is set equal to zero)*[12].

Theorem 2. *If the function f on M^0 obeys* (G_1–G_6)[13] *and satisfies Equation* (2.8) *then f is given by* (2.7).

In the present paper (in Sect. 3) we prove Theorem 1. The proof of Theorem 2 will be given in the second part of the work.

Consider now the question under what conditions an equilibrium state corresponds to a stationary solution of the Bogoliubov hierarchy equations. This is completely solved by the following.

Theorem 3. *The correlation function of an equilibrium state P is a stationary solution of the Bogoliubov hierarchy equations if and only if*

$$\left\langle v_0, \sum_{q \in \bar{x}} \nabla_q \varrho_P(\bar{x}) \right\rangle = 0 \tag{2.9}$$

where v_0 is the vector which appears on the right hand side of (2.7).

Condition (2.9) means the invariance of the state P with respect to the group of transformations $\{S_t^{(v_0)}, t \in R^1\}$ of the space M generated by the shifts in R^ν

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[11] As we shall show later, if the conditions (G_1–G_2) and (G_4–G_6) hold, then the function $\varrho_P(\bar{x})$ has the required smoothness properties for its substitution in the equation (1.9).

[12] Theorem 1 holds if one replaces the condition (G_3) by an appropriate condition of decreasing and reformulates correspondingly the condition (G_5).

[13] In the proof of Theorem 2 we use only the conditions (G_1), (G_2b), (G_3) and the relation

$$\lim_{\operatorname{diam} \bar{x} \to \infty} |f(\bar{x})| = 0$$

following from (G_5).

along the vector v_0. By definition,

$$S_t^{(v_0)}X=\{(q,v)\in R^\nu\times R^\nu:(q-tv_0,v)\in X\}\,,\qquad X\in M\,.$$

There exists a hypothesis that the condition (2.9) can be violated only for $\nu>2$ (for $\nu=1$ the invariance property of P follows from that of the potential U, see [31, 32]; for $\nu>1$ it is proved if $\mu\leqq C(\beta)<\infty$, see [33]). Theorem 3 extends the result of Gallavotti [17] to the multiphase case.[14]

Proof of Theorem 3. In the proof given below we use repeatedly such formal manipulations as change of the order of integration, the derivation under the sign of integral etc. Similar manipulations are used in a more general situation in the proof of Theorem 1. Their validity follows from a set of auxiliary lemmas formulated in Section 3 and proved in Section 4. These lemmas refer to an arbitrary Gibbs random field whose generating function obeys (G_1–G_6). Here we need them only in the particular case of an equilibrium state corresponding to the Hamiltonian H.

By Lemma 3.3 (ii) for any $\bar{x}\in M^0$, $\bar{x}\neq\emptyset$ and any $(q,v)\in\bar{x}$

$$\nabla_q\varrho_P(\bar{x})=\int\limits_M \nabla_q\exp[-h_0(\bar{x})-h_0(\bar{x}|Y)]dP(Y) \tag{2.10}$$

and

$$\nabla_v\varrho_P(\bar{x})=\int\limits_M \nabla_v\exp[-h_0(\bar{x})-h_0(\bar{x}|Y)]dP(Y) \tag{2.11}$$

where index 0 means that $h_0(\bar{x})$ and $h_0(\bar{x}|Y)$ correspond to f_0 given by (2.7).

In addition to (1.8) and (1.13) let

$$D=\{X\in M\colon \min_{q'\in X,\,q'\neq q}|q-q'|>d_0 \ \text{ for any } \ q\in X\}$$

and

$$D(\bar{x})=\{X\in D\colon \min_{q\in\bar{x},\,q'\in X}|q-q'|>d_0\}\,,\qquad \bar{x}\neq\emptyset\,.$$

By Lemma 3.1 (iii) the integrands in (2.10) and (2.11) are equal to zero as $\bar{x}\cup Y\notin D$.

Let now

$$M_{\bar{x}}=\{X\in M\colon X\cap\bar{x}\neq\emptyset\}\,,\qquad \bar{x}\neq\emptyset\,.$$

It follows from the definition of the measure λ [see (1.1)] that

$$\lambda(M^0\cap M_{\bar{x}})=0\,. \tag{2.12}$$

Then due to the local absolute continuity of P

$$P(M_{\bar{x}})=0 \tag{2.13}$$

Note that $\bar{x}\cup Y\in D$ and $Y\notin M_{\bar{x}}$ if and only if $\bar{x}\in D^0$ and $Y\in D(\bar{x})$. Therefore we can replace M by $D(\bar{x})$ in the integrals (2.10) and (2.11) and for $\bar{x}\in D^0$ we have

$$\begin{aligned}\{\varrho_P(\bar{x}),H(\bar{x})\}&=\int\limits_{D(\bar{x})}\{\exp[-h_0(\bar{x})-h_0(\bar{x}|Y)]\,,\ H(\bar{x})\}dP(Y)\\&=\int\limits_{D(\bar{x})}\{\exp[-\beta H_{v_0,\mu}(\bar{x})-\beta U(\bar{x}|Y)]\,,\ H(\bar{x})\}dP(Y)\,,\end{aligned}$$

[14] In this theorem we use only the C^1-smoothness property of the potential U instead of condition (I_2).

where

$$H_{v_0,\mu}(\bar{x}) = \tfrac{1}{2} \sum_{v' \in \bar{x}} \langle v' - v_0, v' - v_0 \rangle + U(\bar{x}) + \mu n(\bar{x})$$
$$= H(\bar{x}) - \sum_{v' \in \bar{x}} \langle v', v_0 \rangle + (\mu + \tfrac{1}{2} \langle v_0, v_0 \rangle) n(\bar{x}) \tag{2.14}$$

[see (1.7), (1.10), and (1.11)],

$$U(\bar{x}|Y) = \lim_{\bar{y} \to Y} U(\bar{x}|\bar{y})$$

and

$$U(\bar{x}|\bar{y}) = \sum_{z \in \bar{y}} U(\bar{x}|z) = \sum_{q \in \bar{x},\, q' \in \bar{y}} U(|q - q'|)$$

[see (1.12)].

One can rewrite the last integral as

$$- \int_{D(\bar{x})} \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)]$$
$$\cdot \left(-\beta \left\{ \sum_{v \in \bar{x}} \langle v, v_0 \rangle ,\ U(\bar{x}) \right\} + \beta \left\{ U(\bar{x}|Y) ,\ \sum_{v \in \bar{x}} \langle v, v \rangle \right\} \right) dP(Y)$$
$$= - \int_{D(\bar{x})} \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)]$$
$$\cdot \sum_{(q,v) \in \bar{x}} (\beta \langle v_0, \nabla_q U(\bar{x}) \rangle + \beta \langle v, \nabla_q U(\bar{x}|Y) \rangle) dP(Y) \tag{2.15}$$

and divide it into two terms, the first of which is

$$- \int_{D(\bar{x})} \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)]$$
$$\cdot \sum_{q \in \bar{x}} \langle \beta v_0, \nabla_q U(\bar{x}) + \nabla_q U(\bar{x}|Y) \rangle dP(Y)$$
$$= - \int_{D(\bar{x})} \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)]$$
$$\cdot \sum_{q \in \bar{x}} \langle v_0, \nabla_q [\beta H_{v_0,\mu}(\bar{x}) + \beta U(\bar{x}|Y)] \rangle dP(Y)$$
$$= \int_{D(\bar{x})} \sum_{q \in \bar{x}} \langle v_0, \nabla_q \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)] \rangle dP(Y) .$$

By the above arguments, this integral equals

$$\int_M \sum_{q \in \bar{x}} \langle v_0, \nabla_q \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)] \rangle dP(Y)$$
$$= \left\langle v_0, \sum_{q \in \bar{x}} \int_M \nabla_q \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)] dP(Y) \right\rangle$$
$$= \left\langle v_0, \sum_{q \in \bar{x}} \nabla_q \varrho_P(\bar{x}) \right\rangle . \tag{2.16}$$

The second term arising from (2.15) is

$$-\beta \int_{D(\bar{x})} \exp[-\beta H_{v_0,\mu}(\bar{x}) - \beta U(\bar{x}|Y)]$$
$$\cdot \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x}|Y) \rangle dP(Y) . \tag{2.17}$$

Let Ω be a Borel subset of R^ν and

$$D_\Omega(\bar{x})=\{X\in M: X_\Omega\in D^0(\bar{x})\}. \tag{2.18}$$

If $\bar{x}\in M(\Omega)$ is fixed and Ω is bounded but sufficiently large then the integrand in (2.17) is well defined on $D_\Omega(\bar{x})$. Under the same assumptions, $D_\Omega(\bar{x})\setminus D(\bar{x})\subset M\setminus D$.

Using the definition of the Gibbs measure [see (2.1)] one can easily prove that (I_1) (see Sect. 1) implies

$$P(D)=1. \tag{2.19}$$

Hence $P(D_\Omega(\bar{x})\setminus D(\bar{x}))=0$ and we may rewrite (2.17) for sufficiently large Ω as

$$-\beta\int_M \chi_{D_\Omega(\bar{x})}(Y)\exp[-\beta H_{v_0,\mu}(\bar{x})-\beta U(\bar{x}|Y)] \cdot\sum_{(q,v)\in\bar{x}}\langle v-v_0, \nabla_q U(\bar{x}|Y)\rangle dP(Y). \tag{2.20}$$

Condition (I_3) (see Sect. 1) and the definition of $D_\Omega(\bar{x})$ [see (2.18)] imply that the integrand in (2.20) depends only on Y_Ω (as Ω is large enough). Thus formula (1.3) is applicable and (2.20) is equal to

$$-\beta\int_{M(\Omega)\cap D^0(\bar{x})}\exp[-\beta H_{v_0,\mu}(\bar{x})-\beta U(\bar{x}|\bar{y})] \cdot\sum_{(q,v)\in\bar{x}}\langle v-v_0, \nabla_q U(\bar{x}|\bar{y})\rangle p_\Omega(\bar{y})d\lambda(\bar{y}).$$

On account of the relation

$$p_\Omega(\bar{x}\cup\bar{y})=p_\Omega(\bar{y})\exp[-\beta H_{v_0,\mu}(\bar{x})-\beta U(\bar{x}|\bar{y})], \qquad \bar{x}\in M(\Omega), \qquad \bar{y}\in M(\Omega)\cap D^0(\bar{x})$$

[see (2.4) and condition (I_4)], the last integral equals

$$-\beta\int_{M(\Omega)\cap D^0(\bar{x})} p_\Omega(\bar{x}\cup\bar{y})\sum_{(q,v)\in\bar{x}}\langle v-v_0, \nabla_q U(\bar{x}|\bar{y})\rangle d\lambda(\bar{y}).$$

Thus for sufficiently large Ω

$$\{\varrho_P(\bar{x}), H(\bar{x})\}=\left\langle v_0, \sum_{q\in\bar{x}}\nabla_q\varrho_P(\bar{x})\right\rangle -\beta\int_{M(\Omega)\cap D^0(\bar{x})} p_\Omega(\bar{x}\cup\bar{y})\sum_{(q,v)\in\bar{x}}\langle v-v_0, \nabla_q U(\bar{x}|\bar{y})\rangle d\lambda(\bar{y}). \tag{2.21}$$

Furthermore, by Lemma 3.3 (ii), (2.19) and Lemma 3.1 (iii),

$$\begin{aligned}&\int_{M_1\cap D^0(\bar{x})}\{\varrho_P(\bar{x}\cup z), U(\bar{x}|z)\}dz\\ &=-\int_{M_1\cap D^0(\bar{x})}\int_{D(\bar{x}\cup z)}\exp[-\beta H_{v_0,\mu}(\bar{x}\cup z)-\beta U(\bar{x}\cup z|Y)]\\ &\quad\cdot\{\beta H_{v_0,\mu}(\bar{x}\cup z)+\beta U(\bar{x}\cup z|Y), U(\bar{x}|z)\}dP(Y)dz\\ &=\beta\int_{M_1\cap D^0(\bar{x})}\int_M \exp[-\beta H_{v_0,\mu}(\bar{x}\cup z)-\beta U(\bar{x}\cup z|Y)]\\ &\quad\cdot\sum_{(q,v)\in\bar{x}\cup z}\langle v-v_0, \nabla_q U(\bar{x}|z)\rangle dP(Y)dz.\end{aligned}$$

Using Lemma 3.5, divide this integral into two terms. The first of them has the form

$$\beta \int_{M_1 \cap D^0(\bar{x})} \int_M \exp[-\beta H_{v_0,\mu}(\bar{x} \cup (q,v)) - \beta U(\bar{x} \cup (q,v) | Y)]$$
$$\cdot \langle v - v_0, \nabla_q U(\bar{x} | (q,v)) \rangle dP(Y) dq dv$$

and actually is equal to zero. To see this, change the order of integration and integrate first over dv. The formula for $H_{v_0,\mu}(\bar{x})$ [see (2.14)] implies that the probability distribution of a single velocity is Gaussian with mean v_0. This gives the equality claimed. The second term equals

$$\beta \int_{M_1 \cap D^0(\bar{x})} \int_M \exp[-\beta H_{v_0,\mu}(\bar{x} \cup z) - \beta U(\bar{x} \cup z | Y)]$$
$$\cdot \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | z) \rangle dP(Y) dz$$
$$= \beta \int_{M_1 \cap D^0(\bar{x})} \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | z) \rangle \varrho_P(\bar{x} \cup z) dz$$
$$= \beta \int_{M_1 \cap D^0(\bar{x})} \int_{M(\Omega)} p_\Omega(\bar{x} \cup \bar{y} \cup z) \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | z) \rangle d\lambda(\bar{y}) dz \tag{2.22}$$

[here we used (2.5) and (1.5)]. Note that from condition (I_1) and from (2.4)–(2.5) it follows that

$$p_\Omega(\bar{x}) = \varrho_P(\bar{x}) = 0\,, \qquad \bar{x} \notin D^0\,, \qquad \bar{x} \neq \emptyset\,. \tag{2.23}$$

Thus we can integrate on the right hand side of (2.22) over $M(\Omega) \cap D^0(\bar{x})$ [on $d\lambda(\bar{y})$]. Then applying Lemma 1.1 to the function

$$\Phi(\bar{y}, z) = \begin{cases} p_\Omega(\bar{x} \cup \bar{y} \cup z) \sum\limits_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | z) \rangle\,, & z \in M_1(\Omega) \cap D^0(\bar{x})\,, \\ 0\,, \quad \text{otherwise}\,, \end{cases}$$

we can represent (2.22) in the form

$$\beta \int_{M(\Omega) \cap D^0(\bar{x})} \sum_{z \in \bar{y}} p_\Omega(\bar{x} \cup \bar{y}) \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | z) \rangle d\lambda(\bar{y})$$
$$= \beta \int_{M(\Omega) \cap D^0(\bar{x})} p_\Omega(\bar{x} \cup \bar{y}) \sum_{(q,v) \in \bar{x}} \langle v - v_0, \nabla_q U(\bar{x} | \bar{y}) \rangle d\lambda(\bar{y})\,.$$

Comparing the last integral with (2.21), we conclude that

$$\{\varrho_P(x), H(\bar{x})\} + \int_{M_1 \cap D^0(\bar{x})} \{\varrho_P(\bar{x} \cup z)\,,\ U(\bar{x} | z)\} dz$$
$$= \langle v_0, \sum_{q \in \bar{x}} \nabla_q \varrho_P(\bar{x}) \rangle\,.$$

Q.E.D.

3. Proof of Theorem 1

As we already said, the proof of Theorem 1 is based on a number of auxiliary statements, which have been used partially in Section 2, in the proof of Theorem 3. We formulate these statements as Lemmas 3.1–3.5 and prove them in Section 4.

Let

$$D_d^+(\bar{x};\,Y)=\{\bar{z}\in D^0:\bar{z}\subset\bar{x}\cup Y,\ \bar{z}\cap\bar{x}\neq\emptyset,\ \bar{z}\cap Y\neq\emptyset,\ \max_{q\in\bar{z}\cap\bar{x},\,q'\in\bar{z}\cap Y}|q-q'|\geqq d\},$$
$$\bar{x}\in D^0,\qquad Y\in D(\bar{x}),\qquad d>d_0.$$

Lemma 3.1. *Let a function f satisfy the conditions* (G_1–G_6). *Then*

(i) *for any $\bar{x}$, Y, $\bar{x}\cup Y\in D$, $d\geqq d_2$ (see* (G_5)*) and $(q,v)\in\bar{x}$*

$$\sum_{\bar{z}\in D_d^+(\bar{x},Y)}(|f(\bar{z})|+|\nabla_q f(\bar{z})|+|\nabla_v f(\bar{z})|)\leqq c_3\sum_{k\geqq[d]}\Psi(k)k^{\nu(n_0-1)-1}\tag{3.1}$$

where c_3 depends only on $n(\bar{x})$,

(ii) *the limit* $\lim_{\bar{y}\to Y} h(\bar{x}|\bar{y})=h(\bar{x}|Y)$ *is finite if $\bar{x}\cup Y\in D$ and equals $+\infty$ otherwise, moreover*

$$\exp[-h(\bar{x}|Y)]\leqq c_4,\qquad \bar{x}\in M^0,\qquad Y\in M,\tag{3.2}$$

where c_4 depends only on $n(\bar{x})$,

(iii) *for any $Y\in M$ the function* $\exp[-h(\bar{x}|Y)]$ (*resp.*, $\exp[-h(\bar{x})-h(\bar{x}|Y)]$) *is of class C^1 at each point $\bar{x}\in D$, $\bar{x}\neq\emptyset$ (resp., $\bar{x}\in M^0$, $\bar{x}\neq\emptyset$) and*

$$\nabla\exp[-h(\bar{x}|Y)]=\begin{cases}-\exp[-h(\bar{x}|Y)]\displaystyle\sum_{\bar{z}\in\bar{x}\cup Y,\,\bar{z}\cap\bar{x}\neq\emptyset,\,\bar{z}\cap Y\neq\emptyset}\nabla f(\bar{z}), & \bar{x}\cup Y\in D,\\ 0,\quad \bar{x}\in D^0, & \bar{x}\cup Y\in D,\end{cases}\tag{3.3}$$

(resp., $\nabla\exp[-h(\bar{x})-h(\bar{x}|Y)]$

$$=\begin{cases}\exp[-h(\bar{x})]\nabla\exp[-h(\bar{x}|Y)]+(\nabla\exp[-h(\bar{x})])\exp[-h(\bar{x}|Y)], & \bar{x}\cup Y\in D,\\ 0, & \bar{x}\cup Y\notin D)\end{cases}\tag{3.4}$$

where ∇ denotes the gradients ∇_q or ∇_v for any $(q,v)\in\bar{x}$ (of course, the meaning of ∇ must be the same inside a given equality) [15].

From now on we shall assume (without specifying this every time again) that P is a Gibbs random field with generating function f obeying (G_1–G_6). As above, the condition (G_1) implies relations (2.19) and (2.23) and the local absolute continuity of P implies equality (2.13).

For any Borel set $\Omega\subset R^\nu$ and any $\bar{x}\in M(\Omega)$ let

$$R_\Omega(\bar{x})=\int_{M(\Omega^c)}\exp[-h(\bar{x}|Y)]dP(Y)\tag{3.5}$$

[the integral exists by (3.2)].

Lemma 3.2. *If Ω is bounded and $\bar{x}\in M(\Omega)\cap D^0$ then $R_\Omega(\bar{x})>0$. Furthermore, $R_\Omega(\emptyset)=P(M(\Omega^c))=P_\Omega(\emptyset)>0$.*

Lemma 3.3. (i) *For any bounded Borel set $\Omega\subset R^\nu$ the function $R_\Omega(\bar{x})$ (resp., the local density $p_\Omega(\bar{x})$) belongs to the class C^1 at each point $\bar{x}\in M(\Omega)\cap D^0$ (resp.,*

[15] The symbol ∇ will also be used with the same meaning further on.

$\bar{x} \in M(\Omega)$, $\bar{x} \neq \emptyset$) *and*

$$\nabla R_\Omega(\bar{x}) = \int_{M(\Omega^c)} \nabla \exp[-h(\bar{x}|Y)] dP(Y), \qquad \bar{x} \in M(\Omega) \cap D^0 \tag{3.6}$$

(*resp.*,

$$\nabla p_\Omega(\bar{x}) = \int_{M(\Omega^c)} \nabla \exp[-h(\bar{x}) - h(\bar{x}|Y)] dP(Y), \ \bar{x} \in M(\Omega)\Big), \tag{3.7}$$

(ii) *the correlation function* $\varrho_P(\bar{x})$ *belongs to the class* C^1 *at each point* $\bar{x} \in M^0$, $\bar{x} \neq \emptyset$, *and*

$$\nabla \varrho_P(\bar{x}) = \int_M \nabla \exp[-h(\bar{x}) - h(\bar{x}|Y)] dP(Y), \qquad \bar{x} \in M^0, \tag{3.8}$$

(iii) *for any* $\bar{x} \in M^0$, $q \in R^\nu$

$$\lim_{|v| \to \infty} \varrho_P(\bar{x} \cup (q, v)) = 0. \tag{3.9}$$

Let us turn immediately to the proof of Theorem 1. For any bounded Borel set $\Omega \subset R^\nu$ and $\bar{x} \in M(\Omega) \cap D^0$, comparing (2.4) and (3.5), using Lemma 3.2, and on account of (1.6), we obtain

$$\exp[-h(\bar{x})] = [R_\Omega(\bar{x})]^{-1} p_\Omega(\bar{x}) = [R_\Omega(\bar{x})]^{-1} \int_{M(\Omega)} (-1)^{n(\bar{y})} \varrho_P(\bar{x} \cup \bar{y}) d\lambda(\bar{y}). \tag{3.10}$$

Writing $[R_\Omega(\bar{x})]^{-1}$ in the form

$$[R_\Omega(\bar{x})]^{-1} = [P_\Omega(\emptyset)]^{-1} [1 + Q_\Omega(\bar{x})] \tag{3.11}$$

we have

$$\exp[-h(\bar{x})] = [P_\Omega(\emptyset)]^{-1} \int_{M(\Omega)} (-1)^{n(\bar{y})} \varrho_P(\bar{x} \cup \bar{y}) d\lambda(\bar{y}) + [P_\Omega(\emptyset)]^{-1} p_\Omega(\bar{x}) Q_\Omega(\bar{x}).$$

By Lemmas 3.2 and 3.3 (i) $Q_\Omega(\bar{x})$ and $p_\Omega(\bar{x})$ are of class C^1 at each point

$\bar{x} \in M(\Omega) \cap D^0$ and

$$\{\exp[-h(\bar{x})], \ H(\bar{x})\} = [P_\Omega(\emptyset)]^{-1} \left\{ \int_{M(\Omega)} (-1)^{n(\bar{y})} \varrho_P(\bar{x} \cup \bar{y}) d\lambda(\bar{y}), \ H(\bar{x}) \right\}$$
$$+ [P_\Omega(\emptyset)]^{-1} \{p_\Omega(\bar{x}) Q_\Omega(\bar{x}), \ H(\bar{x})\}. \tag{3.12}$$

Let us consider the sequence of the balls

$$\Omega_n = \{q \in R^\nu : |q| \leqq n\}, \qquad n = 1, 2, \dots.$$

For any $\bar{x} \in M^0$ there exists, of course, $n_{\bar{x}}$ such that $\bar{x} \in M(\Omega_n)$ for $n \geqq n_{\bar{x}}$. Now we will show that for fixed $\bar{x} \in D^0$

$$\lim_{n \to \infty} [P_{\Omega_n}(\emptyset)]^{-1} \{p_{\Omega_n}(\bar{x}) Q_{\Omega_n}(\bar{x}), \ H(\bar{x})\} = 0. \tag{3.13}$$

For this it is sufficient to prove that

$$\lim_{n \to \infty} [P_{\Omega_n}(\emptyset)]^{-1} |\nabla(p_{\Omega_n}(\bar{x}) Q_{\Omega_n}(\bar{x}))| = 0, \qquad \bar{x} \in D^0. \tag{3.14}$$

On account of (3.10) and (3.11),

$$[P_{\Omega_n}(\emptyset)]^{-1}\nabla(p_{\Omega_n}(\bar{x})Q_{\Omega_n}(\bar{x}))$$
$$=\nabla[(\exp[-h(\bar{x})])(1-[P_{\Omega_n}(\emptyset)]^{-1}R_{\Omega_n}(\bar{x}))]\,,\qquad \bar{x}\in M(\Omega_n)\cap D^0\,,$$

and (3.14) will follow if we prove that

$$\lim_{n\to\infty}(1-[P_{\Omega_n}(\emptyset)]^{-1}R_{\Omega_n}(\bar{x}))=\lim_{n\to\infty}[P_{\Omega_n}(\emptyset)]^{-1}|\nabla R_{\Omega_n}(\bar{x})|=0\,,\qquad \bar{x}\in D^0\,. \tag{3.15}$$

Now go back to the definition of $R_{\Omega_n}(\bar{x})$ [see (3.5)]. We have

$$1-[P_{\Omega_n}(\emptyset)]^{-1}R_{\Omega_n}(\bar{x})=1-[P_{\Omega_n}(\emptyset)]^{-1}\int\limits_{M(\Omega_n^c)}\exp[-h(\bar{x}|Y)]dP(Y)$$
$$=[P_{\Omega_n}(\emptyset)]^{-1}\int\limits_{M(\Omega_n^c)}(1-\exp[-h(\bar{x}|Y)])dP(Y)\,.$$

On account of (2.19), one can integrate here only over $M(\Omega_n^c)\cap D$. If $Y\in M(\Omega_n^c)\cap D$ and $n>\max\limits_{q\in\bar{x}}|q|+d_2$ then, by (3.1),

$$h(\bar{x}|Y)\leqq c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,,\qquad \bar{x}\in D^0\,, \tag{3.16}$$

where $k_n=n-\max\limits_{q\in\bar{x}}|q|$. Since the right hand side of (3.16) vanishes if $n\to\infty$ [see (G_5)], we obtain for sufficiently large n the estimate

$$[P_{\Omega_n}(\emptyset)]^{-1}\left|\int\limits_{M(\Omega_n^c)\cap D}(1-\exp[-h(\bar{x}|Y)])dP(Y)\right|$$
$$\leqq 2[P_{\Omega_n}(\emptyset)]^{-1}\int\limits_{M(\Omega_n^c)\cap D}|h(\bar{x}|Y)|dP(Y)\leqq 2\frac{P(M(\Omega_n^c)\cap D)}{P_{\Omega_n}(\emptyset)}c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}$$
$$=2c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,. \tag{3.17}$$

Furthermore, by (3.6), (2.19), and (3.3), if $n>\max\limits_{q\in\bar{x}}|q|+d_2$,

$$\nabla R_{\Omega_n}(\bar{x})=\int\limits_{M(\Omega_n^c)}\nabla\exp[-h(\bar{x}|Y)]dP(Y)=\int\limits_{M(\Omega_n^c)\cap D}\nabla\exp[-h(\bar{x}|Y)]dP(Y)$$
$$=\int\limits_{M(\Omega_n^c)\cap D}\exp[-h(\bar{x}|Y)]\sum_{\bar{z}\subset\bar{x}\cup Y,\,\bar{z}\cap\bar{x}\neq\emptyset,\,\bar{z}\cap Y\neq\emptyset}\nabla f(\bar{z})dP(Y)\,.$$

Using (3.1) and (3.2), we obtain from this for $n>\max\limits_{q\in\bar{x}}|q|+d_2$ the estimate

$$[P_{\Omega_n}(\emptyset)]^{-1}|\nabla R_{\Omega_n}(\bar{x})|\leqq\frac{P(M(\Omega_n^c))}{P_{\Omega_n}(\emptyset)}(c_3\exp c_3)\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}$$
$$=(c_3\exp c_3)\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,. \tag{3.18}$$

From (3.17) and (3.18) we deduce (3.15) and hence (3.13).

Now return to (3.12). Using (3.13) we have

$$\{\exp[-h(\bar{x})],H(\bar{x})\}$$
$$=\lim_{n\to\infty}[P_{\Omega_n}(\emptyset)]^{-1}\left\{\int\limits_{M(\Omega_n)}(-1)^{n(\bar{y})}\varrho_P(\bar{x}\cup\bar{y})d\lambda(\bar{y}),H(\bar{x})\right\},\qquad \bar{x}\in D^0\,. \tag{3.19}$$

Lemma 3.4. *For any bounded Borel set $\Omega\subset R^\nu$ and $\bar{x}\in M(\Omega)$*

$$\nabla \int_{M(\Omega)} (-1)^{n(\bar{y})}\varrho_P(\bar{x}\cup\bar{y})d\lambda(\bar{y}) = \int_{M(\Omega)} (-1)^{n(\bar{y})}\nabla\varrho_P(\bar{x}\cup\bar{y})d\lambda(\bar{y}).$$

Note that by (2.23) the function $\varrho_P(\bar{x}\cup\bar{y})$ vanishes for $\bar{x}\cup y\notin D^0$. On account of this and using Lemma 3.1 and (2.12), we transform the Poisson brackets on the right hand side of (3.19) as follows:

$$\begin{aligned}\left\{\int_{M(\Omega_n)} (-1)^{n(\bar{y})}\varrho_P(\bar{x}\cup\bar{y})d\lambda(\bar{y}), H(\bar{x})\right\} &= \int_{M(\Omega_n)} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}), H(\bar{x})\}d\lambda(\bar{y}) \\ &= \int_{M(\Omega_n)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}), H(\bar{x})\}d\lambda(\bar{y}) \\ &= \int_{M(\Omega_n)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}[\{\varrho_P(\bar{x}\cup\bar{y}), H(\bar{x}\cup\bar{y})\} \\ &\quad -\{\varrho_P(\bar{x}\cup\bar{y}), U(\bar{x}|\bar{y})\} \\ &\quad -\{\varrho_P(\bar{x}\cup\bar{y}), H(\bar{y})\}]d\lambda(\bar{y}). \qquad (3.20)\end{aligned}$$

Lemma 3.5. *For any bounded Borel set $\Omega\subset R^\nu$ and $\bar{x}\in M(\Omega)\cap D^0$ the following integrals converge*

$$\int_{M(\Omega)\cap D^0(\bar{x})} \int_{M_1\cap D^0(\bar{x})} \sum_{(q,v)\in\bar{x}\cup z} |\nabla_v\varrho_P(\bar{x}\cup\bar{y}\cup z)|\,|\nabla_q U(\bar{x}|z)|dzd\lambda(\bar{y}), \qquad (3.21)$$

$$\begin{aligned}&\int_{M(\Omega)\cap D^0(\bar{x})} \int_{M_1(\Omega)} \int_{M_1\cap D^0(z)} [|\nabla_v\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z')| + |\nabla_{v'}\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z')|] \\ &\cdot|U'(|q-q'|)|dz'dzd\lambda(\bar{y}), \quad z=(q,v), \quad z'=(q',v'), \qquad (3.22)\end{aligned}$$

and

$$\int_{M(\Omega)\cap D^0(\bar{x})} \int_{M(\Omega)} |\nabla_q\varrho_P(\bar{x}\cup\bar{y}\cup z)|\cdot|v|dzd\lambda(\bar{y}), \quad z=(q,v). \qquad (3.23)$$

Fix $\bar{x}\in M^0$ and a bounded Borel set $\Omega\subset R^\nu$, and consider the function

$$\Phi_1(\bar{y},z) = \begin{cases}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{x}|z)\}, & \bar{y}\in M(\Omega)\cap D^0(\bar{x}),\ z\in M_1\cap D^0(\bar{x}), \\ 0, & \text{otherwise}.\end{cases}$$

Since the integral (3.21) converges, one can use Lemma 1.1. We choose $n > \max_{q\in\bar{x}}|q| + d_1$ and set $\Omega=\Omega_n$. Then on account of the condition (I_4) of Section 1, $U(\bar{x}|z)=0$ for $z\notin M(\Omega_n)$. Finally we obtain

$$\begin{aligned}&\int_{M(\Omega_n)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{x}|\bar{y})\}d\lambda(\bar{y}) \\ &= \int_{M(\Omega_n)\cap D^0(\bar{x})} \sum_{z\in\bar{y}} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{x}|z)\}d\lambda(\bar{y}) \\ &= -\int_{M(\Omega_n)\cap D^0(\bar{x})} \int_{M_1(\Omega_n)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{x}|z)\}dzd\lambda(\bar{y}) \\ &= -\int_{M(\Omega_n)\cap D^0(\bar{x})} \int_{M_1\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{x}|z)\}dzd\lambda(\bar{y}). \qquad (3.24)\end{aligned}$$

Now fix $\bar{x}\in M^0$, $z'=(q',v')\in M_1$ and a bounded Borel set $\Omega\subset R^\nu$. Let

$$\Phi_2(\bar{y},z)=\begin{cases}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z'),\ U(|q-q'|)\}\,,\ \bar{y}\in M(\Omega)\cap D^0(z')\,,\\ z=(q,v)\in M_1\cap D^0(z')\,,\\ 0\,,\quad \text{otherwise}\,.\end{cases}$$

The convergence of the integral (3.22) implies that

$$\int\limits_{M_1}\ \int\limits_{M(\Omega)\cap D^0(z')}\ \int\limits_{M_1(\Omega)\cap D^0(z')}|\{\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z'),\ U(|q-q'|)\}|dzd\lambda(\bar{y})dz'<+\infty\,,$$
$$z=(q,v)\,,\qquad z'=(q',v')\,.\tag{3.25}$$

Hence for almost all (w.r.t. m_1) $z'\in M_1$

$$\int\limits_{M(\Omega)\cap D^0(z')}\ \int\limits_{M_1(\Omega)\cap D^0(z')}|\{\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z'),\ U(|q-q'|)\}|dzd\lambda(\bar{y})<\infty\,.$$

This means that for almost all $z'\in M_1$ we can apply Lemma 1.1, to the function $\Phi_2(\bar{y},z)$. Then we have

$$\begin{aligned}&\int\limits_{M(\Omega)\cap D^0(z')}(-1)^{n(y)}\int\limits_{M_1(\Omega)\cap D^0(z')}\{\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z'),\ U(|q-q'|)\}dzd\lambda(\bar{y})\\ &=-\int\limits_{M(\Omega)\cap D^0(z')}\ \sum_{z=(q,v)\in\bar{y}}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z'),\ U(|q-q'|)\}d\lambda(\bar{y})\\ &=-\int\limits_{M(\Omega)\cap D^0(z')}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z'),\ U(\bar{y}|z')\}d\lambda(\bar{y})\,.\end{aligned}\tag{3.26}$$

Using (3.25), we can integrate in the last expression over dz'. First integrate over whole M_1. This gives

$$\begin{aligned}&\int\limits_{M_1}\ \int\limits_{M(\Omega)\cap D^0(z')}\ \int\limits_{M_1(\Omega)\cap D^0(z')}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z'),\ U(|q-q'|)\}dzd\lambda(\bar{y})dz'\\ &=-\int\limits_{M_1}\ \int\limits_{M(\Omega)\cap D^0(z')}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z'),\ U(\bar{y}|z')\}d\lambda(\bar{y})dz'\,.\end{aligned}\tag{3.27}$$

Both integrals in (3.27) are absolutely convergent. Hence, one can choose any order of integration on the both sides of (3.27). Integrating the function $\{\varrho_P(\bar{x}\cup\bar{y}\cup z\cup z'), U(|q-q'|)\}$ over dv and dv' and using the statement (iii) of Lemma 3.3 we obtain zero. Thus

$$\begin{aligned}&\int\limits_{M(\Omega)}(-1)^{n(\bar{y})}\int\limits_{M_1\cap D^0(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{y}|z)\}dzd\lambda(\bar{y})\\ &=\int\limits_{M(\Omega)\cap D^0(\bar{x})}(-1)^{n(\bar{y})}\int\limits_{M_1\cap D^0(\bar{x}\cup\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{y}|z)\}dzd\lambda(\bar{y})=0\,.\end{aligned}$$

By the last equality and (3.24), for $n>\max\limits_{q\in\bar{x}}|q|+d_1$,

$$\begin{aligned}&\int\limits_{M(\Omega_n)\cap D^0(\bar{x})}(-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{x}|y)\}d\lambda(\bar{y})\\ &=\int\limits_{M(\Omega_n)\cap D^0(\bar{x})}(-1)^{n(\bar{y})}\int\limits_{M_1\cap D^0(\bar{x}\cup\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{x}\cup\bar{y}|z)\}dzd\lambda(\bar{y})\,.\end{aligned}\tag{3.28}$$

Now integrate (3.26) over $z' \in M_1(\Omega)$. By repeating the above arguments we obtain

$$\int_{M(\Omega)} (-1)^{n(\bar{y})} \int_{M_1(\Omega)\cap D^0(\bar{y})} \{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{y}|z)\} dz d\lambda(\bar{y}) = 0. \tag{3.29}$$

The absolute convergence of the last integral enables us to apply Lemma 1.1 to the function

$$\Phi_3(\bar{y}, z) = \begin{cases} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{y}\setminus z|z)\}, & \bar{y}\in M(\Omega)\cap D^0, \quad z\in\bar{y}, \\ 0, \quad \text{otherwise}. \end{cases}$$

On account of (3.29), we have

$$\begin{aligned} &-\int_{M(\Omega)\cap D^0} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{y})\} d\lambda(\bar{y}) \\ &= \tfrac{1}{2}\int_{M(\Omega)\cap D^0} (-1)^{n(\bar{y})} \sum_{z\in\bar{y}} \{\varrho_P(\bar{x}\cup\bar{y}),\ U(\bar{y}\setminus z|z)\} d\lambda(\bar{y}) = 0. \end{aligned} \tag{3.30}$$

Finally, using the convergence of the integral (3.23) we apply Lemma 1.1 to the function

$$\Phi_4(\bar{y}, z) = \begin{cases} (-1)^{n(\bar{y})}\langle \nabla_q \varrho_P(\bar{x}\cup\bar{y}), v\rangle, & \bar{y}\in M(\Omega), \quad z=(q,v)\in\bar{y}, \\ 0, \quad \text{otherwise}. \end{cases}$$

And so we obtain the relation

$$\begin{aligned} &\int_{M_1(\Omega)} \int_{M(\Omega)} (-1)^{n(\bar{y})}\langle \nabla_q \varrho_P(\bar{x}\cup\bar{y}\cup z), v\rangle d\lambda(\bar{y}) dz \\ &= -\int_{M(\Omega)} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ \tfrac{1}{2}\sum_{v\in\bar{y}} \langle v, v\rangle\} d\lambda(\bar{y}) \end{aligned}$$

which together with (3.30) shows that

$$\begin{aligned} &-\int_{M(\Omega)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\{\varrho_P(\bar{x}\cup\bar{y}),\ H(\bar{y})\} d\lambda(\bar{y}) \\ &= \int_{M(\Omega)} \int_{M_1(\Omega)} (-1)^{n(\bar{y})}\langle \nabla_q \varrho_P(\bar{x}\cup\bar{y}\cup z), v\rangle dz d\lambda(\bar{y}). \end{aligned} \tag{3.31}$$

Comparing (3.20), (3.28) and (3.31), we obtain for $n > \max_{q\in\bar{x}} |q| + d_1$,

$$\begin{aligned} &\left\{\int_{M(\Omega_n)} (-1)^{n(\bar{y})} \varrho_P(\bar{x}\cup\bar{y}) d\lambda(\bar{y}),\ H(\bar{x})\right\} \\ &= \int_{M(\Omega_n)\cap D^0(\bar{x})} (-1)^{n(\bar{y})}\Big[\{\varrho_P(\bar{x}\cup\bar{y}),\ H(\bar{x}\cup\bar{y})\} \\ &\quad + \int_{M_1\cap D^0(\bar{x}\cup\bar{y})} \{\varrho_P(\bar{x}\cup\bar{y}\cup z),\ U(\bar{x}\cup\bar{y}|z)\} dz\Big] d\lambda(\bar{y}) \\ &\quad + \int_{M_1(\Omega_n)} \int_{M(\Omega_n)} (-1)^{n(\bar{y})}\langle \nabla_q \varrho_P(\bar{x}\cup\bar{y}\cup z), v\rangle d\lambda(\bar{y}) dz. \end{aligned}$$

Now we use the hypothesis that ϱ_P is a stationary solution of the Bogoliubov equations. Hence,

$$\left\{\int\limits_{M(\Omega_n)} (-1)^{n(\bar{y})}\varrho_P(\bar{x}\cup\bar{y})d\lambda(\bar{y}),\ H(\bar{x})\right\}$$
$$= \int\limits_{M_1(\Omega_n)}\int\limits_{M(\Omega_n)} (-1)^{n(\bar{y})}\langle \nabla_q\varrho_P(\bar{x}\cup\bar{y}\cup z), v\rangle d\lambda(\bar{y})dz\,,\qquad z=(q,v)\,. \tag{3.32}$$

Using Lemma 3.4 and (1.6) we represent the right-hand side of (3.32) as

$$\int\limits_{M_1(\Omega_n)}\left\langle \nabla_q\int\limits_{M(\Omega_n)} (-1)^{n(\bar{y})}\varrho_P(\bar{x}\cup\bar{y}\cup z)d\lambda(\bar{y}), v\right\rangle dz$$
$$= \int\limits_{M_1(\Omega_n)} \langle \nabla_q p_{\Omega_n}(\bar{x}\cup z), v\rangle dz\,,\qquad z=(q,v)\,. \tag{3.33}$$

Let us introduce the notations:

$$q=(q^1,\dots,q^\nu)\,,\qquad v=(v^1,\dots,v^\nu)\,,\qquad q_j, v_j\in R^1\,,\qquad j=1,\dots,\nu\,,$$
$$\partial_i M_1(\Omega_n)=\{q,v):q\in\Omega_n,\ q^i=0,\ v\in R^\nu\}\,,$$
$$d_iq=\prod_{\substack{1\leqq j\leqq \nu,\\ j\neq i}} dq^j\,,\qquad (q^\pm)^i=\pm\Big(n^2-\sum_{\substack{1\leqq j\leqq \nu,\\ j\neq i}}(q^j)^2\Big)^{1/2}\,,$$
$$q_i^\pm=q_i^\pm(q)=(q^1,\dots,q^{i-1},(q^\pm)^i,q^{i+1},\dots,q^\nu)\,,\qquad z_i^\pm=(q_i^\pm,v)\,,\qquad i=1,\dots,\nu\,.$$

Then we can write

$$\int\limits_{M_1(\Omega_n)} \langle \nabla_q p_{\Omega_n}(\bar{x}\cup z), v\rangle dz$$
$$=\sum_{i=1}^{\nu}\int\limits_{\partial_i M_1(\Omega_n)} v^i\,[p_{\Omega_n}(\bar{x}\cup z_i^+)-p_{\Omega_n}(\bar{x}\cup z_i^-)]d_iq dv\,,\qquad z=(q,v)\,. \tag{3.34}$$

Now turning to (3.19) and taking into account (3.32), (3.33), and (3.34), we obtain

$$-\{\exp[-h(\bar{x})]\,,\ H(\bar{x})\}=\exp[-h(\bar{x})]\,\{h(\bar{x}),\ H(\bar{x})\}$$
$$=\lim_{n\to\infty}\sum_{i=1}^{\nu}\frac{1}{P_{\Omega_n}(\emptyset)}\int\limits_{\partial_i M_1(\Omega_n)} v^i[p_{\Omega_n}(\bar{x}\cup z_i^+)-p_{\Omega_n}(\bar{x}\cup z_i^-)]d_iq dv\,,\qquad \bar{x}\in D^0\,. \tag{3.35}$$

Let us represent the local density $p_{\Omega_n}(\bar{x}\cup z_i^\pm)$ in the form

$$p_{\Omega_n}(\bar{x}\cup z_i^\pm)=\exp[-h(\bar{x})]\,R'_{\Omega_n}(\bar{x};z_i^\pm)\,, \tag{3.36}$$

where [see (2.3)]

$$R'_{\Omega_n}(\bar{x};z_i^\pm)=\int\limits_{M(\Omega_n^c)}\exp[-f(z_i^\pm)-h(\bar{x}|z_i^\pm)-h(\bar{x}\cup z_i^\pm|Y)]dP(Y)\,. \tag{3.37}$$

Then after substituting (3.36) into (3.35) we have

$$\{h(\bar{x}),\ H(\bar{x})\}$$
$$=\lim_{n\to\infty}\sum_{i=1}^{\nu}\frac{1}{P_{\Omega_n}(\emptyset)}\int\limits_{\partial_i M_1(\Omega_n)} v^i\,[R'_{\Omega_n}(\bar{x};z_i^\pm)-R'_{\Omega_n}(\bar{x};z_i^-)]d_iq dv\,,\qquad \bar{x}\in D^0\,. \tag{3.38}$$

Now we shall prove that the right hand side of (3.38) does not depend on $\bar{x}$. Of course, it is sufficient to verify that for $i=1,2,\ldots,$

$$\lim_{n\to\infty}\frac{1}{P_{\Omega_n}(\emptyset)}\int_{\partial_i M_1(\Omega_n)} v^i[R'_{\Omega_n}(\bar{x};z_i^{\pm})-R'_{\Omega_n}(\emptyset;z_i^{\pm})]d_iqdv=0\,. \tag{3.39}$$

First of all we note that the integrand in the right-hand side of (3.37) vanishes if $Y\notin D(z_i^{\pm})$. Hence we can integrate in (3.37) only over $M(\Omega_n^c)\cap D(z_i^{\pm})$. Further, from the definition of R'_{Ω_n}, we obtain the following inequality

$$\begin{aligned}&|R'_{\Omega_n}(\bar{x};z_i^{\pm})-R'_{\Omega_n}(\emptyset;z_i^{\pm})|\leqq \int_{M(\Omega_n^c)\cap D(z_i^{\pm})}\exp[-f(z_i^{\pm})-h(z_i^{\pm}|Y)]\\&\cdot|1-\exp[-h(\bar{x}|z_i^{\pm})-h(\bar{x}|Y)-h(\bar{x}|z_i^{\pm}|Y)]|dP(Y)\,,\end{aligned}$$

where

$$h(\bar{x}|z_i^{\pm}|Y)=\lim_{\bar{y}\to Y}\sum_{\bar{w}\subset\bar{x}\cup\bar{y},\,\bar{w}\cap\bar{x}\neq\emptyset,\,\bar{w}\cap\bar{y}\neq\emptyset} f(\bar{w}\cup z_i^{\pm})\,.$$

If $n>\max_{q\in\bar{x}}|q|+d_2$ then, by Lemma 3.1 (i), for any $Y\in M(\Omega_n^c)\cap D(z_i^{\pm})$

$$|h(\bar{x}|z_i^{\pm})+h(\bar{x}|Y)+h(\bar{x}|z_i^{\pm}|Y)|\leqq 3c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,,$$

where, as above, $k_n=n-\max_{q\in\bar{x}}|q|$. The right hand side of the last estimate vanishes if $n\to\infty$ [see (G_5)]. Hence, one can state that for sufficiently large n and any $Y\in M(\Omega_n^c)\cap D(z_i^{\pm})$

$$|1-\exp[-h(\bar{x}|z_i^{\pm})-h(\bar{x}|Y)-h(\bar{x}|z_i^{\pm}|Y)]|\leqq 6c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,.$$

The function $\exp[-h(z_i^{\pm}|Y)]$ is, by Lemma 3.1 (ii), uniformly bounded in both its variables. From what we have said above we obtain the following estimates

$$\begin{aligned}&|R'_{\Omega_n}(\bar{x},z_i^{\pm})-R'_{\Omega_n}(\emptyset,z_i^{\pm})|\\&\leqq c_3P(M(\Omega_n^c))\exp[-f(z_i^{\pm})]\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\,,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{P_{\Omega_n}(\emptyset)}\left|\int_{\partial_i M_1(\Omega_n)} v^i[R'_{\Omega_n}(\bar{x},z_i^{\pm})-R'_{\Omega_n}(\emptyset,z_i^{\pm})]d_iqdv\right|\\&\leqq c_3\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}\int_{\partial_i M_1(\Omega_n)}|v^i|\exp[-f(q_i^{\pm},v)]d_iqdv\,.\end{aligned} \tag{3.40}$$

According to condition (G_2, c), the right hand side of (3.40) does not exceed

$$c_5\sum_{k\geqq k_n}\Psi(k)k^{\nu(n_0-1)-1}n^{\nu-1}$$

where c_5 depends only on $n(\bar{x})$. From condition (G_5) we see that this expression vanishes if $n\to\infty$. Thus the relation (3.39) is proved.

Now (3.38) may be written as

$$\{h(\bar{x}),\ H(\bar{x})\}=c_6\,,\qquad \bar{x}\in D^0\,,\qquad \bar{x}\neq\emptyset$$

where c_6 does not depend on $\bar{x}$. Putting here $\bar{x}=x=(q,v)\in M_1$ and $v=0$, we obtain that $c_6=0$. Thus

$$\{h(\bar{x}),\ H(\bar{x})\}=0\,,\qquad \bar{x}\in D^0\,,\qquad \bar{x}\neq\emptyset\,.$$

Finally, note that

$$\{f(\bar{x}),\ H(\bar{x})\}+\sum_{y\in\bar{x}}\{f(\bar{x}\setminus y),\ U(\bar{x}\setminus y|y)\}=\sum_{\bar{y}\subseteq\bar{x}}(-1)^{n(\bar{x}\setminus\bar{y})}\{h(\bar{y}),\ H(\bar{y})\}\,. \tag{3.41}$$

This completes the proof of Theorem 1.

4. Proof of the Auxiliary Lemmas

We shall use the following terminology: the points of M and M^0 will be called the configurations of particles, and the pairs $x=(q,v)\in R^\nu\times R^\nu$—the particles.

Proof of Lemma 3.1. (i) Let $\bar{x}\in M^0$, $Y\in M$, $\bar{x}\cup Y\in D$ and $d\geqq d_2$. First, suppose that $\bar{x}\cap Y=\emptyset$, and therefore, $Y\in D(\bar{x})$. For any $d'\geqq d$ denote

$$D_{d'}^{-}(\bar{x};\,Y)=\left\{\bar{z}\in D^0:\bar{z}\subset\bar{x}\cup Y,\bar{z}\cap\bar{x}\neq\emptyset,\bar{z}\cap Y\neq\emptyset,\quad \max_{q\in\bar{z}\cap\bar{x},\,q'\in\bar{z}\cap\bar{y}}|q-q'|<d'\right\} \tag{4.1}$$

and

$$D_{d,d'}(\bar{x};\,Y)=D_d^{+}(\bar{x};\,Y)\cap D_{d'}^{-}(\bar{x};\,Y)\,. \tag{4.2}$$

Our first goal is to estimate the cardinality of $D_{d,d'}(\bar{x};\,Y)$ denoted by $|D_{d,d'}(\bar{x};\,Y)|$[16]. For this fix an arbitrary particle $x=(q,v)$ and integers k,n,n_1 such that $k\geqq[d]$, $1\leqq n_1<n\leqq n_0$ and $n-n_1\leqq n(\bar{x})$ and consider the set

$$E_k(n,n_1,x)$$
$$=\left\{\bar{z}\in D_{k,k+1}(\bar{x};\,Y):\bar{z}\ni x,n(\bar{z})=n,n(\bar{z}\cap Y)=n_1,\,k\leqq\max_{q'\in\bar{z}\cap Y}|q-q'|<k+1\right\}. \tag{4.3}$$

Clearly, $D_{k,k+1}(\bar{x};\,Y)\subset\bigcup\limits_{n,n_1,x}E_k(n,n_1,x)$ and hence $|D_{k,k+1}(\bar{x};\,Y)|\leqq\sum\limits_{n,n_1,x}|E_k(n,n_1,x)|$.

To estimate $|E_k(n,n_1,x)|$ we note that for any pair of particles (q',v'), $(q'',v'')\in\bar{x}\cup Y$, $|q'-q''|>d_0$. Hence, the number of particles $(q',v')\in\bar{x}\cup Y$ satisfying the condition $k\leqq|q-q'|<k+1$ is less then $c_7k^{\nu-1}$ where c_7 is a constant. By the same bound one can estimate the number of the particles $(q',v')\in Y$ maximizing the distance $|q-q'|$ in (4.3). Furthermore the number N of particles $(q'',v'')\in Y$ satisfying the condition $|q-q''|<k+1$ is less than c_7k^ν. Then the number of ways of choosing the intersection $\bar{z}\cap Y$

$$\binom{N}{n_1-1}=\frac{N(N-1)\ldots(N-n_1+2)}{(n_1-1)!}\leqq\frac{(c_7k^\nu)^{n_1-1}}{(n_1-1)!}\,.$$

Finally, the number of ways of choosing the intersection containing the given particle x is

$$\binom{n(\bar{x})}{n-n_1-1}=\frac{n(\bar{x})\,(n(\bar{x})-1)\ldots(n(\bar{x})-n+n_1+2)}{(n-n_1-1)!}\,.$$

[16] A similar notation will be used below for any set of configurations. For the number of particles in a given configuration $\bar{x}\in M^0$ we preserve the notation (1.7).

Thus we find that

$$|E_k(n, n_1, x)| \leqq \frac{c_7 k^{\nu-1}(c_7 k^\nu)^{n_1-1}[n(\bar{x})]^{n-n_1-1}}{(n_1-1)!\,(n-n_1-1)!} \leqq c_7^{n_0-1}[n(\bar{x})]^{n(\bar{x})}k^{\nu(n_0-1)-1},$$

and

$$|D_{k,k+1}(\bar{x}; Y)| \leqq n_0^2 c_7^{n_0-1}[n(\bar{x})]^{n(\bar{x})}k^{\nu(n_0-1)-1} = c_3' k^{\nu(n_0-1)-1}$$

where c_3' depends only on $n(\bar{x})$.

Note that, by condition (G_3), a configuration $\bar{z}\in D_d^+(\bar{x}; Y)$ gives a non-zero contribution in the sum $\sum_{\bar{z}\in D^+(\bar{x},Y)} (|f(\bar{z})| + |\nabla_q f(\bar{z})| + |\nabla_v f(\bar{z})|)$ so long as $n(\bar{z}) \leqq n_0$ and, by condition (G_5), a single term in this sum is bounded by 3Ψ (diam $\bar{z}$) [see (2.6)] where Ψ is a non-increasing function. Therefore,

$$\sum_{\bar{z}\in D_d^+(\bar{x}; Y)} (|f(\bar{z})| + |\nabla_q f(\bar{z})| + |\nabla_v f(\bar{z})|)$$
$$\leqq 3 \sum_{k \geqq [d]} \sum_{\bar{z}\in D_{k,k+1}(\bar{x}; Y)} \Psi(k) \leqq 3c_3' \sum_{k\geqq [d]} \Psi(k)k^{\nu(n_0-1)-1}.$$

Consider now the case where $\bar{x}\cup Y\in D$ but $\bar{x}\cap Y \neq \emptyset$. Then, as we remarked above, $Y\setminus\bar{x}\in D(\bar{x})$ and

$$D_d^+(\bar{x}; Y) \subseteq D_d^+(\bar{x}; Y\setminus\bar{x}) \cup \{\bar{z}\subseteq\bar{x} : \operatorname{diam}\bar{z} \geqq d\}.$$

Using the stated estimate for $\bar{x}$ and $Y\setminus\bar{x}$ and condition (G_5) one can find that in this case the left-hand side of the last inequality does not exceed

$$3c_3' \sum_{k\geqq [d]} \Psi(k)k^{\nu(n_0-1)-1} + 3\cdot 2^{n(\bar{x})}\Psi(d).$$

Clearly, this bound does not exceed

$$c_3 \sum_{k\geqq [d]} \Psi(k)k^{\nu(n_0-1)-1}$$

where c_3 depends only on $n(\bar{x})$. So (i) is proved.

(ii) First let $\bar{x}\cup Y\in D$. To prove the existence of $\lim_{\bar{y}\to Y} h(\bar{x}|\bar{y})$ it is sufficient to find for any $\varepsilon>0$ a number $d>0$ such that $|h(\bar{x}|\bar{y}) - h(\bar{x}|\bar{y}')| < \varepsilon$ provided $Y_{\Omega_d} \subseteq \bar{y} \subset Y$ and $Y_{\Omega_d} \subseteq \bar{y}' \subset Y$, $\Omega_d = \{q\in R^\nu : |q| \leqq d\}$. Using the statement (i) and the convergence of the series, in the right-hand side of (3.1), one can easily verify, that such a d may be choosen from the conditions

$$d > d_2 + \max_{q\in\bar{x}} |q|, \qquad c_3 \sum_{k\geqq [d]} \Psi(k)k^{\nu(n_0-1)-1} < \varepsilon$$

where

$$d' = d - \max_{q\in\bar{x}} |q|.$$

Next we prove the estimate (3.2). By (3.1),

$$\begin{aligned}\exp[-h(\bar{x}|Y)] &= \exp\left[-\sum_{\bar{z}\in D_{\bar{d}_2}^-(\bar{x};\,Y)} f(\bar{z}) - \sum_{\bar{z}\in D_{\bar{d}_2}^+(\bar{x};\,Y)} f(\bar{z})\right] \\ &\leqq \exp\left[-\sum_{\bar{z}\in D_{\bar{d}_2}^-(\bar{x},\,Y)} f(\bar{z})\right] \exp\left[c_3 \sum_{k\geqq [d_2]} \Psi(k)k^{\nu(n_0-1)-1}\right].\end{aligned} \tag{4.4}$$

Using the above arguments one can show that $|D_{\bar{d}_2}^-(\bar{x};\,Y)|$ is bounded by a constant depending only on $n(\bar{x})$. Moreover, each factor $\exp[-f(\bar{z})]$ contributing to the first exponential on the right-hand side of (4.4) is bounded according to condition (G_2). The bound (3.2) follows from this in the case where $\bar{x}\cup Y\in D$.

Let $\bar{x}\cup Y\notin D$. Then one can find a $\bar{z}\subset\bar{x}\cup Y$ such that $\bar{z}\cap\bar{x}\neq\emptyset$, $\bar{z}\cap Y\neq\emptyset$ and $f(\bar{z})=+\infty$. Hence, $h(\bar{x}|\bar{y})=+\infty$ for any $\bar{y}\subset Y$ such that $\bar{z}\cap Y\subseteqq\bar{y}$, and $h(\bar{x}|Y)=+\infty$. This proves (ii).

Remark 4.1. By similar arguments one can prove that for any $\bar{x},\bar{y}\in M^0$ and $Y\in M$

$$\exp\left[-\sum_{\substack{\bar{z}\subset\bar{x}\cup Y:\,\bar{z}\cap\bar{x}\neq\emptyset,\,\bar{z}\cap Y\neq\emptyset\\ \bar{z}\neq\bar{y}}} f(\bar{z})\right]\leqq c_4\,.$$

(iii) First consider the case when $\bar{x}\in D^0$ and $Y\in D(\bar{x})$. In this case the smoothness of $f(\bar{x})$ [see (G_2, a)] implies that of $\exp[-h(\bar{x})]$ and we have only to prove the smoothness of $\exp[-h(\bar{x}|Y)]$ which follows from the similar property of $h(\bar{x}|Y)$.

We shall use a theorem on smoothness properties of the limit function (see e.g. [34], Th. 111) on account of the fact that $h(\bar{x}|Y)$ is, by definition, the limit of functions of the class C^1.

Fix $x_1=(q_1,v_1)$ and denote $\bar{x}_1=\bar{x}\setminus x_1$. Clearly there exists $\delta>0$ such that if $(q,v)\in R^\nu\times R^\nu$ and $|q-q_1|<\delta$, $\bar{x}_1\cup(q,v)\cup Y\in D$. We shall show that for any $\varepsilon>0$ there exists $\bar{y}\subset Y$ such that for all $q\in R^\nu$ satisfying the condition $|q-q_1|<\delta$, for all $v\in R^\nu$ and all $\bar{y}'\subset Y$ containing $\bar{y}$, the following holds:

$$|\nabla h(\bar{x}_1\cup(q,v)|\bar{y}')-\nabla h(\bar{x}_1\cup(q,v)|\bar{y})|<\varepsilon\,, \tag{4.5}$$

where ∇, as usual, is ∇_q or ∇_v.

Let

$$\bar{y}=\{(q'',v'')\in Y:\max_{q'\in\bar{x}}|q'-q''|\leqq d+\delta\}$$

where $d>d_2$ is such that $c_3\cdot\sum_{k\geqq[d]}\Psi(k)k^{\nu(n_0-1)-1}<\varepsilon$. Then for $|q-q_1|<\delta$ and $\bar{y}\subseteqq\bar{y}'\subset Y$

$$|\nabla h(\bar{x}_1\cup(q,v)|\bar{y}')-\nabla h(\bar{x}_1\cup(q,v)|\bar{y})|\leqq\sum_{\bar{z}\in D_d^+(\bar{x}_1\cup(q,v);\,Y)}|\nabla f(\bar{z})|\,.$$

Applying the bound (3.1) we obtain (4.5). The existence and continuity of the gradients $\nabla h(\bar{x}|Y)$ as formulae (3.3) and (3.4) follow from this immediately.

If $\bar{x}\cup Y\in D$, but $Y\notin D(\bar{x})$ then $\bar{x}\cap Y\neq\emptyset$, $\bar{x}\in D^0$ and $Y\setminus\bar{x}\in D(\bar{x})$. In this case we can apply the proved statement to the function $h(\bar{x}|Y\setminus\bar{x})$ and then use the identity

$$h(\bar{x}|Y)=h(\bar{x}|Y\setminus\bar{x})+h(\bar{x}\cap Y)+h(\bar{x}\cap Y|\bar{x}\setminus Y)\,.$$

Suppose now, that $\bar{x} \cup Y \notin D$. On account of the last identity we can restrict ourselves to the case $\bar{x} \cap Y = \emptyset$. The gradients

$$\nabla_v \exp[-h(\bar{x}|Y)], \qquad \nabla_v \exp[-h(\bar{x}) - h(\bar{x})|Y)], \qquad v \in \bar{x},$$

exist and vanish since $\exp[-h(\bar{x}|Y)] = 0$ and the same holds for any change of the velocity v. Now let us prove that the gradient $\nabla_q \exp[-h(\bar{x}|Y)]$ (resp., $\nabla_q \exp[-h(\bar{x}) - h(\bar{x}|Y)]$), $q \in \bar{x}$, exists and vanishes for $\bar{x} \in D^0$ (resp., $\bar{x} \in M^0$). Since the arguments in the both cases are completely similar we consider only the first of them.

If one can find $q' \in \bar{x}$ and $q'' \in Y$ such that $0 < |q' - q''| < d_0$ then the statement is trivial since the configuration $(\bar{x} \setminus (q, v)) \cup (\tilde{q}, v) \cup Y \notin D$ for any $\tilde{q} \in R^\nu$ sufficiently close to q, and consequently $\exp[-h((\bar{x} \setminus (q, v)) \cup (\tilde{q}, v)|Y)] = 0$. A similar argument is applicable to the case when $Y \notin D$. Thus we have to consider only the case where $\bar{x} \in D^0$, $Y \in D$ and $\min\limits_{q' \in \bar{x},\, q'' \in Y} |q' - q''| = d_0$. If, in addition, for a given $q \in \bar{x}$,

$$\min_{q'' \in Y} |q - q''| > d_0$$

then $\nabla_q \exp[-h(\bar{x}|Y)] = 0$ by the above arguments. Finally fix $x = (q, v) \in \bar{x}$ and consider the case when $\min\limits_{q'' \in Y} |q - q''| = d_0$.

Let $q = (q^1, \ldots, q^\nu)$, $q^i \in R^1$, $i = 1, \ldots, \nu$. Fix i and put for $\tau \in R^1$

$$q_\tau^i = q^i + \tau, \qquad (q_\tau)_i = (q^1, \ldots, q^{i-1}, q_\tau^i, q^{i+1}, \ldots, q^\nu),$$
$$(x_\tau)_i = ((q_\tau)_i, v), \qquad (\bar{x}_\tau)_i = (\bar{x} \setminus x) \cup (x_\tau)_i.$$

It is not hard to see that there are only four possibilities: (a) for any sufficiently small $\tau \neq 0$ the configuration $(\bar{x}_\tau)_i \cup Y \in D$, (b) for any sufficiently small $\tau > 0$, $(\bar{x}_\tau)_i \cup Y \in D$ but $(\bar{x}_{-\tau})_i \cup Y \notin D$, (c) for any sufficiently small $\tau > 0$, $(\bar{x}_\tau)_i \cup Y \notin D$ but $(\bar{x}_{-\tau})_i \cup Y \in D$, (d) for any sufficiently small $\tau \neq 0$, $(\bar{x}_\tau)_i \cup Y \notin D$. By (ii),

$$\exp[-h((\bar{x}_t)_i|Y)] = 0$$

for any sufficiently small $t < 0$ in the cases (b) and (d) and for any sufficiently small $t > 0$ in the cases (c) and (d). Hence, in the case (d) the derivative

$$\partial/\partial q^i \exp[-h(\bar{x}|Y)] = 0,$$

and in the cases (b) and (c) the same is true for the left and right derivatives respectively. This shows that the derivative $\partial/\partial q_t^i \exp[-h((\bar{x}_t)_i|Y)]$ exists: in the case (a) for all sufficiently small $t \neq 0$, in the case (b) for all sufficiently small $t > 0$ and in the case (c) for all sufficiently small $t < 0$.

Thus we have to prove in the cases (a), (b), and (c) that all derivatives mentioned above approach zero as $t \to 0$. For definiteness consider the case (a) and put $t > 0$.

Using formula (3.3), we can write

$$\frac{\partial}{\partial q_t^i}\exp[-h((\bar{x}_t)_i|Y)] = -\exp[-h((\bar{x}_t)_i|Y)] \sum_{\substack{\bar{z}\subset(\bar{x}_t)_i\cup Y,\, \bar{z}\cap Y\neq\emptyset,\\ \bar{z}\ni(x_t)_i}} \frac{\partial}{\partial q_t^i} f(\bar{z})$$

$$= -\sum_{\bar{z}\in D^+_{\bar{d}_2}((\bar{x}_t)_i;\, Y),\, \bar{z}\in(x_t)_i} \exp\left[-\sum_{\bar{z}'\in D^+_{\bar{d}_2}((\bar{x}_t)_i,\, Y)} f(\bar{z}') - \sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t)_i;\, Y)} f(\bar{z}')\right]\frac{\partial}{\partial q_t^i} f(\bar{z})$$

$$-\sum_{\bar{z}\in D_{\bar{d}_2}((\bar{x}_t)_i;\, Y),\, \bar{z}\ni(x_t)_i} \exp\left[-\sum_{\bar{z}'\in D^+_{\bar{d}_2}((\bar{x}_t)_i;\, Y)} f(\bar{z}') - \sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t)_i;\, Y)} f(\bar{z}')\right]\frac{\partial}{\partial q_t^i} f(\bar{z}). \tag{4.6}$$

We shall consider each of the two terms on the right-hand side of (4.6) separately. For the brevity the index i will be omitted. By (3.1), the first term is bounded by

$$c_3 \sum_{k\geq[d_2]} \Psi(k)k^{\nu(n_0-1)-1} \exp\left[c_3 \sum_{k\geq[d_2]} \Psi(k)k^{\nu(n_0-1)-1}\right] \exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y)} f(\bar{z}')\right]$$

$$= c_8 \exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y)} f(\bar{z}')\right],$$

where c_8 does not depend on t for sufficiently small $t>0$. Furthermore, the number of factors in the product $\exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y)} f(\bar{z}')\right]$ is bounded uniformly in t if $t>0$ is sufficiently small. Each factor $\exp[-f(\bar{z}')]$ is bounded by condition (G_2), and at least one of them vanishes as $t\to 0+$ by condition (G_4) (viz., the factor $\exp[-f(((q_t), v)\cup(q'', v''))]$ where $q''\in Y$ is such that $|q-q''|=d_0$). Hence, the first additive term on the right hand side of (4.6) vanishes as $t\to 0+$.

Let us pass to the second term on the right hand side of (4.6). We represent it in the form

$$\sum_{\substack{\bar{z}\in D_{\bar{d}_2}((\bar{x}_t);\, Y),\\ \bar{z}\;(x_t)}} \exp\left[-\sum_{\bar{z}'\in D^+_{\bar{d}_2}((\bar{x}_t);\, Y)} f(\bar{z}')\right]$$

$$\cdot\exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y),\, \bar{z}'\neq\bar{z}} f(\bar{z}')\right]\frac{\partial}{\partial q_t}\exp[-f(\bar{z})].$$

The number of additive terms in the external sum is uniformly bounded for sufficiently small $t>0$. Hence we have to consider the single term

$$\exp\left[-\sum_{\bar{z}'\in D^+_{\bar{d}_2}((\bar{x}_t);\, Y)} f(\bar{z}')\right]\exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y),\, \bar{z}'\neq\bar{z}\,.} f(\bar{z}')\right]$$

$$\cdot\frac{\partial}{\partial q_t}\exp[-f(\bar{z})],$$

which, by (3.1), is less than

$$c_9\left|\frac{\partial}{\partial q_t}\exp[-f(\bar{z})]\right|\exp\left[-\sum_{\bar{z}'\in D_{\bar{d}_2}((\bar{x}_t);\, Y),\, \bar{z}'\neq\bar{z}} f(\bar{z}')\right]$$

where c_9 does not depend on t for sufficiently small $t>0$. The number of factors $\exp[-f(\bar{z}')]$ is bounded uniformly in t if $t>0$ is sufficiently small. Any such factor is bounded by condition (G_2b); the same is true for the factor $|\partial/\partial q_t \exp[-f(\bar{z})]|$. Finally note that there is at least one among these factors which vanishes as $t\to 0+$ by the condition (G_4). This is either the factor

$$|\partial/\partial q_t \exp[-f(((q_t), v)\cup(q'', v''))]| \quad \text{if} \quad \bar{z}=((q_t), v)\cup(q'', v'')$$

or

$$\exp[-f(((q_t), v)\cup(q'', v''))] \quad \text{if} \quad \bar{z}\neq((q_t), v)\cup(q'', v'') .$$

Here $(q'', v'')\in Y$ is the particle mentioned above. Thus the second term on the right hand side of (4.6) vanishes as $t\to 0+$. Hence $\lim_{t\to 0+} \partial/\partial q_t \exp[-h(\bar{x}_t|Y)]=0$ and statement (iii) is proved.

Proof of Lemma 3.2. Let Ω be a bounded Borel set. Using the equality (2.19) we can write

$$R_\Omega(\bar{x})=\int_{M(\Omega^c)\cap D} \exp[-h(\bar{x}|Y)]dP(Y), \qquad \bar{x}\in M(\Omega). \tag{4.7}$$

Since $R_{\Omega_1}\leqq R_{\Omega_2}$ if $\Omega_1\supseteqq\Omega_2$, it is sufficient to prove that $R_\Omega(\bar{x})>0$ for $\Omega=\Omega_n$ where $n>\max_{q\in\bar{x}}|q|+d_2$. In this case the conditions $\bar{x}\in D^0$, $Y\in M(\Omega_n^c)\cap D$ imply that $\bar{x}\cup Y\in D$, and so, by (3.1),

$$|h(\bar{x}|Y)|\leqq \sum_{\bar{z}\in D_n^+(\bar{x};\,Y)} |f(\bar{z})|\leqq c_3 \sum_{k\geqq n} \Psi(k)k^{\nu(n_0-1)-1}<\infty .$$

From this it follows that the integrand in (4.7) is strictly positive. Finally note that according to the definition of the Gibbs random field [see (2.1)],

$$P(\emptyset/\mathfrak{B}(\Omega^c))(X)=\Xi_\Omega^{-1}(X_{\Omega^c})>0$$

almost everywhere (w.r.t. P) and therefore

$$P_\Omega(\emptyset)=P(M(\Omega^c))=P(M(\Omega^c)\cap D)>0 .$$

This completes the proof of Lemma 3.2.

Proof of Lemma 3.3 (i). The proof of formulae (3.6) and (3.7) is identical and we consider only the first of them. We use a theorem on the derivation of a integral depending on a parameter (see [34], Th. 114). Thus we have to prove that for any fixed $\bar{x}\in D^0$, $|\nabla \exp[-h(\bar{x}'|Y)]|$ is bounded uniformly in $\bar{x}'\in D^0$ sufficiently close to $\bar{x}$ and all $Y\in M$. It is sufficient to consider the case $\bar{x}'\cup Y\in D$ since in the opposite case $\nabla \exp[-h(\bar{x}'|Y)]=0$ [see (3.3)].

Fix $\bar{x}\in D^0$ and $x=(q,v)\in\bar{x}$. Fix further $\delta>0$ such that, if $x'=(q',v')\in M_1$ and $|q-q'|<\delta$ then $\bar{x}'\in D^0$ where $\bar{x}'=(\bar{x}\setminus x)\cup x'$. We shall show that for any such $\bar{x}'$ and any $Y\in M$

$$|\nabla' \exp[-h(\bar{x}'|Y)]|<c_{10} \tag{4.8}$$

where ∇' is $\nabla_{q'}$ or $\nabla_{v'}$ and c_{10} depends only on $n(\bar{x})$.

If $\bar{x}' \cup Y \in D$ then by (3.3)

$$\begin{aligned}&\nabla' \exp[-h(\bar{x}'|Y)]\\ &\leqq \exp[-h(\bar{x}'|Y)] \sum_{\bar{z}\in D^{+}_{\bar{d}_2}(\bar{x}';\,Y)} |\nabla' f(\bar{z})| + \exp[-h(\bar{x}'|\bar{y})] \sum_{\bar{z}\in D^{-}_{\bar{d}_2}(\bar{x}';\,Y)} |\nabla' f(\bar{z})|\,.\end{aligned}$$

By (3.1) and (3.2), the first term on the right hand side is bounded by

$$c_4 \exp\left[c_3 \sum_{k\geqq[d_2]} \Psi(k) k^{\nu(n_0-1)-1}\right].$$

To estimate the second term we represent it in the form

$$\sum_{\bar{z}\in D_{\bar{d}_2}(\bar{x}';\,Y)} \exp[-h(\bar{x}'|Y)] \exp[-f(\bar{z})] |\nabla' \exp[-f(\bar{z})]|\,.$$

The number of terms in this sum is uniformly bounded in $\bar{x}'$ and Y satisfying the conditions $\bar{x}' \in D^0$, $\bar{x}' \cup Y \in D$. By Remark 4.1 and condition (G_2b), the product of the first two factors in every additive term of the last sum is bounded by a constant depending only on $n(\bar{x})$. Other factors are bounded by the same condition. This gives the bound (4.8).

Using the formula (3.6) and the bound (4.8), we obtain, by Lebesgue's convergence theorem, the continuity of the gradients $\nabla R_\Omega(\bar{x})$. The same property of $p_\Omega(\bar{x})$ is proved similarly.

Remark 4.2. For any $\bar{x} \in D^0$, $Y \in M$ and $\bar{y} \in M^0$ the following estimate holds

$$\left|\nabla \exp\left[-\sum_{\bar{z}\subset \bar{x}\cup Y,\, \bar{z}\cap\bar{x}\neq\emptyset,\, \bar{z}\cap Y\neq\emptyset,\, \bar{z}\neq\bar{y}} f(\bar{z})\right]\right| \leqq c_{10}\,.$$

This estimate is proved by the same arguments as (4.8).

(ii) The smoothness of the correlation function ϱ_P and the formula (3.8) are proved by the same arguments as above. Note that it follows from (3.4) and (3.8) that

$$\nabla \varrho_P(\bar{x}) = 0\,, \tag{4.9}$$

if $\bar{x} \notin D^0$.

(iii) By (2.5),

$$\begin{aligned}&\varrho_P(\bar{x}\cup(q,v)) = \exp[-f(q,v)]\\ &\cdot \exp[-h(\bar{x}) - h(\bar{x}|(q,v))] \int_M \exp[-h(\bar{x}\cup(q,v)|Y)] dP(Y)\end{aligned}$$

As $|v|\to\infty$, the first factor tends to zero [see (G_2c)] and the other factors are bounded according to (G_2b) and (3.2). This gives equality (3.9). Lemma 3.3 is proved.

Proof of Lemma 3.4. We use again the theorem on the derivation of an integral depending on a parameter. We want to prove that $|\nabla \varrho_P(\bar{x}\cup\bar{y})|$ is bounded by a function depending only on $\bar{y}$ whose integral over $M(\Omega)$ [on $d\lambda(\bar{y})$] is finite. We shall show that if $\bar{x}\cup\bar{y} \in M(\Omega)$, then

$$|\nabla \varrho_P(\bar{x}\cup\bar{y})| \leqq c_{11} \exp\left[-\sum_{(q,v)\in\bar{y}} f(q,v)\right] \tag{4.10}$$

where c_{11} depends only on Ω. The estimate (4.10) implies the result since by the definition of the measure λ [see (1.1)]

$$\int_{M(\Omega)} \exp\left[-\sum_{(q,v)\in y} f(q,v)\right] d\lambda(\bar{y}) = \exp\left[\int_{\Omega}\int_{R^1} \exp[-f(q,v)]dvdq\right],^{17} \tag{4.11}$$

and the right hand side is bounded by condition (G_2c).

On account of (4.9), we shall restrict ourselves to the case $\bar{x}\cup\bar{y}\in M(\Omega)\cap D^0$. By (3.8), statement (iii) of Lemma 3.1 and bounds (3.2) and (4.8), we then obtain

$$\begin{aligned}
|\nabla \varrho_P(\bar{x}\cup\bar{y})| &= \int_M \nabla \exp[-h(\bar{x}\cup\bar{y}) - h(\bar{x}\cup\bar{y}|Y)]dP(Y)\\
&\leqq |\nabla \exp[-h(\bar{x}\cup\bar{y})]| \int_M \exp[-h(\bar{x}\cup\bar{y}|Y)]dP(Y)\\
&\quad + \exp[-h(\bar{x}\cup\bar{y})] \int_M |\nabla \exp[-h(\bar{x}\cup\bar{y}|Y)]|dP(Y)\\
&\leqq c_4|\nabla \exp[-h(\bar{x}\cup\bar{y})]| + c_{10}\exp[-h(\bar{x}\cup\bar{y})]\,.
\end{aligned} \tag{4.12}$$

Note that c_4 and c_{10} in (4.12) depend in general on $n(\bar{x}\cup\bar{y})$. But if $\bar{x}\cup\bar{y}\in M(\Omega)\cap D^0$ then $n(\bar{x}\cup\bar{y})$, of course, does not exceed some constant depending only on Ω. So one can think of c_4 and c_{10} in (4.12) depending only on Ω. Furthermore,

$$\begin{aligned}
&|\nabla \exp[-h(\bar{x}\cup\bar{y})]| = |\nabla \exp[-h(\bar{x}\setminus\bar{y}) - h(\bar{x}\setminus\bar{y}|\bar{y})]| \exp[-h(\bar{y})]\\
&\leqq (\exp[-h(\bar{x}\setminus\bar{y}|\bar{y})]|\nabla \exp[-h(\bar{x}\setminus\bar{y})]| + |\nabla \exp[-h(\bar{x}\setminus\bar{y}|\bar{y})]| \exp[-h(\bar{x}\setminus\bar{y})])\\
&\quad \cdot \exp\left[-\sum_{\bar{z}\subseteq\bar{y},\, n(\bar{z})>1} f(\bar{z})\right] \exp\left[-\sum_{(q,v)\in\bar{y}} f(q,v)\right],
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\exp[-h(\bar{x}\cup\bar{y})]\\
&= \exp[-h(\bar{x}\setminus\bar{y})] \exp[-h(\bar{x}\setminus\bar{y}|\bar{y})] \exp\left[-\sum_{\bar{z}\subseteq\bar{y},\, n(\bar{z})>1} f(\bar{z})\right] \exp\left[-\sum_{(q,v)\in\bar{y}} f(q,v)\right].
\end{aligned} \tag{4.14}$$

Using condition (G_2b) and the boundedness of $n(\bar{x}\cup\bar{y})$ as Ω is fixed, one can verify that the right hand sides of (4.13) and (4.14) do not exceed

$$c'_{11} \exp\left[-\sum_{(q,v)\in\bar{y}} f(q,v)\right]$$

where c'_{11} depends only on Ω. On account of (4.12) we obtain (4.10). Thus Lemma 3.4 is proved.

Proof of Lemma 3.5. Using (3.8) and Fubiny's theorem we conclude that the convergence of the integral (3.21) follows from that of the integral

$$\begin{aligned}
&\int_{M(\Omega)\cap D^0(\bar{x})} \int_M \int_{M_1\cap D^0(\bar{x}\cup\bar{y})} \sum_{(q,v)\in\bar{x}\cup z} |\nabla_v \exp[-h(\bar{x}\cup\bar{y}\cup z) - h(\bar{x}\cup\bar{y}\cup z|Y)]|\\
&\cdot |\nabla_q U(\bar{x}|z)| dz dP(Y) d\lambda(\bar{y})\,.
\end{aligned} \tag{4.15}$$

Using (2.13) one can integrate in (4.15) over the set $M^c_{\bar{x}\cup\bar{y}} = M\setminus M_{\bar{x}\cup\bar{y}}$ instead of M. The number of additive terms in the integrand equals $n(\bar{x})+1$. Hence we

[17] If $\bar{y}=\emptyset\in M_0$, then the integrand on the left hand side of this equality is set to be equal to zero.

can consider these terms separately. There are two possible cases: either $(q, v) \in \bar{x}$, or $(q, v) = z$. Since the arguments in the both cases are essentially the same, we restrict ourselves to the second one

For $\bar{y} \in D^0(\bar{x})$, $z \in M_1 \cap D^0(\bar{x} \cup \bar{y})$ and $Y \in M^c_{\bar{x} \cup \bar{y}}$ one can rewrite the integrand in the form

$$\exp[-h(\bar{x} \cup \bar{y}) - h(\bar{x} \cup \bar{y} | Y)] |\nabla_v \exp[-f(q, v) - h((q, v) | \bar{x} \cup \bar{y} \cup Y)]| \, |\nabla_q U(\bar{x} | (q, v))| .$$

The factor $\exp[-h(\bar{x} \cup \bar{y}) - h(\bar{x} \cup \bar{y} | Y)]$ in this product does not depend on z and may be taken outside the internal integral. By Remarks 4.1 and 4.2 the product of two remaining factors does not exceed

$$\begin{aligned} &\sum_{q' \in \bar{x}} |U'(|q-q'|)| \\ &\cdot \Bigg(|\nabla_v \exp[-f(q, v) - f((q, v) \cup (q', v'))]| \exp\Bigg[-\sum_{\bar{z} \subset \bar{x} \cup \bar{y} \cup Y, \bar{z} \neq (q', v')} f(\bar{z} \cup (q, v))\Bigg] \\ &\quad + \exp[-f(q, v) - f((q, v) \cup (q', v'))] \cdot \Bigg|\nabla_v \exp\Bigg[-\sum_{\bar{z} \subset \bar{x} \cup \bar{y} \cup Y, \bar{z} \neq (q', v')} f(\bar{z} \cup (q, v))\Bigg]\Bigg|\Bigg) \\ &\quad \leqq \sum_{q' \in \bar{x}} |U'(|q-q'|)| \\ &\cdot (c_4 |\nabla_v \exp[-f(q, v) - f((q, v) \cup (q', v'))]| + c_{10} \exp[-f(q, v) - f((q, v) \cup (q', v'))]). \end{aligned}$$

Condition (G_6) implies the convergence of the integral

$$\begin{aligned} &\int_{M_1 \cap D^0(\bar{x} \cup \bar{y})} |U'(|q-q'|)| \\ &\cdot (c_4 |\nabla_v \exp[-f(q, v) - f((q, v) \cup (q', v'))]| \\ &\quad + c_{10} \exp[-f(q, v) - f((q, v) \cup (q', v'))]) dq dv \end{aligned}$$

for all $q' \in \bar{x}$. Hence to prove the convergence of the integral (4.15) [and therefore (3.21)] it is sufficient to show that

$$\int_{M(\Omega) \cap D^0(\bar{x})} \int_M \exp[-h(\bar{x} \cup \bar{y}) - h(\bar{x} \cup \bar{y} | Y)] dP(Y) d\lambda(\bar{y}) < \infty . \tag{4.16}$$

According to (3.2) $\exp[-h(\bar{x} \cup \bar{y} | Y)] \leqq c_4$, where the constant c_4 may be considered as depending only on Ω. Now the function $\exp[-h(\bar{x} \cup \bar{y})]$, is, like in the proof of Lemma 3.4, less than $c_{12} \exp\Big[-\sum_{(q, v) \in \bar{y}} f(q, v)\Big]$ where c_{12} depends only on Ω. Substituting these bounds in to (4.16) we obtain, by (4.11), that the integral on the left hand side converges.

We now pass to the proof of the convergence of the integral (3.19). As above one can consider the integrals corresponding to each of two additive terms in the square brackets separately. Since both integrals may be studied similarly we shall prove the existence of one of them, namely

$$\begin{aligned} &\int_{M_1} \int_{M(\Omega)} \int_{M_1(\Omega) \cap D^0(q', v')} |\nabla_v \varrho_P(\bar{x} \cup \bar{y} \cup (q, v) \cup (q', v'))| \\ &\cdot |U'(|q-q'|)| dq dv d\lambda(\bar{y}) dq' dv' . \end{aligned} \tag{4.17}$$

Using (3.8) and Fubiny's theorem one can verify that the convergence of the integral (4.17) follows from that of

$$\int_{M(\Omega)} \int_{M} \int_{M_1} \int_{M_1(\Omega)\cap D^0(q',v')} |\nabla_v \exp[-h(\bar{x}\cup\bar{y}\cup(q,v)\cup(q',v'))$$
$$-h(\bar{x}\cup\bar{y}\cup(q,v)\cup(q',v')|Y)]|$$
$$\cdot |U'(|q-q'|)| dq dv dq' dv' dP(Y) d\lambda(\bar{y}) \tag{4.18}$$

By (2.12), (2.13), (2.19), and (3.4), we can replace the regions $M_1(\Omega)\cap D^0(q',v')$, M_1, M, and $M(\Omega)$ in (4.18) by $M_1(\Omega)\cap D^0(\bar{x}\cup\bar{y}\cup(q',v')\cup Y)$, $M_1\cap D^0(\bar{x}\cup\bar{y}\cup Y)$, $D(\bar{x}\cup\bar{y})$, and $M(\Omega)\cap D^0(\bar{x})$ respectively. Then the integrand may be written in the form

$$\exp[-h(\bar{x}\cup\bar{y})-h(\bar{x}\cup\bar{y}|Y)]$$
$$\cdot |\nabla_v \exp[-f(q,v)-f(q',v')-f((q,v)\cup(q',v'))-h((q,v)\cup(q',v')|\bar{x}\cup\bar{y}\cup Y)]|$$
$$\cdot |U'(|q-q'|)| .$$

The first factor does not depend on q, v, q', v' and may be taken outside the double internal integral in (4.18). By (3.2) and (4.8), the second factor does not exceed

$$\exp[-f(q',v')]$$
$$\cdot (|\nabla_v \exp[-f(q,v)-f((q,v)\cup(q',v'))]| \exp[-h((q,v)\cup(q',v')|\bar{x}\cup\bar{y}\cup Y)]$$
$$+\exp[-f(q,v)-f((q,v)\cup(q',v'))]\cdot|\nabla_v \exp[-h((q,v)\cup(q',v')|\bar{x}\cup\bar{y}\cup Y)]|)$$
$$\leqq \exp[-f(q',v')]$$
$$\cdot (c_4|\nabla_v \exp[-f(q,v)-f((q,v)\cup(q',v'))]|+c_{10}\exp[-f(q,v)-f((q,v)\cup(q',v'))]) .$$

Here c_4 and c_{10} are constants.

Let us consider now the integral

$$\int_{M_1} \int_{M_1(\Omega)\cap D^0((q',v'))} \exp[-f(q',v')]$$
$$\cdot (c_4|\nabla_v \exp[-f(q,v)-f((q,v)\cup(q',v'))]|+c_{10}\exp[-f(q,v)-f((q,v)\cup(q',v'))])$$
$$\cdot |U'(|q-q'|)| dq dv dq' dv' . \tag{4.19}$$

Condition (I_4) enables us to replace M_1 in (4.19) by $M_1(\Omega')$ where Ω' is a large bounded Borel set containing Ω. On account of this fact and the conditions (G_2c) and (G_6), it is not hard to see that the integral (4.19) converges. Then we apply the bound (4.16) and this completes the proof of the convergence of the integral (3.22).

Finally we prove the convergence of the integral (3.23). As above it is sufficient to show that the following integral exists:

$$\int_{M(\Omega)\cap D^0(\bar{x})} \int_{M} \int_{M_1(\Omega)\cap D^0(\bar{x}\cup\bar{y})} |\nabla_q \exp[-h(\bar{x}\cup\bar{y}\cup(q,v))-h(\bar{x}\cup\bar{y}\cup(q,v)|Y)]|$$
$$\cdot |v| dq dv dP(Y) d\lambda(\bar{y}) . \tag{4.20}$$

It follows from (3.2) and (4.8) that the integrand in (4.20) does not exceed

$$\begin{aligned}&\exp[-h(\bar{x}\cup\bar{y})-h(\bar{x}\cup\bar{y}|Y)]\,(|\nabla_q\exp[-f(q,v)]|\,\exp[-h((q,v)|\bar{x}\cup\bar{y}\cup Y)]\\&+\exp[-f(q,v)]\,|\nabla_q\exp[-h((q,v)|\bar{x}\cup\bar{y}\cup Y)]|)|v|\\&\leqq\exp[-h(\bar{x}\cup\bar{y})-h(\bar{x}\cup\bar{y}|Y)]\,(c_4|\nabla_q\exp[-f(q,v)]|+c_{10}\exp[-f(q,v)])\,|v|\,.\end{aligned}$$

By condition (G_2c)

$$\int_{M_1(\Omega)\cap D^0(\bar{x}\cup\bar{y})}(c_4|\nabla_q\exp[-f(q,v)]+c_{10}\exp[-f(q,v)])\,|v|dqdv<\infty\,.$$

On account of (4.16) we obtain from this that the integral (4.20) is finite. Lemma 3.5 is proved.

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